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# Leveraging Noisy Observations in Zero-Sum Games

Emmanouil-Marios Athanasakos and Samir M. Perlaza

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## Leveraging Noisy Observations in Zero-Sum Games

Emmanouil-Marios Athanasakos and Samir M. Perlaza

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**Abstract:** This report studies an instance of zero-sum games in which one player (the leader) commits to its opponent (the follower) to choose its actions by sampling a given probability measure (strategy). The actions of the leader are observed by the follower as the output of an arbitrary channel. In response to that, the follower chooses its action based on its current information, that is, the leader's commitment and the corresponding noisy observation of its action. Within this context, the equilibrium of the game with noisy action observability is shown to always exist and the necessary conditions for its uniqueness are identified. Interestingly, the noisy observations have important impact on the cardinality of the follower's set of best responses. Under particular conditions, such a set of best responses is proved to be a singleton almost surely. The proposed model captures any channel noise with a density with respect to the Lebesgue measure. As an example, the case in which the channel is described by a Gaussian probability measure is investigated.

**Key-words:** Zero Sum Games, Gaussian Channel, Nash Equilibrium, Lebesgue measure

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## Comment Exploiter les Observations Bruitées dans les Jeux à Somme Nulle

**Résumé :** Ce rapport étudie une instance de jeux à somme nulle dans laquelle un joueur (le leader) s'engage envers son adversaire (le suiveur) à choisir ses actions en échantillonnant une mesure de probabilité donnée (stratégie). Les actions du leader sont observées par le suiveur en tant que sortie d'un canal arbitraire. En réponse à cela, le suiveur choisit son action en fonction de ses informations actuelles, c'est-à-dire l'engagement du leader et l'observation bruitée correspondante de son action. Dans ce contexte, on montre que l'équilibre du jeu avec une observabilité bruitée des actions existe toujours, et les conditions nécessaires à son unicité sont identifiées. De manière intéressante, les observations bruitées ont un impact important sur la cardinalité de l'ensemble des meilleures réponses du suiveur. Dans des conditions particulières, un tel ensemble de meilleures réponses est prouvé être singleton presque sûrement. Le modèle proposé capture tout bruit de canal avec une densité par rapport à la mesure de Lebesgue. À titre d'exemple, le cas où le canal est décrit par une mesure de probabilité gaussienne est examiné.

**Mots-clés :** Jeux à somme nulle, Canal Gaussian, et équilibre de Nash, Mesure de Lebesgue

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## 1 Introduction

Zero-sum games (ZSGs) have proven to be a valuable tool across an array of disciplines, ranging from machine learning (ML) to economics. ZSGs can be utilized to proficiently model interactions between mutually adversarial decision makers (players) and have taken on a pivotal role in the advancement of techniques that enhance the robustness of machine learning algorithms against adversarial attacks. Game theoretic adversarial learning/training techniques are introduced as a competitive game between the learner and the adversary in [1] and [2]. Additionally, the fundamental concept of generative adversarial networks (GANs) draws inspiration from a two-player zero-sum game played between the discriminator and the generator [3–6].

The majority of the research at the intersection of game theory and adversarial ML utilizes the properties of ZSGs to forecast game results based on mixed strategies rather than individual actions. To put it simply, the core premise of the Stackelberg equilibrium (SE) [7] in mixed strategies assumes that the leader’s chosen strategy is completely observable by the follower, while actions remain unobservable. Conversely, in the context of the SE in pure strategies, the assumption is that the actions are perfectly observable, rendering the concept of committing to a strategy irrelevant [8–10]. Regardless, these assumptions are rarely fulfilled in the majority of ML applications and can be challenging to substantiate their realism. Indeed, data is acquired through a data acquisition system, which is prone to quantization errors, additive noise, and distortions arising from data transmission and storage. These supplementary imperfections are not necessarily attributable solely to adversarial intent but rather stem from the inherent characteristics of transmission channels. This work makes a step towards this direction and studies new equilibrium concepts. This work extends the game theoretic framework that was introduced in [11] by considering arbitrary channels. Driven by the growing relevance of game theory in the progress ML [12–14], particularly in the context of GANs and adversarial learning, this work presents the subsequent contributions:

- Under perfect observation of the commitment (strategy of the leader), a new game formulation is introduced in which the follower observes a noisy version of actions played by the leader. In order to capture the impact of imperfect data acquisition, leader’s actions are the inputs of an arbitrary channel, while the observations of the follower are the outputs of the channel.
- The formulation is not restricted to specific types of noise distributions; rather, it is general enough to facilitate any noise distribution that can be described by a probability density function (pdf).
- The set of best responses of the follower is characterized and found that is a singleton almost surely. Moreover, the cases in which the noise is beneficial or irrelevant for the follower are investigated and the optimal commitments of the leader are characterized.
- The equilibrium is shown to always exist and the key conditions that affect its uniqueness are identified.
- An example for the case in which the channel is an additive white Gaussian noise channel (AWGN) is investigated and the equilibrium solution is characterized.

## 2 Preliminaries and Notation

### 2.1 Game Formulation

Consider a two-player ZSG in normal form in which Player 1 and Player 2 choose their actions from the sets  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , respectively. These sets are countable, and for all  $k \in \{1, 2\}$ ,  $\mathcal{A}_k \triangleq \{a_{k,1}, a_{k,2}, \dots, a_{k,m_k}\}$  exhibits cardinality  $m_k > 1$ . Let the function

$$u : \mathcal{A}_1 \times \mathcal{A}_2 \rightarrow \mathbb{R}, \quad (1)$$

be such that if Player 1 plays  $a_{1,i}$  and Player 2 plays  $a_{2,j}$ , the payoff achieved by both players is  $u(a_{1,i}, a_{2,j})$ . Note that the function  $u$  in (1) is a simple function as its image is formed by the elements in the set  $\{u(a_{1,i}, a_{2,j}) : (i, j) \in \{1, 2, \dots, m_1\} \times \{1, 2, \dots, m_2\}\}$ . For the ease of notation, for all  $(i, j) \in \{1, 2, \dots, m_1\} \times \{1, 2, \dots, m_2\}$  consider that  $u_{i,j} \triangleq u(a_{1,i}, a_{2,j})$ . Player 1 and Player 2 choose their actions to maximize and minimize their payoffs, respectively.

When the game is played with commitments and one of the players observes the action played by its counterpart subject to noise, the notion of memoryless channels in the sense of [15, 16] is utilized. Let  $\Delta(\tilde{\mathcal{A}}_2|\mathcal{A}_2)$  be the set of all probability measures on  $\tilde{\mathcal{A}}_2$  conditioned on the elements of  $\mathcal{A}_2$ . The channel is represented by  $P_{\tilde{\mathcal{A}}_2|\mathcal{A}_2} \in \Delta(\tilde{\mathcal{A}}_2|\mathcal{A}_2)$ , for some given set  $\tilde{\mathcal{A}}_2$ . More specifically, given a channel input  $b \in \mathcal{A}_2$ , the channel output is  $\tilde{b} \in \tilde{\mathcal{A}}_2$  with probability  $P_{\tilde{\mathcal{A}}_2|\mathcal{A}_2=b}(\tilde{b})$ . A strategy for Player 1, denoted by  $P_{\mathcal{A}_1|\tilde{\mathcal{A}}_2} \in \Delta(\mathcal{A}_1|\tilde{\mathcal{A}}_2)$ , is a probability measure on  $\mathcal{A}_1$  conditioned on the elements of  $\tilde{\mathcal{A}}_2$ . Hence, when the channel output is  $\tilde{b}$ , Player 1 chooses its actions by sampling the probability measure  $P_{\mathcal{A}_1|\tilde{\mathcal{A}}_2=\tilde{b}} \in \Delta(\mathcal{A}_1)$ , where  $\Delta(\mathcal{A}_1)$  denotes the set of all probability measures on the set  $\mathcal{A}_1$ . Alternatively, a strategy for Player 2 is a probability measure denoted by  $P_{\mathcal{A}_2} \in \Delta(\mathcal{A}_2)$ . That is, Player 2 chooses its actions by sampling  $P_{\mathcal{A}_2}$ .

The extended ZSG that includes the noisy channel  $P_{\tilde{\mathcal{A}}_2|\mathcal{A}_2}$  is represented by the tuple:

$$\mathcal{G}(u, P_{\tilde{\mathcal{A}}_2|\mathcal{A}_2}) \triangleq (\mathcal{A}_1, \mathcal{A}_2, u, P_{\tilde{\mathcal{A}}_2|\mathcal{A}_2}). \quad (2)$$

The game  $\mathcal{G}(u, P_{\tilde{\mathcal{A}}_2|\mathcal{A}_2})$  is played in three stages. In the first stage, Player 2 announces its strategy  $P_{\mathcal{A}_2} \in \Delta(\mathcal{A}_2)$  to Player 1 and commits to choose its actions by using such a strategy. In the second stage, Player 2 plays action  $b \in \mathcal{A}_2$  with probability  $P_{\mathcal{A}_2}(b)$  and Player 1 observes the channel output  $\tilde{b} \in \tilde{\mathcal{A}}_2$  with probability  $P_{\tilde{\mathcal{A}}_2|\mathcal{A}_2=b}(\tilde{b})$ . In the final stage, Player 1 plays action  $a \in \mathcal{A}_1$ , with probability  $P_{\mathcal{A}_1|\tilde{\mathcal{A}}_2=\tilde{b}}(a)$  and both players obtain equal payoffs  $u(a, b)$ .

