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On Stability of Nonlinear Armsbuilding Security Games*

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Abstract

We formulate a security game in a context of mixed armament acquisition, involving a finite set of Nations in strategic relationship, with utility functions which are nonlinear and non-differentiable on the boundary of their sets of definition. Since we want to study the long-term effect of the Nations' investment in nuclear weapons, we focus on the steady-state analysis of the game. This requires us to extend the classical results from Rosen on compact-concave games, to unbounded concave games, relying on the uniform coercivity property of the Nations' utility functions. In addition, we prove the existence and uniqueness, under mild assumptions, of an interior point Nash Equilibrium solution of the game. To learn the interior point Nash Equilibrium, we provide a decentralized proximal point-type algorithm, and derive a closed form expression of its convergence rate. Simulations are performed in case of a duopoly, illustrating the utility improvement and stabilizing effects of nuclear armaments, by comparison with the conventional-only setting. We subsequently generalize this result to an arbitrarily large, but finite, number of Nations. The problem is extended further by allowing arms trades between Nations, split into suppliers on the one side, and recipients – interpreted as opponents – on the other side. It is widely recognized that arms trades can have negative externalities on the international system's security, and there are benefits to cooperate. Forming an export coalition enables the socialization of the operational costs among the coalition members and reallocation according to an *ex-post* cost allocation mechanism, while ensuring the maximization of the sum of the coalition members' security functions. For a given coalition structure, we develop a partial equilibrium model of the international arms market. We prove the existence and uniqueness of a market equilibrium in arms trade prices and supply. Finally, we derive conditions on the exports, arms trade prices, and cost allocation mechanism guaranteeing the stability of the export coalition, thus highlighting the viability of a common defense policy for the export coalition.

Keywords: Security Game • Nash Equilibrium • Uniform Coercivity • Learning in Games • Stability

1 Introduction

The primary goal of Nations is to maximize their perceived security, which requires each Nation to develop an effective deterrence strategy [20]. In classical game theoretic models, deterrence as a military strategy is based upon a standard, iterated two-player prisoner's dilemma model by which cooperation can be assured if a rational player understands the punishment or cost of defection [19, 5, 35]. Models in this area are often inspired by the USA and USSR-allied, or at least Western vs Eastern blocs, nuclear arms race, and formulate the security dilemma in arms procurement as a two-player noncooperative game in normal form [38, 30]. However, many of the actions taken by Nations in pursuit of the maximization of their perceived security – such as arms procurement and the development of new military technologies – might, in turn, lead to a decrease rather than an increase in their perceived security. This dilemma is explained by the fact that the increase in military capability of one Nation might be interpreted as a risk that the arming Nation will use it to perform an attack in the future. In this context, in order to re-establish the balance of power, the other Nations could either increase their own military capabilities or initiate a preemptive strike. Choosing the first option may result in a security spiral, in which several Nations are tied in an arms race, responding to increases in arms procurement and defense expenditure by others by arming themselves more and more

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heavily, and may lead to war in the long run. Such a situation is known as the security dilemma [16, 17], and is one possible outcome of arms race models.

Standard literature dealing with arms race models [40, 31] focuses on the dynamic process of the arms race over the time or on fitting real data collected over the years using this model. In this setting, game theory is a useful tool to model interactions between Nations with conflicting interests. The classical literature dealing with nuclear strategy using game theory focuses mainly on discrete decision space repeated games [29] and differential games. Within repeated games, iterated prisoners' dilemmas between two Nations [25, 24] have been extensively studied. At each iteration, each Nation has a choice between a high or low level of arms. In the static version, each Nation's dominant strategy is to choose a high level of arms. The Nash Equilibrium outcome of the static prisoner dilemma model for arms race is therefore that both Nations choose high. As a result, the static game outcome is worse for both Nations than if both had chosen a low level of arms. However, in reality, the game is not played once and for all, but is an ongoing series of decisions, i.e., the prisoner dilemma is played repeatedly by the two Nations. This opens the possibility for cooperation to emerge through rewards and punishment strategies, e.g., tit-for-tat [13].

A somewhat equivalent framework stems from Richardson-type models of arms race [23, 36], which provide parametrized systems of differential equations to represent the evolution of the weapons stockpile. These equations can be derived as closed-loop Nash Equilibria of linear-quadratic differential games that implement an interpretable reasoning of players [36, 26, 27, 37]. However, military strategies that hinge on nuclear weapons imply that Nations' evaluation of the effect of each other's actions should be nonlinear, and Richardson-type's models fail to represent accurately the specific nature of nuclear armaments [9, 11]. Indeed, contrary to conventional war, nuclear warfare allows for significant preemptive and retaliatory strikes whose magnitude depend nonlinearly on the initial stockpiles. A two-stage game model may show that a second-strike capability decreases as the exponential of the ratio of the initial stockpiles [8].

If the goal of game-theoretic models is to quantify the outcomes of situations of conflict between armed Nations, non-proliferation policies and containment treaties are often designed with the goal to stabilize the rivalry between Nations and reduce strategic risks. For example, during the Cold War, several agreements between the USA and the USSR, such as the intermediate-range nuclear forces (INF) treaty, helped balance nuclear-able weapons so that each side could survive a preemptive nuclear attack with a sufficiently large stockpile of ballistic missiles to launch a retaliatory strike, in the context of nuclear strategies heavily influenced by the concept of MAD (mutually assured destruction) and strategic stability. In its most traditional sense, strategic stability refers to two complementary characteristics of the international system: (i) the absence of an incentive to alter military posture in response to a state of vulnerability, whether in peacetime or times of crisis, (ii) a tendency to return to stable relations after a period of escalation [3, 34]. The meaning of this concept has gradually evolved into a set of norms, rules and procedures that minimize the risk of one state rapidly gaining a strategic advantage over another [3].

Nowadays, 9 countries are thought to possess nuclear weapons. Within them, only 5 – the USA, Russia, China, France and the United Kingdom – have signed the treaty on the non-proliferation of nuclear weapons (NPT), the second most widely ratified treaty after the United Nations charter. Under the NPT, each of the 191 parties 'undertakes to pursue negotiations in good faith on effective measures relating to the cessation of the nuclear arms race at an early date and to nuclear disarmament, and on a treaty on general and complete disarmament under strict and effective international control' [1]. The security architecture is completed by the comprehensive nuclear-test-ban treaty (CTBT) with 177 parties, which prohibits nuclear tests and as such limits the possibility to develop nuclear weapons*.

1.1 Problem Statement

We model the interactions among Nations possessing both conventional and nuclear weapons, i.e., mixed armament, using noncooperative game theory. Contrary to most papers which focus on the dynamics of arms procurement, we focus on the steady-state analysis of the game, which captures the long-term solution of the dynamical model, and is formulated as a one-shot (static) game. On the quantitative side, a first difficulty appears in modeling the Nations' utility functions such that they reflect the security perceived at a Nation-wide level – thus, depending on the Nation's armament strategy and on that of its rivals. This

*Currently, while it is already enforced by parties, the CTBT has not yet entered into force because 8 out of 44 Annex 2 Nations have not ratified it.

leads us to define utility functions which are nonlinear and non-differentiable on the boundary of their sets of definition. A second difficulty arises from this modeling choice and requires that we extend classical results from noncooperative game theory, to characterize the game equilibria. On the qualitative side, we aim to assess the effects of long-term investments in nuclear weapons on the efficiency and strategic stability of the Nations’ international system. To be more specific, we aim to prove that, for an arbitrarily large, but finite, number of Nations, mutual possession of nuclear weapons has a utility-enhancing effect on the utility of Nations competing for mixed armaments.

The problem is extended further by allowing arms trades between Nations, split into suppliers on the one side, and recipients – interpreted as opponents – on the other side. Indeed, world production and exports of advanced weapons systems are concentrated in the hands of a small number of Nations, called suppliers, which are self-sufficient for their government procurement and may export to recipients, often engaged in regional antagonisms. For example, NATO is the classic form of cooperation, with the alliance also enforcing partial cooperation in exports [22]. However, suppliers’ exports by increasing recipients’ military capability can reduce the suppliers’ own security. These spillovers can be internalized among the suppliers, through different market designs. Forming an export coalition enables the socialization of the operational costs among the coalition members, while ensuring the maximization of the sum of the coalition members’ security functions. We assume that suppliers do not anticipate the impact of their strategies on the arms trade prices. Instead, an international market operator is responsible for the definition of the arms trade prices. In this perfectly competitive market, we aim to understand the impact of arms trade price and exports on the Nations’ security functions, and determine conditions for a common defense policy for the export coalition to emerge.

1.2 Main Contributions

We propose a multipolar security model which addresses the specific logic of nuclear strategies in a context involving mixed armament, i.e., both conventional and nuclear weapons. The model is formulated in a dynamic setting, but since we want to study the stabilizing and security dilemma reduction potential of mixed armament, we focus on the one-shot game steady-state formulation, which captures the long-term effects of the competition among Nations. This paper provides three important contributions to the state of the art. First, considering a new way to model the security perceived at the Nation-wide level, that takes inspiration from [8], we extend the classical results from Rosen [33] on compact-concave games to unbounded concave games relying on the uniform coercivity property of the Nations’ utility functions. This allows us, in a second step, to prove the existence of an interior point Nash Equilibrium solution of the one-shot game. The proof of uniqueness of the interior point Nash Equilibrium follows, under mild assumption on the game parameters. We provide a decentralized proximal point-type algorithm to learn the interior point Nash Equilibrium, and derive a closed form expression of its convergence rate. Focusing on the interpretation of the outcome of our armsbuilding model, we determine numerically that nuclear weapons have utility improvement and stabilizing effects, by comparison with the convention-only setting. We subsequently generalize this result to an arbitrarily large, but finite, number of Nations.

However, our armsbuilding model ignores the arms trades that may occur between Nations split into suppliers on the one side, and recipients on the other side. Though multinational arms export control has been debated for a long time, it is well known that there are benefits from the international coordination of controls. However, the literature on the economic analysis of these issues is scarce, partly because the market for arms is somewhat unusual. We contribute to fill the gap by developing a partial equilibrium model of the international arms market, with suppliers on the one side, split into allies, i.e., members of an export coalition, or neutrals, and recipients on the other side. For a coalition structure defined *ex-ante*, we prove the existence and uniqueness of a market equilibrium in arms trade prices and supply. Finally, we derive conditions on the exports, arms trade prices, and cost allocation mechanism guaranteeing the stability of the export coalition.

Paper Organization

This paper is organized as follows. In Section 1, we provide an overview of main game theoretic approaches for arms procurement competition modeling and how their outcomes can be used to design containment treaties

with stabilizing effects. We also introduce the problem statement, and main contributions. In Section 2, the armsbuilding model is formulated as a one-shot noncooperative game. Conditions for the existence and uniqueness of an interior point Nash Equilibrium solution of this game are studied in Section 3. Additionally, we prove the variational stability of the interior point Nash Equilibrium, and analyze the convergence rate of a decentralized proximal point-type algorithm to learn the interior point Nash Equilibrium. Analytical results and numerical illustrations in case of a duopoly are discussed in Section 4, highlighting the stabilizing effect of investments in nuclear armaments. We subsequently generalize this result for an arbitrarily large, but finite, number of Nations. In Section 5, the armsbuilding model is extended by allowing arms exports between suppliers, being either members of a joint armament export coalition – made of a finite number of allies – or neutrals, and recipients. For a coalition structure determined *ex-ante*, we develop a partial equilibrium model of the international arms market, which is subsequently analyzed, and provide conditions for the stability of the coalition. We conclude in Section 6.

Notation

Throughout the paper, we rely on the following conventions. $\mathbb{E}_X[f]$ specifies that the expectation of the function f is taken with respect to the random variable X (when X is clear from the context, we abbreviate the notation to $\mathbb{E}[f]$). The stack of the N vectors x_1, \dots, x_N is denoted $x \stackrel{\text{def}}{=} \text{col}(x_1, \dots, x_N)$. x_{-n} is the stack of the vectors containing the decision variables of all the agents in \mathcal{N} except n . The collection of N variables x_1, \dots, x_N is denoted $(x_n)_{n=1, \dots, N}$. x^T is the transpose of the vector x . $|\mathcal{X}|$ refers to the cardinal of the set \mathcal{X} . $\mathbf{1}_{\mathcal{X}}(\cdot)$ represents the characteristic function of the set \mathcal{X} , which equals 1 in \mathcal{X} , 0 otherwise. *nuc* and *conv* denote the classes of weapons, which can be nuclear and conventional respectively. $\llbracket 1, K \rrbracket$ is the set of integers between 1 and $K \in \mathbb{N}^*$. F_n denotes the gradient of Nation n 's utility function J_n with respect to its own action. The pseudo-gradient F is the stack of the $(F_n)_n$. H^τ is the pseudo-Hessian for class τ armament. In addition, the main notations which are used throughout the paper are summarized in Table 1.

2 The Mixed Armament Competition Model

The international system is first and foremost made up of Nations, that we will indifferently call players. In international relations [2], an organization is said to be unitary if and only if its decisions contain all the information useful for its interaction with the international system. Moreover, an organization is said to be rational if its decision-making process consists in maximizing its (well-ordered) preferences through its actions, given the information available to it and its anticipation of what other organizations might do. Hence, in our work, Nations can be represented as rational unitary players. Note that no hypothesis is made a priori on the order of the preferences.

We aim to study the strategies of Nations to invest in nuclear and conventional weapons in order to maximize their security while minimizing their storage and R&D costs. We start by placing ourselves in a framework of discrete time evolution. Nations seek to ensure their own security within a certain time horizon (finite or infinite), and in anticipation of a crisis. They can acquire conventional and nuclear weapons to maximize their security, while bearing storage and R&D costs. We choose a realist framework [41], and assume that Nations are primarily defensive and secondarily aggressive. This means that they seek through conventional and nuclear armaments to ensure their own security, and that their marginal gain from arming themselves increases with the levels of adversary armaments.

2.1 Dynamics of Armsbuilding

Let \mathcal{N} be a finite set of N Nations, and T be the horizon of the game (possibly infinite). We propose a model with continuous decision spaces, i.e., the Nations (as players) do not decide whether to attack or not, but they decide on the power drawn from their investment in nuclear and conventional weapons. Models with continuous decision spaces prove themselves more realistic [6, 20, 36, 37]. We might assume that the power drawn for each category of armament depends linearly on the Nation's arms production and that the coefficients relating the arms production to the power drawn out of it differ between the categories of weapons. In absence of numerical values for these coefficients, we will not consider this level of details in the present work. In what follows, $\tau = \{\text{nuc}\}$ or $\{\text{conv}\}$ and denotes nuclear or conventional.

Table 1: Table of notation.

Sets	
\mathcal{N}	Set of N Nations
$\sigma(n)$	Set of opponents to Nation n
\mathcal{U}_n	Nation n 's strategy set
\mathcal{U}_{-n}	Joint strategy set of all the Nations except n
\mathcal{U}	Joint strategy set
\mathcal{N}_A	Set of allies
\mathcal{N}_N	Set of neutrals
\mathcal{N}_R	Set of recipients
$\tilde{\mathcal{U}}_n$	Nation n 's strategy set including exports
Variables	
$u_n^\tau(t)$	Class τ power units supplied by n at t
x_n^τ	Class τ power units supplied by n
$x_{\sigma(n)}^\tau$	Target value of class τ adverse armament
x_n	Stack of $x_n^\tau, \tau \in \{\text{nuc}; \text{conv}\}$
x_{-n}^τ	Class τ power units supplied by all Nations except n
W_n	Number of weapons held by Nation n
W_n^r	Number of remaining weapons for Nation n after a single preemptive strike
$x_n^{*\tau}$	Class τ power units supplied by n at equilibrium
$x_n^{(k)}$	Nation n 's power units supplied at iteration k
χ_n	Nation n 's export in conventional power units
p	Allies' arms trading unit price
\bar{p}	Neutrals' arms trading unit price
x^R	Updated recipients' araments levels
Parameters	
τ	Nuclear or Conventional
N	Number of Nations
T	Horizon of the dynamic game
ν	Obsolescence rate
ρ_n	Nation n 's discounting rate
α^τ	Normalized weight for security perceived from class τ weapons
λ_n	Nation n 's weapons accuracy
ξ_n	Nation n 's threat threshold capturing its peacefulness
C_n^{conv}	Marginal unitary cost for the power drawn from conventional weapons for n
C_n^{nuc}	Marginal unitary cost for the power drawn from nuclear weapons for n
θ_n	Proximal Point method parameter for n
C^A	Marginal unitary cost for the coalition
P	Arms trading price cap
γ	Allies coalition cost allocation
ζ_0	Recipients' demand for allies intercept
ζ	Recipients' demand for allies price coefficient
η	Recipients' demand for allies substitutability coefficient
$\bar{\zeta}_0$	Recipients' demand for neutrals intercept
$\bar{\zeta}$	Recipients' demand for neutrals price coefficient
$\bar{\eta}$	recipients' demand for neutral substitutability coefficient
Functions	
$J_n(\cdot)$	Nation n 's utility function
$f_n(\cdot)$	Nation n 's security function
$f_n^\tau(\cdot)$	Nation n 's security function for class τ weapons
$c_n^S(\cdot)$	Nation n 's storage cost
$c_n^D(\cdot)$	Nation n 's development cost
$\phi(\cdot)$	concave function
J_n^0	Nation n 's utility evaluated in the conventional-only setting
J_n^*	Nation n 's utility evaluated in the interior point Nash Equilibrium
$J_A(\cdot)$	allies coalition's utility function
$\tilde{J}_n(\cdot)$	Nation n 's utility function under arms trading
$c^S(\cdot)$	coalition cost function
$d(\cdot)$	recipients' demand for weapons imports from allies
$\bar{d}(\cdot)$	recipients' demand for weapons imports from neutrals

Let $u_n^\tau(t)$ be the power drawn from the quantity of weapons of class τ produced by Nation $n \in \mathcal{N}$ and $x_n^\tau(t)$ be the power drawn from its stockpile of weapons of class τ at time period t . We define $u_n(t) \stackrel{\text{def}}{=} (u_n^\tau(t))_\tau$, the column vector which contains the power drawn from the production of Nation n for each class of weapons, $u(t) \stackrel{\text{def}}{=} \text{col}((u_n(t))_n)$ be the stack of the power drawn from the weapons production by the N Nations. Similarly, $x_n(t) \stackrel{\text{def}}{=} (x_n^\tau(t))_\tau$ is the column vector which contains the power drawn from stockpile of Nation n for each class of weapons, and $x(t) \stackrel{\text{def}}{=} \text{col}((x_n(t))_n)$ be the stack of the power drawn from the stockpiles of the N Nations. Furthermore, we define the following sequences: $x \stackrel{\text{def}}{=} (x(t))_t$ and $u_n \stackrel{\text{def}}{=} (u_n(t))_t$. The dynamics of Nation n 's power drawn from stockpile of class τ weapons is defined as follows:

$$\forall n \in \mathcal{N}, \quad x_n^\tau(t+1) = x_n^\tau(t) + u_n^\tau(t) - \nu x_n^\tau(t), \quad (1)$$

with an obsolescence rate $\nu \geq 0$. The players' utility at time period t writes:

$$j_n(u_n(t), x(t)) = f_n(x(t)) - c_n^S(x_n(t)) - c_n^D(u_n(t)), \quad (2)$$

where $f_n(\cdot)$ is the security function of player n , $c_n^S(\cdot)$ its storage (operational) cost and $c_n^D(\cdot)$ its development (R&D) cost. The players may have local constraints, e.g., finite budget, maximum capacity of production, etc., which may evolve dynamically. We denote $\mathcal{U}_n(t) \subset \mathbb{R}_+^2$ the set of $u_n(t)$ that preserve these constraints, i.e., the feasibility set of player n . Each player $n \in \mathcal{N}$ maximizes the sum J_n over time of its future utility, with discounting rate $\rho_n \in]0; 1[$, $n \in \mathcal{N}$:

$$V_n(x) = \max_{u_n(t) \in \mathcal{U}_n(t), \forall t} J_n(u_n, x), \quad (3)$$

where $J_n(u_n, x) \stackrel{\text{def}}{=} \sum_{t=0}^T \rho_n^t j_n(u_n(t), x(t))$.

