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# The $T$ -coercivity approach for solving Stokes problem: stabilization of finite element pairs\*

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## Abstract

Using the  $T$ -coercivity theory as advocated in [14], we propose a new variational formulation of Stokes problem. With this new formulation, unstable finite element pairs can be stabilized. In addition, the numerical scheme is easy to implement, and a better approximation of the velocity and the pressure is observed numerically, especially when the viscosity is small.

*Keywords:* Stokes problem,  $T$ -coercivity, pressure robust variational formulation

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## 1. Introduction

The Stokes problem describes the steady state of incompressible Newtonian flows. They are derived from the Navier–Stokes equations [21]. With regard to numerical analysis, the study of Stokes problem helps to build an appropriate approximation of the Navier–Stokes equations. We propose here to write a new variational formulation of Stokes problem using the  $T$ -coercivity theory, following §2.3.2 in [14]. In Section 2, we recall the  $T$ -coercivity theory as written in [15]. In Section 3 we apply it to the continuous Stokes Problem. We prove basic  $T$ -coercivity by finding an operator  $T$  such that the

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\*Reformulation via  $T$ -coercivity for Stokes

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global variational formulation of Stokes Problem is  $T$ -coercive. In Section 4, we build and analyse a new variational formulation that uses explicitly this operator  $T$ . Then, in Section 5, we introduce conforming discretizations of the new variational formulation, and we study the convergence of the discrete solution to the exact one. As a particular case, convergence is obtained for the unstable finite element pair  $\mathbf{P}^1 \times P^0$ . In Section 6, we focus on the case of non homogeneous boundary conditions. Finally, we provide some numerical experiments in Section 7 to illustrate our points and give some concluding remarks in Section 8.

## 2. $T$ -coercivity

We recall here the  $T$ -coercivity theory as written in [15]. Consider first the variational problem, where  $V$  and  $W$  are two Hilbert spaces and  $f \in V'$ :

$$\text{Find } u \in V \text{ such that } \forall v \in W, a(u, v) = \langle f, v \rangle_V. \quad (2.1)$$

Classically, we know that Problem (2.1) is well-posed if  $a(\cdot, \cdot)$  satisfies the stability and the solvability conditions of the so-called Banach–Nečas–Babuška (BNB) Theorem (see a.e. [20, Thm. 25.9]). For some models, one can also prove the well-posedness using the  $T$ -coercivity theory. We refer to [15] for Helmholtz-like problems, see [25], [17] and [22] for the neutron diffusion equation, and to [16] for the magnetostatic problem.

**Definition 1** (Basic  $T$ -coercivity). *Let  $V$  and  $W$  be two Hilbert spaces and  $a(\cdot, \cdot)$  be a continuous and bilinear form over  $V \times W$ . It is  $T$ -coercive if*

$$\exists T \in \mathcal{L}(V, W), \text{ bijective}, \exists \alpha > 0, \forall v \in V, |a(v, Tv)| \geq \alpha \|v\|_V^2. \quad (2.2)$$

It is proved in [15, 14] that the  $T$ -coercivity condition is equivalent to the stability and solvability conditions of the BNB Theorem. Whereas the BNB theorem relies on an abstract inf–sup condition,  $T$ -coercivity uses explicit inf–sup operators, both at the continuous and discrete levels.

**Theorem 1** (well-posedness). *Let  $a(\cdot, \cdot)$  be a continuous and bilinear form. Suppose that the form  $a(\cdot, \cdot)$  is  $T$ -coercive. Then Problem (2.1) is well-posed.*

### 3. Stokes problem

Let  $\Omega$  be a connected bounded domain of  $\mathbb{R}^d$ ,  $d = 2, 3$ , with a polygonal ( $d = 2$ ) or Lipschitz polyhedral ( $d = 3$ ) boundary  $\partial\Omega$ . We consider the generalized Stokes problem:

$$\text{Find } (\mathbf{u}, p) \text{ such that } \begin{cases} -\nu \Delta \mathbf{u} + \mathbf{grad} p = \mathbf{f}, \\ \operatorname{div} \mathbf{u} = g. \end{cases} \quad (3.1)$$

with Dirichlet boundary conditions for the velocity  $\mathbf{u}$  and a normalization condition for the pressure  $p$ :  $\int_{\Omega} p = 0$ . If the Dirichlet boundary conditions are homogeneous, we write  $(3.1)_H$ , and  $(3.1)_{NH}$  else.

The vector field  $\mathbf{u}$  represents the velocity of the fluid and the scalar field  $p$  represents its pressure divided by the fluid density which is supposed to be constant. The first equation of (3.1) corresponds to the momentum balance equation and the second one corresponds to the conservation of the mass. The constant parameter  $\nu > 0$  is the kinematic viscosity of the fluid.

Let us provide some definition and reminders. Let us set  $\mathbf{L}^2(\Omega) = (L^2(\Omega))^d$ ,  $\mathbf{H}_0^1(\Omega) = (H_0^1(\Omega))^d$ ,  $\mathbf{H}^{-1}(\Omega) = (H^{-1}(\Omega))^d$  its dual space and  $L_{zmv}^2(\Omega) = \{q \in L^2(\Omega) \mid \int_{\Omega} q = 0\}$ .

The natural function space for the velocity is  $\mathbf{H}^1(\Omega)$  and if homogeneous boundary conditions are prescribed, it is  $\mathbf{H}_0^1(\Omega)$ , while for the pressure the natural function space is  $L_{zmv}^2(\Omega)$ . The data is  $(\mathbf{f}, g)$ . The vector field  $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$  represents a body forces divided by the fluid density. The scalar field  $g \in L_{zvm}^2(\Omega)$  is some abstract data that will be useful for the analysis of the case of non homogeneous Dirichlet boundary conditions.

Let us first recall Poincaré-Steklov inequality:

$$\exists C_{PS} > 0 \mid \forall v \in H_0^1(\Omega), \quad \|v\|_{L^2(\Omega)} \leq C_{PS} \|\mathbf{grad} v\|_{\mathbf{L}^2(\Omega)}. \quad (3.2)$$

Thanks to this result, in  $H_0^1(\Omega)$ , the semi-norm is equivalent to the natural norm, so that the scalar product reads  $(v, w)_{H_0^1(\Omega)} = (\mathbf{grad} v, \mathbf{grad} w)_{\mathbf{L}^2(\Omega)}$  and the norm is  $\|v\|_{H_0^1(\Omega)} = \|\mathbf{grad} v\|_{\mathbf{L}^2(\Omega)}$ . Let  $\mathbf{v}, \mathbf{w} \in \mathbf{H}_0^1(\Omega)$ . We denote by  $(v_i)_{i=1}^d$  (resp.  $(w_i)_{i=1}^d$ ) the components of  $\mathbf{v}$  (resp.  $\mathbf{w}$ ), and we set  $\mathbf{Grad} \mathbf{v} = (\partial_j v_i)_{i,j=1}^d \in \mathbb{L}^2(\Omega)$ , where  $\mathbb{L}^2(\Omega) = [L^2(\Omega)]^{d \times d}$ . We have:

$$(\mathbf{Grad} \mathbf{v}, \mathbf{Grad} \mathbf{w})_{\mathbb{L}^2(\Omega)} = (\mathbf{v}, \mathbf{w})_{\mathbf{H}_0^1(\Omega)} = \sum_{i=1}^d (v_i, w_i)_{H_0^1(\Omega)}$$

and:

$$\|\mathbf{v}\|_{\mathbf{H}_0^1(\Omega)} = \left( \sum_{j=1}^d \|v_j\|_{H_0^1(\Omega)}^2 \right)^{1/2} = \|\mathbf{Grad} \mathbf{v}\|_{\mathbb{L}^2(\Omega)}.$$

Let us set  $\mathbf{V} = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) \mid \operatorname{div} \mathbf{v} = 0\}$ . The vector space  $\mathbf{V}$  is a closed subset of  $\mathbf{H}_0^1(\Omega)$ . We denote by  $\mathbf{V}^\perp$  the orthogonal of  $\mathbf{V}$  in  $\mathbf{H}_0^1(\Omega)$ . We recall that [21, cor. I.2.4]:

**Proposition 1.** *The operator  $\operatorname{div} : \mathbf{H}_0^1(\Omega) \rightarrow L^2(\Omega)$  is an isomorphism of  $\mathbf{V}^\perp$  onto  $L_{zmv}^2(\Omega)$ . We call  $C_{\operatorname{div}} \geq 1$  the constant such that:*

$$\forall p \in L_{zmv}^2(\Omega), \exists! \tilde{\mathbf{v}}_p \in \mathbf{V}^\perp \mid \operatorname{div} \tilde{\mathbf{v}}_p = p \text{ and } \|\tilde{\mathbf{v}}_p\|_{\mathbf{H}_0^1(\Omega)} \leq C_{\operatorname{div}} \|p\|_{L^2(\Omega)}. \quad (3.3)$$

In the above, we note that since

$$\forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \quad \|\mathbf{v}\|_{\mathbf{H}_0^1(\Omega)}^2 = \|\mathbf{curl} \mathbf{v}\|_{\mathbb{L}^2(\Omega)}^2 + \|\operatorname{div} \mathbf{v}\|_{L^2(\Omega)}^2,$$

one has necessarily  $C_{\operatorname{div}} \geq 1$ .

The variational formulation of Problem (3.1)<sub>H</sub> reads:

Find  $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L_{zmv}^2(\Omega)$  such that

$$\begin{cases} \nu(\mathbf{u}, \mathbf{v})_{\mathbf{H}_0^1(\Omega)} - (p, \operatorname{div} \mathbf{v})_{L^2(\Omega)} = \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}_0^1(\Omega)} & \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega); \\ (q, \operatorname{div} \mathbf{u})_{L^2(\Omega)} = (g, q)_{L^2(\Omega)} & \forall q \in L_{zmv}^2(\Omega). \end{cases} \quad (3.4)$$

Classically, one proves that Problem (3.4) is well-posed using Poincaré-Steklov inequality (3.2) and Prop. 1. Check for instance the proof of [21, Thm. I.5.1].

Let us set  $\mathcal{X} = \mathbf{H}_0^1(\Omega) \times L_{zmv}^2(\Omega)$  which is a Hilbert space which we endow with the following norm:

$$\|(\mathbf{v}, q)\|_{\mathcal{X}, \nu} = \left( \|\mathbf{v}\|_{\mathbf{H}_0^1(\Omega)}^2 + \nu^{-2} \|q\|_{L^2(\Omega)}^2 \right)^{1/2}. \quad (3.5)$$

We define the following bilinear symmetric and continuous form:

$$\begin{cases} a_0 : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R} \\ (\mathbf{u}', p') \times (\mathbf{v}, q) \mapsto \nu(\mathbf{u}', \mathbf{v})_{\mathbf{H}_0^1(\Omega)} - (p', \operatorname{div} \mathbf{v})_{L^2(\Omega)} - (q, \operatorname{div} \mathbf{u}')_{L^2(\Omega)} \end{cases}. \quad (3.6)$$

We also define the linear and continuous form:

$$\begin{cases} \ell_0 : \mathcal{X} \rightarrow \mathbb{R} \\ (\mathbf{v}, q) \mapsto \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}_0^1(\Omega)} - (g, q)_{L^2(\Omega)} \end{cases}. \quad (3.7)$$

We can write Problem (3.1)<sub>H</sub> in an equivalent way as follows:

$$\text{Find } (\mathbf{u}, p) \in \mathcal{X} \text{ s.t. } a_0((\mathbf{u}, p), (\mathbf{v}, q)) = \ell_0((\mathbf{v}, q)) \quad \forall (\mathbf{v}, q) \in \mathcal{X}. \quad (3.8)$$

Let us prove that Problem (3.8) is well-posed using basic  $T$ -coercivity.

**Proposition 2.** *The bilinear form  $a_0(\cdot, \cdot)$  is  $T$ -coercive:*

$$\begin{aligned} \exists T \in \mathcal{L}(\mathcal{X}), \text{ bijective}, \exists \alpha > 0, \forall (\mathbf{u}', p') \in \mathcal{X}, \\ a_0((\mathbf{u}', p'), T((\mathbf{u}', p')))) \geq \alpha \|(\mathbf{u}', p')\|_{\mathcal{X}, \nu}^2. \end{aligned} \quad (3.9)$$

*Proof.* We follow here the proof given in [4, 11]. Let us consider  $(\mathbf{u}', p') \in \mathcal{X}$  and let us build  $(\mathbf{v}^*, q^*) = T(\mathbf{u}', p') \in \mathcal{X}$  satisfying (2.2) (with  $V = \mathcal{X}$ ). We need three main steps.