Therefore, the expected payoff obtained by the players is determined by the function  $v : \Delta(\mathcal{A}_1|\tilde{\mathcal{A}}_2) \times \Delta(\mathcal{A}_2) \rightarrow \mathbb{R}$ , such that given the strategy  $P_{\mathcal{A}_1|\tilde{\mathcal{A}}_2}$  of Player 1 and the strategy  $P_{\mathcal{A}_2}$  of Player 2, the expected payoff is

$$v(P_{\mathcal{A}_1|\tilde{\mathcal{A}}_2}, P_{\mathcal{A}_2}) = \int_{\mathcal{A}_2} \left( \int_{\tilde{\mathcal{A}}_2} \left( \int_{\mathcal{A}_1} u(a, b) dP_{\mathcal{A}_1|\tilde{\mathcal{A}}_2=\tilde{b}}(a) \right) dP_{\tilde{\mathcal{A}}_2|\mathcal{A}_2=b}(\tilde{b}) \right) dP_{\mathcal{A}_2}(b). \quad (3)$$

While the sets  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are assumed to be countable, the set  $\tilde{\mathcal{A}}_2$  is assumed to be a subset of  $\mathbb{R}$ . For all  $j \in \{1, \dots, m_2\}$ , the channel  $P_{\tilde{\mathcal{A}}_2|\mathcal{A}_2}$  satisfies

$$P_{\tilde{\mathcal{A}}_2|\mathcal{A}_2=a_{2,j}} \lll \lambda, \quad (4)$$

where  $\lll$  denotes mutual absolute continuity; and  $\lambda$  denotes the Lebesgue measure on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ .

Under these assumptions, the expected payoff function  $v$  in (3) satisfies

$$v(P_{A_1|\tilde{A}_2}, P_{A_2}) = \int_{\tilde{A}_2} \left( \sum_{i=1}^{m_1} P_{A_1|\tilde{A}_2=\tilde{b}}(a_{1,i}) \left( \sum_{j=1}^{m_2} u_{i,j} P_{A_2}(a_{2,j}) \frac{dP_{\tilde{A}_2|A_2=a_{2,j}}}{dP_{\tilde{A}_2|A_2=a_{2,1}}}(\tilde{b}) \right) \right) dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b}), \quad (5)$$

where for all  $j \in \{1, \dots, m_2\}$ , the function  $\frac{dP_{\tilde{A}_2|A_2=a_{2,j}}}{dP_{\tilde{A}_2|A_2=a_{2,1}}} : \tilde{A}_2 \rightarrow (0, +\infty)$  is the Radon-Nikodym derivative of  $P_{\tilde{A}_2|A_2=a_{2,j}}$  with respect to  $P_{\tilde{A}_2|A_2=a_{2,1}}$ .

## 2.2 Equilibrium

The necessary mathematical objects for characterizing the solution concept of the extended game  $\mathcal{G}(u, P_{\tilde{A}_2|A_2})$  in (2) are defined in this section.

**Definition 2.1** (Best Responses of Player 1) Given a commitment  $P \in \Delta(\mathcal{A}_2)$  and a channel output  $\tilde{b} \in \tilde{A}_2$ , the set of best responses of Player 1, is determined by the correspondence  $\text{BR}_1 : \Delta(\mathcal{A}_2) \times \tilde{A}_2 \rightarrow \mathcal{P}(\Delta(\mathcal{A}_1))$ , where  $\mathcal{P}(\Delta(\mathcal{A}_1))$  denotes the power set of  $\mathcal{A}_1$ , such that

$$\text{BR}_1(P, \tilde{b}) = \arg \max_{Q \in \Delta(\mathcal{A}_1)} \sum_{i=1}^{m_1} Q(a_{1,i}) \left( \sum_{j=1}^{m_2} u_{i,j} P(a_{2,j}) \frac{dP_{\tilde{A}_2|A_2=a_{2,j}}}{dP_{\tilde{A}_2|A_2=a_{2,1}}}(\tilde{b}) \right). \quad (6)$$

A strategy  $P_{A_1|\tilde{A}_2} \in \Delta(\mathcal{A}_1|\tilde{A}_2)$  of Player 1 is a best response to the commitment  $P \in \Delta(\mathcal{A}_2)$ , if for all  $\tilde{b} \in \tilde{A}_2$  the probability measure  $P_{A_1|\tilde{A}_2=\tilde{b}} \in \text{BR}_1(P, \tilde{b})$ .

Let the function  $\hat{v} : \Delta(\mathcal{A}_2) \rightarrow \mathbb{R}$  be such that for a given commitment  $P \in \Delta(\mathcal{A}_2)$ ,

$$\begin{aligned} \hat{v}(P) &= \max_{Q_{A_1|\tilde{A}_2} \in \Delta(\mathcal{A}_1|\tilde{A}_2)} v(Q_{A_1|\tilde{A}_2}, P), \\ \text{s.t. } \forall \tilde{b} \in \tilde{A}_2 \quad & Q_{A_1|\tilde{A}_2=\tilde{b}} \in \text{BR}_1(P, \tilde{b}), \end{aligned} \quad (7)$$

where the function  $v$  is defined in (5) and the correspondence  $\text{BR}_1$  in (6).

Accordingly, the set of optimal commitments is the set of minimizers of the function  $\hat{v}$  given in (7). This observation leads to the following notion of equilibrium.

**Definition 2.2** (Equilibrium) The strategies  $P_{A_1|\tilde{A}_2}^\dagger \in \Delta(\mathcal{A}_1|\tilde{A}_2)$  and  $P_{A_2}^\dagger \in \Delta(\mathcal{A}_2)$  for Player 1 and Player 2, respectively, are said to form an equilibrium of the game  $\mathcal{G}(u, P_{\tilde{A}_2|A_2})$  in (2) if

$$P_{A_2}^\dagger \in \arg \min_{P \in \Delta(\mathcal{A}_2)} \hat{v}(P) \quad (8)$$

and for all  $\tilde{b} \in \tilde{A}_2$

$$P_{A_1|\tilde{A}_2=\tilde{b}}^\dagger \in \text{BR}_1(P_{A_2}^\dagger, \tilde{b}), \quad (9)$$

where the function  $\hat{v}$  is defined in (7) and the correspondence  $\text{BR}_1$  is defined in (6).

### 2.3 Existing Special Cases of the Game $\mathcal{G}(u, P_{\tilde{\mathcal{A}}_2|A_2})$

Consider that the channel  $P_{\tilde{\mathcal{A}}_2|A_2}$  is such that for all  $\tilde{b} \in \tilde{\mathcal{A}}_2$ ,  $P_{\tilde{\mathcal{A}}_2|A_2=a_{2,j}}(\tilde{b}) = P_{\tilde{\mathcal{A}}_2|A_2=a_{2,r}}(\tilde{b})$ , with  $(j, r) = \{1, \dots, m_2\}^2$  and  $j \neq r$ . This is the case in which the channel input and output are independent. In such case, the game  $\mathcal{G}(u, P_{\tilde{\mathcal{A}}_2|A_2})$  simplifies to the game denoted by

$$\mathcal{G}(u) \triangleq (\mathcal{A}_1, \mathcal{A}_2, u). \quad (10)$$

When players simultaneously choose their actions in the absence of commitment the game in (10) has been studied in [17]. The solution concept is the Nash Equilibrium (NE), as introduced in [18].

When there exist a deterministic one-to-one mapping between the inputs and the outputs of the channel, the game  $\mathcal{G}(u, P_{\tilde{\mathcal{A}}_2|A_2})$  is reduced to the game studied in [19] and [20], [21]. This is the case in which for all  $a_{2,j} \in \mathcal{A}_2$ , with  $j = \{1, \dots, m_2\}$ , it holds that  $P_{\tilde{\mathcal{A}}_2|A_2=a_{2,j}}(\tilde{b}) = 1$ . That is, regardless the commitment, Player 1 can always select an optimal action in response to Player 2's action.

When the commitment is the only information that Player 1 has regarding the action played by Player 2, *i.e.*, the channel input and output are independent, the game  $\mathcal{G}(u, P_{\tilde{\mathcal{A}}_2|A_2})$  is equivalent to the one investigated in [22] and [23], and the solution concept is the SE in mixed strategies.

Other special cases of the game  $\mathcal{G}(u, P_{\tilde{\mathcal{A}}_2|A_2})$  are studied in [24] and [11]. In the former, the strategies of the players are restricted to pure strategies and the sets  $\mathcal{A}_2$  and  $\tilde{\mathcal{A}}_2$  are finite and assumed to be identical. In the latter, the analysis is generalized to include the consideration of mixed strategies.

## 3 Main Results

For the rest of this work, it is assumed that for all  $(i, l) \in \{1, \dots, m_1\}^2$ ,

$$\lambda \left( \left\{ \tilde{b} \in \tilde{\mathcal{A}}_2 : \sum_{j=1}^{m_2} (u_{i,j} - u_{l,j}) P(a_{2,j}) f_{\tilde{\mathcal{A}}_2|A_2=a_{2,j}}(\tilde{b}) = 0 \right\} \right) = 0, \quad (11)$$

where  $f_{\tilde{\mathcal{A}}_2|A_2=a_{2,j}}$  is the corresponding pdf of the measure  $P_{\tilde{\mathcal{A}}_2|A_2=a_{2,j}}$ .

The assumption in (11) is always satisfied when for all  $(i, l) \in \{1, \dots, m_1\}^2$ , it holds that  $\prod_{j=1}^{m_2} (u_{i,j} - u_{l,j}) P(a_{2,j}) > 0$ . In [11], it has been shown that imposing such condition restricts the game  $\mathcal{G}(u)$  to exhibit a unique NE in mixed strategies.

The following lemma describes the set of best responses of Player 1.