We aim to study this model in the long run, with $\nu = 0$. Thence, we shall examine stable dynamics of armsbuilding, i.e., when solutions of equation (1) have a bounded limit at infinity which is asymptotically stable. It is worth noting that dynamic stability is at the heart of the concept of strategic stability. In what follows, we denote $x_n \stackrel{\text{def}}{=} \lim_{t \rightarrow +\infty} x_n(t)$ player n 's steady state, with $\lim_{t \rightarrow +\infty} u(t) = 0$. Using a slight abuse of notation, the static model is given by:

$$V_n(x) = \max_{x_n \in \mathcal{U}_n} J_n(x) \quad \text{where} \quad J_n(x) \stackrel{\text{def}}{=} j_n(0, x), \quad (4)$$

with the decision variable now being Nation n 's power drawn from its stockpiles, $x_n \in \mathcal{U}_n \subset \mathbb{R}_+^2$. Let $x_{-n} \stackrel{\text{def}}{=} \text{col}((x_m)_{m \neq n})$, $\forall n$ be the vector that contains the stack of the power drawn from the stockpiles of weapons of all Nations in \mathcal{N} except n .

2.2 Strategic Relationship

For each Nation, the international system is divided into opponents, neutrals and allies (commercial or military). A commercial alliance is a binding agreement allowing transfer of military capabilities between parties. A military alliance is a binding agreement establishing a coalition of military interests and capabilities between parties. To simplify the setting, we will not consider alliances in the following except in Section 5. In what follows, we make the assumption that the decisions of Nations are not affected by those of neutrals.

We now want to structure the set of Nations, by taking into account how their decisions impact those of the others. For each Nation $n \in \mathcal{N}$, the set of opponents $\sigma(n) \subset \mathcal{N}$ is defined by a binary relation of hostility, represented by the hostility function $\sigma(\cdot)$. It is reasonable to assume that a Nation is never its own opponent and that any opponent of a Nation must treat the considered Nation as an opponent in return. Therefore, we assume that the relation of hostility is irreflexive and symmetric, i.e.,

$$\begin{aligned} \forall n \in \mathcal{N}, & & n &\notin \sigma(n), \\ \forall n, m \in \mathcal{N}, & & m \in \sigma(n) &\iff n \in \sigma(m). \end{aligned}$$

We introduce the strategic relationship as the reflexive-transitive closure of the hostility relationship:



Figure 1: For the purpose of visualization, the international system can be represented as a graph capturing the hostility relationships between the Nations. Formally, n is an opponent of m if and only if $m \in \sigma(n)$, which materializes through the existence of an oriented edge linking $n \rightarrow m$. In **1a** and **1b**, the international system is represented as a complete graph involving two and three Nations respectively. In **1c** the graph is semi-bipartite, whereas in **1d**, the graph has two cliques capturing local-global interactions.

Definition 1 (Strategic relation) We say that two Nations n and m are in strategic relationship and write $n \sim m$ if a (possibly degenerate) sequence of hostility relationships connects n to m , i.e., if there exists $k \in \mathbb{N}$ such that $m \in \sigma^k(n) \stackrel{\text{def}}{=} \sigma(\{n\})$.

Lemma 1 The strategic relationship is an equivalence relation.

Proof. By definition, the strategic relationship is symmetric. It is transitive because $\forall n, m, l \in \mathcal{N}$ such that $n \neq m$, $m \neq l$, $n \neq l$, $n \sim m$ and $m \sim l$ mean that there exist $k, k' \in \mathbb{N}$ such that $m \in \sigma^k(n)$ and $l \in \sigma^{k'}(m)$. Then, by composition of the strategic relationship, we get that $l \in \sigma^{k+k'}(n)$. Therefore, the strategic relationship is transitive. Now, $\forall n \in \mathcal{N}$, $\sigma^0(n) = \{n\}$, thus the strategic relationship is an equivalence relation. \square

Lemma 1 implies that the decisions of Nations belonging to the same equivalence class are independent of the decisions of the Nations out of it. Hence, we can split the set of players into a finite number of equivalence classes. This allows to decompose the main noncooperative game into the same number of noncooperative games, that can be solved independently. In what follows, we assume this procedure has already been done, and focus on a game in which the N Nations share strategic relations:

Assumption 1 All Nations in \mathcal{N} are in strategic relationship.

In particular, if the game is nontrivial and consists of at least two players, then every Nation has an opponent. Illustrations of graph representations of the international system capturing hostility relationships are provided in Figure 1. Though no graph based arguments are used in the rest of the paper, such representations are interesting to visualize the organization of the international system relying on the binary hostility relationship and provide qualitative interpretations.

2.3 Target Value

For Nation $n \in \mathcal{N}$, the target value of adverse armament of class τ is denoted by $x_{\sigma(n)}^\tau$, which is a function of x_{-n}^τ . Later, we will call it Nation n target value. This value represents the level of threat caused by the set of opponents. Several modeling choices are possible. For instance, Nations preparing for a one-off crisis will evaluate the target value as the maximum level of armaments of class τ of the set of opponents. In any case, it must satisfy the following assumption:

Assumption 2 There exist $\kappa^-, \kappa^+ > 0$ input coefficients that do not depend on the power drawn from the weapon stockpiles, such that

$$\kappa^- \max_{m \in \sigma(n)} x_m^\tau \leq x_{\sigma(n)}^\tau \leq \kappa^+ \max_{m \neq n} x_m^\tau.$$

Remark 2.1 Target values defined as the maximum, the sum or the mean over $\sigma(n)$ of the power drawn from the weapon stockpiles satisfy Assumption 2

$$\begin{aligned} \max_{m \in \sigma(n)} x_m^\tau &\leq \max_{m \in \sigma(n)} x_m^\tau \leq \max_{m \neq n} x_m^\tau, \\ \max_{m \in \sigma(n)} x_m^\tau &\leq \sum_{m \in \sigma(n)} x_m^\tau \leq N \max_{m \neq n} x_m^\tau, \\ \frac{1}{N} \max_{m \in \sigma(n)} x_m^\tau &\leq \frac{1}{|\sigma(n)|} \sum_{m \in \sigma(n)} x_m^\tau \leq \max_{m \neq n} x_m^\tau. \end{aligned}$$

Let us discuss briefly the interpretation of Assumption 2. The right-hand side of the inequality ensures that Nations use armaments levels of others to compute the level of threat they face, while the left hand side ensures that Nations do not ignore the greatest opponent in the computation of the threat they face.

2.4 Security Functions

The perceived security of a Nation $n \in \mathcal{N}$ is a function of the power drawn from its own weapon stockpile x_n and of its target value $x_{\sigma(n)}$, where $x_{\sigma(n)}$ is the column vector which contains the target value of Nation n in each weapon class. As Nations seek to maximize their perceived security, it increases with their effective weapon stockpile and decreases with the opponents' effective weapon stockpiles. Using deterrence terminology, the effective stockpile value can be defined as the maximum available quantity of armaments, a Nation is both credible, and capable to use in a potential conflict. We shall assume that Nations have a credible threat in using the military capabilities we consider. However, the effectiveness of the weapon stockpiles may depend on the quality and type of armaments as well as on the military doctrine of each Nation. We distinguish between two classes of armaments: nuclear (strategic) weapons allowing for preemptive strikes, and conventional (tactical) weapons which can be used on the battlefield. Therefore, Nation n 's security function can be decomposed as follows:

$$f_n(x_n, x_{-n}) = \sum_{\tau} \alpha^\tau f_n^\tau(x_n, x_{-n}), \quad \text{where } \sum_{\tau} \alpha^\tau = 1, \alpha^\tau \geq 0, \forall \tau. \quad (5)$$

2.4.1 Extrinsic and Intrinsic Perceived Security

For each class τ of armaments, we decompose f_n^τ into two parts: the extrinsic part, which represents the loss of perceived security due to the environment, and the intrinsic part, which represents the gain of perceived security due to the possession of armaments of class τ . The extrinsic perceived security of Nation n brought by class τ armaments coincides with its opponents' effective stockpile of class τ . Now, as Nations are primarily defensive and secondarily aggressive, the intrinsic perceived security of Nation n brought by class τ armaments is modeled as a linear function of Nation n 's effective stockpile value of armaments of class τ . Its slope is a concave function $\phi(\cdot)$ of the weapons they are supposed to deter. Nuclear strategies are generally dual-purpose, contrary to conventional strategies. Thus, the slope is $\phi\left(\frac{x_{\sigma(n)}^{\text{conv}} + x_{\sigma(n)}^{\text{nuc}}}{\xi_n}\right)$ for nuclear, and $\phi\left(\frac{x_{\sigma(n)}^{\text{conv}}}{\xi_n}\right)$ for conventional, where $\xi_n > 0$ is a threat threshold which captures Nation n 's peacefulness. To go further we need to evaluate the effective stockpile value for nuclear and conventional armaments.

2.4.2 Effective Stockpiles

Consider first conventional weapons. It is reasonable to assume that Nations are able to use at some point in a potential conflict all of the conventional weapons they possess. Therefore, Nation n 's power drawn from effective stockpile of conventional weapons is equal to the power drawn from its actual stockpile of conventional weapons x_n^{conv} .

Now, consider nuclear weapons. Depending on multiple factors, Nations can be cautious and prepare for an adversarial first strike by seeking a second-strike capability, or they can hinge on their preemptive first-strike capability to deter their counterparts. Therefore, stochastic multi-stage games can be used to model the succession of strikes [8]. We use this setting to derive the closed form expression of the expected

number of remaining weapons in case of a conflict between two armed Nations, after a single preemptive strike.

Following [8], we consider two Nations called n and $m \in \mathcal{N}, n \neq m$. Nation n holds $W_n \in \mathbb{N}$ weapons. It is attacking Nation m , which holds $W_m \in \mathbb{N}^*$ weapons. Nation n aims to destroy with a preemptive strike as much as possible of Nation m 's weapons. Assume that Nation n 's weapons have an accuracy $\lambda_n > 0$, i.e., each weapon has a probability $p_n = 1 - e^{-\lambda_n}$ of destroying its target. Assume that both weapons and targets are indistinguishable. Let W_m^r be the random variable that gives the number of remaining weapons for Nation m after a single preemptive strike of Nation n . Let $K \stackrel{\text{def}}{=} \lfloor \frac{W_n}{W_m} \rfloor$ and $w \stackrel{\text{def}}{=} W_n - KW_m$.

Proposition 1 *After an optimal single preemptive strike by Nation n , the expected number of remaining weapons for Nation m is:*

$$\begin{aligned} \mathbb{E}[W_m^r] &= (W_m - w) \exp(-K\lambda_n) + w \exp(-(K+1)\lambda_n) \\ &= W_m \exp\left(-\lambda_n \frac{W_n}{W_m}\right), \quad \text{if } K \in \mathbb{N}. \end{aligned}$$

Proof. Let $\mathcal{W}_n = \llbracket 1, W_n \rrbracket$ and $\mathcal{W}_m = \llbracket 1, W_m \rrbracket$. Let $\pi(\cdot)$ be the weapon target assignment plan from Nation n on targets belonging to Nation m , i.e., for all $k \in \mathcal{W}_m$, let $\pi(k) \in \mathcal{W}_n$ be the number of weapons assigned by Nation n to target k . We order the targets such that $\pi(\cdot)$ is decreasing. Notice that W_m^r is a function of $\pi(\cdot)$, but to simplify the notation we omit this dependence. Nation n has to solve the following optimization problem:

$$\min_{\pi(\cdot)} \mathbb{E}[W_m^r], \quad \text{s.t.} \quad \sum_{k \in \mathcal{W}_m} \pi(k) = W_n, \quad (6)$$

which consists in finding a minimizer of a function over a non-empty finite set, whence the minimizer exists. We argue that the (unique) minimizer is given by

$$\pi(k) = \begin{cases} K+1 & \text{if } 1 \leq k \leq W_n - KW_m, \\ K & \text{else.} \end{cases} \quad (7)$$

First, consider a strike on a single target $k \in \mathcal{W}_m$. Up to a reordering, let $(X_l)_{l \in \llbracket 1, \pi(k) \rrbracket}$ be a collection of Bernoulli independent and identically distributed random variables with parameter p_n . $X_l = 1$ means that weapon l destroys its target k , with probability $p_n = 1 - e^{-\lambda_n}$. Define the random variable Y_k as

$$Y_k \stackrel{\text{def}}{=} \sum_{l=1}^{\pi(k)} X_l.$$

Target k remains intact if and only if $Y_k = 0$, with probability $\mathbb{P}(Y_k = 0) = e^{-\pi(k)\lambda_n}$. Whence, the number of remaining weapons held by Nation m is given by

$$W_m^r = \sum_{k \in \mathcal{W}_m} \mathbf{1}_{\{Y_k=0\}}.$$

Therefore, the expected number of remaining weapons held by Nation m is given by

$$\mathbb{E}[W_m^r] = \sum_{k \in \mathcal{W}_m} \mathbb{E}[\mathbf{1}_{\{Y_k=0\}}] = \sum_{k \in \mathcal{W}_m} \mathbb{P}(Y_k = 0) = \sum_{k \in \mathcal{W}_m} e^{-\pi(k)\lambda_n}.$$

Let $\pi(\cdot)$ be a minimizer of problem (6). Let us show that $\pi(0) \leq \pi(W_m) + 1$ using on a proof by contradiction. Assume the converse: $\pi(0) \geq \pi(W_m) + 2$. As $\pi(\cdot)$ is decreasing, there exists l, l' such that

$$\pi(0) = \pi(l) > \pi(l+1) \geq \pi(l') = \pi(W_m)$$

and that l' is minimal. Define the *ad hoc* function $\pi^\sharp(\cdot)$ by

$$\forall k \in \mathcal{W}_m, \quad \pi^\sharp(k) \stackrel{\text{def}}{=} \begin{cases} \pi(l) - 1 & \text{if } k = l, \\ \pi(l') + 1 & \text{if } k = l', \\ \pi(k) & \text{else.} \end{cases}$$

Note that π^\sharp is decreasing and sums to W_n . Indeed, by assumption, $\pi(l) \geq \pi(l') + 2$. Whence, one can compute:

$$\mathbb{E}[W_m^r]_{|\pi^\sharp} - \mathbb{E}[W_m^r]_{|\pi} = (e^{\lambda_n} - 1) \left(e^{-\lambda_n \pi(l)} - e^{-\lambda_n (\pi(l') + 1)} \right).$$

Now, because $\pi(l') + 1 = \pi(W_m) + 1 < \pi(0) = \pi(l)$, it holds that

$$\mathbb{E}[W_m^r]_{|\pi^\sharp} - \mathbb{E}[W_m^r]_{|\pi} < 0,$$

which contradicts that π be minimal. Whence, any minimizer π must satisfy $\pi(0) \leq \pi(W_m) + 1$. Now the only decreasing function summing to W_n is given by (7). In turn, the minimum is:

$$\mathbb{E}[W_m^r] = (W_m - w) \exp(-K\lambda_n) + w \exp(-(K+1)\lambda_n).$$

Now, if $W_n = KW_m$, i.e., $w = 0$, then $\mathbb{E}[W_m^r] = W_m \exp(-K\lambda_n)$. □

Nation n 's opponents have $x_{\sigma(n)}^\tau$ weapons of class τ . Thus Proposition 1 shows that the effective power drawn from stockpile value of class τ weapons, after a single preemptive strike by Nation n with nuclear weapons, is given by $x_{\sigma(n)}^\tau \exp\left(-\lambda_n \frac{x_n^{\text{conv}}}{x_{\sigma(n)}^\tau}\right)$.