1. According to Prop. 1, there exists  $\tilde{\mathbf{v}}_{p'} \in \mathbf{V}^\perp$  such that:

$$\operatorname{div} \tilde{\mathbf{v}}_{p'} = p' \text{ in } \Omega \text{ and } \|\tilde{\mathbf{v}}_{p'}\|_{\mathbf{H}_0^1(\Omega)} \leq C_{\operatorname{div}} \|p'\|_{L^2(\Omega)}. \quad (3.10)$$

Let us set  $(\mathbf{v}^*, q^*) := (\lambda \mathbf{u}' - \nu^{-1} \tilde{\mathbf{v}}_{p'}, -\lambda p')$ , with  $\lambda > 0$ . We obtain:

$$a_0((\mathbf{u}', p'), (\mathbf{v}^*, q^*)) = \nu \lambda \|\mathbf{u}'\|_{\mathbf{H}_0^1(\Omega)}^2 + \nu^{-1} \|p'\|_{L^2(\Omega)}^2 - (\mathbf{u}', \tilde{\mathbf{v}}_{p'})_{\mathbf{H}_0^1(\Omega)}. \quad (3.11)$$

2. In order to bound the last term of (3.11), we use Young inequality and then inequality (3.10), so that for all  $\eta > 0$ :

$$(\mathbf{u}', \tilde{\mathbf{v}}_{p'})_{\mathbf{H}_0^1(\Omega)} \leq \frac{\eta}{2} \|\mathbf{u}'\|_{\mathbf{H}_0^1(\Omega)}^2 + \frac{\eta^{-1}}{2} (C_{\operatorname{div}})^2 \|p'\|_{L^2(\Omega)}^2. \quad (3.12)$$

3. Using the bound (3.12) in (3.11) and choosing  $\eta = \nu \lambda$ , we get:

$$a_0((\mathbf{u}', p'), (\mathbf{v}^*, q^*)) \geq \nu \left( \frac{\lambda}{2} \|\mathbf{u}'\|_{\mathbf{H}_0^1(\Omega)}^2 + \nu^{-2} \left( 1 - \frac{\lambda^{-1}}{2} (C_{\operatorname{div}})^2 \right) \|p'\|_{L^2(\Omega)}^2 \right).$$

Consider now  $\lambda = (C_{\operatorname{div}})^2$ . We obtain:

$$a_0((\mathbf{u}', p'), (\mathbf{v}^*, q^*)) \geq \nu C_{\min} \|(\mathbf{u}', p')\|_{\mathcal{X}, \nu}^2 \text{ where } C_{\min} = \frac{1}{2} \min((C_{\operatorname{div}})^2, 1) = \frac{1}{2}.$$

We obtain (3.9) with  $\alpha = \nu C_{\min}$ . The operator  $T$  such that  $T((\mathbf{u}', p')) = (\mathbf{v}^*, q^*) := ((C_{\text{div}})^2 \mathbf{u}' - \nu^{-1} \tilde{\mathbf{v}}_{p'}, -(C_{\text{div}})^2 p')$  is linear and continuous:

$$\begin{aligned} \|T((\mathbf{u}', p'))\|_{\mathcal{X}, \nu}^2 &:= \|\mathbf{v}^*\|_{\mathbf{H}_0^1(\Omega)}^2 + \nu^{-2} \|q^*\|_{L^2(\Omega)}^2 \\ &\leq 2(C_{\text{div}})^4 \|\mathbf{u}'\|_{\mathbf{H}_0^1(\Omega)}^2 + 2\nu^{-2} \|\tilde{\mathbf{v}}_{p'}\|_{\mathbf{H}_0^1(\Omega)}^2 + (C_{\text{div}})^4 \nu^{-2} \|p'\|_{L^2(\Omega)}^2, \\ &\leq 2(C_{\text{div}})^4 \|\mathbf{u}'\|_{\mathbf{H}_0^1(\Omega)}^2 + ((2(C_{\text{div}})^2 + (C_{\text{div}})^4) \nu^{-2} \|p'\|_{L^2(\Omega)}^2), \\ &\leq (C_{\max})^2 \|(\mathbf{u}', p')\|_{\mathcal{X}, \nu}^2, \end{aligned}$$

where  $C_{\max} = C_{\text{div}} \max(\sqrt{2}C_{\text{div}}, (2 + (C_{\text{div}})^2)^{1/2})$ .

The injectivity of the operator  $T$  is clear. Given  $(\mathbf{v}^*, q^*) \in \mathcal{X}$ , choosing  $(\mathbf{u}', p') = ((C_{\text{div}})^{-2} \mathbf{v}^* - (C_{\text{div}})^{-4} \nu^{-1} \tilde{\mathbf{v}}_{q^*}, -(C_{\text{div}})^{-2} q^*)$  yields  $T((\mathbf{u}', p')) = (\mathbf{v}^*, q^*)$ . Hence, the operator  $T \in \mathcal{L}(\mathcal{X})$  is bijective.  $\square$

**Remark 1.** Let us observe that if we define  $T((\mathbf{u}', p')) := (\lambda \mathbf{u}' - \nu^{-1} \tilde{\mathbf{v}}_{p'}, -\lambda p')$ , then one can check that one has a bijective operator which leads to  $T$ -coercivity as soon as  $\lambda > \frac{1}{4}(C_{\text{div}})^2$ . One can have even more flexibility in the choice of  $T$  by choosing a different factor in front of  $\tilde{\mathbf{v}}_{p'}$ , and then choosing  $\lambda$  accordingly.

We can now prove the following result for the generalized Stokes problem:

**Theorem 2.** *Problem (3.8) is well-posed, so it admits one and only one solution for any  $(\mathbf{f}, g) \in \mathbf{H}^{-1}(\Omega) \times L^2_{zmv}(\Omega)$ . Writing  $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_\perp$  with  $\mathbf{u}_0 \in \mathbf{V}$  and  $\mathbf{u}_\perp \in \mathbf{V}^\perp$ , the solution is such that:*

$$\forall \mathbf{f} \in \mathbf{H}^{-1}(\Omega), \forall g \in L^2_{zmv}(\Omega) \quad \begin{cases} \|\mathbf{u}_\perp\|_{\mathbf{H}_0^1(\Omega)} \leq C_{\text{div}} \|g\|_{L^2(\Omega)}, \\ \|\mathbf{u}_0\|_{\mathbf{H}_0^1(\Omega)} \leq \nu^{-1} \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}, \\ \|p\|_{L^2(\Omega)} \leq C_{\text{div}} \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} + \nu C_{\text{div}}^2 \|g\|_{L^2(\Omega)}. \end{cases} \quad (3.13)$$

*Proof.* According to Prop. 2, the continuous bilinear form  $a_0(\cdot, \cdot)$  is  $T$ -coercive. Hence, according to Theorem 1, Problem (3.8) is well-posed. Let us now derive (3.13). Consider  $(\mathbf{u}, p)$  the unique solution of Problem (3.8), where  $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_\perp$  with  $\mathbf{u}_0 \in \mathbf{V}$  and  $\mathbf{u}_\perp \in \mathbf{V}^\perp$ . Choosing  $\mathbf{v} = 0$ , we obtain that

$\operatorname{div} \mathbf{u}_\perp = g$ , so that  $\|\mathbf{u}_\perp\|_{\mathbf{H}_0^1(\Omega)} \leq C_{\operatorname{div}} \|g\|_{L^2(\Omega)}$ . Now, choosing  $\mathbf{v} = \mathbf{u}_0$  and using orthogonality, we have:  $\nu \|\mathbf{u}_0\|_{\mathbf{H}_0^1(\Omega)}^2 = \langle \mathbf{f}, \mathbf{u}_0 \rangle_{\mathbf{H}_0^1(\Omega)} \leq \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} \|\mathbf{u}_0\|_{\mathbf{H}_0^1(\Omega)}$ , so that:  $\|\mathbf{u}_0\|_{\mathbf{H}_0^1(\Omega)} \leq \nu^{-1} \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)}$ . Next, we choose, in (3.8),  $q = 0$  and  $\mathbf{v} = -\tilde{\mathbf{v}}_p \in \mathbf{V}^\perp$ , where  $\operatorname{div} \tilde{\mathbf{v}}_p = p$  and  $\|\tilde{\mathbf{v}}_p\|_{\mathbf{H}_0^1(\Omega)} \leq C_{\operatorname{div}} \|p\|_{L^2(\Omega)}$  (see Prop. 1). Since  $\mathbf{u}_0 \in \mathbf{V}$ , it holds that  $(\mathbf{u}_0, \tilde{\mathbf{v}}_p)_{\mathbf{H}_0^1(\Omega)} = 0$ . This gives:

$$\begin{aligned} \|p\|_{L^2(\Omega)}^2 &= (p, \operatorname{div} \tilde{\mathbf{v}}_p)_{L^2(\Omega)} = -\langle \mathbf{f}, \tilde{\mathbf{v}}_p \rangle_{\mathbf{H}_0^1(\Omega)} + \nu \langle \Delta \mathbf{u}_\perp, \tilde{\mathbf{v}}_p \rangle_{\mathbf{H}_0^1(\Omega)}, \\ &\leq \left( \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} + \nu \|\mathbf{u}_\perp\|_{\mathbf{H}_0^1(\Omega)} \right) \|\tilde{\mathbf{v}}_p\|_{\mathbf{H}_0^1(\Omega)} \\ &\leq C_{\operatorname{div}} \left( \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} + \nu C_{\operatorname{div}} \|g\|_{L^2(\Omega)} \right) \|p\|_{L^2(\Omega)}, \end{aligned}$$

so that:  $\|p\|_{L^2(\Omega)} \leq C_{\operatorname{div}} \|\mathbf{f}\|_{\mathbf{H}^{-1}(\Omega)} + \nu C_{\operatorname{div}}^2 \|g\|_{L^2(\Omega)}$ .  $\square$

In the following sections, we will consider that  $g = 0$ , so that we solve the classical incompressible Stokes model, except in §6.

#### 4. New variational formulations

We solve the classical incompressible Stokes model, with  $g = 0$ .

##### 4.1. Explicit $T$ -coercivity

Let  $\lambda > \frac{1}{4}(C_{\operatorname{div}})^2$ . In remark 1, we introduced the operator  $T \in \mathcal{L}(\mathcal{X})$  defined by:

$$\begin{cases} T : \mathcal{X} & \rightarrow \mathbb{R} \\ (\mathbf{v}, q) & \mapsto (\lambda \mathbf{v} - \nu^{-1} \tilde{\mathbf{v}}_q, -\lambda q) \end{cases}, \quad (4.1)$$

where  $\tilde{\mathbf{v}}_q \in \mathbf{V}^\perp$  is given by (3.3):  $\operatorname{div} \tilde{\mathbf{v}}_q = q$  and  $\|\tilde{\mathbf{v}}_q\|_{\mathbf{H}_0^1(\Omega)} \leq C_{\operatorname{div}} \|q\|_{L^2(\Omega)}$ . We now write the variational formulation (3.8) with test function  $T((\mathbf{v}, q))$  instead of  $(\mathbf{v}, q)$ . Let us define  $\tilde{a}_\lambda((\mathbf{u}', p'), (\mathbf{v}, q)) = a_0((\mathbf{u}', p'), T(\mathbf{v}, q))$ . We have:

$$\begin{aligned} \tilde{a}_\lambda((\mathbf{u}', p'), (\mathbf{v}, q)) &= \nu \lambda (\mathbf{u}', \mathbf{v})_{\mathbf{H}_0^1(\Omega)} - (\mathbf{u}', \tilde{\mathbf{v}}_q)_{\mathbf{H}_0^1(\Omega)} \\ &\quad - \lambda (p', \operatorname{div} \mathbf{v})_{L^2(\Omega)} + \nu^{-1} (p', q)_{L^2(\Omega)} \\ &\quad + \lambda (q, \operatorname{div} \mathbf{u}')_{L^2(\Omega)}. \end{aligned} \quad (4.2)$$

According to Prop. 2, we have the...

**Proposition 3.** *The bilinear form (4.2) is coercive.*



Introducing  $\ell_\lambda((\mathbf{v}, q)) := \ell_0(T(\mathbf{v}, q))$ , we can propose a first new variational formulation to Problem (3.1)<sub>H</sub>, which reads:

$$\text{Find } (\mathbf{u}, p) \in \mathcal{X} \text{ s.t. } \tilde{a}_\lambda((\mathbf{u}, p), (\mathbf{v}, q)) = \ell_\lambda((\mathbf{v}, q)) \quad \forall (\mathbf{v}, q) \in \mathcal{X}. \quad (4.3)$$

Using the bijectivity of  $T$ , it is obvious that (4.3) is equivalent to (3.8), so well-posedness of (4.3) is a direct consequence of the well-posedness of (3.8), and vice versa.

Below, we provide a direct proof of why solutions to (4.3) actually solve Problem (3.1)<sub>H</sub>. To that aim, we need to provide an "explicit" expression of  $\ell_\lambda((\mathbf{v}, q))$ , which we will prove useful later on. We recall that:

$$\begin{aligned} \forall \mathbf{f}' \in \mathbf{H}^{-1}(\Omega), \quad \exists! (z_{\mathbf{f}'}, \mathbf{w}_{\mathbf{f}'}) \in L^2_{zmv}(\Omega) \times \mathbf{V} \mid \\ \mathbf{f}' = \mathbf{grad} z_{\mathbf{f}'} + \mathbf{curl} \mathbf{curl} \mathbf{w}_{\mathbf{f}'}, \end{aligned} \quad (4.4)$$

where  $(\mathbf{w}_{\mathbf{f}'}, z_{\mathbf{f}'})$  satisfy (3.1)<sub>H, ν=1</sub> with data  $\mathbf{f}'$ . We have in the sense of distributions:

$$-\Delta(\cdot) = -\mathbf{curl} \mathbf{curl}(\cdot) + \mathbf{grad} \operatorname{div}(\cdot). \quad (4.5)$$

**Proposition 4.** *Let  $\mathbf{f}' \in \mathbf{H}^{-1}(\Omega)$ . Given,  $q \in L^2_{zmv}$ , let  $\tilde{\mathbf{v}}_q \in \mathbf{V}^\perp$  be defined by (3.3). We have:*

$$\langle \mathbf{f}', \tilde{\mathbf{v}}_q \rangle_{\mathbf{H}_0^1(\Omega)} = -(z_{\mathbf{f}'}, q)_{L^2(\Omega)}. \quad (4.6)$$

*Proof.* Let  $\mathbf{f}'$  be decomposed as in (4.4).

On the one hand, integrating by parts twice and using (4.5), we get:

$$\begin{aligned} -\langle \mathbf{curl} \mathbf{curl} \mathbf{w}_{\mathbf{f}'}, \tilde{\mathbf{v}}_q \rangle_{\mathbf{H}_0^1(\Omega)} &= -\langle \mathbf{curl} \mathbf{curl} \tilde{\mathbf{v}}_q, \mathbf{w}_{\mathbf{f}'} \rangle_{\mathbf{H}_0^1(\Omega)}, \\ &= -\langle \Delta \tilde{\mathbf{v}}_q, \mathbf{w}_{\mathbf{f}'} \rangle_{\mathbf{H}_0^1(\Omega)} - \langle \mathbf{grad} q, \mathbf{w}_{\mathbf{f}'} \rangle_{\mathbf{H}_0^1(\Omega)}, \\ &= (\tilde{\mathbf{v}}_q, \mathbf{w}_{\mathbf{f}'})_{\mathbf{H}_0^1(\Omega)} + (q, \operatorname{div} \mathbf{w}_{\mathbf{f}'})_{L^2(\Omega)}, \\ &= 0, \end{aligned}$$

resp. since  $\operatorname{div} \tilde{\mathbf{v}}_q = q$ ,  $\tilde{\mathbf{v}}_q \in \mathbf{V}^\perp$ ,  $\mathbf{w}_{\mathbf{f}'} \in \mathbf{V}$ , and  $\operatorname{div} \mathbf{w}_{\mathbf{f}'} = 0$ .