**Lemma 3.1** *Given a commitment  $P \in \Delta(\mathcal{A}_2)$  and a channel output  $\tilde{b} \in \tilde{\mathcal{A}}_2$ , the set of best responses of Player 1 is*

$$\text{BR}_1(P, \tilde{b}) \triangleq \{Q \in \Delta(\mathcal{A}_1) : Q(a) > 0 \text{ if } a \in \mathcal{S}(P, \tilde{b})\}, \quad (12)$$

where

$$\mathcal{S}(P, \tilde{b}) \triangleq \arg \max_{a \in \mathcal{A}_1} \left\{ \sum_{j=1}^{m_2} u(a, a_{2,j}) P(a_{2,j}) \frac{dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,j}}}{dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,1}}}(\tilde{b}) \right\}. \quad (13)$$

*Proof:* The proof of is presented in Appendix A.  $\blacksquare$

The preceding lemma indicates the fact that given a commitment  $P \in \Delta(\mathcal{A}_2)$  and a channel output  $\tilde{b} \in \tilde{\mathcal{A}}_2$ , Player 1's optimal strategy is to concentrate the probability measure  $P_{A_1|\tilde{A}_2=\tilde{b}}$  over the set  $\mathcal{S}(P, \tilde{b}) \subseteq \mathcal{A}_1$ , with  $\mathcal{S}(P, \tilde{b})$  in (13). If  $|\mathcal{S}(P, \tilde{b})| = 1$ , there is a unique best response  $P_{A_1|\tilde{A}_2=\tilde{b}}^* \in \text{BR}_1(P, \tilde{b})$  that assigns probability one to the singleton  $\mathcal{S}(P, \tilde{b})$ . The set of best responses of Player 1 might also contain infinitely many strategies. In the specific case in which  $\text{BR}_1(P, \tilde{b}) = \Delta(\mathcal{A}_1)$ , then as observed in [11], Player 1 opts for indifferently choosing its actions. Interestingly, under assumptions (4) and (11), for each channel output, Player 1 deterministically chooses a specific action almost surely, which will be discussed in the sequel.

For all  $i \in \{1, \dots, m_1\}$  and for all  $P \in \Delta(\mathcal{A}_2)$ , let the set  $\mathcal{H}_i(P)$  be

$$\mathcal{H}_i(P) \triangleq \left\{ \tilde{b} \in \tilde{\mathcal{A}}_2 : i \in \arg \max \left\{ \sum_{j=1}^{m_2} u_{i,j} P(a_{2,j}) \frac{dP_{\tilde{A}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b})} \right\} \right\} \setminus \mathcal{H}_{i-1}(P), \quad (14)$$

with  $\mathcal{H}_0(P) \triangleq \emptyset$ . Let also for all  $(i, l) \in \{1, \dots, m_1\}^2$ , with  $i \neq l$ , the set  $\mathcal{H}_{i,l}$  be

$$\mathcal{H}_{i,l}(P) \triangleq \left\{ \tilde{b} \in \tilde{\mathcal{A}}_2 : \sum_{j=1}^{m_2} u_{i,j} P(a_{2,j}) \frac{dP_{\tilde{A}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,k}}(\tilde{b})} = \sum_{j=1}^{m_2} u_{l,j} P(a_{2,j}) \frac{dP_{\tilde{A}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b})} \right\}. \quad (15)$$

Note that  $\mathcal{H}_1(P), \mathcal{H}_2(P), \dots, \mathcal{H}_{m_1}(P)$  form a partition of  $\tilde{\mathcal{A}}_2$ . The following theorem presents a property exhibited by the set  $\mathcal{H}_{i,l}(P)$  in (15).

**Theorem 3.1** *Given a commitment  $P \in \Delta(\mathcal{A}_2)$ , under assumptions (4) and (11), if for all  $(i, l) \in \{1, \dots, m_1\}^2$ , with  $i \neq l$ , and for all  $a_{2,j} \in \mathcal{A}_2$*

$$(u_{i,j} - u_{l,j}) P(a_{2,j}) \neq 0, \quad (16)$$

*then it holds that*

$$P_{\tilde{\mathcal{A}}_2}(\mathcal{H}_{i,l}(P)) = 0, \quad (17)$$

*where  $\mathcal{H}_{i,l}(P)$  is defined in (15); and the probability measure  $P_{\tilde{\mathcal{A}}_2}$  is such that for all measurable subsets  $\mathcal{A}$  of  $\tilde{\mathcal{A}}_2$ , it holds that*

$$P_{\tilde{\mathcal{A}}_2}(\mathcal{A}) = \int P_{\tilde{A}_2|A_2=a}(\mathcal{A}) dP(a). \quad (18)$$

*Proof:* The proof of is presented in Appendix B.  $\blacksquare$

A first observation from Theorem 3.1, is that if Player 2's commitment is in strict mixed strategies, then the elements of the set  $\mathcal{H}_{i,l}(P)$  are observed with probability zero with respect to  $P_{\tilde{\mathcal{A}}_2}$  in (18). Note that an element  $\tilde{b}$  of the set  $\mathcal{H}_{i,l}(P)$  is the output of the channel for which actions  $a_{1,i}$  and  $a_{1,l}$  might be both best responses for Player 1 to the commitment  $P$  and the channel output  $\tilde{b}$ . Hence, under the assumptions of Theorem 3.1, the set  $\mathcal{S}(P, \tilde{b})$  is always a singleton and thus  $|\text{BR}_1(P, \tilde{b})| = 1$ . On the other hand, if (16) does not hold, then the set of best responses of Player 1 includes infinitely many strategies. This indicates that Player 1 indifferently chooses its actions from the set  $\mathcal{S}(P, \tilde{b})$ . This is formalized by the following lemma.

**Lemma 3.2** *Under assumptions (4) and (11), given a commitment  $P \in \Delta(\mathcal{A}_2)$  that satisfies (16), for all  $\tilde{b} \in \tilde{\mathcal{A}}_2$  and for all  $P_{A_1|\tilde{A}_2} \in \text{BR}_1(P, \tilde{b})$ , with  $\text{BR}_1(P, \tilde{b})$  in (6), it holds that for all  $i \in \{1, \dots, m_1\}$*

$$P_{A_1|\tilde{A}_2=\tilde{b}}(a_{1,i}) = \begin{cases} 1, & \text{if } \tilde{b} \in \mathcal{H}_i(P) \\ 0, & \text{otherwise,} \end{cases} \quad (19)$$

where  $\mathcal{H}_i(P)$  is defined in (14) and  $a_{1,i} \in \mathcal{A}_1$ . Moreover, the set of best responses satisfies

$$P_{\tilde{\mathcal{A}}_2} \left( \left\{ \tilde{b} \in \tilde{\mathcal{A}}_2 : \left| \text{BR}_1 \left( P, \tilde{b} \right) \right| = 1 \right\} \right) = 1, \quad (20)$$

*Proof:* The proof of is presented in Appendix C. ■

The following theorem characterizes the equilibrium of the extended game  $\mathcal{G} \left( u, P_{\tilde{\mathcal{A}}_2|A_2} \right)$  in (2).

**Theorem 3.2** *Under assumptions (4) and (11), and if (16) holds for all  $j \in \{1, \dots, m_2\}$  and for all  $(i, l) \in \{1, \dots, m_1\}^2$ , with  $i \neq l$ , then the game  $\mathcal{G} \left( u, P_{\tilde{\mathcal{A}}_2|A_2} \right)$  possesses a unique equilibrium almost surely with respect to  $P_{\tilde{\mathcal{A}}_2}$ .*

*Proof:* The proof is presented in Appendix D. ■

The game  $\mathcal{G} (u)$  in (10) holds significant importance due to the fact that its the expected payoff can be used as reference point when analyzing the game in (2). Let the function  $\omega : \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_2) \rightarrow \mathbb{R}$  represent the expected payoff of game  $\mathcal{G} (u)$  such that

$$\omega(P_{A_1}, P_{A_2}) = \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} P_{A_1}(a_{1,i}) u_{i,j} P_{A_2}(a_{2,j}). \quad (21)$$

Let the function  $\hat{u} : \Delta(\mathcal{A}_2) \rightarrow \mathbb{R}$  be such that for all commitments  $P \in \Delta(\mathcal{A}_2)$ ,

$$\hat{u}(P) = \max_{Q \in \Delta(\mathcal{A}_1)} \omega(Q, P), \quad (22)$$

where the function  $\omega$  is defined in (21). The term  $\hat{u}(P)$  in (22) represents the expected payoff of the game  $\mathcal{G} (u)$  in (10), when Player 2 commits to play strategy  $P$ . Note that, the minimum value over all commitments  $P$  of the function  $\hat{u}$  is the payoff at the NE. The following lemma compares the payoffs at the equilibria of the games  $\mathcal{G} (u)$  in (10) and  $\mathcal{G} \left( u, P_{\tilde{\mathcal{A}}_2|A_2} \right)$  in (2).

**Lemma 3.3** *Let the probability measures  $\left( P_{A_1|\tilde{\mathcal{A}}_2}^\dagger, P_{A_2}^\dagger \right)$  form an equilibrium of the game  $\mathcal{G} \left( u, P_{\tilde{\mathcal{A}}_2|A_2} \right)$  in (2) and let the pair of strategies  $\left( P_{A_1}^*, P_{A_2}^* \right) \in \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_2)$  be an NE of the game  $\mathcal{G} (u)$  in (10). Then for all commitments  $P \in \Delta(\mathcal{A}_2)$ , it holds that*

$$\hat{u}(P_{A_2}^*) \leq v \left( P_{A_1|\tilde{\mathcal{A}}_2}^\dagger, P_{A_2}^\dagger \right) \leq \hat{v}(P) \leq \min_j \max_i u_{i,j}, \quad (23)$$

where the functions  $\hat{u}$ ,  $\hat{v}$  and  $v$  are given in (22), (7) and (5), respectively.

*Proof:* The proof is presented in Appendix E. ■

Lemma 3.3 unveils the fact that the payoff of the game  $\mathcal{G} \left( u, P_{\tilde{\mathcal{A}}_2|A_2} \right)$  at the equilibrium is lower bounded by the payoff of the game  $\mathcal{G} (u)$  at the NE and upper bounded by the payoff of such a game at the SE in pure strategies. The former is the case in which Player 1 does not observe the actions of Player 2 (e.g. very noisy channel,  $I(P_{\tilde{\mathcal{A}}_2|A_2}; P_{A_2}) = 0$ ); and the latter corresponds to the case in which Player 1 perfectly observes the actions of its opponent (e.g. channel with zero noise,  $I(P_{\tilde{\mathcal{A}}_2|A_2}; P_{A_2}) = H(P_{A_2})$ ).