2.4.3 Generic Security Function

Combining all of the above, we can define a generic security function for both conventional and nuclear armaments. Let $a, b \in \{0, 1\}$. Define Nation n 's security function for conventional armaments as:

$$f_n^{\text{conv}}(x_n, x_{-n}) = x_n^{\text{conv}} \exp\left(-\lambda_{\sigma(n)} \frac{x_{\sigma(n)}^{\text{nuc}}}{x_n^{\text{conv}}}\right)^a \phi\left(\frac{x_{\sigma(n)}^{\text{conv}}}{\xi_n}\right) - x_{\sigma(n)}^{\text{conv}} \exp\left(-\lambda_n \frac{x_n^{\text{nuc}}}{x_{\sigma(n)}^{\text{conv}}}\right)^b, \quad (8)$$

and Nation n 's security function for nuclear armaments as:

$$\begin{aligned} f_n^{\text{nuc}}(x_n, x_{-n}) = & x_n^{\text{nuc}} \exp\left(-\lambda_{\sigma(n)} \frac{x_{\sigma(n)}^{\text{nuc}}}{x_n^{\text{nuc}}}\right)^a \phi\left(\frac{x_{\sigma(n)}^{\text{conv}} + x_{\sigma(n)}^{\text{nuc}}}{\xi_n}\right) \\ & - x_{\sigma(n)}^{\text{nuc}} \exp\left(-\lambda_n \frac{x_n^{\text{nuc}}}{x_{\sigma(n)}^{\text{nuc}}}\right)^b, \end{aligned} \quad (9)$$

where $\lambda_{\sigma(n)}$ is a measure of the accuracy of the nuclear weapons of Nation n 's opponents. In general, Nations might choose among four doctrine choices, shaping their security functions. Indeed, each Nation will choose whether it relies on its vulnerable second-strike capability ($a=1$) or not ($a=0$); and whether it relies on its preemptive first strike capability ($b=1$) or not ($b=0$). We focus on the case $(a, b) = (0, 1)$, more representative of modern nuclear military capabilities. Therefore, applying (5), the security function of Nation $n \in \mathcal{N}$ will be given by:

$$\begin{aligned} f_n(x_n, x_{-n}) = & \alpha^{\text{nuc}} \left[x_n^{\text{nuc}} \phi\left(\frac{x_{\sigma(n)}^{\text{conv}} + x_{\sigma(n)}^{\text{nuc}}}{\xi_n}\right) - x_{\sigma(n)}^{\text{nuc}} \exp\left(-\lambda_n x_n^{\text{nuc}} / x_{\sigma(n)}^{\text{nuc}}\right) \right] \\ & + \alpha^{\text{conv}} \left[x_n^{\text{conv}} \phi\left(\frac{x_{\sigma(n)}^{\text{conv}}}{\xi_n}\right) - x_{\sigma(n)}^{\text{conv}} \exp\left(-\lambda_n x_n^{\text{nuc}} / x_{\sigma(n)}^{\text{conv}}\right) \right], \end{aligned} \quad (10)$$

with $\alpha^{\text{nuc}} + \alpha^{\text{conv}} = 1$, $\alpha^{\text{nuc}}, \alpha^{\text{conv}} \geq 0$, λ_n is Nation n 's nuclear armaments accuracy, and $\xi_n > 0$ is a Nation-specific constant which models Nation n 's peacefulness. Remind that $\phi(\cdot)$ is a concave, strictly increasing function. From now on, the following assumption is made about $\phi(\cdot)$. This assumption provides a framework for the modeling of international relations consistent with the theory of realism assumed in this work [41].

Assumption 3 *The function $\phi(\cdot)$ is concave and strictly increasing. It holds that $\phi(0) = 0$, $\phi'(0) = 1$, $\phi^{(3)}(0) > 0$ and that $\phi(s) = o(s)$ at infinity.*

For technical purposes, let $k > 0$ and define the truncated function $\phi^k : s \mapsto \min\{\phi(s), \phi(k)\}$. Furthermore, let us denote $J_n^k(\cdot)$ the utility function defined as in $J_n(\cdot)$, where $\phi(\cdot)$ has been substituted with $\phi^k(\cdot)$.

Remark 2.2 In what follows, we will choose $\phi : s \mapsto \ln(1 + s)$ in numerical computations. Indeed, this choice of $\phi(\cdot)$ checks Assumption 3. Also observe that provided $\phi(\cdot)$ checks Assumption 3, $\phi^k(\cdot)$ will also satisfy Assumption 3 and be bounded.

Now, let us introduce a differential operator notation that will come in rather convenient later on. Define the directional derivative of $J_n(\cdot)$ along x_n by the following expression.

$$\partial_n^{\text{out}} J_n(x) \stackrel{\text{def}}{=} \frac{\langle \nabla_{x_n} J_n(x), x_n \rangle}{\|x_n\|}. \quad (11)$$

Also, observe that under Assumption 2 and 3, the following inequality holds:

$$\partial_n^{\text{out}} J_n(x_n, x_{-n}) \leq \bar{\lambda} + \phi\left(\frac{\kappa_+ \sqrt{2}}{\bar{\xi}} \|\bar{x}\|\right) - \bar{C} \|x_n\|, \quad \forall n \in \mathcal{N}, \quad (12)$$

where the notations listed below are in use:

$$\bar{C} = \min_{m \in \mathcal{N}, \tau} C_m^\tau, \quad \bar{\xi} = \max_{m \in \mathcal{N}} \xi_m, \quad \bar{\lambda} = \max_{m \in \mathcal{N}} \lambda_m, \quad \bar{x}^\tau = \max_{m \in \mathcal{N}} x_m^\tau \quad \text{and} \quad \bar{x} = (\bar{x}^\tau)_\tau$$

An analogous inequality holds for $(J_n^k(\cdot), \phi^k(\cdot))$:

$$\partial_n^{\text{out}} J_n^k(x_n, x_{-n}) \leq \bar{\lambda} + \phi^k\left(\frac{\kappa_+ \sqrt{2}}{\bar{\xi}} \|\bar{x}\|\right) - \bar{C} \|x_n\|, \quad \forall n \in \mathcal{N}, \quad (13)$$

We observe that $f_n(\cdot, x_{-n})$ is a \mathcal{C}^∞ function on $(\mathbb{R}_+^*)^2$. It is also well-defined on $\partial(\mathbb{R}_+^*)^2 \setminus \{0\}$. Moreover, around this set, it is continuous and bounded thus one can extend it by continuity at 0 by the value 0. However, it is not differentiable on $\partial(\mathbb{R}_+^*)^2$. Therefore, we can not apply directly standard results from game theory which require that the players' utility functions be defined on a closed-convex domain and be differentiable. The analysis of the one-shot game will therefore require to deal, first, with the degenerate case of the boundary and, second, to develop theoretical tools to study utility functions defined on open unbounded domains.

3 Analysis of the One-Shot Game

Let $\mathcal{U} \stackrel{\text{def}}{=} \prod_n \mathcal{U}_n$. We define $\mathcal{G} \stackrel{\text{def}}{=} (\mathcal{N}, \mathcal{U}, (J_n)_n)$ as the one-shot (static) game involving a set \mathcal{N} of N armed Nations. For all $n \in \mathcal{N}$, Nation n 's utility function is derived from (4), with target value satisfying Assumption 2. The security functions are defined in (5), and we assume quadratic cost functions, $c_n^S(\cdot)$, $\forall n$:

$$c_n^S(x_n) = \frac{1}{2} C_n^{\text{conv}} (x_n^{\text{conv}})^2 + \frac{1}{2} C_n^{\text{nuc}} (x_n^{\text{nuc}})^2,$$

where $C_n^{\text{conv}}, C_n^{\text{nuc}} > 0$ are storage marginal unitary costs for conventional and nuclear weapons respectively, assuming $c_n^D(0) = 0$. Nation n 's utility function takes the form: $J_n(x_n, x_{-n}) = f_n(x) - c_n^S(x_n)$. We will analyze the outcome of the noncooperative game \mathcal{G} relying on the classical Nash Equilibrium, as solution concept.

Definition 2 (Nash Equilibrium) A Nash Equilibrium $x^* = (x_n^*)_n$ of \mathcal{G} is a vector of power drawn from the stockpiles of weapons, such that

$$J_n(x^*) \geq J_n(x_n, x_{-n}^*), \quad \forall x_n \in \mathcal{U}_n, \quad \forall n \in \mathcal{N}.$$

We recall below the definition of a concave game.

Definition 3 (Concave N -player game) Let $E \stackrel{\text{def}}{=} \prod_{n=1}^N E_n$ be a product of Euclidean spaces $(E_n)_n$. Let $\mathcal{U} \subset E$ be a convex subset of E and for all $n \in \mathcal{N}$ assume the utility functions $J_n : \mathcal{U} \rightarrow \mathbb{R}$ are continuous. The game \mathcal{G} , where each player $n \in \mathcal{N}$ solves the parametrized optimization problem:

$$\max_{x_n \in E_n} J_n(x_n, x_{-n}) \quad \text{s.t. } x \in \mathcal{U}, \quad \forall n \in \mathcal{N},$$

is called a concave game if for all $n \in \mathcal{N}$, $x_n \mapsto J_n(x_n, x_{-n})$ is a concave function for each fixed value of x_{-n} such that $x \in \mathcal{U}$. If \mathcal{U} is compact, then it is called a compact-concave game.

A classical result [33, Theorem 1] ensures the existence of Nash Equilibria for concave N -player games whose joint strategy set is compact and convex. In the follow up, we extend this result to the case where: (i) the joint strategy set is a closed convex; and, (ii) the players' utility functions satisfy a coercivity property. Finally, we prove the existence of an interior point Nash Equilibrium for the N -player static game \mathcal{G} .

Lemma 2 $J_n(\cdot)$ is strictly concave in x_n for all $n \in \mathcal{N}$.

Proof. Taking the second-order derivatives of $J_n(\cdot)$ with respect to x_n , we get that

$$H_{n,n}(x) = \begin{pmatrix} \frac{\partial^2 J_n(x)}{\partial x_n^{\text{nuc}} \partial x_n^{\text{nuc}}} & \frac{\partial^2 J_n(x)}{\partial x_n^{\text{nuc}} \partial x_n^{\text{conv}}} \\ \frac{\partial^2 J_n(x)}{\partial x_n^{\text{conv}} \partial x_n^{\text{nuc}}} & \frac{\partial^2 J_n(x)}{\partial x_n^{\text{conv}} \partial x_n^{\text{conv}}} \end{pmatrix} = \begin{pmatrix} -C_n^{\text{nuc}} - \lambda_n^2 \sum_{\tau} \alpha^\tau / x_{\sigma(n)}^\tau e^{-\lambda_n x_n^{\text{nuc}}} & 0 \\ 0 & -C_n^{\text{conv}} \end{pmatrix}.$$

Therefore, $H_{n,n}(x)$ is negative definite. Thence, J_n is strictly concave in x_n . \square

Remark 3.1 Note that contrary to the finite-horizon game involving a finite number of time steps in the dynamic game, decision variables cannot be normalized in the one-shot game. Indeed, as we study the equilibria in the long-run, i.e., steady states, without requiring a priori budget constraints, there is no reason that Nash Equilibria take bounded values when time goes to infinity, e.g., in the case of a security spiral.

The following proposition deals with the *degenerate cases* where, at a Nash Equilibrium, a Nation would disarm and give up a weapons class. We show that this kind of solution requires all Nations to disarm for the same weapons class.

Proposition 2 (Disarmament) Let x be a Nash Equilibrium. Let $\tau \in \{\text{nuc}; \text{conv}\}$ and $n \in \mathcal{N}$. The following statements hold true:

1. If $x_{\sigma(n)}^\tau = 0$, then for all $m \in \sigma(n)$, $x_m^\tau = 0$.
2. If $x_n^\tau = 0$, then for all $m \in \mathcal{N}$, $x_m^\tau = 0$.

Proof. To prove the first statement, let $x_{\sigma(n)}^\tau = 0$. Then, for all $m \in \sigma(n)$,

$$0 \leq x_m^\tau \leq \max_{\ell \in \sigma(n)} x_\ell^\tau \leq \frac{x_{\sigma(n)}^\tau}{\kappa^-} = 0, \text{ by Assumption 2.}$$

Therefore, $\forall m \in \sigma(n)$, $x_m^\tau = 0$. Now, we want to prove the second statement. First, let us show that if $x_n^\tau = 0$, then $\forall m \in \sigma(n)$, $x_m^\tau = 0$ by contraposition. Assume that there exists $m \in \sigma(n)$ such that $x_m^\tau > 0$, which implies $x_{\sigma(n)}^\tau > 0$ using the first statement. Now, as $x_{\sigma(n)}^\tau > 0$, a direct computation yields a lower bound on the gradient of the security function at $(x_n^{\text{nuc}}, x_n^{\text{conv}}) \in (\mathbb{R}_+)^2$:

$$\frac{\partial f_n}{\partial x_n^\tau}(x_n, x_{-n}) \geq \alpha^\tau \phi\left(\frac{x_{\sigma(n)}^\tau}{\xi_n}\right) > 0,$$

because $\phi(\cdot)$ is strictly increasing and $\phi(0) = 0$. Note that, at $x_n^\tau = 0$, $\frac{\partial c_n^S}{\partial x_n^\tau}(x_n) = C_n^\tau x_n^\tau = 0$. Therefore,

$$\frac{\partial J_n}{\partial x_n^\tau}(x_n, x_{-n}) = \frac{\partial f_n}{\partial x_n^\tau}(x_n, x_{-n}) - \frac{\partial c_n^S}{\partial x_n^\tau}(x_n) = \frac{\partial f_n}{\partial x_n^\tau}(x_n, x_{-n}) > 0.$$

Hence, all strategies with $x_n^\tau = 0$ are dominated, i.e., if x is a Nash Equilibrium, we have that $x_n^\tau > 0$. Therefore, if $x_n^\tau = 0$, then $\forall m \in \sigma(n)$, $x_m^\tau = 0$.

Now, assume that $x_n^\tau = 0$. Then, by induction, $\forall k \in \mathbb{N}$, $\forall m \in \sigma^k(n)$, $x_m^\tau = 0$. As \mathcal{N} is a strategic relationship class, $\mathcal{N} = \sigma^{|\mathcal{N}|}(n)$. Therefore, $x_m^\tau = 0$. \square

Let $\mathcal{U}^\dagger \stackrel{\text{def}}{=} \mathcal{U} \cap (\mathbb{R}_+^*)^{2N}$ and $\mathcal{U}_n^\dagger \stackrel{\text{def}}{=} \mathcal{U}_n \cap (\mathbb{R}_+^*)^2$. As a consequence of Proposition 2, there exists a trivial Nash Equilibrium which coincides with a “general and complete disarmament” strategy ($x = 0$). In the following sections, we prove the existence and uniqueness of interior points Nash Equilibria $x^* \in \mathcal{U}^\dagger$. An interior point Nash Equilibrium does not lie on the boundary of $(\mathbb{R}_+^*)^2$.

3.1 Existence of a Nash Equilibrium

It is well-known that every compact-concave game admits a Nash Equilibrium. When J_n is (jointly) continuous in x , concave and continuously differentiable in x_n for all $x_{-n} \in \mathcal{U}_{-n} \stackrel{\text{def}}{=} \prod_{m \neq n} \mathcal{U}_m$, existence of a Nash Equilibrium is guaranteed for bounded \mathcal{U}_n [6, Theorem 4.4]. When \mathcal{U}_n is closed but not bounded, intuition is given about the existence of a Nash Equilibrium under coercivity assumptions, e.g., by requiring that $-J_n$ is coercive in x_n , that is $\lim_{\|x_n\| \rightarrow +\infty} J_n(x_n, x_{-n}) = +\infty, \forall x_{-n} \in \mathcal{U}_{-n}, n \in \mathcal{N}$.

The coercivity assumption may not be sufficient, and we need to impose uniform coercivity to prove the existence of a Nash Equilibrium in that setting. Let consider the following counter-example: $N = 2$, $\mathcal{U}_1 = \mathcal{U}_2 = \mathbb{R}$, $J_n(x_n, x_{-n}) = (x_n - x_{-n} - 1)^2$. For all n and all y , $J_n(\cdot, y)$ is coercive and convex, and J_n is continuous. However, the problem has no Nash Equilibrium. The problem is that we need to show that we can restrict ourselves to a compact joint space and that simple coercivity does not allow us to get rid of the dependence of the sublevel sets on the parameters fixed by the opponents.

In this section, we introduce the notion of uniform coercivity, and prove the uniqueness of the interior point Nash Equilibrium under some additional conditions. Another way to prove this latter result has been introduced in [6, Proposition 4.1] for two-person games, by sharpening the classical notion of fixed point and relying on a contraction mapping argument.

As the utility functions of the game \mathcal{G} are concave in their own strategy space, we can build a compact-concave game \mathcal{G}' by imposing constraints to \mathcal{G} , ensuring the existence of a Nash Equilibrium for \mathcal{G}' . Let $\mathcal{U}' \subset \mathcal{U}$ be a compact-convex set.

Lemma 3 *Assume that a Nash Equilibrium is reached at an interior point of \mathcal{U}' . Then, since the players' utility functions are concave in their own strategy space, it is also a Nash Equilibrium for the game with strategy space \mathcal{U} .*

Proof. The proof relies directly on the concavity of the utility functions. \square

Using this property, we can show under further assumptions on the asymptotic behavior of the utility functions that the boundedness of \mathcal{U} is not required.

Definition 4 *Let $\Psi : \mathcal{U} \subset E \rightarrow \mathbb{R}$ be a continuous function. We say that $\Psi(\cdot)$ is uniformly coercive in x_n over E_n if and only if there exists $r_n > 0$ and $g : E_n \rightarrow \mathbb{R}$ such that:*

$$\begin{aligned} \forall x \in \mathcal{U}, \quad \partial_n^{\text{out}} \Psi(x) &\geq g(x_n), \\ \forall y \in E_n, \quad \|y\| \geq r_n &\implies g(y) > 0, \end{aligned}$$

where $\partial_n^{\text{out}} \Psi(x) = \frac{\langle \nabla_{x_n} \Psi(x), x_n \rangle}{\|x_n\|}$ is the directional derivative of $\Psi(\cdot)$ along x_n .