On the other hand, integrating by parts once, we have:

$$\langle \mathbf{grad} z_{\mathbf{f}'}, \tilde{\mathbf{v}}_q \rangle_{\mathbf{H}_0^1(\Omega)} = -(z_{\mathbf{f}'}, q)_{L^2(\Omega)},$$

so the claim follows. □

Hence, using (4.6), we conclude that the right-hand-side  $\ell_\lambda((\mathbf{v}, q))$  writes:

$$\begin{aligned}\ell_\lambda((\mathbf{v}, q)) &= \lambda \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}_0^1(\Omega)} - \nu^{-1} \langle \mathbf{f}, \tilde{\mathbf{v}}_q \rangle_{\mathbf{H}_0^1(\Omega)}, \\ &= \lambda \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}_0^1(\Omega)} + \nu^{-1} (z_{\mathbf{f}}, q)_{L^2(\Omega)}.\end{aligned}$$

With this expression, we can now prove directly why solutions to (4.3) actually solve Problem (3.1)<sub>H</sub>.

**Theorem 3.** *The solution to (4.3) solves Problem (3.1)<sub>H</sub>.*

*Proof.* The bilinear form  $\tilde{a}_\lambda(\cdot, \cdot)$  is continuous and coercive. Let  $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ , and let  $z_{\mathbf{f}}$  be given by (4.4), the linear form  $\ell_\lambda(\cdot)$  is continuous. According to Lax-Milgram Theorem, Problem (4.3) is well-posed.

Choosing  $(\mathbf{v}, 0)$  as test function in (4.3), we obtain that  $-\nu \Delta \mathbf{u} + \mathbf{grad} p = \mathbf{f}$  in the sense of distributions.

Then, choosing  $\mathbf{v} = \tilde{\mathbf{v}}_{\text{div } \mathbf{u}}$  defined by (3.3) and  $q = \lambda \nu \text{div } \mathbf{u}$  in (4.3), we have:

$$\begin{aligned}& \nu \lambda (\mathbf{u}, \tilde{\mathbf{v}}_{\text{div } \mathbf{u}})_{\mathbf{H}_0^1(\Omega)} - (\mathbf{u}, \tilde{\mathbf{v}}_q)_{\mathbf{H}_0^1(\Omega)} \\ & - \lambda (p, \text{div } \mathbf{u})_{L^2(\Omega)} + \lambda (p, \text{div } \mathbf{u})_{L^2(\Omega)} + \lambda^2 \nu \|\text{div } \mathbf{u}\|_{L^2(\Omega)}^2 \\ & = \lambda \langle \mathbf{f}, \tilde{\mathbf{v}}_{\text{div } \mathbf{u}} \rangle_{\mathbf{H}_0^1(\Omega)} + \lambda (z_{\mathbf{f}}, \text{div } \mathbf{u})_{L^2(\Omega)}.\end{aligned}$$

Hence, we obtain that:

$$\lambda \nu \|\text{div } \mathbf{u}\|_{L^2(\Omega)}^2 = \langle \mathbf{f}, \tilde{\mathbf{v}}_{\text{div } \mathbf{u}} \rangle_{\mathbf{H}_0^1(\Omega)} + (z_{\mathbf{f}}, \text{div } \mathbf{u})_{L^2(\Omega)} = 0,$$

due to Prop. 4. The solution  $(\mathbf{u}, p)$  to Problem (4.3) satisfies Problem (3.1)<sub>H</sub>.  $\square$

#### 4.2. Using orthogonality

Going back to the original Problem (3.1)<sub>H</sub>, we note that all solutions  $(\mathbf{u}', p')$  to Problem (4.3) are such that  $\mathbf{u}' \in \mathbf{V}$ . So, in the statement of Problem (4.3), the term  $(\mathbf{u}', \tilde{\mathbf{v}}_q)_{\mathbf{H}_0^1(\Omega)}$  can be removed from the expression (4.2) of the bilinear form  $\tilde{a}_\lambda$ : as a matter of fact, according to (3.3), for all  $q \in L_{zmv}^2(\Omega)$ ,  $(\mathbf{u}', \tilde{\mathbf{v}}_q)_{\mathbf{H}_0^1(\Omega)} = 0$  because  $\tilde{\mathbf{v}}_q \in \mathbf{V}^\perp$ . Interestingly, in this manner one stabilizes the expression (3.11) in the proof of Proposition 2, because the cross term vanishes there: no treatment is required and the



Notice that we can write problem (4.8) with two equations as:

$$\begin{cases} \text{Find } (\mathbf{u}, p) \in \mathcal{X} \text{ s.t. for all } (\mathbf{v}, q) \in \mathcal{X} \\ (i) \quad \nu \lambda(\mathbf{u}, \mathbf{v})_{\mathbf{H}_0^1(\Omega)} - \lambda(p, \operatorname{div} \mathbf{v})_{L^2(\Omega)} = \lambda\langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}_0^1(\Omega)}, \\ (ii) \quad \lambda(q, \operatorname{div} \mathbf{u})_{L^2(\Omega)} + \nu^{-1}(p, q)_{L^2(\Omega)} = \nu^{-1}(z_{\mathbf{f}}, q)_{L^2(\Omega)}. \end{cases} \quad (4.9)$$

This new variational formulation appears as a stabilized variational formulation, in the sense of §II.1.2 in [21], pages 120-123. It can be solved once  $z_f$ , or a suitable approximation of  $z_f$ , is available. This suggests to use (4.9) as a *post processing step* as follows:

- Compute  $z_f$  by solving numerically the Stokes problem (3.4) with  $\nu = 1$ , and data  $\mathbf{f}$ ;
- Solve (4.9) with the data  $\mathbf{f}$  and the computed  $z_f$ .

Notice that we can recover that the solution  $\mathbf{u}$  to (4.9) belongs to  $\mathbf{V}$  in a simple manner: let us split  $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_\perp$ , where  $(\mathbf{u}_0, \mathbf{u}_\perp) \in \mathbf{V} \times \mathbf{V}^\perp$ , so that  $\operatorname{div} \mathbf{u} = \operatorname{div} \mathbf{u}_\perp$ . Choosing  $\mathbf{v} = \mathbf{u}_\perp$  in (4.9)-(i), and  $q = \operatorname{div} \mathbf{u}$  in (4.9)-(ii), we obtain:

$$\begin{cases} (i) \quad \nu \|\mathbf{u}_\perp\|_{\mathbf{H}_0^1(\Omega)}^2 - (p, \operatorname{div} \mathbf{u})_{L^2(\Omega)} = \langle \mathbf{f}, \mathbf{u}_\perp \rangle_{\mathbf{H}_0^1(\Omega)} = -(z_{\mathbf{f}}, \operatorname{div} \mathbf{u})_{L^2(\Omega)}, \\ (ii) \quad \lambda \|\operatorname{div} \mathbf{u}\|_{L^2(\Omega)}^2 + \nu^{-1}(p, \operatorname{div} \mathbf{u})_{L^2(\Omega)} = \nu^{-1}(z_{\mathbf{f}}, \operatorname{div} \mathbf{u})_{L^2(\Omega)}. \end{cases} \quad (4.10)$$

Summing (4.10)-(i) and (4.10)-(ii) times  $\nu$ , we obtain:

$$\nu \left( \|\mathbf{u}_\perp\|_{\mathbf{H}_0^1(\Omega)}^2 + \lambda \|\operatorname{div} \mathbf{u}\|_{L^2(\Omega)}^2 \right) = 0.$$

Hence,  $\mathbf{u}_\perp = 0$  and  $\operatorname{div} \mathbf{u} = 0$ .

#### 4.3. Without orthogonality

Let us briefly consider what would happen if we were to consider a right inverse of the divergence operator, that is different from the one proposed in Prop. 1. For example, with values in the whole of  $\mathbf{H}_0^1(\Omega)$  (not restricted to  $\mathbf{V}^\perp$ ). Such an operator  $q \mapsto \bar{\mathbf{v}}_q$  is considered in Section 3.1 of [12], in the case of a 2D domain with a smooth boundary. For this operator, let  $\bar{C}_{\operatorname{div}}$  be a constant such that:

$$\forall q \in L_{zmv}^2(\Omega), \exists \bar{\mathbf{v}}_q \in \mathbf{H}_0^1(\Omega) \mid \operatorname{div} \bar{\mathbf{v}}_q = q \text{ and } \|\bar{\mathbf{v}}_q\|_{\mathbf{H}_0^1(\Omega)} \leq \bar{C}_{\operatorname{div}} \|q\|_{L^2(\Omega)}. \quad (4.11)$$

In general,  $\bar{\mathbf{v}}_q$  does not belong to  $\mathbf{V}^\perp$ , cf. [12].

Let  $\lambda > \frac{1}{4}(\bar{C}_{\text{div}})^2$  (cf. remark 1).  $T$ -coercivity can be obtained as before, with the operator

$$\begin{cases} \bar{T} : \mathcal{X} & \rightarrow \mathbb{R} \\ (\mathbf{v}, q) & \mapsto (\lambda \mathbf{v} - \nu^{-1} \bar{\mathbf{v}}_q, -\lambda q) \end{cases}, \quad (4.12)$$

Define  $\bar{a}_\lambda((\mathbf{u}', p'), (\mathbf{v}, q)) = a_0((\mathbf{u}', p'), \bar{T}(\mathbf{v}, q))$ . We have:

$$\begin{aligned} \bar{a}_\lambda((\mathbf{u}', p'), (\mathbf{v}, q)) &= \nu \lambda (\mathbf{u}', \mathbf{v})_{\mathbf{H}_0^1(\Omega)} - (\mathbf{u}', \bar{\mathbf{v}}_q)_{\mathbf{H}_0^1(\Omega)} \\ &\quad - \lambda (p', \text{div } \mathbf{v})_{L^2(\Omega)} + \nu^{-1} (p', q)_{L^2(\Omega)} \\ &\quad + \lambda (q, \text{div } \mathbf{u}')_{L^2(\Omega)}. \end{aligned} \quad (4.13)$$

However, for  $\mathbf{u}' \in \mathbf{V}$  (that is for solutions  $(\mathbf{u}', p)$  to Problem (4.3)), the term  $(\mathbf{u}', \bar{\mathbf{v}}_q)_{\mathbf{H}_0^1(\Omega)}$  can no longer be removed from the expression (4.2) of the bilinear form  $\bar{a}_\lambda$ . As a matter of fact one has

$$(\mathbf{u}', \bar{\mathbf{v}}_q)_{\mathbf{H}_0^1(\Omega)} = (\mathbf{curl } \mathbf{u}', \mathbf{curl } \bar{\mathbf{v}}_q)_{\mathbf{L}^2(\Omega)}.$$

Hence, in the non-orthogonal case, one has to choose an ad hoc constant  $\lambda > \frac{1}{4}(\bar{C}_{\text{div}})^2$  to ensure  $T$ -coercivity (cf. Prop. 2).

Similarly in the expression of the right-hand side  $\bar{\ell}_\lambda$ .

**Proposition 6.** *Let  $\mathbf{f}' \in \mathbf{H}^{-1}(\Omega)$  be decomposed as in (4.4). Given,  $q \in L^2_{zmv}(\Omega)$ , let  $\bar{\mathbf{v}}_q \in \mathbf{H}_0^1(\Omega)$  be defined by (4.11). We have:*

$$\langle \mathbf{f}', \bar{\mathbf{v}}_q \rangle_{\mathbf{H}_0^1(\Omega)} = -(z_{\mathbf{f}'}, q)_{L^2(\Omega)} - (\mathbf{curl } \mathbf{w}_{\mathbf{f}'}, \mathbf{curl } \bar{\mathbf{v}}_q)_{\mathbf{L}^2(\Omega)}. \quad (4.14)$$

*Proof.* Straightforward.  $\square$

To conclude on the use of explicit  $T$ -coercivity in the non-orthogonal case, we observe that the problem, if split as in (4.9), leads to a more intricate variational formulation, which reads

$$\begin{cases} \text{Find } (\mathbf{u}, p) \in \mathcal{X} \text{ s.t. for all } (\mathbf{v}, q) \in \mathcal{X} \\ (i)' \quad \nu \lambda (\mathbf{u}, \mathbf{v})_{\mathbf{H}_0^1(\Omega)} - \lambda (p, \text{div } \mathbf{v})_{L^2(\Omega)} = \lambda \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}_0^1(\Omega)}, \\ (ii)' \quad - (\mathbf{u}, \bar{\mathbf{v}}_q)_{\mathbf{H}_0^1(\Omega)} + \lambda (q, \text{div } \mathbf{u})_{L^2(\Omega)} + \nu^{-1} (p, q)_{L^2(\Omega)} \\ \quad \quad \quad = \nu^{-1} (z_{\mathbf{f}}, q)_{L^2(\Omega)} + \nu^{-1} (\mathbf{curl } \mathbf{w}_{\mathbf{f}}, \mathbf{curl } \bar{\mathbf{v}}_q)_{\mathbf{L}^2(\Omega)}. \end{cases}$$

Regarding the right-hand side in (ii)', it requires the knowledge of both  $z_f$  and of  $\mathbf{w}_{\mathbf{f}}$  (or of suitable approximations): a *two-step* procedure could be used

as before (see the end of §4.2). One needs to evaluate also  $(\mathbf{u}, \bar{\mathbf{v}}_q)_{\mathbf{H}_0^1(\Omega)} = (\mathbf{curl} \mathbf{u}, \mathbf{curl} \bar{\mathbf{v}}_q)_{\mathbf{L}^2(\Omega)}$  in the left-hand side of (ii'). The latter part requires some knowledge of the right inverse of the divergence operator, and it does not seem to be achievable numerically at a reasonable cost.