## 4 Example: Gaussian Channel

Thus far, the probability measure that describes the channel  $P_{\tilde{A}_2|A_2}$  of the game  $\mathcal{G}(u, P_{\tilde{A}_2|A_2})$  in (2) is not yet specified. This generality signifies a key feature of the theoretical model, which captures several data processing impairments. Moreover, while the specific game solution concept relies on the characteristics of the channel, the generality of the game formulation presented in Section 2 highlights its capability to effectively address any type of noise which has a density with respect to the Lebesgue measure.

In this section, the channel is assumed to be an AWGN channel and the action sets of Player 1 and Player 2 have the same cardinality and contain two actions. In this scenario, the channel is such that for all  $a_{2,j} \in \mathcal{A}_2$  and all measurable subsets  $\mathcal{A}$  of  $\tilde{\mathcal{A}}_2$

$$P_{\tilde{A}_2|A_2=a_{2,j}}(\mathcal{A}) = \frac{1}{\sqrt{2\pi\sigma_j^2}} \int_{\mathcal{A}} \exp\left(-\frac{1}{2\sigma_j^2}(\tilde{b} - \mu_j - a_{2,j})^2\right) d\lambda(\tilde{b}), \quad (24)$$

where  $\lambda$  is the Lebesgue measure in  $\mathbb{R}$ ;  $\mu_j - a_{2,j}$  and  $\sigma_j^2$  are the mean and the variance, respectively. From (24), it holds that the Radon-Nikodym derivative of  $P_{\tilde{A}_2|A_2=a_{2,j}}$  with respect to  $P_{\tilde{A}_2|A_2=a_{2,1}}$ , and for all  $\tilde{b} \in \tilde{\mathcal{A}}_2$  is

$$\frac{dP_{\tilde{A}_2|A_2=a_{2,2}}}{dP_{\tilde{A}_2|A_2=a_{2,1}}}(\tilde{b}) = \exp\left(-\left(\frac{(a_{2,2}^2 - a_{2,1}^2)}{2\sigma^2} + \frac{\tilde{b}(a_{2,1} - a_{2,2})}{\sigma^2}\right)\right), \quad (25)$$

where it is assumed that  $\mu_j = 0$  and  $\sigma_j = \sigma > 0$ , with  $j \in \{1, 2\}$ . From (15), for all  $P_{A_2} \in \Delta(\mathcal{A}_2)$  the set  $\mathcal{H}_{i,l}(P_{A_2})$ , with  $(i, l) \in \{1, 2\}^2$ , satisfies

$$\begin{aligned} \mathcal{H}_{1,2}(P_{A_2}) &\triangleq \left\{ \tilde{b} \in \tilde{\mathcal{A}}_2 : \frac{dP_{\tilde{A}_2|A_2=a_{2,2}}}{dP_{\tilde{A}_2|A_2=a_{2,1}}}(\tilde{b}) = \frac{(u_{1,1} - u_{2,1})P_{A_2}(a_{2,1})}{(u_{2,2} - u_{1,2})P_{A_2}(a_{2,2})} \right\} \\ &= \left\{ \frac{\sigma^2}{a_{2,1} - a_{2,2}} \left( \log \left( \frac{(u_{1,1} - u_{2,1})P_{A_2}(a_{2,1})}{(u_{2,2} - u_{1,2})P_{A_2}(a_{2,2})} \right) \right. \right. \\ &\quad \left. \left. + D\left(P_{\tilde{A}_2|A_2=a_{2,2}} \| P_{\tilde{A}_2|A_2=a_{2,1}}\right) \right) \right\}, \end{aligned} \quad (26)$$

where  $D$  denotes the Kullback-Leibler divergence. Note that the set  $\mathcal{H}_{1,2}(P_{A_2})$  is a singleton. This said, it holds that  $\lambda(\mathcal{H}_{1,2}(P_{A_2})) = 0$  and thus,  $P_{\tilde{A}_2}(\mathcal{H}_{1,2}(P_{A_2})) = 0$ , which demonstrates the validity of Theorem 3.1.

Let the complementary cumulative distribution function (ccdf) of the Gaussian pdf be

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} \exp\left(-\frac{z^2}{2}\right) dz. \quad (28)$$

Using this notation, the following lemma provides an explicit expression for the function  $\hat{v}$  in (7)

**Lemma 4.1** *Given a commitment  $P_{A_2} \in \Delta(\mathcal{A}_2)$ , if for all  $a_{2,j} \in \mathcal{A}_2$ , the probability measure  $P_{\tilde{A}_2|A_2=a_{2,j}}$  satisfies (24), then it holds*

$$\hat{v}(P_{A_2}) = \max_{Q_{A_1|\tilde{A}_2} \in \text{BR}_1(P_{A_2}, \tilde{b})} v(Q_{A_1|\tilde{A}_2}, P_{A_2}) = \sum_{j=1}^{m_2} P_{A_2}(a_{2,j}) \left( u_{2,j} + (u_{1,j} - u_{2,j}) Q\left(\frac{b_{P_{A_2}}^* - a_{2,j}}{\sigma}\right) \right), \quad (29)$$

where function  $v$  is defined in (5);  $b_{P_{A_2}}^* \in \mathcal{H}_{1,2}(P_{A_2})$ ; and function  $Q$  is in (28).

*Proof:* The proof is presented in Appendix F. ■

In Fig. 1, the strategies  $(P_{A_1}^*, P_{A_2}^*)$  with  $P_{A_1}^*(a_{1,1}) = 1 - P_{A_1}^*(a_{1,2}) = 0.23$  and  $P_{A_2}^*(a_{2,1}) = 1 - P_{A_2}^*(a_{2,2}) = 0.45$  form the unique NE of the game  $\mathcal{G}(u)$  whose payoff is  $\hat{u}(P_{A_2}^*)$  (red star). Examining the behavior of the function  $\hat{v}$  in (7) for a range of noise variance values  $\sigma^2$ , it is observed that; (i) the equilibrium of the game  $\mathcal{G}(u, P_{\bar{A}_2|A_2})$  is always lower bounded by the equilibrium of  $\hat{u}$  (dashed blue line in the zoomed subfigure) as Lemma 3.3 revealed; (ii) the function  $\hat{v}$  is monotonically decreasing with respect to  $\sigma^2$ ; (iii) letting  $\sigma^2$  grow to infinity, results in the equivalence of the equilibrium of  $\mathcal{G}(u, P_{\bar{A}_2|A_2})$  and the NE of game  $\mathcal{G}(u)$  (green curve approaching the red star); and (iv) letting  $\sigma^2$  to be close to zero, the equilibrium payoff of the game  $\mathcal{G}(u, P_{\bar{A}_2|A_2})$  in (2) becomes identical to the SE in pure strategies of the game  $\mathcal{G}(u)$  in (10) which is the min-max solution in pure strategies (red line).

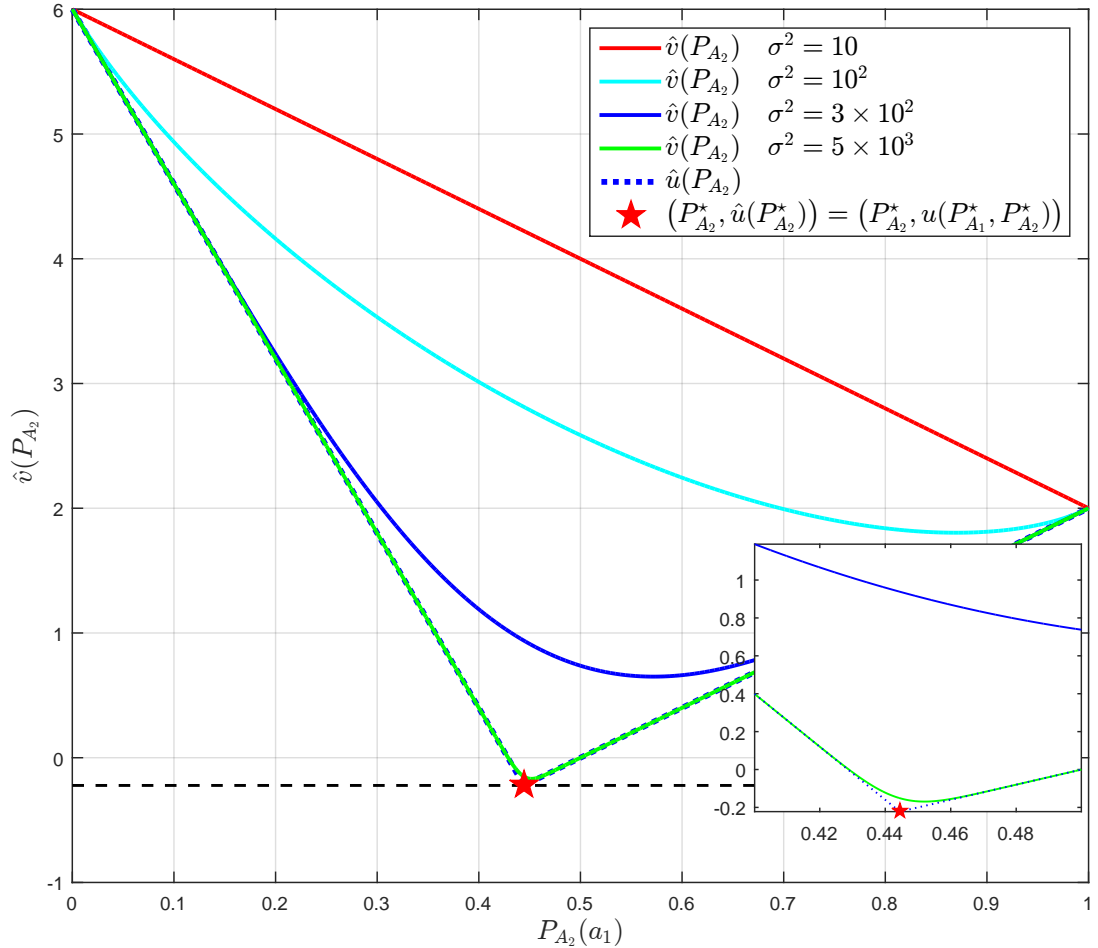


Figure 1: Plots of  $\hat{v}$  in (7) and  $\hat{u}$  in (22) as a function of the commitment  $P_{A_2}$ . The payoff matrix is  $u = (8, -6; -2, 2)$ . The channel is the AWGN defined in (24) with variance  $\sigma_1^2 = \sigma_2^2 = \sigma^2$ ;  $\mu_1 = \mu_2 = 0$ ; and  $-a_{2,1} = a_{2,2} = 10^2$ .