Using (12) and that $\phi^k(\cdot)$ are bounded, we check that the opposite of utility functions $-J_n^k(\cdot)$, $\forall n$ are uniformly coercive over their own strategy space.

Remark 3.2 *By coercive, one generally means that a function grows rapidly at infinity. To get the idea, consider Definition 4 with a function Ψ defined on the real line and assume, e.g., that g is bounded from below by a positive constant, i.e., there exists $g_0 > 0$ such that for all $s \in \mathbb{R}$, $g(s) > g_0$. Thus, it holds that for all $s \in \mathbb{R}$, $|\Psi'(s)| > g_0$ whence $\Psi(s) \xrightarrow{|s| \rightarrow +\infty} +\infty$.*

Proposition 3 *Let $\mathcal{G} = (\mathcal{N}, \mathcal{U}, (J_n)_n)$ be a concave game. Assume $\mathcal{U} \subset E$ is closed and $-J_n(\cdot)$ is uniformly coercive over its own strategy space \mathcal{U}_n , $\forall n$. Then, \mathcal{G} admits a Nash Equilibrium.*

Proof. Let $\mathcal{B}_n(r)$ be the closed ball of radius $r > 0$ in E_n . Let $\mathcal{B}(r) \stackrel{\text{def}}{=} \prod_{n=1}^N \mathcal{B}_n(r)$. Hence, $\Gamma(r) \stackrel{\text{def}}{=} \mathcal{B}(r) \cap \mathcal{U} \subset \mathcal{U}$ is the intersection of two closed convex sets and is bounded, thus it is a compact-convex set. Then, \mathcal{G} is a compact-concave game over $\Gamma(r)$ and admits a Nash Equilibrium, denoted by x^* . Now, for all $n \in \mathcal{N}$, $-J_n(\cdot)$ is uniformly coercive over E_n . Hence, there exists $r_n > 0$ and $g_n(\cdot)$ such that

$$\begin{aligned} \forall x \in \mathcal{U}, \quad & -\partial_n^{\text{out}} J_n(x) \geq g_n(x_n), \\ \forall y \in E_n, \quad & \|y\| \geq r_n \implies g_n(y) > 0. \end{aligned}$$

Set $r > \max_n \{r_n\}$. If there existed $n \in \mathcal{N}$ such that $x_n \in \partial \mathcal{B}_n(r)$, then the optimality conditions would write:

$$\partial_n^{\text{out}} J_n(x) \geq 0 \quad \text{with } \|x_n\| = r,$$

Now, $\|x_n\| = r \geq r_n$ thus $\partial_n^{\text{out}} J_n(x) \leq -g_n(x_n) < 0$, which is a contradiction. Therefore, for $r > 0$ large enough, Nash Equilibria of \mathcal{G} are either interior points of $\Gamma(r) \subset \mathcal{U}$ or boundary points of \mathcal{U} . Notice that in the former case, by Lemma 3, they are also interior points Nash Equilibria for the game with strategy space \mathcal{U} . Therefore, the game \mathcal{G} admits a Nash Equilibrium. \square

Since the utility functions $J_n^k(\cdot)$, $n \in \mathcal{N}$ are uniformly coercive over their own strategy space, it results from Proposition 3 that for all $k > 0$, the game $\mathcal{G}^k \stackrel{\text{def}}{=} (\mathcal{N}, \mathcal{U}, (J_n^k)_n)$ admits a Nash Equilibrium, denoted by x^{*k} . One can expect that when k is large, \mathcal{G}^k is essentially the same game as \mathcal{G} and that x^{*k} may serve as an intermediary for proving the existence of Nash Equilibria of the game \mathcal{G} . We formalize this idea below.

Proposition 4 *There exists a Nash Equilibrium solution of \mathcal{G} .*

Proof. From Proposition 3 that for all $k > 0$, the game $\mathcal{G}^k \stackrel{\text{def}}{=} (\mathcal{N}, \mathcal{U}, (J_n^k)_n)$ admits a Nash Equilibrium x^{*k} . Thus the directional derivative of $J_n^k(\cdot)$ at x^{*k} must be non-negative. Define the function $F : \mathbb{R}_+ \rightarrow \mathbb{R}$ by

$$F : z \mapsto \bar{\lambda} + \phi \left(\frac{\kappa_+ \sqrt{2}}{C \bar{\xi}} z \right).$$

Let $z \stackrel{\text{def}}{=} \bar{C} \|\bar{x}^{*k}\|$. From (12), we infer that $z \leq F(z)$. As $\phi(\cdot)$ is strictly increasing, so is F on \mathbb{R}_+ . Thus for all $p \in \mathbb{N}$, $z \leq F^p(z)$. Now, the assumptions on ϕ imply that F has a unique fixed point over \mathbb{R}_+^* . Denote this value by z_0 . By definition, z_0 depends only on ϕ , $\bar{\lambda}$, $\frac{\kappa_+}{C \bar{\xi}}$. Furthermore, $F^p(z)$ converges to z_0 as p goes to infinity. Therefore, $z \leq z_0$.

Let $r > z_0 / \bar{C}$ and $k > \sqrt{2} \frac{\kappa_+}{\bar{\xi}} r$. Hence, one can check that the games $\tilde{\mathcal{G}}^k = (\mathcal{N}, \mathcal{U} \cap \mathcal{B}(r), (J_n^k)_n)$ and $\tilde{\mathcal{G}} = (\mathcal{N}, \mathcal{U} \cap \mathcal{B}(r), (J_n)_n)$ are identical and as such they share their Nash Equilibria. Now, let x^* a Nash Equilibrium of $\tilde{\mathcal{G}}$. As $\|x^*\| \leq z / \bar{C} \leq z_0 / \bar{C} < r$, x^* does not lie on $\partial \mathcal{B}(r)$ thus it is a Nash Equilibrium of \mathcal{G} . \square

Theorem 1 *The game \mathcal{G} admits an interior point Nash Equilibrium $x^* \in \mathcal{U}^\dagger$ under the sufficient conditions:*

$$C_n^{\text{nuc}} \xi_n < 2\kappa^- \alpha^{\text{nuc}}, \quad C_n^{\text{conv}} \xi_n < \kappa^- \alpha^{\text{conv}}.$$

Proof. Let $\varepsilon > 0$ and $\Gamma_\varepsilon = \bigcap_{n,\tau} \{x_n^\tau \geq \varepsilon\} \subset \mathcal{U}^\dagger$. From Proposition 3, as Γ_ε is a closed convex set, the game $(\mathcal{N}, \Gamma_\varepsilon, (J_n)_n)$ admits a Nash equilibrium x^* . Reasoning by contradiction, suppose that $x^* \in \partial \Gamma_\varepsilon$. This means that there exists $n \in \mathcal{N}$ and $\tau \in \{\text{nuc}; \text{conv}\}$ such that $x_n^{*\tau} = \varepsilon$. Hence, using first order Taylor-Lagrange expansion, at least one of the following inequalities holds true:

$$\begin{aligned} \frac{\partial J_n}{\partial x_n^{\text{conv}}}(x^*) &\geq (\kappa^- \alpha^{\text{conv}} - C_n^{\text{conv}} \xi_n) \frac{\varepsilon}{\xi_n} + o(\varepsilon), \\ \frac{\partial J_n}{\partial x_n^{\text{nuc}}}(x^*) &\geq (2\kappa^- \alpha^{\text{nuc}} - C_n^{\text{nuc}} \xi_n) \frac{\varepsilon}{\xi_n} + o(\varepsilon). \end{aligned}$$

If \mathcal{G} parameters are chosen such that $\kappa^- \alpha^{\text{conv}} > C_n^{\text{conv}} \xi_n$ and $2\kappa^- \alpha^{\text{nuc}} > C_n^{\text{nuc}} \xi_n$, then for ε small enough, one has the inequality

$$\frac{\partial J_n}{\partial x_n^\tau}(x^*) > 0,$$

which contradicts the necessary optimality condition

$$\frac{\partial J_n}{\partial x_n^\tau}(x^*) \leq 0.$$

Therefore, $x^* \in \mathcal{U}^\dagger$. Hence, $\mathcal{G}^\dagger \stackrel{\text{def}}{=} (\mathcal{N}, \mathcal{U}^\dagger, (J_n)_n)$ admits the same Nash Equilibrium x^* , which is also an interior point Nash Equilibrium of \mathcal{G} . □

3.2 Uniqueness of the Interior Point Nash Equilibrium

Assumption 4 *The following inequalities hold:*

$$\forall n \in \mathcal{N}, \quad \begin{cases} 1 - \frac{1}{x_{\sigma(n)}^{\text{nuc}}} \sum_{m \neq n} \frac{\partial x_{\sigma(n)}^{\text{nuc}}}{\partial x_m^{\text{nuc}}} \geq 0, \\ C_n^{\text{nuc}} - \frac{\alpha^{\text{nuc}}}{\xi_n} \sum_{m \neq n} \frac{\partial x_{\sigma(n)}^{\text{nuc}}}{\partial x_m^{\text{nuc}}} \geq 0. \end{cases}$$

In the rest of the paper, we assume that Assumption 4 holds. Notice that Assumption 4 can be related to the sufficient conditions of Theorem 1 guaranteeing the existence of a unique interior point Nash Equilibrium solution of \mathcal{G} .

Proposition 5 *The game \mathcal{G}^\dagger is strongly monotone.*

Proof. Let F_n be the gradient of player n 's utility with respect to its own actions. It is defined by

$$F_n \stackrel{\text{def}}{=} \begin{bmatrix} \frac{\partial J_n(x)}{\partial x_n^{\text{nuc}}} & \frac{\partial J_n(x)}{\partial x_n^{\text{conv}}} \end{bmatrix}^T.$$

Whence, we can compute each component:

$$\forall x_n \in \mathcal{U}, \quad \begin{cases} \frac{\partial J_n(x)}{\partial x_n^{\text{conv}}} = \alpha^{\text{conv}} \phi(x_{\sigma(n)}^{\text{conv}}/\xi_n) - C_n^{\text{conv}} x_n^{\text{conv}} \\ \frac{\partial J_n(x)}{\partial x_n^{\text{nuc}}} = \alpha^{\text{nuc}} \phi\left(\frac{x_{\sigma(n)}^{\text{conv}} + x_{\sigma(n)}^{\text{nuc}}}{\xi_n}\right) + \lambda_n \sum_{\tau} \alpha^\tau \exp\left(-\lambda_n \frac{x_n^{\text{nuc}}}{x_{\sigma(n)}^\tau}\right) - C_n^{\text{nuc}} x_n^{\text{nuc}} \end{cases}$$

which implies that $J_n(\cdot)$ reaches its maximum in the variable x_n^{conv} at $\frac{\alpha^{\text{conv}}}{C_n^{\text{conv}}} \phi\left(\frac{x_{\sigma(n)}^{\text{conv}}}{\xi_n}\right)$, if this value is admissible, or, else, at the border of the interval of definition of x_n^{conv} .

Consider the pseudo-Hessian matrix H^{nuc} of the players' utilities considering only nuclear weapons where each n^{th} row and m^{th} column component is given as $H_{n,m}^{\text{nuc}} = \frac{\partial F_n^{\text{nuc}}}{\partial x_m^{\text{nuc}}}$, $n, m \in \mathcal{N}$, i.e., in details by (40). One can check that $-H^{\text{nuc}}$ is a Z -matrix as its-off diagonal entries are negative. Under Assumption 4, we observe that $-H^{\text{nuc}}$ is strictly diagonally dominant thus $-H^{\text{nuc}}$ is an M -matrix, i.e., a Z -matrix with eigenvalues whose real parts are nonnegative [42]. Thus, H^{nuc} is negative definite. Furthermore, letting $F \stackrel{\text{def}}{=} \text{col}((F_n)_n)$, as $\frac{\partial F_n}{\partial x_n^{\text{conv}}}$ is constant for all $n \in \mathcal{N}$, the inequality

$$(y - x)^T [F(y) - F(x)] < 0, \quad \forall x \neq y$$

follows from [33, Theorem 6] and the game \mathcal{G}^\dagger is strongly monotone. □

Theorem 2 *The game \mathcal{G} has a unique interior point Nash Equilibrium $x^* \in \mathcal{U}^\dagger$.*

Proof. Let us consider that there exists two Nash Equilibria x^* and x^\sharp solutions of the game \mathcal{G}^\dagger . As both x^* and x^\sharp are Nash Equilibria they must satisfy the stationarity condition. Mutliplying the first order condition with $(y - x^*)$ at point x^* and $(y - x^\sharp)$ at point x^\sharp , we get:

$$(y - x^*)^T F(x^*) = 0, \quad \forall y \in \mathcal{U}^\dagger, \quad (14a)$$

$$(y - x^\sharp)^T F(x^\sharp) = 0, \quad \forall y \in \mathcal{U}^\dagger. \quad (14b)$$

Taking (14a) in $y = x^*$ and (14b) in $y = x^\sharp$ and summing the two above equations, we get:

$$(x^\sharp - x^*)^T (F(x^\sharp) - F(x^*)) = 0,$$

which contradicts the fact that the game \mathcal{G}^\dagger is strongly monotone from Proposition 5. Therefore, the game \mathcal{G}^\dagger admits a unique Nash Equilibrium. \square

Definition 5 (Variational Stability [14]) *An equilibrium point $x^* \in \mathcal{U}^\dagger$ is said to be variationally stable (or simply stable) if there exists a neighborhood \mathcal{V} of x^* such that $(y - x^*)^T F(y) \leq 0$ for all $y \in \mathcal{V}$. In particular, if this property holds for all $y \in \mathcal{U}^\dagger$, we say that x^* is globally stable.*

Proposition 6 *The interior point Nash Equilibrium solution of \mathcal{G}^\dagger is stable.*

Proof. We proceed by contradiction. Assume that there exists \tilde{y} such that $(\tilde{y} - x^*)^T F(\tilde{y}) > 0$. Then, relying on the strong monotonicity of the game \mathcal{G}^\dagger proved in Proposition 5, $(\tilde{y} - x^*)^T F(x^*) > 0$. But, by definition, $F(x^*) = 0$. There $(y - x^*)^T F(y) \leq 0, \forall y \neq x^*$, which proves the stability of x^* .

Variational Stability is a sufficient condition for decentralized equilibrium learning methods to converge.

3.3 Decentralized Nash Equilibrium Learning

When more than two Nations are involved, algorithmic methods need to be developed to compute Nash Equilibria. The historical values of the weapons stockpiles for each Nation is assumed to be known by the N Nations. However, each Nation computes its strategy through a local update relying on the historical values of the Nations' stockpiles. Furthermore, each Nation's utility function depends on parameters that are private to the Nation. Therefore, a decentralized method needs to be implemented to learn the interior point Nash Equilibrium. To that purpose, we check below Lipschitz continuity of the pseudo-gradient $F(x)$ that will be needed later to guarantee the convergence of the decentralized algorithm.

Proposition 7 *The pseudo-gradient $F(x)$ of the game \mathcal{G} is Lipschitz continuous in $x \in \mathcal{U}^\dagger$.*

Proof. We need to prove that each coordinate of the pseudo-gradient $F(x)$ is Lipschitz continuous. Therefore, we aim to prove the Lipschitz continuity of $F_n(x)$. For the first component x_n^{conv} , we rely on the analytical expression of $\frac{\partial J_n}{\partial x_n^{\text{conv}}}(\cdot)$ given in Proposition 5 proof, it is easy to check that

$$\left| \frac{\partial J_n}{\partial x_n^{\text{conv}}}(u, x_{-n}) - \frac{\partial J_n}{\partial x_n^{\text{conv}}}(v, x_{-n}) \right| \leq C_n^{\text{conv}} |u - v|, \quad \forall x_{-n} \in \prod_{m \neq n} \mathcal{U}_m^\dagger, \quad \forall n \in \mathcal{N}.$$

For the second component x_n^{nuc} , we first need to prove that there exists a constant $C > 0$ such that $C \leq x_{\sigma(n)}^\tau$. Proceeding by contradiction, this would mean that $x_n = 0, \forall n$ for the max case which is impossible because $x \in \mathcal{U}^\dagger$, and there exists at least one $n \in \mathcal{N}$ such that $x_n < 0$ or $x_n = 0, \forall n$ for the mean and sum cases, which is also impossible by definition of \mathcal{U}^\dagger . Using this lower bound and using the Mean Value Theorem, we check that

$$\left| \frac{\partial J_n}{\partial x_n^{\text{nuc}}}(u, x_{-n}) - \frac{\partial J_n}{\partial x_n^{\text{nuc}}}(v, x_{-n}) \right| \leq \left(C_n^{\text{nuc}} + \lambda_n \sum_{\tau} \alpha^\tau \frac{\lambda_n}{C} \right) |u - v|, \quad \forall x_{-n} \in \prod_{m \neq n} \mathcal{U}_m^\dagger, \quad \forall n \in \mathcal{N}.$$

\square

Under Propositions 5 and 7, various gradient-based algorithms can be proposed to compute the interior point Nash Equilibrium solution of \mathcal{G} and are proved to converge [12]. We will implement a regularization algorithm for monotone game, called proximal point method (PPM). PPM is an alternative method to distributed gradient descent schemes, whose interest lies in the fact that it avoids having to coordinate the players in their steplength choice.