## 5. Discretization

We solve the classical incompressible Stokes model, with  $g = 0$ .

### 5.1. A foreword on classical discretizations

To fix ideas, let us consider a discretization to solve the classical variational formulation (3.4) (conforming Taylor-Hood finite elements [29], non-conforming Crouzeix-Raviart finite elements [18], etc.). Let us call the (conforming) discrete spaces  $\mathbf{V}_{0,h} \subset \mathbf{H}_0^1(\Omega)$  and  $Q_h \subset L_{zmv}^2(\Omega)$ . Then to prove the discrete  $T$ -coercivity, we need to state the discrete counterpart to Proposition 1. To do so, we can build a family of linear operators  $\Pi_c^h : \mathbf{H}_0^1(\Omega) \rightarrow \mathbf{V}_{0,h}$ , known as Fortin operators, such that (see a.e. [10, §8.4.1]):

$$\exists C_c \mid \forall h, \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega) \quad \|\mathbf{Grad} \Pi_c^h \mathbf{v}\|_{\mathbf{L}^2(\Omega)} \leq C_c \|\mathbf{Grad} \mathbf{v}\|_{\mathbf{L}^2(\Omega)}, \quad (5.1)$$

$$\forall h, \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \forall q_h \in Q_h \quad (\operatorname{div} \Pi_c^h \mathbf{v}, q_h)_{L^2(\Omega)} = (\operatorname{div} \mathbf{v}, q_h)_{L^2(\Omega)}. \quad (5.2)$$

In the conforming case, we rely on the bilinear form  $a_0(\cdot, \cdot)$  to state the discrete variational formulation. On the other hand, using a nonconforming discretization, although we will not use the bilinear form  $a_0(\cdot, \cdot)$  to exhibit the discrete variational formulation, we will need a similar operator to (5.1)-(5.2) to prove the discrete  $T$ -coercivity [24].

### 5.2. Discretizations

Consider  $(\mathcal{T}_h)_h$  a simplicial triangulation sequence of  $\Omega$ . For all  $D \subset \mathbb{R}^d$ , and  $k \in \mathbb{N}$ , we call  $P^k(D)$  the set of order  $k$  polynomials on  $D$ ,  $\mathbf{P}^k(D) = (P^k(D))^d$ , and we consider the broken polynomial space:

$$P_{disc}^k(\mathcal{T}_h) = \{q \in L^2(\Omega); \quad \forall T \in \mathcal{T}_h, q|_T \in P^k(T)\}.$$

We use the notation  $P^0(\mathcal{T}_h)$  for  $P_{disc}^0(\mathcal{T}_h)$ .

We define the spaces of  $P^k$ -Lagrange functions:

$$\begin{aligned} V_h^k &:= \{v_h \in H^1(\Omega); \quad \forall T \in \mathcal{T}_h, v_h|_T \in P^k(T)\} \\ V_{0,h}^k &:= \{v_h \in V_h^k; \quad v_h|_{\partial\Omega} = 0\}. \end{aligned}$$

We set  $\mathbf{V}_h^k := (V_h^k)^d$  and  $\mathbf{V}_{0,h}^k := (V_{0,h}^k)^d$ . We call  $Q_h^k := P_{disc}^k(\mathcal{T}_h) \cap L_{zmv}^2(\Omega)$ . Notice that  $\mathbf{V}_{0,h}^k \times Q_h^{k'} \subset \mathbf{H}_0^1(\Omega) \times L_{zmv}^2(\Omega)$  and  $\text{div } \mathbf{V}_{0,h}^k \subset Q_h^{k-1}$ .

It is well known that discretizing (3.4) with  $\mathbf{P}^k - P_{disc}^{k-1}$  finite element,  $k \geq 1$ , is not stable on all shape regular meshes [10]. A wide range of strategies to get round this problem have been explored for years. Below is a quick overview of these strategies:

- The discrete velocity space can be enriched see [8, 3, 6, 13, 26] for  $k = 1$ .
- The discrete variational formulation can be stabilized, see [9, 7, 31, 2] for  $k = 1$ , [1] for  $k = 2$ .
- The mesh can be designed in such a way that the discrete inf-sup condition applies, see [28, 30, 5] for  $k > 1$ .

Let  $k \geq 1$ . The (conforming) discretization of Problem (4.9) with  $\mathbf{P}^k - P_{disc}^{k-1}$  finite element reads:

$$\left\{ \begin{array}{l} \text{Find } (\mathbf{u}_h, p_h) \in \mathbf{V}_{0,h}^k \times Q_h^{k-1} \text{ s.t. for all } (\mathbf{v}_h, q_h) \in \mathbf{V}_{0,h}^k \times Q_h^{k-1} : \\ \nu \lambda(\mathbf{u}_h, \mathbf{v}_h)_{\mathbf{H}_0^1(\Omega)} - \lambda(p_h, \text{div } \mathbf{v}_h)_{L^2(\Omega)} = \lambda(\mathbf{f}, \mathbf{v}_h)_{\mathbf{H}_0^1(\Omega)}, \\ \lambda(q_h, \text{div } \mathbf{u}_h)_{L^2(\Omega)} + \nu^{-1} (p_h, q_h)_{L^2(\Omega)} = \nu^{-1} (z_{\mathbf{f}}, q_h)_{L^2(\Omega)}. \end{array} \right. \quad (5.3)$$

Let us write  $\mathbf{f} = -\nu \Delta \mathbf{u} + \mathbf{grad } z_{\mathbf{f}}$  (i.e.  $p = z_{\mathbf{f}}$ ), so that Problem (5.3) reads:

$$\left\{ \begin{array}{l} \text{Find } (\mathbf{u}_h, p_h) \in \mathbf{V}_{0,h}^k \times Q_h^{k-1} \text{ s.t. for all } (\mathbf{v}_h, q_h) \in \mathbf{V}_{0,h}^k \times Q_h^{k-1} : \\ \nu \lambda(\mathbf{u}_h, \mathbf{v}_h)_{\mathbf{H}_0^1(\Omega)} - \lambda(p_h, \text{div } \mathbf{v}_h)_{L^2(\Omega)} = \nu \lambda(\mathbf{u}, \mathbf{v}_h)_{\mathbf{H}_0^1(\Omega)} \\ \quad \quad \quad - \lambda(z_{\mathbf{f}}, \text{div } \mathbf{v}_h)_{L^2(\Omega)}, \\ \lambda(q_h, \text{div } \mathbf{u}_h)_{L^2(\Omega)} + \nu^{-1} (p_h, q_h)_{L^2(\Omega)} = \nu^{-1} (z_{\mathbf{f}}, q_h)_{L^2(\Omega)}. \end{array} \right. \quad (5.4)$$

### 5.3. Convergence when $z_{\mathbf{f}}$ is known

Let  $\Pi_{h,cg}$  be the  $\mathbf{L}^2(\Omega)$ -orthogonal projection operator from  $\mathbf{L}^2(\Omega)$  to  $\mathbf{V}_{0,h}^k$  and  $\Pi_{h,dg}$  be the  $L^2(\Omega)$ -orthogonal projection operator from  $L^2(\Omega)$  to  $P_{disc}^{k-1}(\mathcal{T}_h)$ . We have the

**Proposition 7.** *Let  $(\mathbf{u}, p) \in \mathcal{X}$  be the solution to Problem (4.9) and  $(\mathbf{u}_h, p_h) \in \mathcal{X}_h$  be the solution to Problem (5.3). We have the following error estimate:*

$$\|(\mathbf{u}_h - \mathbf{u}, p_h - \Pi_{h,dg} z_{\mathbf{f}})\|_{\mathcal{X}, \sqrt{\lambda\nu}} \leq (\sqrt{d\lambda} + 1) \|\mathbf{u} - \Pi_{h,cg} \mathbf{u}\|_{\mathbf{H}_0^1(\Omega)}. \quad (5.5)$$

Notice that the error estimate for the velocity is fully independent of the pressure. Hence, the (conforming) discretization of Problem (4.9) with  $\mathbf{P}^k - P_{disc}^{k-1}$  is a so called *pressure robust method* for which the discrete velocity  $\mathbf{u}_h$  tends to  $\mathbf{u}$  independently of  $\nu$ . The discrete pressure  $p_h$  tends to  $\Pi_{h,dg} z_{\mathbf{f}}$  all the faster the smaller  $\nu$  is. Interestingly, we remark that if  $\mathbf{u} = 0$ , then  $\Pi_{h,cg} \mathbf{u} = 0$ , so that  $\mathbf{u}_h = 0$  and  $p_h = \Pi_{h,dg} z_{\mathbf{f}}$ , even for  $k = 1$  (see Section 7.3). Let us now prove Proposition 7.

*Proof.* Setting  $z_{\mathbf{f},h} = \Pi_{h,dg} z_{\mathbf{f}}$ , we have:  $(z_{\mathbf{f}}, \operatorname{div} \mathbf{v}_h)_{L^2(\Omega)} = (z_{\mathbf{f},h}, \operatorname{div} \mathbf{v}_h)_{L^2(\Omega)}$  and  $(z_{\mathbf{f}}, q_h)_{L^2(\Omega)} = (z_{\mathbf{f},h}, q_h)_{L^2(\Omega)}$  for all  $\mathbf{v}_h \in \mathbf{V}_{0,h}^k$ , and all  $q_h \in Q_h^{k-1}$ . Summing both equations of Problem (5.4) and reshuffling terms, it comes:

$$\left\{ \begin{array}{l} \text{Find } (\mathbf{u}_h, p_h) \in \mathbf{V}_{0,h}^k \times Q_h^{k-1} \text{ s.t. for all } (\mathbf{v}_h, q_h) \in \mathbf{V}_{0,h}^k \times Q_h^{k-1} : \\ \nu \lambda (\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h)_{\mathbf{H}_0^1(\Omega)} + \nu^{-1} (p_h - z_{\mathbf{f},h}, q_h)_{L^2(\Omega)} \\ = \lambda (p_h - z_{\mathbf{f},h}, \operatorname{div} \mathbf{v}_h)_{L^2(\Omega)} - \lambda (q_h, \operatorname{div} \mathbf{u}_h)_{L^2(\Omega)}. \end{array} \right. \quad (5.6)$$

Choosing  $\mathbf{v}_h = \mathbf{u}_h - \Pi_{h,cg} \mathbf{u} = (\mathbf{u}_h - \mathbf{u}) + (\mathbf{u} - \Pi_{h,cg} \mathbf{u})$  and  $q_h = p_h - z_{\mathbf{f},h}$  in (5.6), and dividing by  $\lambda \nu$ , we obtain:

$$\begin{aligned} \|\mathbf{u}_h - \mathbf{u}\|_{\mathbf{H}_0^1(\Omega)}^2 + (\mathbf{u}_h - \mathbf{u}, \mathbf{u} - \Pi_{h,cg} \mathbf{u})_{\mathbf{H}_0^1(\Omega)} + \lambda^{-1} \nu^{-2} \|p_h - z_{\mathbf{f},h}\|_{L^2(\Omega)}^2 \\ = -\nu^{-1} (p_h - z_{\mathbf{f},h}, \operatorname{div} \Pi_{h,cg} \mathbf{u})_{L^2(\Omega)}. \end{aligned}$$

Hence, using Cauchy-Schwarz inequality, we get:

$$\begin{aligned} & \|\mathbf{u}_h - \mathbf{u}\|_{\mathbf{H}_0^1(\Omega)}^2 + \lambda^{-1} \nu^{-2} \|p_h - z_{\mathbf{f},h}\|_{L^2(\Omega)}^2 \\ & \leq \|\mathbf{u}_h - \mathbf{u}\|_{\mathbf{H}_0^1(\Omega)} \|\mathbf{u} - \Pi_{h,cg} \mathbf{u}\|_{\mathbf{H}_0^1(\Omega)} + \nu^{-1} \|p_h - z_{\mathbf{f},h}\|_{L^2(\Omega)} \|\operatorname{div} \Pi_{h,cg} \mathbf{u}\|_{L^2(\Omega)}. \end{aligned}$$

Observe that  $\|\mathbf{u}_h - \mathbf{u}\|_{\mathbf{H}_0^1(\Omega)}$  is bounded by  $\|(\mathbf{u}_h - \mathbf{u}, p_h - z_{\mathbf{f},h})\|_{\mathcal{X}, \sqrt{\lambda} \nu}$ , while  $\nu^{-1} \|p_h - z_{\mathbf{f},h}\|_{L^2(\Omega)}$  is bounded by  $\sqrt{\lambda} \|(\mathbf{u}_h - \mathbf{u}, p_h - z_{\mathbf{f},h})\|_{\mathcal{X}, \sqrt{\lambda} \nu}$ . Finally  $\|\operatorname{div} \Pi_{h,cg} \mathbf{u}\|_{L^2(\Omega)} \leq \sqrt{d} \|\mathbf{u} - \Pi_{h,cg} \mathbf{u}\|_{\mathbf{H}_0^1(\Omega)}$ , so we have:

$$\|(\mathbf{u}_h - \mathbf{u}, p_h - z_{\mathbf{f},h})\|_{\mathcal{X}, \sqrt{\lambda} \nu} \leq (\sqrt{d} \lambda + 1) \|\mathbf{u} - \Pi_{h,cg} \mathbf{u}\|_{\mathbf{H}_0^1(\Omega)}.$$

□

#### 5.4. Convergence when $z_{\mathbf{f}}$ is not known

If  $z_{\mathbf{f}}$  is not known explicitly, let us assume that we have at hand an approximation  $\tilde{z}_{\mathbf{f},h} \in Q_h^{k-1}$  and use it to solve the following approximation of



Problem (4.9):

$$\begin{cases} \text{Find } (\tilde{\mathbf{u}}, \tilde{p}) \in \mathcal{X} \text{ s.t. for all } (\mathbf{v}, q) \in \mathcal{X} : \\ \nu \lambda (\tilde{\mathbf{u}}, \mathbf{v})_{\mathbf{H}_0^1(\Omega)} - \lambda (\tilde{p}, \operatorname{div} \mathbf{v})_{L^2(\Omega)} = \lambda \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}_0^1(\Omega)}, \\ \lambda (q, \operatorname{div} \tilde{\mathbf{u}})_{L^2(\Omega)} + \nu^{-1} (\tilde{p}, q)_{L^2(\Omega)} = \nu^{-1} (\tilde{z}_{\mathbf{f},h}, q)_{L^2(\Omega)}. \end{cases} \quad (5.7)$$