In Fig. 2, the game  $\mathcal{G}(u)$  exhibits a unique equilibrium in pure strategies and the NE payoff of the game  $\mathcal{G}(u)$  is  $\hat{u}(P_{A_2}^*)$  (red star). Here, the equilibrium payoffs of both games are identical. In both examples it holds that  $\hat{u}(P_{A_2}^*) \leq v(P_{A_1}^\dagger|_{\bar{A}_2}, P_{A_2}^\dagger)$ , which signifies that even subject to noise, the equilibrium payoff increases when Player 1 observes a noisy version of the action played by Player 2.

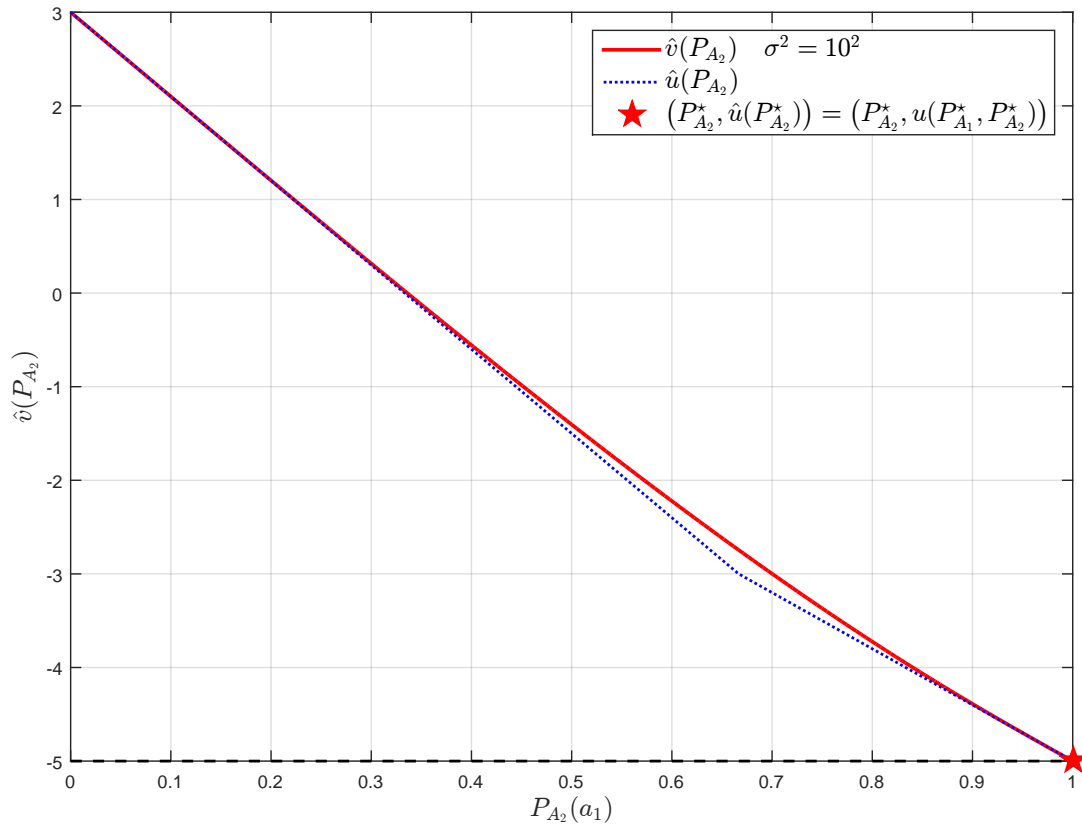


Figure 2: Plots of  $\hat{v}$  in (7) and  $\hat{u}$  in (22) as a function of the commitment  $P_{A_2}$ . The payoff matrix is  $u = (-5, 1; -6, 3)$ . The channel is the AWGN defined in (24) with variance  $\sigma_1^2 = \sigma_2^2 = \sigma^2$ ;  $\mu_1 = \mu_2 = 0$ ; and  $-a_{2,1} = a_{2,2} = 10^2$ .

## Appendices

### A Proof of Lemma 3.1

Let  $P_{A_1|\tilde{A}_2} \in \Delta(\mathcal{A}_1|\tilde{\mathcal{A}}_2)$  and  $P \in \Delta(\mathcal{A}_2)$  be the strategies of Player 1 and Player 2, respectively. Then, the expected payoff given in (5) satisfies the following:

$$\begin{aligned} & v\left(P_{A_1|\tilde{A}_2}, P\right) \\ &= \int_{\tilde{\mathcal{A}}_2} \left( \sum_{i=1}^{m_1} P_{A_1|\tilde{A}_2=\tilde{b}}(a_{1,i}) \left( \sum_{j=1}^{m_2} u_{i,j} P(a_{2,j}) \frac{dP_{\tilde{A}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b})} \right) \right) dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b}) \\ &\leq \int_{\tilde{\mathcal{A}}_2} \left( \max_i \left\{ \sum_{j=1}^{m_2} u_{i,j} P(a_{2,j}) \frac{dP_{\tilde{A}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b})} \right\} \right) dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b}), \end{aligned} \quad (30)$$

where the equality in (30) is obtained by concentrating the probability measure  $P_{A_1|\tilde{A}_2}$  over the set

$$\arg \max_{i \in \{1, \dots, m_1\}} \left\{ \sum_{j=1}^{m_2} u_{i,j} P(a_{2,j}) \frac{dP_{\tilde{A}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b})} \right\}, \quad (31)$$

which completes the proof.

### B Proof of Theorem 3.1

Assume that for a pair  $(i, l) \in \{1, \dots, m_1\}^2$ , with  $i \neq l$ , and given a commitment  $P \in \Delta(\mathcal{A}_2)$  the set  $\mathcal{H}_{i,l}(P)$  in (15) is empty, then it follows directly that the equality in (17) holds, and thus there is nothing to prove. Otherwise, there exists a  $\tilde{b} \in \mathcal{H}_{i,l}(P)$  such that

$$\sum_{j=1}^{m_2} (u_{i,j} - u_{l,j}) P(a_{2,j}) \frac{dP_{\tilde{A}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b})} = 0, \quad (32)$$

Suppose that  $1 \leq |\mathcal{H}_{i,l}(P)| < +\infty$ , *i.e.*, the set  $\mathcal{H}_{i,l}(P)$  is finite. Hence, it holds that  $\lambda(\mathcal{H}_{i,l}(P)) = 0$ , which implies from (4) that the equality in (17) holds.

Alternatively, assume that  $|\mathcal{H}_{i,l}(P)| = +\infty$ , then it holds that for all  $(i, l) \in \{1, \dots, m_1\}^2$ , with  $i \neq l$ , the equality in (32) satisfies

$$\sum_{j=1}^{m_2} u_{i,j} P(a_{2,j}) \frac{dP_{\tilde{A}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b})} \frac{dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b})}{d\lambda} = \sum_{j=1}^{m_2} u_{l,j} P(a_{2,j}) \frac{dP_{\tilde{A}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b})} \frac{dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b})}{d\lambda}, \quad (33)$$

where  $\lambda$  denotes the Lebesgue measure on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ . Equality in (33) is equivalent to

$$\sum_{j=1}^{m_2} u_{i,j} P(a_{2,j}) f_{\tilde{A}_2|A_2=a_{2,j}}(\tilde{b}) = \sum_{j=1}^{m_2} u_{l,j} P(a_{2,j}) f_{\tilde{A}_2|A_2=a_{2,j}}(\tilde{b}), \quad (34)$$

where  $f_{\tilde{A}_2|A_2=a_{2,j}}$  is the corresponding probability density function of the measure  $P_{\tilde{A}_2|A_2=a_{2,j}}$ . Note that, the expression in (34) satisfies

$$\sum_{j=1}^{m_2} (u_{i,j} - u_{l,j}) P(a_{2,j}) f_{\tilde{A}_2|A_2=a_{2,j}}(\tilde{b}) = 0, \quad (35)$$

which implies that for all  $P \in \Delta(\mathcal{A}_2)$  the set  $\mathcal{H}_{i,l}(P)$  defined in (15) can be written as

$$\mathcal{H}_{i,l}(P) \triangleq \left\{ \tilde{b} \in \tilde{\mathcal{A}}_2 : \sum_{j=1}^{m_2} u_{i,j} P(a_{2,j}) f_{\tilde{\mathcal{A}}_2|A_2=a_{2,j}}(\tilde{b}) = \sum_{j=1}^{m_2} u_{l,j} P(a_{2,j}) f_{\tilde{\mathcal{A}}_2|A_2=a_{2,j}}(\tilde{b}) \right\}. \quad (36)$$

Therefore, from (11), it can be concluded that for all  $(i,l) \in \{1, \dots, m_1\}^2$ , with  $i \neq l$ , it holds that  $\lambda(\mathcal{H}_{i,l}(P)) = 0$ . Moreover, from (4) it holds that the probability measure  $P_{\tilde{\mathcal{A}}_2}$  is mutually absolutely continuous with the Lebesgue measure, which implies that the probability of the existence of a  $\tilde{b}$  satisfying (35) is zero and thus the set  $\mathcal{H}_{i,l}(P)$  in (36) is negligible with respect to the probability measure  $P_{\tilde{\mathcal{A}}_2}$ , which concludes the proof.