At iteration k , $x^{(k)}$ is the solution of a Variational Inequality (VI) of the type:

$$(y - x)^T F^{(k)}(x) \leq 0, \quad \forall y \in \mathcal{U}^\dagger,$$

with

$$F^{(k)}(x) = F(x) - \theta_n (x - x^{(k)}),$$

and $\theta_n > 0$ is a regularization parameter which implicitly determines iteration bounds to reach a prescribed error level [28]. In this setting, the regularization parameters are not required to be the same for all the players, because we want that our scheme allows the players to select their steplength independently and do not insist on central prescription of such parameters. Extensions of iterative proximal point method for monotone games where each player can independently select and adapt its algorithm parameter after each iteration [18] exist, but will not be considered in the current version of the work. In practice, the PPM algorithm can be implemented by solving at each iteration k :

$$x_n^{(k+1)} = \arg \max_{u \in \mathcal{U}_n^\dagger} \left(J_n(u, x_{-n}^{(k)}) - \theta_n \|u - x_n^{(k)}\|^2 \right), \quad \forall n \in \mathcal{N}. \quad (15)$$

Algorithm 1 Compute a step of the PPM algorithm

```

1: procedure PPM.STEP( $x_n, \sigma(n), \theta_n$ )
2:   Compute( $x_{\sigma(n)}$ )
3:   return Solve  $\left( \arg \max_{u \in \mathcal{U}_n^\dagger} \left( J_n(u, x_{-n}^{(k)}) - \theta_n \|u - x_n^{(k)}\|^2 \right) \right)$ 
4: end procedure

```

Algorithm 2 Compute an ε approximation of a Nash Equilibria given an Instance I of the model

```

1: procedure PPM.NASH( $I$ )
2:   repeat
3:     temp  $\leftarrow X$ 
4:     for  $n \in \mathcal{N}$  do
5:        $x_n \leftarrow$  PPM.STEP( $x_n, \sigma(n), \theta_n$ )
6:     end for
7:   until  $\|\mathbf{temp} - X\|_1 \leq \varepsilon, \quad \forall n \in \mathcal{N}$ 
8:   return  $X$ 
9: end procedure

```

The following assumption imposes upper bounds on ratios of the strategy updates, at each iteration, to keep the strategies in an admissible range and guarantee the convergence of the PPM algorithm.

Assumption 5 For any player $n \in \mathcal{N}$, there exist $T_n^{nuc}, T_n^{conv} \in [0, +\infty)$ such that for all $k \in \mathbb{N}$,

$$\frac{x_n^{conv,(k)}}{x_n^{nuc,(k)}} \leq T_n^{nuc} \quad \text{and} \quad \frac{x_{\sigma(n)}^{conv,(k)}}{x_n^{conv,(k)}} \leq T_n^{conv}.$$

Relying on Assumption 3, $\phi(u) \leq u, \forall u \in \mathcal{U}$.

Proposition 8 (Convergence Rate) Let Assumption 5 hold. Then, if $\max\{\alpha^{nuc}, \alpha^{conv}\} \leq \min_{n \in \mathcal{N}}\{\xi_n C_n^{conv}\}$, we obtain the following expression for the PPM convergence rate:

$$\|x_n^{(k)} - x_n^*\| \leq \frac{\max\{T_n^{conv} \alpha^{conv}, (1 + T_n^{nuc}) \alpha^{nuc}\} + 2\theta_n \xi_n}{C_n^{conv} \xi_n + 2\theta_n \xi_n} \|x_n^{(k-1)} - x_n^*\|, \quad \forall k \in \mathbb{N}, \forall n \in \mathcal{N}.$$

Proof. From PPM update rule (15) for any player $n \in \mathcal{N}$, we get the following first order conditions at iteration k :

$$\begin{aligned} & -C_n^{\text{conv}} x_n^{\text{conv},(k)} + \alpha^{\text{conv}} \phi \left(\frac{x_{\sigma(n)}^{\text{conv},(k-1)}}{\xi_n} \right) - 2\theta_n (x_n^{\text{conv},(k)} - x_n^{\text{conv},(k-1)}) = 0 \\ & -C_n^{\text{nuc}} x_n^{\text{nuc},(k)} + \lambda_n \sum_{\tau} \alpha^{\tau} \exp \left(-\lambda_n \frac{x_n^{\text{nuc},(k)}}{x_{\sigma(n)}^{\tau,(k-1)}} \right) + \alpha^{\text{nuc}} \phi \left(\frac{x_{\sigma(n)}^{\text{nuc},(k-1)} + x_{\sigma(n)}^{\text{conv},(k-1)}}{\xi_n} \right) - 2\theta_n (x_n^{\text{nuc},(k)} - x_n^{\text{nuc},(k-1)}) = 0. \end{aligned}$$

Re-arranging, we obtain:

$$\begin{aligned} x_n^{\text{conv},(k)} &= \frac{\alpha^{\text{conv}}}{C_n^{\text{conv}} + 2\theta_n} \phi \left(\frac{x_{\sigma(n)}^{\text{conv},(k-1)}}{\xi_n} \right) + \frac{2\theta_n}{C_n^{\text{conv}} + 2\theta_n} x_n^{\text{conv},(k-1)}, \\ x_n^{\text{nuc},(k)} &= \frac{\alpha^{\text{nuc}}}{C_n^{\text{conv}} + 2\theta_n} \phi \left(\frac{x_{\sigma(n)}^{\text{nuc},(k-1)} + x_{\sigma(n)}^{\text{conv},(k-1)}}{\xi_n} \right) + \frac{2\theta_n}{C_n^{\text{conv}} + 2\theta_n} x_n^{\text{nuc},(k-1)} - \frac{\lambda_n}{C_n^{\text{conv}} + 2\theta_n} \sum_{\tau} \alpha^{\tau} \exp \left(-\lambda_n \frac{x_n^{\text{nuc},(k)}}{x_{\sigma(n)}^{\tau,(k-1)}} \right). \end{aligned}$$

Under Assumption 5, we derive the following upper bound for the conventional part of player n 's strategy:

$$(x_n^{\text{conv},(k)} - x_n^{\text{conv},*}) \leq \frac{T_n^{\text{conv}} \alpha^{\text{conv}} / \xi_n + 2\theta_n}{C_n^{\text{conv}} + 2\theta_n} (x_n^{\text{conv},(k-1)} - x_n^{\text{conv},*}).$$

Similarly, under Assumption 5, we derive the following upper bound for the nuclear part of player n 's strategy:

$$(x_n^{\text{nuc},(k)} - x_n^{\text{nuc},*}) \leq \frac{(1 + T_n^{\text{nuc}}) \alpha^{\text{nuc}} / \xi_n + 2\theta_n}{C_n^{\text{conv}} + 2\theta_n} (x_n^{\text{nuc},(k-1)} - x_n^{\text{nuc},*}).$$

From these two inequalities and by definition of the square of the norm $\|x_n^{(k)} - x_n^*\|^2 = \sum_{\tau} |x_n^{\tau,(k)} - x_n^{\tau,*}|^2$, we infer player n 's convergence rate. \square

Remark 3.3 As in [6, Proposition 4.1], another way to prove the uniqueness of the interior point Nash Equilibrium would be to determine additional conditions to apply a contraction mapping argument to the convergence rate inequality.

Remark 3.4 The linear convergence rate of proximal point algorithms have been well studied in the literature. In particular, its linear convergence rate has been studied in [32], under certain condition. Our contribution is to provide a closed form expression of the convergence rate, capturing its dependence on the game parameters.

Illustrations for an International System of Three Nations: Most arms race models from the literature are formulated as two player games, where players are understood as blocs, e.g., USA versus USSR-allied, West versus East, etc. The primary goal of this use case is to showcase that we can extend the analysis to an N player setting. Considering an international system made of three Nations abstracted as a complete graph as in Figure 1b, we illustrate the convergence rate of the PPM, for each one of the three players and in different values of ξ in Figure 2a. For $\xi = 10$, PPM runs through 7195 iterations before approximated close enough the interior point Nash Equilibrium, as opposed to 16997 for $\xi = 70$. Therefore, bigger ξ values lead to longer convergence times, which coincides with the convergence rate derived in Proposition 8: indeed, $\xi_n \rightarrow +\infty$ implies that $\frac{1}{\xi_n} \rightarrow 0$ and the PPM convergence rate can therefore be approximated by 1 provided C_n^{conv} is chosen small enough, therefore leading to a sublinear (slow) convergence rate. Considering as previously heterogeneous target values, we plot in Figure 2b the players' objective values at equilibrium J_n^* in different ξ (ξ being again homogeneous among the players). First, we note that in case where the only parameter differentiating the players is their target values, the player with sum as target value has always a larger utility at equilibrium, than the other players. Second, we can clearly observe significant improvements for all the players' objective values at equilibrium as ξ increases albeit it is with diminishing returns. The code is publicly available at [7].

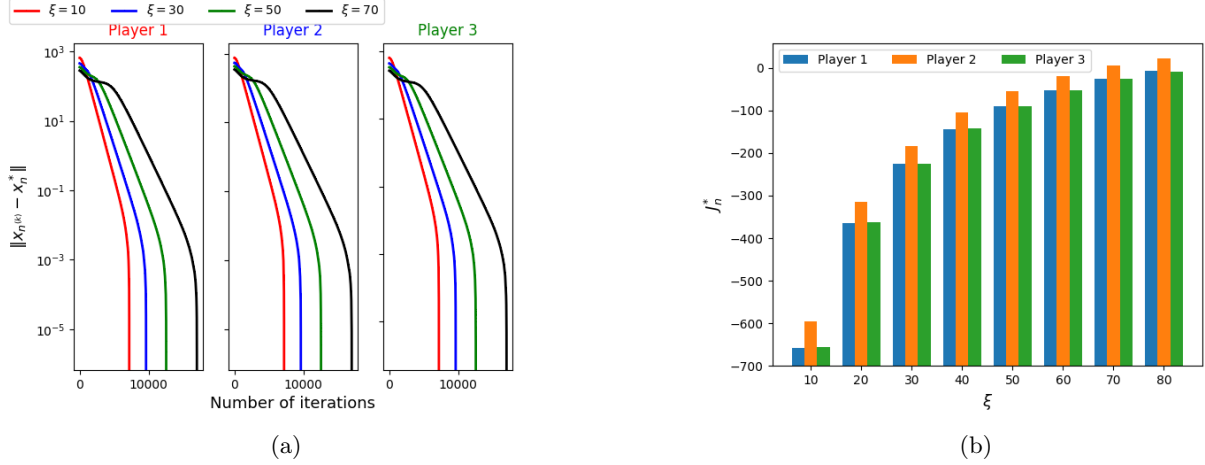


Figure 2: Illustrations of the PPM convergence rates for three Nations, in different values of ξ in Figure 2a. Utility functions' values at the interior point Nash Equilibrium in different values of ξ in Figure 2b. $\xi_n = \xi_m, \forall n, m, n \neq m$. Target functions are chosen heterogeneous, i.e., player 1 considers the maximum, player 2 deals with the sum while player 3 chooses the mean. For the other parameters, we choose $C_n^{\text{conv}} = C_n^{\text{nuc}} = 5 \times 10^{-3}, \lambda_n = 8 \times 10^{-1}, \theta_n = 11 \times 10^{-1}, \forall n \in \mathcal{N}$. The empirical rates match the theory in Subsection 3.3.

4 Strategic Stability Analysis and Simulations

In this section, we show that mutual possession of nuclear weapons has a beneficial effect on the utility of Nations competing for mixed armaments. We argue that, thanks to this utility-enhancing effect, the arms-building logic driven by mutual deterrence minimizes the risk of one state rapidly gaining strategic advantage over another, therefore enforcing strategic stability in its latest meaning [3]. We deal first with the analytical analysis of this problem for the two-player game – or duopoly – before discussing our numerical results. Finally, we focus on the N -player game and demonstrate the utility-improvement effect of nuclear weapons.

4.1 Analytical Analysis in Case of a Duopoly

In the case of a duopoly ($N = 2$), we provide an analytical characterization of the interior point Nash Equilibrium of \mathcal{G} . For all $n, m \in \mathcal{N}$ such that $n \neq m$, we let x_m denote x_{-n} and $\sigma(n) \stackrel{\text{def}}{=} \{m\}$. Then, the interior point Nash Equilibrium of \mathcal{G} is obtained by solving the following system:

$$\max_{x_n \in (\mathbb{R}_+^*)^2} J_n(x_n, x_m), \quad \forall n, m \in \mathcal{N}, n \neq m,$$

leading to the following first order (necessary) optimality conditions:

$$\begin{cases} \frac{\partial J_n}{\partial x_n^{\text{conv}}}(x_n, x_m) = 0, \\ \frac{\partial J_n}{\partial x_n^{\text{nuc}}}(x_n, x_m) = 0, \end{cases} \quad \forall n, m \in \mathcal{N}, n \neq m,$$

which can be written explicitly as the following system of equations:

$$C_n^{\text{conv}} x_n^{\text{conv}} - \alpha^{\text{conv}} \phi\left(\frac{x_m^{\text{conv}}}{\xi_n}\right) = 0, \quad \forall n, m \in \mathcal{N}, n \neq m, \quad (16a)$$

$$C_n^{\text{nuc}} x_n^{\text{nuc}} - \lambda_n \sum_{\tau} \alpha^{\tau} \exp\left(-\frac{\lambda_n x_n^{\text{nuc}}}{x_m^{\tau}}\right) - \alpha^{\text{nuc}} \phi\left(\frac{x_m^{\text{nuc}} + x_m^{\text{conv}}}{\xi_n}\right) = 0, \quad \forall n, m \in \mathcal{N}, n \neq m. \quad (16b)$$

Let $n, m \in \mathcal{N}$ such that $n \neq m$. Define the best-reply map for conventional armaments:

$$\mathcal{R}_n^{\text{conv}} : x \mapsto \frac{\alpha^{\text{conv}}}{C_n^{\text{conv}}} \phi\left(\frac{x}{\xi_n}\right).$$

Then one can rewrite (16a) as $x_n^{\text{conv}} = \mathcal{R}_n^{\text{conv}}(x_m^{\text{conv}})$. Hence, when dealing with two players, the Nash Equilibrium components $(x_n^{\text{conv}})_n$ are found at the intersection $(x_1^{*,\text{conv}}, x_2^{*,\text{conv}})$ of curves $(\mathcal{R}_1^{\text{conv}}(s), s)_{s>0}$ and $(s, \mathcal{R}_2^{\text{conv}}(s))_{s>0}$. Consider the following functional equations with unknowns $\mathcal{R}_n^{\text{nuc}} : x \mapsto y$:

$$C_n^{\text{nuc}}y - \lambda_n \left[\alpha^{\text{nuc}} \exp\left(-\frac{\lambda_n y}{x}\right) + \alpha^{\text{conv}} \exp\left(-\frac{\lambda_n y}{\xi_n z_n}\right) \right] - \alpha^{\text{nuc}} \phi\left(\frac{x}{\xi_n} + z_n\right) = 0, \quad \forall n \in \mathcal{N}. \quad (17)$$

where z_1, z_2 are defined by

$$z_1 \stackrel{\text{def}}{=} \frac{x_2^{*,\text{conv}}}{\xi_1} \quad \text{and} \quad z_2 \stackrel{\text{def}}{=} \frac{x_1^{*,\text{conv}}}{\xi_2}.$$

Then (16b) is equivalent to $\mathcal{R}_n^{\text{nuc}}(x_m^{\text{nuc}}) = x_n^{\text{nuc}}$ by setting $x = x_m^{\text{nuc}}$ and $y = x_n^{\text{nuc}}$. Let $\mu_n = \frac{z_n}{\xi_n}$, $\rho_n = C_n^{\text{nuc}}\xi_n$ and

Consider the change of variables $(X, Y) = \left(\frac{x^{\text{nuc}}}{\xi_n}, C_n^{\text{nuc}}y^{\text{nuc}}\right)$, then (17) recasts as

$$Y - \lambda_n \left[\alpha^{\text{nuc}} \exp\left(-\frac{\lambda_n}{C_n^{\text{nuc}}\xi_n} \frac{Y}{X}\right) + \alpha^{\text{conv}} \exp\left(-\frac{\lambda_n}{C_n^{\text{nuc}}\xi_n} \frac{Y}{z_n}\right) \right] - \alpha^{\text{nuc}} \phi(X + z_n) = 0, \quad \forall n \in \mathcal{N}. \quad (18)$$

Proposition 9 Equation (18) has a unique solution.