Let us write again  $\mathbf{f} = -\nu \Delta \mathbf{u} + \mathbf{grad} z_{\mathbf{f}}$ , so that Problem (5.7) reads:

$$\begin{cases} \text{Find } (\tilde{\mathbf{u}}, \tilde{p}) \in \mathcal{X} \text{ s.t. for all } (\mathbf{v}, q) \in \mathcal{X} : \\ \nu \lambda (\tilde{\mathbf{u}} - \mathbf{u}, \mathbf{v})_{\mathbf{H}_0^1(\Omega)} - \lambda (\tilde{p} - z_{\mathbf{f}}, \operatorname{div} \mathbf{v})_{L^2(\Omega)} = 0, \\ \lambda (q, \operatorname{div} \tilde{\mathbf{u}})_{L^2(\Omega)} + \nu^{-1} (\tilde{p} - \tilde{z}_{\mathbf{f},h}, q)_{L^2(\Omega)} = 0. \end{cases} \quad (5.8)$$

**Proposition 8.** *Let  $(\mathbf{u}, p) \in \mathcal{X}$  be the solution to Problem (4.9) and  $(\tilde{\mathbf{u}}, \tilde{p}) \in \mathcal{X}$  be the solution to Problem (5.7). We have the following estimates:*

$$\begin{cases} \|\tilde{p} - z_{\mathbf{f}}\|_{L^2(\Omega)} \leq \|z_{\mathbf{f}} - \tilde{z}_{\mathbf{f},h}\|_{L^2(\Omega)}, \\ \|\tilde{\mathbf{u}} - \mathbf{u}\|_{\mathbf{H}_0^1(\Omega)} \leq (\sqrt{\lambda} \nu)^{-1} \|z_{\mathbf{f}} - \tilde{z}_{\mathbf{f},h}\|_{L^2(\Omega)}. \end{cases} \quad (5.9)$$

*Proof.* Choosing  $\mathbf{v} = \tilde{\mathbf{u}} - \mathbf{u}$  and  $q = \tilde{p} - z_{\mathbf{f}}$  in (5.8), summing both equations and dividing by  $\lambda \nu$ , it comes:

$$\|\tilde{\mathbf{u}} - \mathbf{u}\|_{\mathbf{H}_0^1(\Omega)}^2 + \lambda^{-1} \nu^{-2} (\tilde{p} - \tilde{z}_{\mathbf{f},h}, \tilde{p} - z_{\mathbf{f}})_{L^2(\Omega)} = 0.$$

Writing  $\tilde{p} - \tilde{z}_{\mathbf{f},h} = (\tilde{p} - z_{\mathbf{f}}) + (z_{\mathbf{f}} - \tilde{z}_{\mathbf{f},h})$ , we obtain:

$$\begin{aligned} \|(\tilde{\mathbf{u}} - \mathbf{u}, \tilde{p} - z_{\mathbf{f}})\|_{\mathcal{X}, \sqrt{\lambda} \nu}^2 &= -\lambda^{-1} \nu^{-2} (z_{\mathbf{f}} - \tilde{z}_{\mathbf{f},h}, \tilde{p} - z_{\mathbf{f}})_{L^2(\Omega)} \\ &\leq \lambda^{-1} \nu^{-2} \|z_{\mathbf{f}} - \tilde{z}_{\mathbf{f},h}\|_{L^2(\Omega)} \|\tilde{p} - z_{\mathbf{f}}\|_{L^2(\Omega)}. \end{aligned}$$

We obtain successively estimates (5.9).  $\square$

The discretization of Problem (5.7) with  $\mathbf{P}^k - P_{disc}^{k-1}$  finite element reads:

$$\begin{cases} \text{Find } (\tilde{\mathbf{u}}_h, \tilde{p}_h) \in \mathbf{V}_{0,h}^k \times Q_h^{k-1} \text{ s.t. for all } (\mathbf{v}_h, q_h) \in \mathbf{V}_{0,h}^k \times Q_h^{k-1} : \\ \nu \lambda (\tilde{\mathbf{u}}_h, \mathbf{v}_h)_{\mathbf{H}_0^1(\Omega)} - \lambda (\tilde{p}_h, \operatorname{div} \mathbf{v}_h)_{L^2(\Omega)} = \lambda \langle \mathbf{f}, \mathbf{v}_h \rangle_{\mathbf{H}_0^1(\Omega)}, \\ \lambda (q_h, \operatorname{div} \tilde{\mathbf{u}}_h)_{L^2(\Omega)} + \nu^{-1} (\tilde{p}_h, q_h)_{L^2(\Omega)} = \nu^{-1} (\tilde{z}_{\mathbf{f},h}, q_h)_{L^2(\Omega)}. \end{cases} \quad (5.10)$$

**Theorem 5.** *Let  $(\mathbf{u}, p) \in \mathcal{X}$  be the solution to Problem (4.9) and  $(\tilde{\mathbf{u}}_h, \tilde{p}_h) \in \mathcal{X}$  be the solution to Problem (5.10). We have the following error estimate:*

$$\begin{aligned} \|(\tilde{\mathbf{u}}_h - \mathbf{u}, \tilde{p}_h - \Pi_{h,dg} z_{\mathbf{f}})\|_{\mathcal{X}, \sqrt{\lambda} \nu} &\leq (\sqrt{d\lambda} + 1) \|\mathbf{u} - \Pi_{h,cg} \mathbf{u}\|_{\mathbf{H}_0^1(\Omega)} \\ &\quad + (\sqrt{\lambda} \nu)^{-1} \|\Pi_{h,dg} z_{\mathbf{f}} - \tilde{z}_{\mathbf{f},h}\|_{L^2(\Omega)}. \end{aligned} \quad (5.11)$$

Suppose that  $\mathbf{u} = 0$ . Then  $\Pi_{h,cg} \mathbf{u} = 0$  and we obtain the estimate below:

$$\|(\tilde{\mathbf{u}}_h, \tilde{p}_h - \Pi_{h,dg} z_{\mathbf{f}})\|_{\mathcal{X}, \sqrt{\lambda} \nu} \leq (\sqrt{\lambda} \nu)^{-1} \|\Pi_{h,dg} z_{\mathbf{f}} - \tilde{z}_{\mathbf{f},h}\|_{L^2(\Omega)}. \quad (5.12)$$

In particular, if  $\tilde{z}_{\mathbf{f},h}$  is a convergent approximation of  $\Pi_{h,dg}z_{\mathbf{f}}$ , the solution  $(\tilde{\mathbf{u}}_h, \tilde{p}_h)$  converges to  $(\mathbf{u}, \Pi_{h,dg}z_{\mathbf{f}})$ . Interestingly, the above with  $k = 1$  corresponds to the  $\mathbf{P}^1 - P^0$  finite element pair, which is known to be unstable for the discretization of the usual variational formulation (3.4) of the Stokes problem. Let us prove Theorem 5.

*Proof.* Setting  $z_{\mathbf{f},h} = \Pi_{h,dg}z_{\mathbf{f}}$ , we have:  $\langle \mathbf{f}, \mathbf{v}_h \rangle_{\mathbf{H}_0^1(\Omega)} = \nu (\mathbf{u}, \mathbf{v}_h)_{\mathbf{H}_0^1(\Omega)} - (z_{\mathbf{f},h}, \operatorname{div} \mathbf{v}_h)_{L^2(\Omega)}$  for all  $\mathbf{v}_h \in \mathbf{V}_{0,h}^k$ . Hence, Problem (5.10) can be written as:

$$\begin{cases} \text{Find } (\tilde{\mathbf{u}}_h, \tilde{p}_h) \in \mathbf{V}_{0,h}^k \times Q_h^{k-1} \text{ s.t. for all } (\mathbf{v}_h, q_h) \in \mathbf{V}_{0,h}^k \times Q_h^{k-1} : \\ \nu \lambda (\tilde{\mathbf{u}}_h - \mathbf{u}, \mathbf{v}_h)_{\mathbf{H}_0^1(\Omega)} - \lambda (\tilde{p}_h - z_{\mathbf{f},h}, \operatorname{div} \mathbf{v}_h)_{L^2(\Omega)} = 0, \\ \lambda (q_h, \operatorname{div} \tilde{\mathbf{u}}_h)_{L^2(\Omega)} + \nu^{-1} (\tilde{p}_h - \tilde{z}_{\mathbf{f},h}, q_h)_{L^2(\Omega)} = 0. \end{cases} \quad (5.13)$$

Choosing  $(\mathbf{v}_h, q_h) = (\tilde{\mathbf{u}}_h - \Pi_{h,cg}\mathbf{u}, \tilde{p}_h - z_{\mathbf{f},h})$  in Problem (5.13), we have:

$$\begin{cases} \nu \lambda (\tilde{\mathbf{u}}_h - \mathbf{u}, \tilde{\mathbf{u}}_h - \Pi_{h,cg}\mathbf{u})_{\mathbf{H}_0^1(\Omega)} - \lambda (\tilde{p}_h - z_{\mathbf{f},h}, \operatorname{div}(\tilde{\mathbf{u}}_h - \Pi_{h,cg}\mathbf{u}))_{L^2(\Omega)} = 0, \\ \lambda (\tilde{p}_h - z_{\mathbf{f},h}, \operatorname{div} \tilde{\mathbf{u}}_h)_{L^2(\Omega)} + \nu^{-1} (\tilde{p}_h - \tilde{z}_{\mathbf{f},h}, \tilde{p}_h - z_{\mathbf{f},h})_{L^2(\Omega)} = 0. \end{cases}$$

Summing both equations and dividing by  $\lambda\nu$ , it now comes:

$$\begin{aligned} & (\tilde{\mathbf{u}}_h - \mathbf{u}, \tilde{\mathbf{u}}_h - \Pi_{h,cg}\mathbf{u})_{\mathbf{H}_0^1(\Omega)} + \lambda^{-1}\nu^{-2} (\tilde{p}_h - \tilde{z}_{\mathbf{f},h}, \tilde{p}_h - z_{\mathbf{f},h})_{L^2(\Omega)} \\ & + \nu^{-1} (\tilde{p}_h - z_{\mathbf{f},h}, \operatorname{div} \Pi_{h,cg}\mathbf{u})_{L^2(\Omega)} = 0. \end{aligned}$$

Noticing that  $\tilde{\mathbf{u}}_h - \Pi_{h,cg}\mathbf{u} = (\tilde{\mathbf{u}}_h - \mathbf{u}) + (\mathbf{u} - \Pi_{h,cg}\mathbf{u})$  and  $\tilde{p}_h - \tilde{z}_{\mathbf{f},h} = (\tilde{p}_h - z_{\mathbf{f},h}) + (z_{\mathbf{f},h} - \tilde{z}_{\mathbf{f},h})$ , we get:

$$\begin{aligned} & \|(\tilde{\mathbf{u}}_h - \mathbf{u}, \tilde{p}_h - z_{\mathbf{f},h})\|_{\mathcal{X}, \sqrt{\lambda\nu}}^2 + (\tilde{\mathbf{u}}_h - \mathbf{u}, \mathbf{u} - \Pi_{h,cg}\mathbf{u})_{\mathbf{H}_0^1(\Omega)} \\ & + \lambda^{-1}\nu^{-2} (z_{\mathbf{f},h} - \tilde{z}_{\mathbf{f},h}, \tilde{p}_h - z_{\mathbf{f},h})_{L^2(\Omega)} + \nu^{-1} (\tilde{p}_h - z_{\mathbf{f},h}, \operatorname{div} \Pi_{h,cg}\mathbf{u})_{L^2(\Omega)} = 0. \end{aligned}$$

Using Cauchy-Schwarz, we deduce that:

$$\begin{aligned} & \|(\tilde{\mathbf{u}}_h - \mathbf{u}, \tilde{p}_h - z_{\mathbf{f},h})\|_{\mathcal{X}, \sqrt{\lambda\nu}}^2 \leq \| \tilde{\mathbf{u}}_h - \mathbf{u} \|_{\mathbf{H}_0^1(\Omega)} \| \mathbf{u} - \Pi_{h,cg}\mathbf{u} \|_{\mathbf{H}_0^1(\Omega)} \\ & + (\sqrt{\lambda\nu})^{-1} \| \tilde{p}_h - z_{\mathbf{f},h} \|_{L^2(\Omega)} \left( (\sqrt{\lambda\nu})^{-1} \| z_{\mathbf{f},h} - \tilde{z}_{\mathbf{f},h} \|_{L^2(\Omega)} + \sqrt{\lambda} \| \operatorname{div} \Pi_{h,cg}\mathbf{u} \|_{L^2(\Omega)} \right). \end{aligned}$$

Since  $\| \tilde{\mathbf{u}}_h - \mathbf{u} \|_{\mathbf{H}_0^1(\Omega)}$  and  $(\sqrt{\lambda\nu})^{-1} \| \tilde{p}_h - z_{\mathbf{f},h} \|_{L^2(\Omega)}$  are both bounded by  $\|(\tilde{\mathbf{u}}_h - \mathbf{u}, \tilde{p}_h - z_{\mathbf{f},h})\|_{\mathcal{X}, \sqrt{\lambda\nu}}$ , and  $\| \operatorname{div} \Pi_{h,cg}\mathbf{u} \|_{L^2(\Omega)} \leq \sqrt{d} \| \mathbf{u} - \Pi_{h,cg}\mathbf{u} \|_{\mathbf{H}_0^1(\Omega)}$ , we obtain the estimates (5.11) and (5.12).  $\square$

## 6. Nonhomogeneous Dirichlet boundary conditions

We solve the classical incompressible Stokes model, with  $g = 0$  and non zero Dirichlet boundary conditions. Hence, we focus now on Problem (3.1)<sub>NH</sub> with  $g = 0$ . The variational formulation of Problem (3.1)<sub>NH</sub> reads:

Find  $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega) \times L^2_{zmv}(\Omega)$  such that

$$\begin{cases} \nu(\mathbf{u}, \mathbf{v})_{\mathbf{H}_0^1(\Omega)} - (p, \operatorname{div} \mathbf{v})_{L^2(\Omega)} = \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}_0^1(\Omega)} & \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ (q, \operatorname{div} \mathbf{u})_{L^2(\Omega)} = 0 & \forall q \in L^2_{zmv}(\Omega), \\ \mathbf{u} = \mathbf{g} \text{ on } \partial\Omega, \end{cases} \quad (6.1)$$

where  $\mathbf{g} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$  is some boundary data such that  $\mathbf{g} \cdot \mathbf{n} \in L^2_{zmv}(\partial\Omega)$ . Let  $\mathbf{u}_{\mathbf{g}} \in \mathbf{H}^1(\Omega)$  be such that  $\mathbf{u}_{\mathbf{g}} = \mathbf{g}$  on  $\partial\Omega$  and  $-\Delta \mathbf{u}_{\mathbf{g}} = 0$  in  $\Omega$ . Then consider  $(\mathbf{u}_0, p) \in \mathbf{H}_0^1(\Omega) \times L^2_{zmv}(\Omega)$  the unique solution to (3.1)<sub>H</sub> with data  $(\mathbf{f} + \nu \Delta \mathbf{u}_{\mathbf{g}}, -\operatorname{div} \mathbf{u}_{\mathbf{g}})$ :

Find  $(\mathbf{u}_0, p) \in \mathcal{X}$  such that for all  $(\mathbf{v}, q) \in \mathcal{X}$

$$a_0((\mathbf{u}_0, p), (\mathbf{v}, q)) = \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}_0^1(\Omega)} + (q, \operatorname{div} \mathbf{u}_{\mathbf{g}})_{L^2(\Omega)}, \quad (6.2)$$

where we use that  $(\mathbf{u}_{\mathbf{g}}, \mathbf{v})_{\mathbf{H}_0^1(\Omega)} = 0$ . Defining  $\mathbf{u} = \mathbf{u}_{\mathbf{g}} + \mathbf{u}_0 \in \mathbf{H}^1(\Omega)$ , we find that  $(\mathbf{u}, p)$  is solution to Problem (6.1). Uniqueness and continuous dependence with respect to the data are easily obtained. The next step is to replace  $(\mathbf{v}, q)$  by  $T((\mathbf{v}, q)) = (\lambda \mathbf{v} - \nu^{-1} \mathbf{v}_q, -\lambda q)$ . One finds by orthogonality that for all  $(\mathbf{v}, q) \in \mathcal{X}$ :

$$\begin{aligned} \nu \lambda (\mathbf{u}_0, \mathbf{v})_{\mathbf{H}_0^1(\Omega)} - \lambda (p, \operatorname{div} \mathbf{v})_{L^2(\Omega)} + \nu^{-1} (p, q)_{L^2(\Omega)} + \lambda (q, \operatorname{div} \mathbf{u}_0)_{L^2(\Omega)} \\ = \lambda \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}_0^1(\Omega)} - \nu^{-1} \langle \mathbf{f}, \mathbf{v}_q \rangle_{\mathbf{H}_0^1(\Omega)} - \lambda (q, \operatorname{div} \mathbf{u}_{\mathbf{g}})_{L^2(\Omega)}. \end{aligned}$$

Hence, a new variational formulation is:

Find  $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega) \times L^2_{zmv}(\Omega)$  such that for all  $(\mathbf{v}, q) \in \mathcal{X}$

$$\begin{aligned} \nu \lambda (\mathbf{u}, \mathbf{v})_{\mathbf{H}_0^1(\Omega)} - \lambda (p, \operatorname{div} \mathbf{v})_{L^2(\Omega)} + \nu^{-1} (p, q)_{L^2(\Omega)} + \lambda (q, \operatorname{div} \mathbf{u})_{L^2(\Omega)} \\ = \lambda \langle \mathbf{f}, \mathbf{v} \rangle_{\mathbf{H}_0^1(\Omega)} + \nu^{-1} (z_{\mathbf{f}}, q)_{L^2(\Omega)}. \end{aligned} \quad (6.3)$$

This is the same variational formulation as the one with homogeneous Dirichlet boundary conditions, except that  $\mathbf{u} \in \mathbf{H}^1(\Omega)$  and  $\mathbf{u}|_{\partial\Omega} = \mathbf{g}$ .

The (conforming) discretization of Problem (6.3) with  $\mathbf{P}^k - P_{disc}^{k-1}$  finite element reads:

$$\begin{cases} \text{Find } (\mathbf{u}_h, p_h) \in \mathbf{V}_h^k \times Q_h^{k-1} \text{ s.t. for all } (\mathbf{v}_h, q_h) \in \mathbf{V}_{0,h}^k \times Q_h^{k-1} \\ \nu \lambda (\mathbf{u}_h, \mathbf{v}_h)_{\mathbf{H}_0^1(\Omega)} - \lambda (p_h, \operatorname{div} \mathbf{v}_h)_{L^2(\Omega)} = \lambda \langle \mathbf{f}, \mathbf{v}_h \rangle_{\mathbf{H}_0^1(\Omega)}, \\ \lambda (q_h, \operatorname{div} \mathbf{u}_h)_{L^2(\Omega)} + \nu^{-1} (p_h, q_h)_{L^2(\Omega)} = \nu^{-1} (z_{\mathbf{f}}, q_h)_{L^2(\Omega)}, \\ \mathbf{u}_h|_{\partial\Omega} = (\Pi_{h,cg} \mathbf{u}_{\mathbf{g}})|_{\partial\Omega}. \end{cases} \quad (6.4)$$

One can study the convergence by introducing  $\mathbf{u}_{0,h} = \mathbf{u}_h - \Pi_{h,cg}\mathbf{u}_g \in \mathbf{V}_{0,h}^k$  and using the previous results for the homogeneous boundary Dirichlet conditions. First, in the estimate (5.9) of Prop. 8, we change the term  $\|z_{\mathbf{f}} - \tilde{z}_{\mathbf{f},h}\|_{L^2(\Omega)}$  in the right-hand side of the two equations into  $\|z_{\mathbf{f}} - \tilde{z}_{\mathbf{f},h}\|_{L^2(\Omega)} + \lambda\nu\|\operatorname{div}(\mathbf{u}_g - \Pi_{h,cg}\mathbf{u}_g)\|_{L^2(\Omega)}$ . Second, in the estimate (5.11) of Theorem 5, we add  $\sqrt{\lambda}\|\operatorname{div}(\mathbf{u}_g - \Pi_{h,cg}\mathbf{u}_g)\|_{L^2(\Omega)}$  to the right-hand side.

## 7. Numerical results

We propose some numerical experiments with  $g = 0$ , depending on whether or not  $z_{\mathbf{f}}$  is known explicitly. In the latter case, we first compute some approximation  $\tilde{z}_{\mathbf{f},h}$ . In principle, either a classical, conforming or nonconforming, discretization can be used to compute  $\tilde{z}_{\mathbf{f},h}$ . The numerical results are obtained on a github platform, implemented in Octave language (see [23] for order 1 and 2 nonconforming finite element methods).

### 7.1. Resolution algorithm

Consider a conforming (resp. nonconforming, cf. [18]) discretization of the classical variational formulation (3.4). Let  $\mathbf{V}_{0,h}$  (resp.  $\mathbf{V}_{0,h}^{nc} \not\subset \mathbf{H}_0^1(\Omega)$ ) be the discrete velocity space, and  $Q_h$  be the discrete pressure space. Let  $\Pi_{h,nc}$  be the  $\mathbf{L}^2(\Omega)$ -orthogonal projection operator from  $\mathbf{L}^2(\Omega)$  to  $\mathbf{V}_{0,h}^{nc}$ . Let  $(\psi_i)_{i=1}^{N_u}$  be a basis of  $\mathbf{V}_{0,h}$  (resp.  $\mathbf{V}_{0,h}^{nc}$ ), and  $(\phi_i)_{i=1}^{N_p}$  be a basis of  $Q_h$ . We set:  $\underline{U} := (\underline{U}_i)_{i=1}^{N_u}$  where  $\mathbf{u}_h := \sum_{i=1}^{N_u} \underline{U}_i \psi_i$ ,  $\underline{P} := (\underline{P}_i)_{i=1}^{N_p}$  where  $p_h := \sum_{i=1}^{N_p} \underline{P}_i \phi_i$ , and  $\underline{F}_u := (\underline{F}_{u,i})_{i=1}^{N_u}$ . Using a conforming discretization,  $\underline{F}_{u,i} = \langle \mathbf{f}, \psi_i \rangle_{\mathbf{H}_0^1(\Omega)}$ . Using a nonconforming discretization, we let  $\ell_{\mathbf{f}} \in \mathcal{L}(\mathbf{V}_{0,h}^{nc}, \mathbb{R})$  be such that  $\forall \mathbf{v}_h \in \mathbf{V}_{0,h}^{nc}$ ,  $\ell_{\mathbf{f}}(\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_{\mathbf{L}^2(\Omega)}$  if  $\mathbf{f} \in \mathbf{L}^2(\Omega)$ ,  $\ell_{\mathbf{f}}(\mathbf{v}_h) = \langle \mathbf{f}, \mathcal{I}_h(\mathbf{v}_h) \rangle_{\mathbf{H}_0^1(\Omega)}$  if  $\mathbf{f} \notin \mathbf{L}^2(\Omega)$ , where  $\mathcal{I}_h : \mathbf{V}_{0,h}^{nc} \rightarrow \mathbf{V}_{0,h}^k$  is for instance an averaging operator [19, §22.4.1]. Then  $\underline{F}_{u,i} = \ell_{\mathbf{f}}(\psi_i)$ . Let  $\mathbb{A} \in \mathbb{R}^{N_u} \times \mathbb{R}^{N_u}$  be the velocity stiffness matrix,  $\mathbb{B} \in \mathbb{R}^{N_p} \times \mathbb{R}^{N_u}$  be the velocity-pressure coupling matrix and  $\mathbb{M} \in \mathbb{R}^{N_p} \times \mathbb{R}^{N_p}$  be the pressure mass matrix.

The linear system to be solved is

$$\left\{ \begin{array}{l} \text{Find } (\underline{U}, \underline{P}) \in \mathbb{R}^{N_u} \times \mathbb{R}^{N_p} \text{ such that:} \\ \nu \mathbb{A} \underline{U} - \mathbb{B}^T \underline{P} = \underline{F}_u \\ \mathbb{B} \underline{U} = 0 \end{array} \right. . \quad (7.1)$$

Let us set  $\mathbb{K} = \mathbb{B} \mathbb{A}^{-1} \mathbb{B}^T \in \mathbb{R}^{N_p} \times \mathbb{R}^{N_p}$ . The matrix  $\mathbb{K} \in \mathbb{R}^{N_p} \times \mathbb{R}^{N_p}$  is a symmetric matrix, furthermore it is positive definite as soon as the kernel of

$\mathbb{B}^T$  is reduced to  $\{0\}$ . When it is the case, the linear system (7.1) can be solved with the help of algorithm (7.3) below (with  $\underline{F}_p = 0$ ).

Consider next the  $\mathbf{P}^k - P_{disc}^{k-1}$  conforming discretization of the variational formulation (5.3) or (5.10). Let  $(\psi_i)_{i=1}^{N_u}$  be the Lagrange basis of  $\mathbf{V}_{0,h}^k$  and  $(\phi_i)_{i=1}^{N_p}$  be the basis of  $Q_h^{k-1}$ . We set:  $\underline{U} := (\underline{U}_i)_{i=1}^{N_u}$  where  $\mathbf{u}_h := \sum_{i=1}^{N_u} \underline{U}_i \psi_i$  and  $\underline{P} := (\underline{P}_i)_{i=1}^{N_p}$  where  $p_h := \sum_{i=1}^{N_p} \underline{P}_i \phi_i$ . We set  $\underline{F}_u := (\underline{F}_{u,i})_{i=1}^{N_u}$  where  $\underline{F}_{u,i} = \langle \mathbf{f}, \psi_i \rangle_{\mathbf{H}_0^1(\Omega)}$  and  $\underline{F}_p := (\underline{F}_{p,i})_{i=1}^{N_p}$  where  $\underline{F}_{p,i} = (z_{\mathbf{f}}, \phi_i)_{L^2(\Omega)}$ , cf. (5.3), or  $\underline{F}_{p,i} = (\tilde{z}_{\mathbf{f},h}, \phi_i)_{L^2(\Omega)}$ , where the discrete pressure  $\tilde{z}_{\mathbf{f},h}$  is an approximation of  $z_{\mathbf{f}}$ , cf. (5.10). The linear system to be solved is

$$\begin{cases} \text{Find } (\underline{U}, \underline{P}) \in \mathbb{R}^{N_u} \times \mathbb{R}^{N_p} \text{ such that:} \\ \nu \lambda \mathbb{A} \underline{U} - \lambda \mathbb{B}^T \underline{P} = \lambda \underline{F}_u \\ \lambda \mathbb{B} \underline{U} + \nu^{-1} \mathbb{M} \underline{P} = \nu^{-1} \underline{F}_p \end{cases} . \quad (7.2)$$

In that case, we set  $\mathbb{K} = \lambda \mathbb{B} \mathbb{A}^{-1} \mathbb{B}^T + \mathbb{M} \in \mathbb{R}^{N_p} \times \mathbb{R}^{N_p}$ , which is automatically a symmetric positive definite matrix.