## C Proof of Lemma 3.2

From (5), it holds that

$$\begin{aligned} & v \left( P_{A_1|\tilde{\mathcal{A}}_2}, P_{A_2} \right) \\ &= \int_{\tilde{\mathcal{A}}_2} \left( \sum_{i=1}^{m_1} P_{A_1|\tilde{\mathcal{A}}_2=\tilde{b}}(a_{1,i}) \left( \sum_{j=1}^{m_2} u_{i,j} P_{A_2}(a_{2,j}) \frac{dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,1}}(\tilde{b})} \right) \right) dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,1}}(\tilde{b}) \\ &= \sum_{l=1}^{m_1} \int_{\mathcal{H}_l(P_{A_2})} \left( \sum_{i=1}^{m_1} P_{A_1|\tilde{\mathcal{A}}_2=\tilde{b}}(a_{1,i}) \left( \sum_{j=1}^{m_2} u_{i,j} P_{A_2}(a_{2,j}) \frac{dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,1}}(\tilde{b})} \right) \right) dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,1}}(\tilde{b}) \end{aligned} \quad (37)$$

$$\leq \sum_{l=1}^{m_1} \int_{\mathcal{H}_l(P_{A_2})} \left( \sum_{i=1}^{m_1} \mathbb{1}_{\{i=l\}} \left( \sum_{j=1}^{m_2} u_{i,j} P_{A_2}(a_{2,j}) \frac{dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,1}}(\tilde{b})} \right) \right) dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,1}}(\tilde{b}) \quad (38)$$

where equality in (37) is due to the fact that  $\bigcup_{l=1}^{m_1} \mathcal{H}_l(P_{A_2}) = \tilde{\mathcal{A}}_2$ ; and the equality in (38) holds when for all  $i \in \{1, \dots, m_1\}$ :

$$P_{A_1|\tilde{\mathcal{A}}_2=\tilde{b}}(a_{1,i}) = \begin{cases} 1, & \text{if } a_{1,i} \in \mathcal{S}(P_{A_2}, \tilde{b}) \text{ and } |\mathcal{S}(P_{A_2}, \tilde{b})| = 1 \\ \gamma_i & \text{if } a_{1,i} \in \mathcal{S}(P_{A_2}, \tilde{b}) \text{ and } |\mathcal{S}(P_{A_2}, \tilde{b})| > 1 \\ 0, & \text{if } a_{1,i} \notin \mathcal{S}(P_{A_2}, \tilde{b}), \end{cases} \quad (39)$$

where  $\gamma_i \in [0, 1]$  are such that  $\gamma_1 + \gamma_2 + \dots + \gamma_{m_1} = 1$ .

Essentially, this results in the characterization of the elements that belong to the correspondence  $\text{BR}_1(P_{A_2}, \tilde{b})$ . This is because from (39) it holds that given a commitment  $P_{A_2} \in \Delta(\mathcal{A}_2)$  and for all  $(i,l) \in \{1, \dots, m_1\}^2$ , with  $i \neq l$

$$P_{\tilde{\mathcal{A}}_2} \left( \left\{ \tilde{b} \in \tilde{\mathcal{A}}_2 : |\mathcal{S}(P_{A_2}, \tilde{b})| > 1 \right\} \right) \quad (40)$$

$$= P_{\tilde{\mathcal{A}}_2} \left( \left\{ \tilde{b} \in \tilde{\mathcal{A}}_2 : \left| \arg \max_{i \in \{1, \dots, m_1\}} \sum_{j=1}^{m_2} u(a_{1,i}, a_{2,j}) P_{A_2}(a_{2,j}) \frac{dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,1}}(\tilde{b})} \right| > 1 \right\} \right) \quad (41)$$

$$\leq P_{\tilde{\mathcal{A}}_2} \left( \left\{ \tilde{b} \in \tilde{\mathcal{A}}_2 : \sum_{j=1}^{m_2} u(a_{1,i}, a_{2,j}) P_{A_2}(a_{2,j}) \frac{dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,1}}(\tilde{b})} = \sum_{j=1}^{m_2} u(a_{1,l}, a_{2,j}) P_{A_2}(a_{2,j}) \frac{dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,1}}(\tilde{b})} \right\} \right) \quad (42)$$

$$= P_{\tilde{\mathcal{A}}_2}(\mathcal{H}_{i,l}(P_{A_2})) \quad (43)$$

$$= 0, \quad (44)$$

where the equality in (41) follows from the definition of the set  $\mathcal{S}(P_{A_2}, \tilde{b})$  in (13); inequality in (42) holds since the set in (41) is a subset of the set in (42); equality in (43) is due to the definition of the set  $\mathcal{H}_{i,l}(P_{A_2})$  in (15); and equality in (44) is due to Theorem 3.1. This shows that the cardinality of Player 1's set of best responses is a singleton almost surely, which completes the proof.

## D Proof of Theorem 3.2

For all commitments  $P_1 \in \Delta(\mathcal{A}_2)$  and  $P_2 \in \Delta(\mathcal{A}_2)$  and for all  $\alpha \in (0, 1)$  the function  $\hat{v}$  in (7) satisfies

$$\hat{v}(\alpha P_1 + (1 - \alpha)P_2) = \max_{Q_{A_1|\tilde{A}_2} \in \Delta(\mathcal{A}_1|\tilde{A}_2)} v(Q_{A_1|\tilde{A}_2}, \alpha P_1 + (1 - \alpha)P_2) \quad (45)$$

$$\begin{aligned} &= \max_{Q_{A_1|\tilde{A}_2} \in \Delta(\mathcal{A}_1|\tilde{A}_2)} \int_{\tilde{A}_2} \left( \sum_{i=1}^{m_1} Q_{A_1|\tilde{A}_2=\tilde{b}}(a_{1,i}) \left( \sum_{j=1}^{m_2} u_{i,j} (\alpha P_1(a_{2,j}) \right. \right. \\ &\left. \left. + (1 - \alpha)P_2(a_{2,j})) \frac{dP_{\tilde{A}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b})} \right) \right) dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b}) \quad (46) \end{aligned}$$

$$\begin{aligned} &= \max_{Q_{A_1|\tilde{A}_2} \in \Delta(\mathcal{A}_1|\tilde{A}_2)} \int_{\tilde{A}_2} \left( \sum_{i=1}^{m_1} Q_{A_1|\tilde{A}_2=\tilde{b}}(a_{1,i}) \left( \alpha \sum_{j=1}^{m_2} u_{i,j} P_1(a_{2,j}) \frac{dP_{\tilde{A}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b})} \right. \right. \\ &\left. \left. + (1 - \alpha) \sum_{j=1}^{m_2} u_{i,j} P_2(a_{2,j}) \frac{dP_{\tilde{A}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b})} \right) \right) dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b}) \quad (47) \end{aligned}$$

$$= \max_{Q_{A_1|\tilde{A}_2} \in \Delta(\mathcal{A}_1|\tilde{A}_2)} \alpha v(Q_{A_1|\tilde{A}_2}, P_1) + (1 - \alpha)v(Q_{A_1|\tilde{A}_2}, P_2) \quad (48)$$

$$\leq \max_{Q_{A_1|\tilde{A}_2} \in \Delta(\mathcal{A}_1|\tilde{A}_2)} \alpha v(Q_{A_1|\tilde{A}_2}, P_1) + \max_{Q_{A_1|\tilde{A}_2} \in \Delta(\mathcal{A}_1|\tilde{A}_2)} (1 - \alpha)v(Q_{A_1|\tilde{A}_2}, P_2) \quad (49)$$

$$= \alpha \hat{v}(P_1) + (1 - \alpha)\hat{v}(P_2), \quad (50)$$

where equality in (48) is due to (7); and the inequality in (49) follows from the fact that the maximum of the sum is at most the sum of the maxima [25]; and the equality in (50) is due to (7). This implies that function  $\hat{v}$  is convex. Moreover, the equality in (49) is obtained if and only if for all  $P \in \Delta(\mathcal{A}_2)$

$$\max_{Q_{A_1|\tilde{A}_2} \in \Delta(\mathcal{A}_1|\tilde{A}_2)} v(Q_{A_1|\tilde{A}_2}, P) = \max_{T \in \Delta(\mathcal{A}_1)} v(T, P), \quad (51)$$

where the function  $v$  is defined in (5).