Proof. Let $n \in \mathcal{N}$. For all $X, Y \geq 0$, define

$$h_n(X, Y) = \begin{cases} Y - \lambda_n \alpha^{\text{nuc}} \exp\left(-\frac{\lambda_n}{C_n^{\text{nuc}}\xi_n} \frac{Y}{X}\right) - \lambda_n \alpha^{\text{conv}} \exp\left(-\frac{\lambda_n}{C_n^{\text{nuc}}\xi_n} \frac{Y}{z_n}\right) - \alpha^{\text{nuc}} \phi(X + z_n) & \text{if } X \neq 0, \\ Y - \lambda_n \exp\left(-\frac{\lambda_n}{C_n^{\text{nuc}}\xi_n} \frac{Y}{z_n}\right) - \alpha^{\text{nuc}} \phi(z_n) & \text{else.} \end{cases}$$

For all $X \geq 0$, $Y \mapsto h_n(X, Y)$ is continuous and strictly increasing over \mathbb{R}_+ . As $h_n(X, 0) \leq -\lambda_n < 0$ and $\lim_{Y \rightarrow +\infty} h_n(X, Y) = +\infty$, it is one-to-one from \mathbb{R}_+ to a set containing 0. Thus it has a unique zero on \mathbb{R}_+ and (18) has a unique solution. \square

For the sake of simplicity, we assume in what follows that $\lambda_n \exp\left(-\frac{\lambda_n}{C_n^{\text{nuc}}\xi_n} \frac{Y}{z_n}\right)$ is negligible, i.e., we assume that preemptive nuclear strikes onto conventional targets achieve little security gain when compared to nuclear targets. Setting it to 0 yields the following equation:

$$Y - \lambda_n \alpha^{\text{nuc}} \exp\left(-\frac{\lambda_n}{C_n^{\text{nuc}}\xi_n} \frac{Y}{X}\right) - \alpha^{\text{nuc}} \phi(X + z_n) = 0, \quad \forall n \in \mathcal{N}. \quad (19)$$

In the following, we provide a closed-form expression of the solution of (19), which relies on the special function $\mathcal{W}(\cdot)$, defined as the principal branch of the Lambert \mathcal{W} -function [10], solution to $w e^w = r$, $r \geq 0$. $\mathcal{W}(\cdot)$ cannot be expressed in terms of elementary functions, although some approximations, bounds, and integral representations, e.g., $\mathcal{W}(x) = \frac{1}{\pi} \int_0^\pi \ln\left(1 + x \frac{\sin t}{t} e^{t \cot t}\right) dt$ are well-known. Finally, for all $X \geq 0$, define

$$G(X, z_n, C_n^{\text{nuc}}\xi_n, \lambda_n) \stackrel{\text{def}}{=} \frac{\alpha^{\text{nuc}}}{C_n^{\text{nuc}}\xi_n} \phi(X + z_n) + \frac{1}{\lambda_n} X \mathcal{W}\left(\frac{\lambda_n^2 \alpha^{\text{nuc}}}{C_n^{\text{nuc}}\xi_n X} \exp\left(-\alpha^{\text{nuc}} \frac{\lambda_n}{C_n^{\text{nuc}}\xi_n} \frac{\phi(X + z_n)}{X}\right)\right).$$

Proposition 10 The functional equation (19) with unknown $X \mapsto Y(X)$ admits a unique solution on $C^\infty(\mathbb{R}_+^*)$ which satisfies:

$$Y = C_n^{\text{nuc}}\xi_n G(X, z_n, C_n^{\text{nuc}}\xi_n, \lambda_n). \quad (20)$$

Proof. Let $n \in \mathcal{N}$. For all $X, Y > 0$, define the *ad hoc* variables w and r by the expressions

$$w = \frac{\lambda_n}{C_n^{\text{nuc}}\xi_n X} (Y - \alpha^{\text{nuc}} \phi(X + z_n)),$$

$$r = \frac{\lambda_n}{C_n^{\text{nuc}}\xi_n X} \lambda_n \alpha^{\text{nuc}} \exp\left(-\frac{\lambda_n}{C_n^{\text{nuc}}\xi_n X} \alpha^{\text{nuc}} \phi(X + z_n)\right).$$

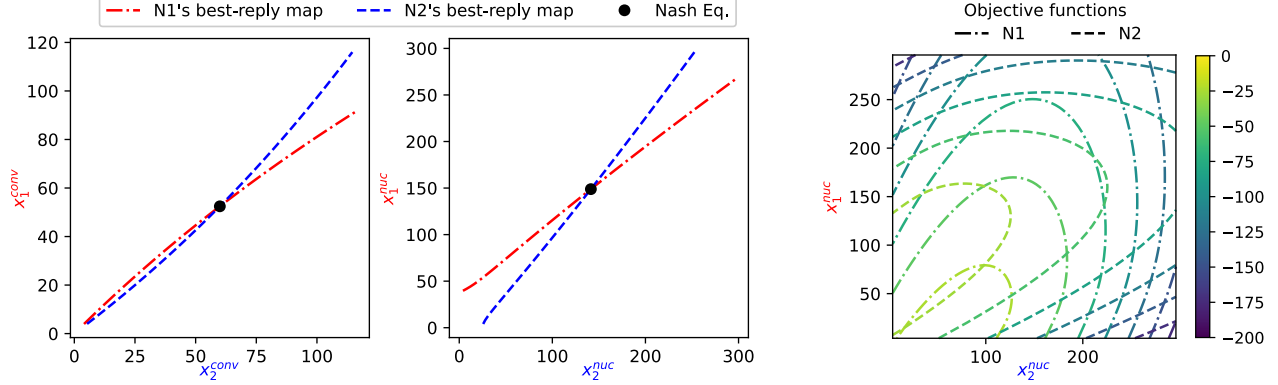


Figure 3: In the special case of the duopoly, the interior point Nash Equilibrium is found graphically at the intersection point of the best-reply maps $(\mathcal{R}_n^{\text{nuc}})_{n=N1,N2}$ of each Nation as defined in (21). The model's parameters are given in Table 2.

One has the expression

$$w - r \exp(-w) = \frac{\lambda_n}{C_n^{\text{nuc}} \xi_n X} \left[Y - \lambda_n \alpha^{\text{nuc}} \exp\left(-\frac{\lambda_n}{C_n^{\text{nuc}} \xi_n} \frac{Y}{X}\right) - \alpha^{\text{nuc}} \phi(X + z_n) \right].$$

Consequently, $(X, Y) \in (\mathbb{R}_+^*)^2$ satisfies (19) if and only if $w e^w = r$, i.e., if and only if $w = \mathcal{W}(r)$, which writes as (20). \square

From Proposition 10, we infer the closed-form expressions of the best-reply maps for nuclear armaments:

$$x_n^{\text{nuc}} = \mathcal{R}_n^{\text{nuc}}(x_m^{\text{nuc}}, x_m^{\text{conv}}) = \xi_n G\left(\frac{x_m^{\text{nuc}}}{\xi_n}, \frac{x_m^{\text{conv}}}{\xi_n}, C_n^{\text{nuc}} \xi_n, \lambda_n\right), \quad \forall n, m \in \mathcal{N}, n \neq m. \quad (21)$$

4.2 Numerical Results and Discussions in Case of a Duopoly

As shown in Figure 3, Nash Equilibria can be obtained numerically when computing the intersection points of the curves drawn by the best-reply maps $\mathcal{R}_n^{\text{nuc}}$ computed as with (21) and $\alpha^{\text{conv}} = \alpha^{\text{nuc}} = \frac{1}{2}$. Note that the security and the utility of both Nations with mixed armaments in a mutual deterrence relationship are greater than with only conventional weapons. Indeed, from Table 2 we can compute the efficiency gap between the social optimum of disarmament and the interior point Nash Equilibrium both evaluated in the international system's social welfare. With only conventional armaments the efficiency gap is of 108 whereas with mixed armaments it drops to 74, representing an inefficiency reduction of 31%. This trend can be interpreted as follows: by investing in nuclear armaments the international system, first, increases its social welfare by comparison with the conventional-only setting, and, second, decreases the distance to the social welfare evaluated in the social optimum of disarmament. Therefore, investments in nuclear armaments have a stabilizing effect – in the sense that they lead to a solution which is closer to the optimum of disarmament, by comparison with the conventional-only setting.

	Nash Equilibrium Strategy $(x_n^{\text{conv}}, x_n^{\text{nuc}})$	Conventional-only		Mixed armaments	
		Utility	<i>Security</i>	Utility	<i>Security</i>
Nation 1	(52, 149)	-53	<i>6</i>	-45	<i>14</i>
Nation 2	(60, 141)	-55	<i>7</i>	-29	<i>33</i>

Table 2: Utility (bold-red) and security (italic-black) functions at Nash Equilibrium. Parameters: $\xi_{N1} = 200$, $\xi_{N2} = 150$, $C_n^{\text{conv}} = C_n^{\text{nuc}} = 5 \times 10^{-3}$, $\lambda_n = 0.7$, $\forall n \in \{N1, N2\}$.

4.3 Generalization to N Nations

When more than two Nations are involved, we aim to show that the utility values of Nations are greater at the mixed-armaments interior point Nash Equilibrium – denoted by x^* – than at the conventional-only interior point Nash Equilibrium – denoted by x^0 .

First, let us recall the optimality conditions at the interior point Nash Equilibrium $x \in \mathcal{U}$:

$$C_n^{\text{conv}} x_n^{\text{conv}} - \alpha^{\text{conv}} \phi\left(\frac{x_{\sigma(n)}^{\text{conv}}}{\xi_n}\right) = 0, \quad (22a)$$

$$C_n^{\text{nuc}} x_n^{\text{nuc}} - \lambda_n \sum_{\tau} \alpha^{\tau} \exp\left(-\frac{\lambda_n x_n^{\text{nuc}}}{x_{\sigma(n)}^{\tau}}\right) - \alpha^{\text{nuc}} \phi\left(\frac{x_{\sigma(n)}^{\text{nuc}} + x_{\sigma(n)}^{\text{conv}}}{\xi_n}\right) = 0. \quad (22b)$$

where $\alpha^{\text{conv}} = 1 - \alpha^{\text{nuc}} \geq 0$ and for all $n \in \mathcal{N}$, C_n^{nuc} , C_n^{conv} , λ_n , ξ_n are fixed non-negative parameters.

In the conventional-only framework, we let $\alpha^{\text{nuc}} \stackrel{\text{def}}{=} 0$ and $\alpha^{\text{conv}} \stackrel{\text{def}}{=} 1$. We call $x^0 \in \mathcal{U}$ the interior point Nash Equilibrium. In the mixed-armaments framework, we call $x^* \in \mathcal{U}$ the interior point Nash Equilibrium. Finally, we define the utility values at both equilibrium by

$$\begin{aligned} J_n^* &\stackrel{\text{def}}{=} J_n(x_n^*, x_{-n}^*), \quad \forall n \in \mathcal{N}, \\ J_n^0 &\stackrel{\text{def}}{=} J_n(x_n^0, x_{-n}^0), \quad \forall n \in \mathcal{N}. \end{aligned}$$

Let $\beta_n > 0$, $\forall n \in \mathcal{N}$ such that $x_n^{*,\text{nuc}} = \beta_n x_{\sigma(n)}^{*,\text{nuc}}$, $\forall n \in \mathcal{N}$. Define

$$\begin{aligned} \rho_n^1 &\stackrel{\text{def}}{=} \frac{\lambda_n + \beta_n^{-1} + \sqrt{(\lambda_n + \beta_n^{-1})^2 - 2x_{\sigma(n)}^{0,\text{conv}}}}{C_n^{\text{nuc}}}, \\ \rho_n^2 &\stackrel{\text{def}}{=} \frac{\lambda_n + \beta_n^{-1} - \sqrt{(\lambda_n + \beta_n^{-1})^2 - 2x_{\sigma(n)}^{0,\text{conv}}}}{C_n^{\text{nuc}}}. \end{aligned}$$

Theorem 3 (Utility Improvement Effect of Nuclear Armaments) *The following inequality holds on the players' utility defined in the context of conventional-only and mixed armaments:*

$$J_n^0 \leq J_n^*, \quad \forall n \in \mathcal{N},$$

provided either the following condition (i) or (ii) is satisfied $\forall n \in \mathcal{N}$ s.t. $x_n^{0,\text{conv}} > x_n^{*,\text{conv}}$:

$$(i) \begin{cases} x_{\sigma(n)}^{0,\text{conv}} < \frac{1}{2} (\lambda_n + \beta_n^{-1})^2 \\ x_n^{*,\text{nuc}} < \rho_n^2 \end{cases} \quad (ii) \begin{cases} x_{\sigma(n)}^{0,\text{conv}} < \frac{1}{2} (\lambda_n + \beta_n^{-1})^2 \\ x_n^{*,\text{nuc}} > \rho_n^1. \end{cases}$$

Proof. We observe that J_n^0 , $\forall n$ is constant in x_n^{nuc} . From Lemma 2, we know that $J_n(\cdot)$ is strictly concave in x_n^{nuc} , $\forall n$. Thus, we infer that for the $n \in \mathcal{N}$ such that $x_n^{*,\text{conv}} \geq x_n^0$, then $J_n^0 \leq J_n^*$.

The problem is for the $n \in \mathcal{N}$ such that $x_n^{*,\text{conv}} < x_n^0$. This implies that

$$J_n^0 > \frac{1}{2} C_n^{\text{conv}} (x_n^{*,\text{conv}})^2 - x_{\sigma(n)}^{0,\text{conv}}.$$

Reasoning again by contradiction, assume that

$$\frac{1}{2} C_n^{\text{conv}} (x_n^{*,\text{conv}})^2 - x_{\sigma(n)}^{0,\text{conv}} > J_n^*. \quad (23)$$

This implies that $J_n^0 < J_n^*$. Checking (23) is equivalent to check $\Psi_n(x_n^{*,\text{nuc}}) \stackrel{\text{def}}{=} J_n^* - \frac{1}{2} C_n^{\text{conv}} (x_n^{*,\text{conv}})^2 + x_{\sigma(n)}^{0,\text{conv}} < 0$. Replacing J_n^* by its closed-form expression and upper bounding $\Psi_n(\cdot)$, we get that $\Psi_n(x_n^{*,\text{nuc}}) < 0$ if

$$\frac{1}{2} C_n^{\text{nuc}} (x_n^{*,\text{nuc}})^2 - (\lambda_n + \beta_n^{-1}) e^{-\lambda_n \beta_n x_n^{*,\text{nuc}}} + x_{\sigma(n)}^{0,\text{conv}} < 0,$$

which is a second-order polynomial inequation in $x_n^{*,\text{nuc}}$ with non-negative discriminant if and only if $x_{\sigma(n)}^{0,\text{conv}} < \frac{1}{2}(\lambda_n + \beta_n^{-1})^2$. In this case, the two real positive roots lie in $\{\rho_n^1, \rho_n^2\}$ and are the two boundary points of the set of solutions of this polynomial inequation. Therefore, if $x_n^{*,\text{nuc}} \in [\rho_n^2; \rho_n^1]$ then $\Psi_n(x_n^{*,\text{nuc}}) < 0$; otherwise $\Psi_n(x_n^{*,\text{nuc}}) \geq 0$. Therefore, for $x_n^{*,\text{nuc}} \notin [\rho_n^2; \rho_n^1]$, $J_n^0 \leq J_n^*$. \square

5 Towards a Common Defense Policy

At the world-wide scale, the production and exports of advanced weapons systems (e.g., fighter jets) are concentrated in the hands of a small number of Nations, called suppliers, which are self-sufficient for their government procurement. Many recipients (interpreted as opponents) are engaged in regional antagonisms and are reliant on imports [22]. However, suppliers' exports by increasing recipients' military capability can reduce the supplier's own security. These spillovers can be internalized among the suppliers, through different market designs. However, military cooperation and collaborative operational costs of weapons stockpiles can only be shared between the members of the alliance. To analyze the market outcome, we develop a partial equilibrium model of the international arms market, involving suppliers which are allies, the rest of the suppliers – interpreted as neutrals – which are not involved in the alliance, and recipients. The model is designed to allow price differentiation between exports from allies and exports from neutrals. Furthermore, we consider conventional armament only, because international trading is only allowed for conventional weapons. The arms trade model developed in this section can be considered independently of the armsbuilding security game analyzed in Sections 2 to 4.

5.1 A Partial Equilibrium Model

We assume that the split into suppliers and recipients is fixed. On the one hand, for the supply side, we consider a set \mathcal{N}_A of N_A allies and a set \mathcal{N}_N of N_N neutrals. On the other hand, we consider a set \mathcal{N}_R of N_R recipients. The split satisfies $\mathcal{N} = \mathcal{N}_A \sqcup \mathcal{N}_N \sqcup \mathcal{N}_R$. In the vocabulary of cooperative game theory, we select one coalition structure among the set of coalition structures, which grows exponentially in the number of Nations involved. We assume one producer per supplier Nation. The alliance takes the form of a joint armament export coalition, which (a) socializes the storage costs among its members, (b) ensures the maximization of the sum of its members' security functions in the face of the recipients which can be seen as potential belligerents. In this setting, the hostility relationships between allies, neutrals, and recipients can be abstracted as a semi-bipartite graph as depicted in Figure 1c.

Following [22], free trade in arms can have negative externalities on the international system's security and there are benefits from the international coordination of controls. We assume that suppliers do not anticipate the impact of their strategies on the arms trade prices. Instead, an international market operator is responsible for the definition of the arms trade prices. The international market operator moves simultaneously with the suppliers. This setting can be formulated as a partial equilibrium problem, with Nash Equilibrium as solution concept [4].

Let p be the price of the unit power drawn from conventional weapons exported by allies in the international market, and \bar{p} by neutrals. We make the assumption that the total demand (in power units) of the recipients is split between the demands from allies $d(p, \bar{p})$ and neutrals $\bar{d}(p, \bar{p})$, and that both $d(\cdot)$ and $\bar{d}(\cdot)$ are linear in the export prices [39]. As expected, an increase in the allies' export price will make the recipients' demand for weapons exported by allies decrease. Substitutability effects of weapons produced by allies and neutrals can be accounted for through the introduction of $\eta, \bar{\eta}$ coefficients capturing the fact the demand for the allies' weapons increases when the price of the neutrals' weapons increases. A similar reasoning applies to neutrals' weapons for export.

Assumption 6 *The demand is linear and accounts for substitutability effects, i.e., there exist $\zeta_0, \zeta, \eta, \bar{\zeta}_0, \bar{\zeta}, \bar{\eta}$ non-negative coefficients such that:*

$$d(p, \bar{p}) = \zeta_0 - \zeta p + \eta \bar{p} \quad \text{and} \quad \bar{d}(p, \bar{p}) = \bar{\zeta}_0 - \bar{\zeta} \bar{p} + \bar{\eta} p.$$

In the present section, only one class of armament is considered, namely “conv”. Hence, for all $n \in \mathcal{N}$, $x_n = (x_n^\tau)_\tau = (x_n^{\text{conv}})$. For the sake of simplicity, only in this section, we shall omit the superscript “conv”

and write x_n for x_n^{conv} . Producer in Nation $n \in \mathcal{N}_A \sqcup \mathcal{N}_N$ supplies x_n units of conventional power to its own government and exports χ_n conventional power units. Let $\tilde{x}_n \stackrel{\text{def}}{=} \text{col}([x_n, \chi_n]) \in \tilde{\mathcal{U}}_n \subset \mathbb{R}_+^2$ and $\tilde{x} \stackrel{\text{def}}{=} \text{col}((\tilde{x}_n)_n)$ be the stack of the $(\tilde{x}_n)_n$.