To solve the coupled velocity-pressure problem (7.1) or (7.2), one relies usually on the three + one steps below (the fourth step being straightforward):

$$\begin{aligned} \text{Prediction:} & \quad \text{Solve in } \underline{U}_\star \text{ such that } \nu \mathbb{A} \underline{U}_\star = \underline{F}_u. \\ \text{Pressure solver:} & \quad \text{Solve in } \underline{P} \text{ such that } \mathbb{K} \underline{P} = \underline{F}_p - \nu \lambda \mathbb{B} \underline{U}_\star. \\ \text{Correction:} & \quad \text{Solve in } \delta \underline{U} \text{ such that } \nu \mathbb{A} \delta \underline{U} = \mathbb{B}^T \underline{P}. \\ \text{Update:} & \quad \underline{U} = \delta \underline{U} + \underline{U}_\star. \end{aligned} \quad (7.3)$$

Indeed, one can check easily that the above computed solution  $(\underline{U}, \underline{P})$  solves (7.1) (resp. (7.2)). The pressure solver with matrix  $\mathbb{K}$  is based on the Uzawa algorithm, which is the conjugate gradient algorithm in the context of the Stokes problem. It can be preconditioned by the inverse of the mass matrix associated to the discrete pressure  $\mathbb{M}$  (see e.g. [27, Lemma 5.9]). Thanks to the uniform discrete inf-sup condition, the number of iterations of the conjugate gradient algorithm is independent of the meshsize.

To solve (7.2), matrices  $\mathbb{M}$  and  $\mathbb{B}$  are kept with all the  $P_{disc}^{k-1}$  degrees of freedom for the discrete pressure. To take into account the zero mean value constraint, at each iteration of the preconditioned conjugate gradient algorithm, the discrete pressure is first computed in  $L^2(\Omega)$ , then orthogonally projected in  $L_{zmv}^2(\Omega)$ . We use the Cholesky factorization (computed once and for all) to solve linear systems with matrix  $\mathbb{A}$ . In addition, for  $k = 2$ , one

can use the lumped mass matrix (for  $k = 1$ , the mass matrix is diagonal). In the numerical results proposed below, we use the same strategy to solve (7.1) with nonconforming Crouzeix-Raviart finite elements [18].

**Remark 2.** We recall that solving the linear system (7.1) via the solver (7.3) is not so straightforward, even with  $\nu = 1$ . At first glance, solving the linear system (7.2) via the solver (7.3) may seem also quite involved. On the other hand, the preconditioned matrix  $\mathbb{M}^{-1/2}\mathbb{K}\mathbb{M}^{-1/2}$  is better conditioned in the latter case thanks to the addition of  $\mathbb{M}$  in the definition of  $\mathbb{K}$ .

## 7.2. Settings

We consider Problem (3.1) with homogeneous or non homogeneous boundary conditions in  $\Omega = (0, 1)^2$ . Let  $(\mathbf{u}, p)$  be the exact solution, and  $(\mathbf{u}_h, p_h)$  be the numerical solution. We compare numerical methods, showing how the coercive  $\mathbf{P}^1 - P^0$  formulation can be used in a post-processing step (i.e. solving (5.7) with  $\tilde{z}_{\mathbf{f},h}$  known), improving then the approximation of the discrete velocity. On Figures 1 to 7, and in Tables 2 to 6 we give the following names to our numerical methods:

- Method with Crouzeix-Raviart (CR): computations are made with the nonconforming Crouzeix-Raviart  $\mathbf{P}_{nc}^1 - P^0$  formulation [18, Example 4], which is not a pressure robust method. We call  $p_{nc}$  the resulting discrete pressure.
- Method with exact pressure (EP): computations are made with the coercive  $\mathbf{P}^1 - P^0$  formulation (5.3) and  $\lambda = 1$ , knowing the exact pressure  $z_{\mathbf{f}}$  (i.e. we solve Problem (5.3)).
- Method with post-processing (Post): computations are made in two steps. In a first step, we approximate the pressure by the CR-method. Then, in a post processing step, we use this numerical pressure as the source term in the EP-method (i.e. we solve Problem (5.10) with  $\tilde{z}_{\mathbf{f},h} = p_{nc}$ ) and  $\lambda = 1$ . Possibly, we iterate eight times the second step, updating  $\tilde{z}_{\mathbf{f},h}$  at each new iteration. The corresponding algorithm reads:

Let  $N_T$  be the number of triangles. For the velocity, the number of unknowns is of order  $N_u \approx 3 N_T$  for the CR-method and  $N_u \approx N_T$  for the EP-method. For the pressure, the number of unknowns is  $N_p = N_T$  for both methods.

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**Algorithm 1** Multi-step resolution

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Solve in  $(\underline{U}, \underline{P})$  Equations (7.1) with  $\mathbf{P}_{nc}^1 - P^0$  formulation.  
**for**  $n \in \{1, \dots, N\}$  **do**  
  Compute  $\underline{F}_p = \mathbb{M}\underline{P}$   
  Solve in  $(\underline{U}, \underline{P})$  Equations (7.2) with  $\mathbf{P}^1 - P^0$  formulation.  
**end for**

---

As a consequence, there are roughly twice as many unknowns for the CR-method than there are for the EP-method. We report in Table 1 the number of unknowns ”# dof” for the numerical tests. In addition, we observe that, with the same stopping criterion, around 30 (resp. 15) iterations of the preconditioned conjugate gradient algorithm are needed to solve the pressure solver of algorithm (7.3) with the CR-method (resp. with the EP-method): this is consistent with Remark 2.

$h$	# dof CR	# dof EP	# dof CR	# dof EP
$1.00 e - 1$	1 048	566	2 184	1 114
$5.00 e - 2$	4 376	2 270	8 064	4 074
$2.50 e - 2$	17 368	8 846	30 464	15 314
$1.25 e - 2$	67 816	34 230	117 664	58 994
$6.25 e - 3$	272 624	136 954	464 448	232 866

Table 1: Number of unknowns: first two test cases (left), last test case (right).

We propose three numerical examples based on manufactured solutions. We first plot the results as error curves for both the velocity and the pressure as a function of the meshsize, and we give the numerically observed average convergence rates. Second, we report the elapsed CPU times for the CR-method and the Post-method. In the Tables, column ”overhead” indicates the ratio between the cost of the second (post processing) step for the Post-method and the cost of the first step (CR-method). Finally, we plot the errors of the two methods as a function of the elapsed CPU times.

### 7.3. Regular manufactured solutions

We postulate that for the EP-method and the Post-method,  $\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)} \leq C_0 \|(\mathbf{u}, p)\|_{\mathcal{X}, \nu} h^2$ , where  $C_0$  is independent of the meshsize. The error estimates for the CR-method are given by [18, Theorems 3, 4, 6]. Notice that the

norm  $\|(\cdot, \cdot)\|_{\mathcal{X}, \nu}$  depends on the viscosity  $\nu$  (3.5) and that the error estimates on the pressure and the velocity are linked. We check the convergence rates and the viscosity dependency of the error estimates. The first test enters the framework of §5.3, see the estimate (5.5).

- **Test case with a vanishing velocity: Figure 1.**

We consider Problem (3.1)<sub>H</sub> with  $\mathbf{f} = \mathbf{grad} p$ , where:  $(\mathbf{u}, p) = (0, x^3 + y^3 - 1/2)$ .

The number of unknowns are reported on Table 1 (left). Figure 1 shows the discrete error values  $\varepsilon_0^\nu(\mathbf{u}_h) := \nu \|\mathbf{u}_h\|_{L^2(\Omega)} / \|p\|_{L^2(\Omega)}$  (left) and  $\varepsilon_0(p_h) := \|\Pi_{h,dg} p - p_h\|_{L^2(\Omega)} / \|p\|_{L^2(\Omega)}$  (right) plotted against the meshsize.

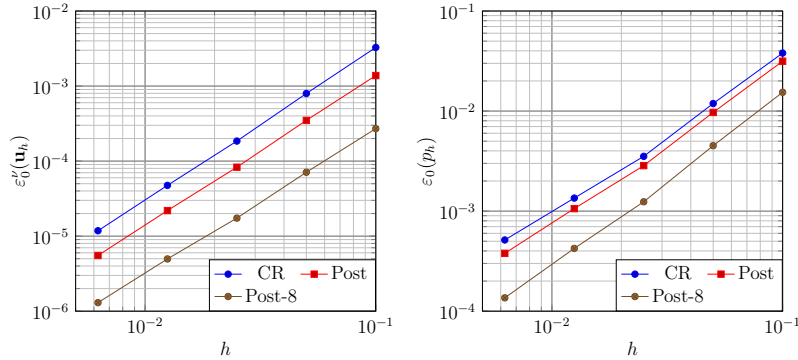


Figure 1: Vanishing velocity. Plots of  $\varepsilon_0^\nu(\mathbf{u}_h)$  and  $\varepsilon_0(p_h)$  for  $\nu = 1$  or  $\nu = 10^{-6}$ .

We note that the discrete errors for the pressure are independent of  $\nu > 0$ . As a matter of fact, since the continuous velocity vanishes, i.e.  $\mathbf{u} = 0$ , the linear system corresponding to the pressure solver step in the algorithm (7.3) does not depend on  $\nu$ . As expected in (5.5), the EP-method returns  $\varepsilon_0^\nu(\mathbf{u}_h) = \mathcal{O}(10^{-15})$  and  $\varepsilon_0(p_h) = \mathcal{O}(10^{-13})$ , so the errors are not reported on Figure 1. For the CR-method, the average convergence rate for the velocity,  $\tau_{\mathbf{u}}$  is 2.03, while the average convergence rate for the pressure,  $\tau_p$  is 1.55, which is better than expected, possibly because the source term is a polynomial of degree 2 which is numerically exactly integrated. For the Post-method, "Post" plots represent the errors after a single iteration of the second step and "Post-8" plots represent the errors after iterating the second step eight times, updating  $\tilde{z}_{\mathbf{f},h}$  at each new iteration. Both post-processings improve the initial computation of the velocity. Using eight iterations allows to improve the accuracy of the discrete pressure. For the Post-method with



a single iteration (resp. eight iterations), the average convergence rate for the velocity,  $\tau_{\mathbf{u}}$  is 1.99 (resp. 1.93), while the average convergence rate for the pressure,  $\tau_p$  is 1.71 (resp. 1.71). Figure 2 shows the discrete error values  $\varepsilon_0''(\mathbf{u}_h)$  (left) and  $\varepsilon_0(p_h)$  (right) plotted against the CPU time.

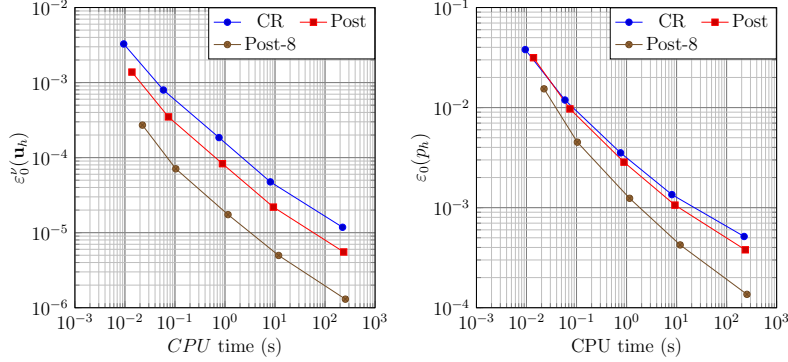


Figure 2: Vanishing velocity. Plots of  $\varepsilon_0''(\mathbf{u}_h)$  and  $\varepsilon_0(p_h)$  for  $\nu = 1$  or  $\nu = 10^{-6}$  against CPU time.

Table 2 below shows the CPU times for the CR-method and the Post-method with either one, or eight, iterations. By design, the Post-method which in-

$h$	CPU CR	CPU Post	overhead	CPU Post-8	overhead
$1.00 e - 1$	$9.46 e - 3$	$1.38 e - 2$	46 %	$2.24 e - 2$	136 %
$5.00 e - 2$	$5.88 e - 2$	$7.40 e - 2$	26 %	$1.04 e - 1$	77 %
$2.50 e - 2$	$7.59 e - 1$	$8.89 e - 1$	17 %	$1.15 e + 0$	52 %
$1.25 e - 2$	$8.06 e + 0$	$9.28 e + 0$	15 %	$1.18 e + 1$	47 %
$6.25 e - 3$	$2.24 e + 2$	$2.36 e + 2$	5 %	$2.55 e + 2$	14 %

Table 2: CPU time (s), vanishing velocity for  $\nu = 10^{-6}$ .

cludes the CR-method as its first step requires more CPU time. However, we notice that the CPU time of the second (post processing) step, the overhead, is only a small fraction of the first step, and also that it decreases dramatically as the meshsize decreases. This can be explained by the fact that on the one hand, there are fewer unknowns and, on the other hand, the pressure solver converges faster (cf. Remark 2). For the Post-method, it seems worth doing eight iterations, especially when the meshsize is small.

- **Test case with a sinusoidal solution: Figures 3-5**

We consider Problem (3.1)<sub>H</sub> with  $\mathbf{f} = -\nu \Delta \mathbf{u} + \mathbf{grad} p$ , where:

$$\mathbf{u} = \begin{pmatrix} (1 - \cos(2\pi x)) \sin(2\pi y) \\ (\cos(2\pi y) - 1) \sin(2\pi x) \end{pmatrix} \text{ and } p = \sin(2\pi x) \sin(2\pi y).$$

The number of unknowns are reported on Table 1 (left). The discrete errors values are  $\varepsilon_0'(\mathbf{u}_h) := \|\Pi_h \mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)} / \|(\mathbf{u}, p)\|_{\mathcal{X}, \nu}$  for the velocity (where  $\Pi_h$  is either equal  $\Pi_{h,nc}$  or  $\Pi_{h,cg}$ ), and  $\varepsilon_0'(p_h) := \nu^{-1} \|\Pi_{h,dg} p - p_h\|_{L^2(\Omega)} / \|(\mathbf{u}, p)\|_{\mathcal{X}, \nu}$  for the pressure.

Figure 3 shows  $\varepsilon_0'(\mathbf{u}_h)$  (left) and  $\varepsilon_0'(p_h)$  (right) plotted against the meshsize, for  $\nu = 1$ . In that case, for the Post-method, there is no need doing eight iterations, so we give numerical results for computations with a single iteration only.