However, from (51) it holds that

$$\max_{Q_{A_1|\tilde{A}_2} \in \Delta(A_1|\tilde{A}_2)} v(Q_{A_1|\tilde{A}_2}, P) - \max_{T \in \Delta(A_1)} v(T, P) \quad (52)$$

$$\begin{aligned} &= \max_{Q_{A_1|\tilde{A}_2} \in \Delta(A_1|\tilde{A}_2)} \int_{\tilde{A}_2} \left( \sum_{i=1}^{m_1} Q_{A_1|\tilde{A}_2=\tilde{b}}(a_{1,i}) \left( \sum_{j=1}^{m_2} u_{i,j} P(a_{2,j}) \frac{dP_{\tilde{A}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b})} \right) \right) dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b}) \\ &- \max_{T \in \Delta(A_1)} \int_{\tilde{A}_2} \left( \sum_{l=1}^{m_1} T(a_{1,l}) \left( \sum_{j=1}^{m_2} u_{l,j} P(a_{2,j}) \frac{dP_{\tilde{A}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b})} \right) \right) dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b}) \end{aligned} \quad (53)$$

$$\begin{aligned} &= \int_{\tilde{A}_2} \max_{Q_{A_1|\tilde{A}_2=\tilde{b}} \in \Delta(A_1)} \left( \sum_{i=1}^{m_1} Q_{A_1|\tilde{A}_2=\tilde{b}}(a_{1,i}) \left( \sum_{j=1}^{m_2} u_{i,j} P(a_{2,j}) \frac{dP_{\tilde{A}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b})} \right) \right) dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b}) \\ &- \int_{\tilde{A}_2} \max_{T \in \Delta(A_1)} \left( \sum_{l=1}^{m_1} T(a_{1,l}) \left( \sum_{j=1}^{m_2} u_{l,j} P(a_{2,j}) \frac{dP_{\tilde{A}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b})} \right) \right) dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b}) \end{aligned} \quad (54)$$

$$\begin{aligned} &= \int_{\tilde{A}_2} \max_{Q_{A_1|\tilde{A}_2=\tilde{b}} \in \Delta(A_1)} \max_{T \in \Delta(A_1)} \left( \sum_{i=1}^{m_1} \sum_{l=1}^{m_1} Q_{A_1|\tilde{A}_2=\tilde{b}}(a_{1,i}) T(a_{1,l}) \right. \\ &\quad \left. \left( \sum_{j=1}^{m_2} (u_{i,j} - u_{l,j}) P(a_{2,j}) \frac{dP_{\tilde{A}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b})} \right) \right) dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b}), \end{aligned} \quad (55)$$

where the equality in (54) is derived by interchanging the max with the integral since for all  $\tilde{b} \in \tilde{A}_2$  it holds that  $Q_{A_1|\tilde{A}_2=\tilde{b}} \in \Delta(A_1)$ .

Hence, from (55) and the condition in (16), it holds that (51) satisfies for all  $P \in \Delta(A_2)$

$$\max_{Q_{A_1|\tilde{A}_2} \in \Delta(A_1|\tilde{A}_2)} v(Q_{A_1|\tilde{A}_2}, P) \neq \max_{T \in \Delta(A_1)} v(T, P). \quad (56)$$

Therefore, the function  $\hat{v}$  is strictly convex in  $\Delta(A_2)$ . This implies that it has a unique minimum and shows that the equilibrium of the game  $\mathcal{G}(u, P_{\tilde{A}_2|A_2})$  is unique. That completes the proof.

## E Proof of Lemma 3.3

Suppose that the pair of strategies  $(P_{A_1}^*, P_{A_2}^*) \in \Delta(A_1) \times \Delta(A_2)$  forms a NE of the game  $\mathcal{G}(u)$  in (10). For all probability measures  $P \in \Delta(A_2)$ , it holds that

$$\hat{u}(P_{A_2}^*) = \max_{Q \in \Delta(A_1)} \min_{P \in \Delta(A_2)} u(Q, P) \quad (57)$$

$$= \min_{P \in \Delta(A_2)} \max_{Q \in \Delta(A_1)} u(Q, P) \quad (58)$$

$$\leq \max_{Q \in \Delta(A_1)} \omega(Q, P) \quad (59)$$

$$= \hat{u}(P), \quad (60)$$

where the equalities in (57) and (58) follow from the min-max theorem [17]; and the last equality is due to the definition of the function  $\hat{u}$  in (22).

Moreover, from (22) it holds that for all  $P \in \Delta(\mathcal{A}_2)$

$$\hat{u}(P) = \max_{Q \in \Delta(\mathcal{A}_1)} \omega(Q, P) \quad (61)$$

$$= \max_{Q_{\mathcal{A}_1|\tilde{\mathcal{A}}_2=\tilde{b}} \in \Delta(\mathcal{A}_1)} \omega(Q_{\mathcal{A}_1|\tilde{\mathcal{A}}_2=\tilde{b}}, P), \quad \forall \tilde{b} \in \tilde{\mathcal{A}}_2 \quad (62)$$

$$\leq \max_{Q_{\mathcal{A}_1|\tilde{\mathcal{A}}_2} \in \Delta(\mathcal{A}_1|\tilde{\mathcal{A}}_2)} v(Q_{\mathcal{A}_1|\tilde{\mathcal{A}}_2}, P) \quad (63)$$

$$= \hat{v}(P), \quad (64)$$

where equality in (62) follows from (21); inequality in (63) is due to the larger optimization domain and the fact that for all  $\tilde{b} \in \tilde{\mathcal{A}}_2$  it holds that  $Q_{\mathcal{A}_1|\tilde{\mathcal{A}}_2=\tilde{b}} \in \Delta(\mathcal{A}_1)$ ; and equality in (64) is due to (7).

Moreover, for all  $P \in \Delta(\mathcal{A}_2)$ , the function in (7) satisfies

$$\hat{v}(P) = \max_{Q_{\mathcal{A}_1|\tilde{\mathcal{A}}_2} \in \Delta(\mathcal{A}_1|\tilde{\mathcal{A}}_2)} v(Q_{\mathcal{A}_1|\tilde{\mathcal{A}}_2}, P) \quad (65)$$

$$= \max_{Q_{\mathcal{A}_1|\tilde{\mathcal{A}}_2} \in \Delta(\mathcal{A}_1|\tilde{\mathcal{A}}_2)} \int_{\tilde{\mathcal{A}}_2} \left( \sum_{j=1}^{m_2} \sum_{i=1}^{m_1} u_{i,j} Q_{\mathcal{A}_1|\tilde{\mathcal{A}}_2=\tilde{b}}(a_{1,i}) P(a_{2,j}) \frac{dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,1}}(\tilde{b})} \right) dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,1}}(\tilde{b}) \quad (66)$$

$$\leq \max_{Q_{\mathcal{A}_1|\tilde{\mathcal{A}}_2} \in \Delta(\mathcal{A}_1|\tilde{\mathcal{A}}_2)} \sum_{j=1}^{m_2} \sum_{i=1}^{m_1} P(a_{2,j}) \max_{l \in \{1, \dots, m_1\}} u_{l,j} \left( \int_{\tilde{\mathcal{A}}_2} Q_{\mathcal{A}_1|\tilde{\mathcal{A}}_2=\tilde{b}}(a_{1,i}) \frac{dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,1}}(\tilde{b})} dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,1}}(\tilde{b}) \right) \quad (67)$$

$$= \max_{Q_{\mathcal{A}_1|\tilde{\mathcal{A}}_2} \in \Delta(\mathcal{A}_1|\tilde{\mathcal{A}}_2)} \sum_{j=1}^{m_2} P(a_{2,j}) \max_{l \in \{1, \dots, m_1\}} u_{l,j} \left( \sum_{i=1}^{m_1} \int_{\tilde{\mathcal{A}}_2} Q_{\mathcal{A}_1|\tilde{\mathcal{A}}_2=\tilde{b}}(a_{1,i}) \frac{dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,1}}(\tilde{b})} dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,1}}(\tilde{b}) \right) \quad (68)$$

$$= \sum_{j=1}^{m_2} P(a_{2,j}) \max_{l \in \{1, \dots, m_1\}} u_{l,j} \left( \sum_{i=1}^{m_1} \int_{\tilde{\mathcal{A}}_2} \max_{Q_{\mathcal{A}_1|\tilde{\mathcal{A}}_2=\tilde{b}} \in \Delta(\mathcal{A}_1)} Q_{\mathcal{A}_1|\tilde{\mathcal{A}}_2=\tilde{b}}(a_{1,i}) \frac{dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,j}}(\tilde{b})}{dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,1}}(\tilde{b})} dP_{\tilde{\mathcal{A}}_2|A_2=a_{2,1}}(\tilde{b}) \right) \quad (69)$$

$$= \sum_{j=1}^{m_2} P(a_{2,j}) \max_{l \in \{1, \dots, m_1\}} u_{l,j}, \quad (70)$$

where the equality in (66) follows from (5) and from [26, Th.1.6.3]; the inequality in (67) holds due to the maximization of the payoff function  $u$  in (1); the equality in (69) is by interchanging the max with the integral since for all  $\tilde{b} \in \tilde{\mathcal{A}}_2$  it holds that  $Q_{\mathcal{A}_1|\tilde{\mathcal{A}}_2=\tilde{b}} \in \Delta(\mathcal{A}_1)$ ; and (70) follow by observing that the summation over  $i$  and the integration over the set  $\tilde{\mathcal{A}}_2$  do not depend on the index  $l$  and therefore are treated as constants with respect to the maximization problem over  $l \in \{1, \dots, m_1\}$ .

Note that from (64) and (60), it holds that for all  $P \in \Delta(\mathcal{A}_2)$

$$\hat{u}(P_{A_2}^*) \leq \hat{v}(P). \quad (71)$$

Moreover, letting  $P = P_{A_2}^\dagger$ , it holds that

$$\hat{u}(P_{A_2}^*) \leq v\left(P_{A_1|\tilde{\mathcal{A}}_2}^\dagger, P_{A_2}^\dagger\right), \quad (72)$$

since  $\hat{v}(P_{A_2}^\dagger) = v\left(P_{A_1|\tilde{\mathcal{A}}_2}^\dagger, P_{A_2}^\dagger\right)$ . From, inequality in (71) and from the fact that

$$\min_{P \in \Delta(\mathcal{A}_2)} \hat{v}(P) = v\left(P_{A_1|\tilde{\mathcal{A}}_2}^\dagger, P_{A_2}^\dagger\right), \quad (73)$$

it holds that

$$v\left(P_{A_1|\tilde{A}_2}^\dagger, P_{A_2}^\dagger\right) \leq \hat{v}(P), \quad (74)$$

with equality if and only if  $P = P_{A_2}^\dagger$ .