Similarly to Section 2.3, the target value $x_{\sigma(n)}$ can be expressed as the maximum, the sum or the mean over \mathcal{N}_R of the updated recipients' armaments levels $x^R \stackrel{\text{def}}{=} \chi^T \Omega + \text{col}((x_m)_{m \in \mathcal{N}_R})^T$ with $\Omega \in \mathbb{R}_+^{(N_A + N_N) \times N_R}$. The matrix Ω determines the mapping of the Nations exports to the import received by each recipient. Indeed, $\chi_n \Omega_{nm}$ is the amount of Nation n 's exports allocated to Nation $m \in \mathcal{N}_R$. Thus, Ω is left-stochastic. Furthermore, for any $n \in \mathcal{N}_A \sqcup \mathcal{N}_N$, $\frac{\partial x_{\sigma(n)}}{\partial \chi_n}$ depends only on $(\Omega_{nm})_{m \in \mathcal{N}_R}$. In the special case where all Nations choose the sum function as their target value, $\frac{\partial x_{\sigma(n)}}{\partial \chi_n} = 1, \forall n \in \mathcal{N}_A \sqcup \mathcal{N}_N$.

Next, we outline the problems posed to each category of players involved in the game: the members of the joint armament export coalition, the neutrals, the international market operator and the recipients.

Members of the Joint Armament Export Coalition: Relying on the alliance definition introduced in (a), (b), the allies coalition's utility takes the form:

$$J^A(\tilde{x}, p) = \sum_{n \in \mathcal{N}_A} f_n(x_n, x_{\sigma(n)}) - c^A(\tilde{x}) + p \sum_{n \in \mathcal{N}_A} \chi_n, \quad (24a)$$

$$f_n(x_n, x_{\sigma(n)}) = x_n \phi\left(\frac{x_{\sigma(n)}}{\xi_n}\right) - x_{\sigma(n)}, \quad (24b)$$

where $c^A(\tilde{x})$ is the coalition total cost, which is socialized among its members, and is defined by the following expression which allows cross-commodity costs between the investments in stockpiles and exports of the coalition members:

$$c^A(\tilde{x}) \stackrel{\text{def}}{=} \frac{1}{2} C^A \left(\sum_{n \in \mathcal{N}_A} (x_n + \chi_n) \right)^2,$$

with $C^A > 0$ the coalition marginal unitary cost. For a fixed coalition structure, the export coalition formation game can be formulated as a two-stage game. In the first stage, assuming coordination among the export coalition members, a coordinator in charge of operating the coalition solves the following optimization problem:

$$\max_{\tilde{x}_n} J^A(\tilde{x}, p), \quad (25a)$$

$$s.t. \quad \tilde{x}_n \in \tilde{\mathcal{U}}_n. \quad (25b)$$

In the second stage, the coordinator defines ex-post the cost allocation mechanism γ among Nations in the export coalition to stabilize the export coalition, i.e., guarantee the formation of the coalition. It is denoted by the collection $(\gamma_n)_{n \in \mathcal{N}_A}$ and it satisfies:

$$\sum_{n \in \mathcal{N}_A} \gamma_n = 1, \quad \gamma_n \geq 0, \quad \forall n \in \mathcal{N}_A.$$

Finally, the allocation received by each supplier Nation $n \in \mathcal{N}_A$ ex-post reads:

$$J_n^A(\tilde{x}, p, \gamma) = f_n(x_n, x_{\sigma(n)}) - \gamma_n c^A(\tilde{x}) + p \chi_n \quad (26)$$

It is noteworthy that J^A does not depend on the cost allocation mechanism and the allocation is efficient by construction:

$$J^A(\tilde{x}, p) = \sum_{n \in \mathcal{N}_A} J_n^A(\tilde{x}, p, \gamma). \quad (27)$$

Neutrals: Assuming Nation n belongs to the set of neutrals, the costs are no longer internalized, therefore leading to the following utility for Nation $n \in \mathcal{N}_N$:

$$J_n^N(\tilde{x}, \bar{p}) = f_n(x_n, x_{\sigma(n)}) - c_n^N(\tilde{x}) + \bar{p} \chi_n,$$

where $c_n^N(\tilde{x}) = c_n^S(x_n + \chi_n) = \frac{1}{2}C_n^{\text{conv}}(x_n + \chi_n)^2$.

Neutrals solve independently the following optimization problem:

$$\max_{\tilde{x}_n} J_n^N(\tilde{x}, \bar{p}), \quad (28a)$$

$$s.t. \quad \tilde{x}_n \in \tilde{\mathcal{U}}_n. \quad (28b)$$

International Market Operator: Simultaneously, the export prices are determined by the international market operator as duals of the supply and demand balance equations. The prices chosen by the price-setting market operator are limited by a finite positive price cap P . Similarly, price floors are possible. We assume a lower limit of 0 for all prices. The market operator's optimization problem reads as follows:

$$\max_{p, \bar{p}} p \left(d - \sum_{n \in \mathcal{N}_A} \chi_n \right) + \bar{p} \left(\bar{d} - \sum_{n \in \mathcal{N}_N} \chi_n \right), \quad (29a)$$

$$s.t. \quad 0 \leq p \leq P, \quad (29b)$$

$$0 \leq \bar{p} \leq P. \quad (29c)$$

Recipients: Assuming Nation n belongs to the set of recipients, its utility reads as follows:

$$J_n^R(\tilde{x}, \bar{p}) = f_n(x_n + \chi^T \Omega e_n, x_{\sigma(n)}) - c_n^S(\tilde{x}_n),$$

where $c_n^S(\tilde{x}_n) = \frac{1}{2}C_n^{\text{conv}}(x_n + \chi_n)^2$ and e_n is the n th element of the standard basis vectors for \mathbb{R}^{N_R} .

Theorem 4 *Let Assumption 6 hold. Then, the partial equilibrium problem leads to a unique solution (x^*, p^*, \bar{p}^*) if and only if*

$$\left(\sum_{m \in \mathcal{N}_A} \frac{\xi_m / \frac{\partial x_{\sigma(n)}}{\partial \chi_n}}{\phi' \left(\frac{x_{\sigma(m)}}{\xi_m} \right)} - \zeta \right) \left(\sum_{m \in \mathcal{N}_N} \frac{\xi_m / \frac{\partial x_{\sigma(n)}}{\partial \chi_n}}{\phi' \left(\frac{x_{\sigma(m)}}{\xi_m} \right)} - \bar{\zeta} \right) - \eta \bar{\eta} \neq 0. \quad (30)$$

Proof. At χ fixed and for $k > 0$, the utility functions $J_n^{N,k}(\cdot)$, $J^{A,k}(\cdot)$ and $J_n^{R,k}(\cdot)$, defined by extension of the definition given just below Assumption 3 with $\phi^k(\cdot)$ instead of $\phi(\cdot)$, are uniformly coercive over their own strategy set in x_n and x . Furthermore, for all $k > 0$, $J_n^{N,k}(\cdot)$, $J^{A,k}(\cdot)$ and $J_n^{R,k}(\cdot)$ are continuous in χ and the feasibility sets of $(\chi_n)_n$ are nonempty and compact. In this setting and by extension of Glicksberg's Theorem [15], we can prove that there exists for all $k > 0$ a Nash Equilibrium solution of the truncated partial equilibrium problem involving $\phi^k(\cdot)$. Using the same reasoning as in the proof of Proposition 4, the existence result can be extended to the partial equilibrium problem.

To complete the above line of reasoning, we aim to compute the closed-form expression of x^*, p^*, \bar{p}^* at equilibrium as functions of χ . At equilibrium, the primal feasibility constraints impose that:

$$\sum_{n \in \mathcal{N}_A} \chi_n = d(p, \bar{p}) \quad \text{and} \quad \sum_{n \in \mathcal{N}_N} \chi_n = \bar{d}(p, \bar{p}).$$

For all $n \in \mathcal{N}_A$, we compute the gradient of the coalition's utility function:

$$\begin{aligned} \frac{\partial J^A}{\partial x_n}(\tilde{x}, p) &= \phi \left(\frac{x_{\sigma(n)}}{\xi_n} \right) - C^A \sum_{m \in \mathcal{N}_A} (x_m + \chi_m), \\ \frac{\partial J^A}{\partial \chi_n}(\tilde{x}, p) &= \frac{\partial x_{\sigma(n)}}{\partial \chi_n} \frac{x_n}{\xi_n} \phi' \left(\frac{x_{\sigma(n)}}{\xi_n} \right) - C^A \sum_{m \in \mathcal{N}_A} (x_m + \chi_m) + p - \frac{\partial x_{\sigma(n)}}{\partial \chi_n}. \end{aligned}$$

From the first order stationarity conditions, we get:

$$x_n^* = \frac{\xi_n / \frac{\partial x_{\sigma(n)}}{\partial \chi_n}}{\phi' \left(\frac{x_{\sigma(n)}}{\xi_n} \right)} \left[\phi \left(\frac{x_{\sigma(n)}}{\xi_n} \right) + \frac{\partial x_{\sigma(n)}}{\partial \chi_n} - p^* \right], \quad \forall n \in \mathcal{N}_A. \quad (31)$$

Now, at equilibrium, apply the primal feasibility constraints and take the mean over \mathcal{N}_A in $\frac{\partial J^A}{\partial x_n}(\tilde{x}, p^*) = 0$:

$$-\sum_{n \in \mathcal{N}_A} x_n^* = d(p^*, \bar{p}^*) - \frac{1}{N_A} \sum_{n \in \mathcal{N}_A} \frac{1}{C^A} \phi\left(\frac{x_{\sigma(n)}}{\xi_n}\right).$$

Finally, substitute x_n^* with its expression from (31):

$$p^* = \frac{d(p^*, \bar{p}^*) + \sum_{m \in \mathcal{N}_A} \frac{\xi_m / \frac{\partial x_{\sigma(m)}}{\partial \chi_m}}{\phi'\left(\frac{x_{\sigma(m)}}{\xi_m}\right)} \left[\phi\left(\frac{x_{\sigma(m)}}{\xi_m}\right) + \frac{\partial x_{\sigma(m)}}{\partial \chi_m} \right] - \frac{1}{N_A} \sum_{n \in \mathcal{N}_A} \frac{1}{C^A} \phi\left(\frac{x_{\sigma(n)}}{\xi_n}\right)}{\sum_{m \in \mathcal{N}_A} \frac{\xi_m / \frac{\partial x_{\sigma(m)}}{\partial \chi_m}}{\phi'\left(\frac{x_{\sigma(m)}}{\xi_m}\right)}}. \quad (32)$$

Similarly, for all $n \in \mathcal{N}_N$, we compute:

$$\begin{aligned} \frac{\partial J_n^N}{\partial x_n}(\tilde{x}, p) &= \phi\left(\frac{x_{\sigma(n)}}{\xi_n}\right) - C_n^{\text{conv}}(x_n + \chi_n), \\ \frac{\partial J_n^N}{\partial \chi_n}(\tilde{x}, p) &= \frac{\partial x_{\sigma(n)}}{\partial \chi_n} \frac{x_n}{\xi_n} \phi'\left(\frac{x_{\sigma(n)}}{\xi_n}\right) - C_n^{\text{conv}}(x_n + \chi_n) + \bar{p} - \frac{\partial x_{\sigma(n)}}{\partial \chi_n}. \end{aligned}$$

From the first order stationarity conditions, we get:

$$x_n^* = \frac{\xi_n / \frac{\partial x_{\sigma(n)}}{\partial \chi_n}}{\phi'\left(\frac{x_{\sigma(n)}}{\xi_n}\right)} \left[\phi\left(\frac{x_{\sigma(n)}}{\xi_n}\right) + \frac{\partial x_{\sigma(n)}}{\partial \chi_n} - \bar{p}^* \right], \quad \forall n \in \mathcal{N}_N. \quad (33)$$

Again, at equilibrium, take the sum over \mathcal{N}_N in $\frac{\partial J_n^N}{\partial x_n}(\tilde{x}, p^*) = 0$ and apply the primal feasibility constraints:

$$-\sum_{n \in \mathcal{N}_N} x_n^* = \bar{d}(p^*, \bar{p}^*) - \sum_{n \in \mathcal{N}_A} \frac{1}{C^A} \phi\left(\frac{x_{\sigma(n)}}{\xi_n}\right). \quad (34)$$

Then, substituting x_n^* with its expression from (33), $\forall n \in \mathcal{N}_N$ in $\frac{\partial J_n^N}{\partial x_n}(\tilde{x}, \bar{p}^*) = 0$, we get:

$$\bar{p}^* = \frac{\bar{d}(p^*, \bar{p}^*) + \sum_{n \in \mathcal{N}_N} \frac{\xi_n / \frac{\partial x_{\sigma(n)}}{\partial \chi_n}}{\phi'\left(\frac{x_{\sigma(n)}}{\xi_n}\right)} \left[\phi\left(\frac{x_{\sigma(n)}}{\xi_n}\right) + \frac{\partial x_{\sigma(n)}}{\partial \chi_n} \right] - \sum_{n \in \mathcal{N}_N} \frac{1}{C_n^{\text{conv}}} \phi\left(\frac{x_{\sigma(n)}}{\xi_n}\right)}{\sum_{n \in \mathcal{N}_N} \frac{\xi_n / \frac{\partial x_{\sigma(n)}}{\partial \chi_n}}{\phi'\left(\frac{x_{\sigma(n)}}{\xi_n}\right)}}. \quad (35)$$

Now, relying on Assumption 6 and (32), as well as the closed-form expressions of p^*, \bar{p}^* detailed in (32), (35), p^*, \bar{p}^* are obtained as solutions of a linear system of two equations with two unknown variables. Making explicit the linear dependencies in p^* and \bar{p}^* , and relying on classical linear algebra, we infer that (32), (35) has a single solution if and only if the determinant of the linear system matrix does not vanish. This gives rise to the equivalent condition (30). \square

Remark 5.1 *However, at equilibrium, the exports $\chi^* = (\chi_n^*)_n$ may not be uniquely defined.*

5.2 Export Coalition Stability

Considering a coalition formation problem leading to a specific coalition structure, we aim to show in this section that there exists a cost allocation mechanism γ , which guarantees the stability of the coalition structure. Because there is only a single coalition with multiple members, the coalition structure is defined

by the indices of the Nations which have joined the Joint Armament Export Coalition and of the rest of the Nations which remain Neutrals forming single player coalition. Therefore, in this section, a coalition structure $\mathcal{S} = (\mathcal{N}_A, \mathcal{N}_N)$ is a couple defining this split.

Consider a fixed coalition structure $\mathcal{S} = (\mathcal{N}_A, \mathcal{N}_N)$ and $(\tilde{x}^*, p^*, \bar{p}^*)$ a Nash Equilibrium associated with it. Let $J_n^{N,*} = J_n^N(\tilde{x}^*, \bar{p}^*)$ be player n 's utility at equilibrium when belonging to the neutrals, and $J_n^{A,*}(\gamma) = J_n^A(\tilde{x}^*, p^*, \gamma)$ be player n 's allocation at equilibrium when belonging to the allies coalition.

Nation n stays in the allies coalition if and only if, at equilibrium, the utility it gets within the allies coalition is larger than the utility it gets being neutral.

Definition 6 (Export Coalition Stability) *Let $(\tilde{x}^*, p^*, \bar{p}^*)$ be a Nash Equilibrium for a given coalition structure $\mathcal{S} = (\mathcal{N}_A, \mathcal{N}_N)$. The allies coalition is stable at $(\tilde{x}^*, p^*, \bar{p}^*)$ if and only if there exists a cost allocation mechanism γ such that the allocation is efficient and the coalition structure is individually rational for all Nations:*

$$J^{A,*} = \sum_{n \in \mathcal{N}_A} J_n^{A,*}(\gamma) \quad (\text{Efficiency}) \quad (36a)$$

$$J_n^{N,*} \leq J_n^{A,*}(\gamma), \quad \forall n \in \mathcal{N}_A. \quad (\text{Individual rationality}) \quad (36b)$$

Remark 5.2 *By construction of the allocation mechanism, the allocation is always efficient, as (27) shows.*

Let $(\tilde{x}^*, p^*, \bar{p}^*)$ be a Nash Equilibrium for a given coalition structure $\mathcal{S} = (\mathcal{N}_A, \mathcal{N}_N)$ and let $n \in \mathcal{N}$. The target value $x_{\sigma(n)}$ is taken at the equilibrium $(\tilde{x}^*, p^*, \bar{p}^*)$ for \mathcal{S} . From the complete semi-bipartite hostility relationship assumed in this section and the target value expression, we infer that $x_{\sigma(n)}$ is the same for all the players in $\mathcal{N}_A \sqcup \mathcal{N}_N$.

Assuming $n \in \mathcal{N}_A$, we set $x_n^{A,*} \stackrel{\text{def}}{=} x_n^*$, which is given by (31), and we define $x_n^{N,*}$ as the level of arms Nation n would produce if it defected *ex post* from the coalition, i.e., it is given as in (33). Conversely, if $n \in \mathcal{N}_N$, we set $x_n^{N,*} \stackrel{\text{def}}{=} x_n^*$, which is given by (33), and we define $x_n^{A,*}$ as the level of arms Nation n would produce if it joined *ex post* the coalition, i.e., it is given as in (31). Thus, in both cases, it holds that

$$x_n^{A,*} \stackrel{\text{def}}{=} \frac{\xi_n / \frac{\partial x_{\sigma(n)}}{\partial \chi_n}}{\phi' \left(\frac{x_{\sigma(n)}}{\xi_n} \right)} \left[\phi \left(\frac{x_{\sigma(n)}}{\xi_n} \right) + \frac{\partial x_{\sigma(n)}}{\partial \chi_n} - p^* \right], \quad \forall n \in \mathcal{N}_A.$$

$$x_n^{N,*} \stackrel{\text{def}}{=} \frac{\xi_n / \frac{\partial x_{\sigma(n)}}{\partial \chi_n}}{\phi' \left(\frac{x_{\sigma(n)}}{\xi_n} \right)} \left[\phi \left(\frac{x_{\sigma(n)}}{\xi_n} \right) + \frac{\partial x_{\sigma(n)}}{\partial \chi_n} - \bar{p}^* \right], \quad \forall n \in \mathcal{N}_N.$$

Lemma 4 *Let $(\tilde{x}^*, p^*, \bar{p}^*)$ be a Nash Equilibrium for a given coalition structure $\mathcal{S} = (\mathcal{N}_A, \mathcal{N}_N)$ and let $n \in \mathcal{N}$. Then, the following statement holds:*

$$f(x_n^{A,*}, x_{\sigma(n)}) \geq f(x_n^{N,*}, x_{\sigma(n)}) \iff x_n^{A,*} \geq x_n^{N,*} \iff p^* \leq \bar{p}^*.$$

Proof. Relying on the definition of the security function for both Allies and Neutrals, we get that

$$f(x_n^{A,*}, x_{\sigma(n)}) - f(x_n^{N,*}, x_{\sigma(n)}) = (x_n^{A,*} - x_n^{N,*}) \phi \left(\frac{x_{\sigma(n)}}{\xi_n} \right).$$

Thus, $f(x_n^{A,*}, x_{\sigma(n)}) \geq f(x_n^{N,*}, x_{\sigma(n)})$ if and only if $x_n^{A,*} \geq x_n^{N,*}$. Now, one can easily check that

$$x_n^{A,*} - x_n^{N,*} = \frac{\xi_n / \frac{\partial x_{\sigma(n)}}{\partial \chi_n}}{\phi' \left(\frac{x_{\sigma(n)}}{\xi_n} \right)} (\bar{p}^* - p^*)$$

Since $\xi_n, \phi' \left(\frac{x_{\sigma(n)}}{\xi_n} \right), \frac{\partial x_{\sigma(n)}}{\partial \chi_n} > 0$, indeed $x_n^{A,*} \geq x_n^{N,*}$ if and only if $p^* \leq \bar{p}^*$, hence the conclusion.