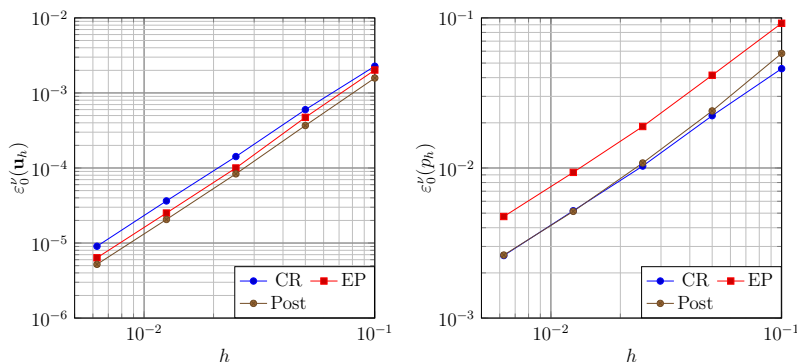


Figure 3: Sinusoidal velocity,  $\nu = 1$ . Plots of  $\varepsilon_0'(\mathbf{u}_h)$  and  $\varepsilon_0'(p_h)$ .

The Post-method and the EP-method give similar velocity errors. The CR-method and the Post-method give similar pressure errors. For a given mesh-size  $h$ , the velocity error is smaller for the EP-method and the Post-method than for the CR-method; and the pressure error is smaller for the CR-method than for the EP-method, but it is obtained at a higher cost. Notice that the Post-method reduces the velocity error without worsening the pressure error. Figure 4 shows  $\varepsilon_0'(\mathbf{u}_h)$  (left) and  $\varepsilon_0'(p_h)$  (right) plotted against the meshsize, now for  $\nu = 10^{-6}$ . For the Post-method, "Post" plots represent the errors after a single iteration of the second step and "Post-8" plots represent the errors after iterating the second step eight times, updating  $\tilde{z}_{\mathbf{f},h}$  at each new iteration. Both post-processings improve the initial computation. Even with eight iterations, the overhead cost remains affordable, especially when the

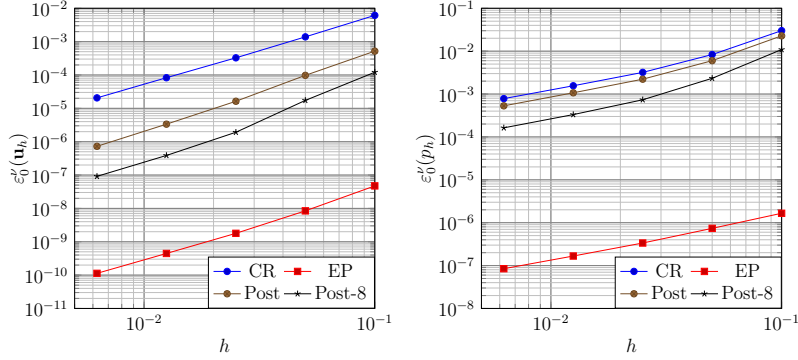


Figure 4: Sinusoidal velocity,  $\nu = 10^{-6}$ . Plots of  $\varepsilon_0^\nu(\mathbf{u}_h)$  and  $\varepsilon_0^\nu(p_h)$ .

meshsize is small.

The EP-method gives much smaller velocity and pressure errors than the CR-method since it is a pressure robust method. For a given meshsize  $h$ , we note that the Post-method allows to reduce the velocity error by a factor larger than 10. The pressure error of the Post-method using one iteration is similar to that of the CR-method, while it is improved using eight iterations. Table 3 shows the average convergence rates for the velocity,  $\tau_{\mathbf{u}}$  and the pressure,  $\tau_p$ .

$\nu$	$\tau$	CR	EP	Post	Post-8
1	$\tau_{\mathbf{u}}$	1.98	2.08	2.06	—
	$\tau_p$	1.03	1.07	1.12	—
$10^{-6}$	$\tau_{\mathbf{u}}$	2.06	2.18	2.37	2.59
	$\tau_p$	1.32	1.07	1.36	1.52

Table 3: Average convergence rates, sinusoidal velocity.

In the case  $\nu = 1$ , the average convergence rates are as expected. In the case  $\nu = 10^{-6}$ , the average convergence rate for the velocity,  $\tau_{\mathbf{u}}$ , is better than expected for the EP-method and the Post-method, and the average convergence rate for the pressure,  $\tau_p$ , is better than expected for the Post-method, probably because the asymptotic convergence regime has not been reached. Table 4 below shows the CPU times for the CR-method and the Post-method with either one, or eight, iterations. As we have used the same meshes, the computation times are similar to those in Table 2, showing once again that

the overhead cost decreases dramatically as the mesh size decreases.

$h$	CPU CR	CPU Post	overhead	CPU Post-8	overhead
$1.00 e - 1$	$1.02 e - 2$	$1.44 e - 2$	41 %	$2.36 e - 2$	131 %
$5.00 e - 2$	$6.01 e - 2$	$7.68 e - 2$	28 %	$1.06 e - 1$	76 %
$2.50 e - 2$	$7.54 e - 1$	$8.87 e - 1$	18 %	$1.16 e + 0$	54 %
$1.25 e - 2$	$7.73 e + 0$	$8.87 e + 0$	15 %	$1.13 e + 1$	46 %
$6.25 e - 3$	$2.22 e + 2$	$2.34 e + 2$	5 %	$2.53 e + 2$	14 %

Table 4: CPU time (s), sinusoidal velocity for  $\nu = 10^{-6}$ .

Figure 5 shows  $\varepsilon_0^v(\mathbf{u}_h)$  (right) and  $\varepsilon_0^v(p_h)$  (left) plotted against the CPU time for the CR-method and the Post-method with a single iteration ("Post" plot) or eight iterations ("Post-8" plot) .

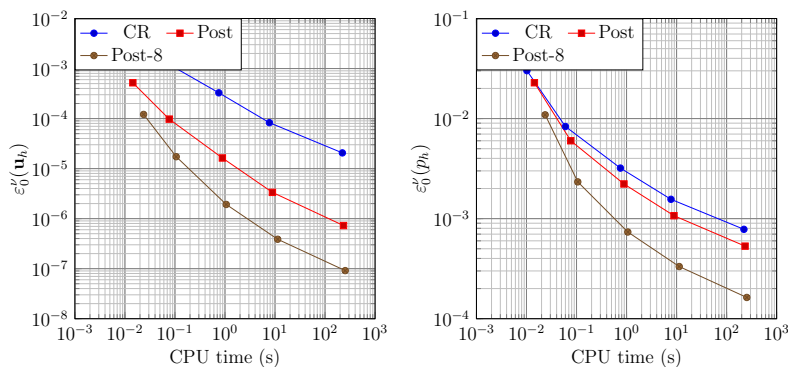


Figure 5: Sinusoidal velocity,  $\nu = 10^{-6}$ . Plots of  $\varepsilon_0^v(\mathbf{u}_h)$  and  $\varepsilon_0^v(p_h)$  against CPU time.

To achieve  $\varepsilon_0^v(\mathbf{u}_h) \lesssim 5 \times 10^{-5}$ , the required CPU time is 1 s for the Post-method with one iteration and 0.1 s for the Post-method with eight iterations, to be compared with more than 200 s with the CR-method.

- **Some observations**

The coercive  $\mathbf{P}^1 - P^0$  formulation (EP-method) gives *pressure robust* results, the obvious limitation being that it requires to know explicitly the potential of the gradient part of the source term. If it is not known, the two step method reduces the velocity error, compared with the calculation carried out using the Crouzeix-Raviart  $\mathbf{P}_{nc}^1 - P^0$  formulation (CR-method). Moreover,

using the second (post processing) step *iteratively* greatly improves the initial result. Finally, the reduction factor is greater the smaller  $\nu$  is.

#### 7.4. Low regularity manufactured solution

Last, we consider Problem (3.1)<sub>NH</sub> with a low regularity solution. Let  $(\rho, \theta)$  be the polar coordinates centered in  $(0.5, 0.5)$ . Let  $\alpha = 0.45$ . We set  $\mathbf{f} = -\nu \Delta \mathbf{u} + \mathbf{grad} p$  where  $(\mathbf{u}, p) = (\rho^\alpha \mathbf{e}_\theta, \rho - \int_\Omega \rho)$ . Results are given for  $\nu = 10^{-6}$ . For the Post-method, we systematically iterate eight times the second step, updating  $\tilde{z}_{\mathbf{f},h}$  at each new iteration. The number of unknowns are reported on Table 1 (right). We used a refined mesh around  $(0.5, 0.5)$ , where the solution is of low regularity. Figure 6 shows the discrete error values  $\varepsilon_0^\nu(\mathbf{u}_h) := \|\Pi_h \mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^2(\Omega)} / \|(\mathbf{u}, p)\|_{\mathcal{X}, \nu}$  (left) and  $\varepsilon_0^\nu(p_h) := \nu^{-1} \|\Pi_{h,dg} p - p_h\|_{L^2(\Omega)} / \|(\mathbf{u}, p)\|_{\mathcal{X}, \nu}$  (right) plotted against the mesh-size.

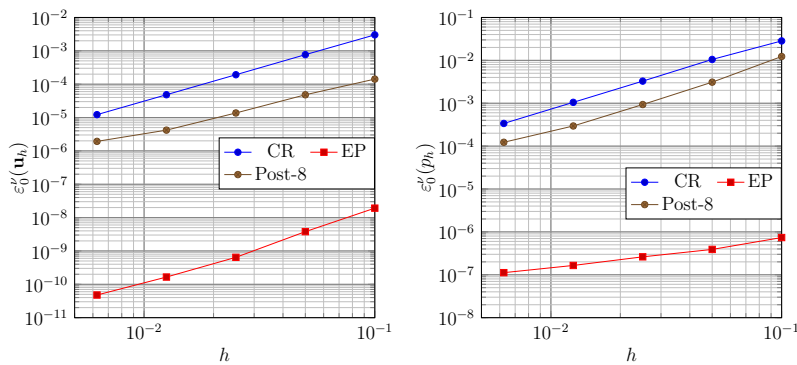


Figure 6: Low regularity velocity,  $\nu = 10^{-6}$ . Plots of  $\varepsilon_0^\nu(\mathbf{u}_h)$  and  $\varepsilon_0^\nu(p_h)$ .

We remark that EP-method shows far better results than the CR-method, and that the Post-method allows again to improve the approximation of the CR-method. Table 5 shows the averaged convergence rates between the successive meshes.

$\tau$	CR	EP	Post-8
$\tau_{\mathbf{u}}$	2.0	1.9	1.5
$\tau_p$	1.6	0.6	1.5

Table 5: Convergence rates, low regularity velocity,  $\nu = 10^{-6}$ .

For the EP-method and the CR-method, we postulate that, asymptotically,  $\tau_{\mathbf{u}} = 1 + \alpha$  and  $\tau_p = \alpha$ . In both cases, the obtained convergence rates are better than expected, which suggest that the asymptotic convergence regime is not reached. It seems that it will be reached faster for the EP-method, probably because it is pressure robust.

Table 6 below shows the CPU times for the CR-method and the Post-method with eight iterations. Again, the overhead cost decreases sharply with the

$h$	CPU CR	CPU Post-8	overhead
$1.00 e - 1$	$2.34 e - 2$	$5.88 e - 2$	123 %
$5.00 e - 2$	$2.42 e - 1$	$3.71 e - 1$	53 %
$2.50 e - 2$	$2.28 e + 0$	$3.46 e - 0$	52 %
$1.25 e - 2$	$2.43 e + 1$	$3.39 e + 1$	39 %
$6.25 e - 3$	$6.86 e + 2$	$7.74 e + 2$	12 %

Table 6: CPU time (s), low regularity velocity for  $\nu = 10^{-6}$ .

meshsize.

Figures 7 show the discrete error values  $\varepsilon_0^\nu(\mathbf{u}_h)$  (left) and  $\varepsilon_0^\nu(p_h)$  (right) against the CPU time for the CR-method and the Post-method with eight iterations.

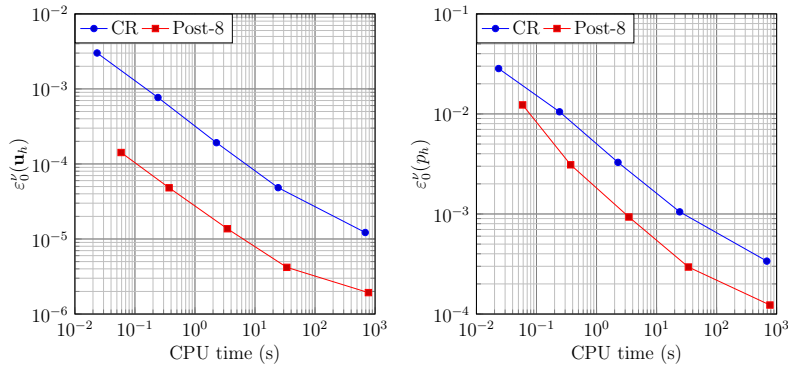


Figure 7: Low regularity velocity,  $\nu = 10^{-6}$ . Plots of  $\varepsilon_0^\nu(\mathbf{u}_h)$  and  $\varepsilon_0^\nu(p_h)$  against CPU time.

To achieve  $\varepsilon_0^\nu(\mathbf{u}_h) \lesssim 2 \times 10^{-5}$ , the CPU time is in the order of 250 s with the CR-method, compared to roughly 2 s with the Post-method with eight iterations.

## 8. Conclusion

We proposed and analysed a new variational formulation of Stokes problem based on  $T$ -coercivity theory. This variational formulation is coercive, and can be discretized with the  $\mathbf{P}^k - P_{disc}^{k-1}$  finite element for all  $k \geq 1$ . To solve the linear system resulting from the discretization, we need to know the pressure, or at least some approximation of it, which in our numerical tests is the discrete pressure obtained using the classical non-conforming method with the Crouzeix-Raviart  $\mathbf{P}_{nc}^1 - P^0$  finite element. This two step method improves the numerical results by notably reducing the errors obtained after the use of the classical method, especially when the viscosity is small. More significantly, the two step method consistently outperforms the classical method in terms of precision with respect to CPU time. Interestingly, this two step approach could be used in principle to solve the time-dependent Navier-Stokes equations with the help of the  $\mathbf{P}^k - P_{disc}^{k-1}$  finite element, as soon as an initial discrete pressure computed with a stable finite element pair is available.

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