Finally, for all  $P \in \Delta(\mathcal{A}_2)$ , the inequality in (70) satisfies

$$\min_{P \in \Delta(\mathcal{A}_2)} \hat{v}(P) \leq \min_{P \in \Delta(\mathcal{A}_2)} \sum_{j=1}^{m_2} P(a_{2,j}) \max_{l \in \{1, \dots, m_1\}} u_{l,j} \quad (75)$$

$$= \min_{j \in \{1, \dots, m_2\}} \max_{i \in \{1, \dots, m_1\}} u_{i,j}. \quad (76)$$

Observing inequalities in (72), (72), (74) and (76) completes the proof.

## F Proof of Lemma 4.1

Let the sets of actions of Player 1 and Player 2 be  $\mathcal{A}_1 = \{a_{1,1}, a_{2,1}\}$  and  $\mathcal{A}_2 = \{a_{1,1}, a_{2,1}\}$ , respectively. Additionally, from (14) it follows that given a commitment  $P_{A_2} \in \Delta(\mathcal{A}_2)$  the subsets  $\mathcal{H}_i(P_{A_2})$  of  $\tilde{\mathcal{A}}_2$ , with  $i \in \{1, 2\}$ , are such that

$$\mathcal{H}_2(P_{A_2}) \triangleq \left\{ \tilde{b} \in \tilde{\mathcal{A}}_2 : \frac{dP_{\tilde{A}_2|A_2=a_{2,2}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,1}}} > \frac{(u_{1,1} - u_{2,1}) P_{A_2}(a_{2,1})}{(u_{2,2} - u_{1,2}) P_{A_2}(a_{2,2})} \right\}; \text{ and} \quad (77)$$

$$\mathcal{H}_1(P_{A_2}) \triangleq \left\{ \tilde{b} \in \tilde{\mathcal{A}}_2 : \frac{dP_{\tilde{A}_2|A_2=a_{2,2}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,1}}} < \frac{(u_{1,1} - u_{2,1}) P_{A_2}(a_{2,1})}{(u_{2,2} - u_{1,2}) P_{A_2}(a_{2,2})} \right\}. \quad (78)$$

The expected payoff function in (5) satisfies

$$\begin{aligned} v\left(P_{A_1|\tilde{A}_2}, P_{A_2}\right) &= P_{A_2}(a_{2,1})u_{2,1} + P_{A_2}(a_{2,2})u_{2,2} + \int_{\mathcal{H}_1(P_{A_2})} P_{A_1|\tilde{A}_2=\tilde{b}}(a_{1,1}) \left( (u_{1,1} - u_{2,1}) P_{A_2}(a_{2,1}) \right. \\ &\quad \left. - (u_{2,2} - u_{1,2}) P_{A_2}(a_{2,2}) \frac{dP_{\tilde{A}_2|A_2=a_{2,2}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,1}}} \right) dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b}) \\ &\quad + \int_{\mathcal{H}_2(P_{A_2})} P_{A_1|\tilde{A}_2=\tilde{b}}(a_{1,1}) \left( (u_{1,1} - u_{2,1}) P_{A_2}(a_{2,1}) \right. \\ &\quad \left. - (u_{2,2} - u_{1,2}) P_{A_2}(a_{2,2}) \frac{dP_{\tilde{A}_2|A_2=a_{2,2}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,1}}} \right) dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b}) \\ &\quad + \int_{\mathcal{H}_{1,2}(P_{A_2})} P_{A_1|\tilde{A}_2=\tilde{b}}(a_{1,1}) \left( (u_{1,1} - u_{2,1}) P_{A_2}(a_{2,1}) \right. \\ &\quad \left. - (u_{2,2} - u_{1,2}) P_{A_2}(a_{2,2}) \frac{dP_{\tilde{A}_2|A_2=a_{2,2}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,1}}} \right) dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b}). \end{aligned} \quad (79)$$

$$\begin{aligned} &= P_{A_2}(a_{2,1})u_{2,1} + P_{A_2}(a_{2,2})u_{2,2} + \int_{\mathcal{H}_1(P_{A_2})} P_{A_1|\tilde{A}_2=\tilde{b}}(a_{1,1}) \left( (u_{1,1} - u_{2,1}) P_{A_2}(a_{2,1}) \right. \\ &\quad \left. - (u_{2,2} - u_{1,2}) P_{A_2}(a_{2,2}) \frac{dP_{\tilde{A}_2|A_2=a_{2,2}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,1}}} \right) dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b}), \end{aligned} \quad (80)$$

where equality in (79) follows from (77), (78) and the fact that  $P_{\tilde{A}_2}(\mathcal{H}_{1,2}(P_{A_2})) = 0$  from Theorem 3.1; and equality in (80) is obtained by concentrating the probability measure  $P_{A_1|\tilde{A}_2=\tilde{b}}$  over the set  $\mathcal{H}_1(P_{A_2})$ .

Specifically, when the probability measure  $P_{\tilde{A}_2|A_2}$  is such that (24) holds and the probability density function satisfies for all  $b \in \mathcal{A}_2$

$$f_{\tilde{A}_2|A_2=b}(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-b)^2}{2\sigma^2}\right), \quad (81)$$

then the function in (7) satisfies

$$\hat{v}(P) = \max_{P_{A_1|\tilde{A}_2} \in \Delta(\mathcal{A}_1|\tilde{\mathcal{A}}_2)} v(P_{A_1|\tilde{A}_2}, P) \quad (82)$$

$$= \max \left\{ P(a_{2,1})u_{2,1} + P(a_{2,2})u_{2,2} + \int_{\mathcal{H}_1(P)} \left( P_{A_1|\tilde{A}_2=\tilde{b}}(a) \left( (u_{1,1} - u_{2,1}) P(a_{2,1}) - (u_{2,2} - u_{1,2}) P(a_{2,2}) \frac{dP_{\tilde{A}_2|A_2=a_{2,2}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b})} \right) \right) dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b}) \right\} \quad (83)$$

$$= P(a_{2,1})u_{2,1} + P(a_{2,2})u_{2,2} + \int_{\mathcal{H}_1(P)} \left( (u_{1,1} - u_{2,1}) P(a_{2,1}) - (u_{2,2} - u_{1,2}) P(a_{2,2}) \frac{dP_{\tilde{A}_2|A_2=a_{2,2}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b})} \right) dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b}) \quad (84)$$

$$= P(a_{2,1})u_{2,1} + P(a_{2,2})u_{2,2} + (u_{1,1} - u_{2,1}) P(a_{2,1}) \int_{\mathcal{H}_1(P)} dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b}) - (u_{2,2} - u_{1,2}) P(a_{2,2}) \int_{\mathcal{H}_1(P)} \left( \frac{dP_{\tilde{A}_2|A_2=a_{2,2}}(\tilde{b})}{dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b})} \right) dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b}) \quad (85)$$

$$= P(a_{2,1})u_{2,1} + P(a_{2,2})u_{2,2} + (u_{1,1} - u_{2,1}) P(a_{2,1}) \int_{\mathcal{H}_1(P)} dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b}) - (u_{2,2} - u_{1,2}) P(a_{2,2}) \int_{\mathcal{H}_1(P)} \exp\left(-\frac{1}{2\sigma^2} \left( (\tilde{b} - a_{2,2})^2 - (\tilde{b} - a_{2,1})^2 \right)\right) dP_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b}) \quad (86)$$

$$= P(a_{2,1})u_{2,1} + P(a_{2,2})u_{2,2} + (u_{1,1} - u_{2,1}) P(a_{2,1}) \int_{b_P^*}^{\infty} f_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b}) d\tilde{b} - (u_{2,2} - u_{1,2}) P(a_{2,2}) \int_{b_P^*}^{\infty} \exp\left(-\frac{1}{2\sigma^2} \left( (\tilde{b} - a_{2,2})^2 - (\tilde{b} - a_{2,1})^2 \right)\right) f_{\tilde{A}_2|A_2=a_{2,1}}(\tilde{b}) d\tilde{b} \quad (87)$$

$$= P(a_{2,1})u_{2,1} + P(a_{2,2})u_{2,2} + \frac{(u_{1,1} - u_{2,1}) P(a_{2,1})}{\sigma\sqrt{2\pi}} \int_{b_P^*}^{\infty} \exp\left(-\frac{1}{2\sigma^2} (\tilde{b} - a_{2,1})^2\right) d\tilde{b} - \frac{(u_{2,2} - u_{1,2}) P(a_{2,2})}{\sigma\sqrt{2\pi}} \int_{b_P^*}^{\infty} \exp\left(-\frac{1}{2\sigma^2} \left( (\tilde{b} - a_{2,2})^2 - (\tilde{b} - a_{2,1})^2 \right)\right) \exp\left(-\frac{1}{2\sigma^2} (\tilde{b} - a_{2,1})^2\right) d\tilde{b} \quad (88)$$

$$= P(a_{2,1}) \left( u_{2,1} + (u_{1,1} - u_{2,1}) Q\left(\frac{b_P^* - a_{2,1}}{\sigma}\right) \right) + P(a_{2,2}) \left( u_{2,2} - (u_{2,2} - u_{1,2}) Q\left(\frac{b_P^* - a_{2,2}}{\sigma}\right) \right) \quad (89)$$

where the equality in (83) follows from (80); equality (84) in is derived by applying Lemma 3.2; equality in (86) is derived by substituting the Radon-Nikodym derivative with (25); equality in (87) follows from replacing the set  $\mathcal{H}_1(P)$  with its corresponding interval  $\tilde{b} \in (b_P^*, +\infty)$ , with  $b_P^* \in \mathcal{H}_{1,2}(P)$  and by substituting the probability measure  $P_{\tilde{A}_2|A_2=b}$  with its corresponding density in (81). Finally, (89) follows by rearranging the terms inside the  $\exp(\cdot)$  and by recognizing that the integrals are actually the complementary cumulative distributions of the Gaussian pdf in(28).

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