□

Lemma 4 means that players increase their security functions by joining the allies coalition instead of remaining neutrals if and only if $p^* \leq \bar{p}^*$.

We now aim at providing sufficient conditions for the joint armament export coalition to be stable at equilibrium. To guarantee the tractability of the results, we assume no cross commodity costs, i.e., $c^A(\tilde{x}) = \frac{1}{2} \sum_{n \in \mathcal{N}_A} (x_n + \chi_n)^2$. Extension allowing cross commodity costs is straightforward.

Theorem 5 *Let $(\tilde{x}^*, p^*, \bar{p}^*)$ be a Nash Equilibrium for a given coalition structure $\mathcal{S} = (\mathcal{N}_A, \mathcal{N}_N)$. The allies coalition is stable at equilibrium $(\tilde{x}^*, p^*, \bar{p}^*)$ if the allocation mechanism satisfies*

$$\gamma_n C^A \leq \frac{(x_n^* + \chi_n^*)^2}{\sum_{m \in \mathcal{N}_A} (x_m^* + \chi_m^*)^2} C_n^{\text{conv}}, \quad \forall n \in \mathcal{N}_A,$$

and either of the two conditions for profitability (i) or (ii) stated below.

$$\begin{aligned} \bar{p}^* \leq p^* \quad \text{and} \quad \chi_n^* &\geq \frac{\xi_n}{\frac{\partial x_{\sigma(n)}}{\partial \chi_n}} \frac{\phi\left(\frac{x_{\sigma(n)}}{\xi_n}\right)}{\phi'\left(\frac{x_{\sigma(n)}}{\xi_n}\right)}, \quad \forall n \in \mathcal{N}_A, \quad (i) \\ p^* \leq \bar{p}^* \quad \text{and} \quad \chi_n^* &\leq \frac{\xi_n}{\frac{\partial x_{\sigma(n)}}{\partial \chi_n}} \frac{\phi\left(\frac{x_{\sigma(n)}}{\xi_n}\right)}{\phi'\left(\frac{x_{\sigma(n)}}{\xi_n}\right)}, \quad \forall n \in \mathcal{N}_A, \quad (ii) \end{aligned}$$

Proof. Let $(\tilde{x}^*, p^*, \bar{p}^*)$ be a Nash Equilibrium for a given coalition structure $\mathcal{S} = (\mathcal{N}_A, \mathcal{N}_N)$ and let $n \in \mathcal{N}_A$.

$$\begin{aligned} J_n^{A,*}(\gamma) - J_n^{N,*} &= f(x_n^{A,*}, x_{\sigma(n)}) - f(x_n^{N,*}, x_{\sigma(n)}) + (p^* - \bar{p}^*) \chi_n^* + \gamma_n c^A(\tilde{x}) - c_n^N(\tilde{x}) \\ &= \left[\chi_n^* - \frac{\xi_n}{\frac{\partial x_{\sigma(n)}}{\partial \chi_n}} \frac{\phi\left(\frac{x_{\sigma(n)}}{\xi_n}\right)}{\phi'\left(\frac{x_{\sigma(n)}}{\xi_n}\right)} \right] (p^* - \bar{p}^*) + \frac{1}{2} C_n^{\text{conv}} (x_n^* + \chi_n^*)^2 - \frac{1}{2} \gamma_n C^A \sum_{m \in \mathcal{N}_A} (x_m^* + \chi_m^*)^2. \end{aligned}$$

Thus provided conditions (i) or (ii) apply, the coalition structure is individually rational:

$$J_n^{A,*}(\gamma) \geq J_n^{N,*}, \quad \forall n \in \mathcal{N}_A.$$

Therefore, the coalition is stable at equilibrium $(\tilde{x}^*, p^*, \bar{p}^*)$. □

Remark 5.3 *The allocation mechanism described in Theorem 5 can only exist if the following cost-compatibility condition $C^A \leq \bar{C}^{\text{conv}}$ is satisfied, where the cost-balanced mean \bar{C}^{conv} is defined by*

$$\bar{C}^{\text{conv}} \stackrel{\text{def}}{=} \frac{\sum_{n \in \mathcal{N}_A} (x_n^* + \chi_n^*)^2 C_n^{\text{conv}}}{\sum_{n \in \mathcal{N}_A} (x_n^* + \chi_n^*)^2},$$

In particular, it implies that $C^A \leq \max_{n \in \mathcal{N}_A} C_n^{\text{conv}}$. Now, assuming this inequality is satisfied, the following allocation mechanism enforces stability under the conditions of Theorem 5:

$$\gamma_n \stackrel{\text{def}}{=} \frac{1}{\bar{C}^{\text{conv}}} \frac{(x_n^* + \chi_n^*)^2}{\sum_{m \in \mathcal{N}_A} (x_m^* + \chi_m^*)^2} C_n^{\text{conv}}, \quad \forall n \in \mathcal{N}_A.$$

Assuming compatibility between marginal unitary costs of the coalition and of members, we have exhibited an adequate allocation mechanism as well as sufficient conditions on export prices and quantities allowing stability to be achieved within the coalition at equilibrium.

5.3 Export Coalition Size

Proposition 11 (Minimum Size of the Export Coalition) Assuming $d(p, \bar{p}) \geq \bar{d}(p, \bar{p})$ and $N_A \geq N_N$, there exists a lower bound on the number of allies guaranteeing that $p^* \leq \bar{p}^*$.

Proof. From Equations (32), (35) closed form expressions, $p^* \leq \bar{p}^*$ can be equivalently reformulated as an implicit equation of the size of the allies coalition:

$$N_A \geq \frac{\sum_{m \in \mathcal{N}_A} \frac{C^A \xi_m}{\phi' \left(\frac{x_{\sigma(m)}^{\text{conv}}}{\xi_m} \right)}}{\sum_{m \in \mathcal{N}_N} \frac{C_m^{\text{conv}} \xi_m}{\phi' \left(\frac{x_{\sigma(m)}^{\text{conv}}}{\xi_m} \right)}} \frac{\sum_{n \in \mathcal{N}_N} \left[C_n^{\text{conv}} \left(\bar{d}(p, \bar{p}) + \sum_{m \in \mathcal{N}_N} \xi_m \frac{\phi \left(\frac{x_{\sigma(m)}^{\text{conv}}}{\xi_m} \right) + 1}{\phi' \left(\frac{x_{\sigma(m)}^{\text{conv}}}{\xi_m} \right)} \right) - \frac{\phi \left(\frac{x_{\sigma(n)}^{\text{conv}}}{\xi_n} \right)}{2} \right]}{\sum_{n \in \mathcal{N}_A} \left[C^A \left(d(p, \bar{p}) + \sum_{m \in \mathcal{N}_A} \xi_m \frac{\phi \left(\frac{x_{\sigma(m)}^{\text{conv}}}{\xi_m} \right) + 1}{\phi' \left(\frac{x_{\sigma(m)}^{\text{conv}}}{\xi_m} \right)} \right) - \frac{\phi \left(\frac{x_{\sigma(n)}^{\text{conv}}}{\xi_n} \right)}{2} \right]}.$$

Relying on the assumptions that $d(p, \bar{p}) \geq \bar{d}(p, \bar{p})$ and $N_A \geq N_N$, we get:

$$\begin{aligned} & \sum_{n \in \mathcal{N}_A} \left[C^A \left(d(p, \bar{p}) + \sum_{m \in \mathcal{N}_A} \xi_m \frac{\phi \left(\frac{x_{\sigma(m)}^{\text{conv}}}{\xi_m} \right) + 1}{\phi' \left(\frac{x_{\sigma(m)}^{\text{conv}}}{\xi_m} \right)} \right) - \frac{\phi \left(\frac{x_{\sigma(n)}^{\text{conv}}}{\xi_n} \right)}{2} \right] \\ & \geq \sum_{n \in \mathcal{N}_N} \left[C_n^{\text{conv}} \left(\bar{d}(p, \bar{p}) + \sum_{m \in \mathcal{N}_N} \xi_m \frac{\phi \left(\frac{x_{\sigma(m)}^{\text{conv}}}{\xi_m} \right) + 1}{\phi' \left(\frac{x_{\sigma(m)}^{\text{conv}}}{\xi_m} \right)} \right) - \frac{\phi \left(\frac{x_{\sigma(n)}^{\text{conv}}}{\xi_n} \right)}{2} \right], \end{aligned}$$

leading to :

$$N_A \geq \frac{N_A C^A \min_{m \in \mathcal{N}_A} \left\{ \xi_m / \phi' \left(\frac{x_{\sigma(m)}^{\text{conv}}}{\xi_m} \right) \right\}}{N_N \max_{m \in \mathcal{N}_N} \left\{ C_m^{\text{conv}} \xi_m / \phi' \left(\frac{x_{\sigma(m)}^{\text{conv}}}{\xi_m} \right) \right\}} \frac{N_N \min_{m \in \mathcal{N}_N} \{ C_m^{\text{conv}} \}}{N_A C^A},$$

and:

$$N_A \geq \frac{\min_{m \in \mathcal{N}_A} \left\{ \xi_m / \phi' \left(\frac{x_{\sigma(m)}^{\text{conv}}}{\xi_m} \right) \right\}}{\max_{m \in \mathcal{N}_N} \left\{ C_m^{\text{conv}} \xi_m / \phi' \left(\frac{x_{\sigma(m)}^{\text{conv}}}{\xi_m} \right) \right\}} \min_{m \in \mathcal{N}_N} \{ C_m^{\text{conv}} \}.$$

□

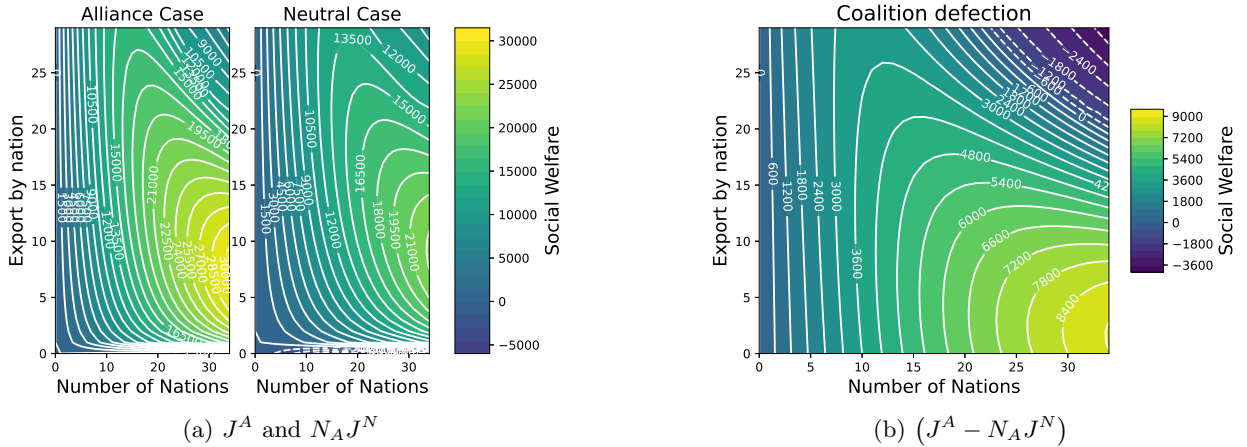


Figure 4: Symmetric game with: $x = 200, \zeta_0 = \bar{\zeta}_0 = 500, \zeta = \bar{\zeta} = 12, \eta = \bar{\eta} = 6, C^A = 10^{-5}, C^{\text{conv}} = 10^{-3}, \xi = 1$.

Illustrations for the Symmetric Game: We consider the symmetric game formulation of the partial equilibrium problem. In Figure 4a, we plot the coalition utility J^A versus the social welfare of the allied nation if they were to leave the coalition $N_A J^N$ with $J^N = J_n^N, \forall n \in \mathcal{N}_A$ as functions of the coalition size and export. In addition, in Figure 4b, we plot $(J^A - N_A J^N)$. We observe that there exists an upper bound on the coalition size and export above which allies increase their utility by leaving the coalition; furthermore, there exist optimal coalition size and export guaranteeing the maximization of the coalition’s utility.

6 Conclusion

We formulate a security game in a context of mixed armament acquisition, involving a finite set of Nations in strategic relationship. Our model incorporates key elements such as resource allocation to R&D and storage, defense capabilities, and geopolitical considerations, to provide a realistic representation of the arms procurement process. To analyze the long-term solution of the dynamical model, we study the steady states of the dynamical game by solving the associated one-shot game. On the theoretical side, we provide two important contributions to the state of the art. First, we extend classical results about compact-concave games to unbounded concave games relying on the uniform coercivity property of the utility functions. Second, our results showcase the stabilizing effect of nuclear armaments and provide possible guidelines for current treaties refinements or for the understanding of various arms competition scenarios, including multilateral arms competitions and asymmetric weaponry. On the theoretical side, our work presents an original and significant contribution, addressing a gap in the literature concerning the stability of games with mixed armaments and nonlinear utility functions. Some of the results, for example the extension of Rosen’s result, may be of independent interest to the broader community.

Allowing arms trades between Nations, we develop a partial equilibrium model of the international arms market. Furthermore, we determine conditions on the exports, arms trade prices, and cost allocation mechanism guaranteeing the stability of the export coalition.

Future research could focus on studying other identified factors such as coalition formation, as a dynamic process, and the role of emerging technologies. Additionally, empirical studies and use case analyses would be valuable to validate the model outcomes and assess its real-world applicability. Finally, we anticipate that our contributions will serve as a valuable tool for policymakers, analysts, and researchers, assisting in the development of effective arms control measures and promoting stability in international relations.

Declarations

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Appendix

A Pseudo-Gradient and Pseudo-Hessian Closed Forms

In the following, we recall standard definitions and closed-form expressions that are used throughout the article.

Pseudo-Gradient and Pseudo-Hessian Definitions

Let $E = \prod_{n \in \mathcal{N}}$ a product of Euclidian spaces. Let \mathcal{U} a smooth subset of E . For all $n \in \mathcal{N}$, let $J_n : \mathcal{U} \rightarrow \mathbb{R}$ be the utility of player $n \in \mathcal{N}$. Finally, define the game $\mathcal{G} = (\prod_{n \in \mathcal{N}} E_n, (J_n)_{n \in \mathcal{N}})$.

Let F_n denote the gradient of player n 's utility with respect to its own actions. It is defined by

$$F_n(x) \stackrel{\text{def}}{=} \nabla_{x_n} J_n(x_n, x_{-n}), \quad \forall x \in \mathcal{U}, \quad \forall n \in \mathcal{N}. \quad (37)$$

The pseudo-gradient F is the stack of the $(F_n)_n$.

Let H denote the pseudo-Hessian matrix of the players' utilities where each n^{th} row and m^{th} column component is given by block as

$$H_{n,m}(x) = \nabla_{x_m} F_n(x), \quad \forall x \in \mathcal{U}, \quad \forall n, m \in \mathcal{N}.$$

Closed-Form Expressions for the Mixed Armament Model

The gradient of player n 's utility with respect to its own actions reads

$$F_n(x) = \begin{pmatrix} \frac{\partial J_n(x)}{\partial x_n^{\text{nuc}}} \\ \frac{\partial J_n(x)}{\partial x_n^{\text{conv}}} \end{pmatrix}.$$

$$\forall x \in \mathcal{U}, \quad \begin{cases} \frac{\partial J_n(x)}{\partial x_n^{\text{conv}}} = \alpha^{\text{conv}} \phi \left(\frac{x_{\sigma(n)}^{\text{conv}}}{\xi_n} \right) - C_n^{\text{conv}} x_n^{\text{conv}} \\ \frac{\partial J_n(x)}{\partial x_n^{\text{nuc}}} = \alpha^{\text{nuc}} \phi \left(\frac{x_{\sigma(n)}^{\text{conv}} + x_{\sigma(n)}^{\text{nuc}}}{\xi_n} \right) + \lambda_n \sum_{\tau} \alpha^{\tau} \exp \left(-\lambda_n \frac{x_n^{\text{nuc}}}{x_{\sigma(n)}^{\tau}} \right) - C_n^{\text{nuc}} x_n^{\text{nuc}} \end{cases} \quad (38)$$

For $\tau \in \{\text{nuc}; \text{conv}\}$, let H^τ denote the pseudo-Hessian matrix of the players' utilities with respect to class- τ armaments, where each n^{th} row and m^{th} column component is given as

$$H_{n,m}^\tau(x) = \frac{\partial^2}{\partial x_n^\tau \partial x_m^\tau} J_n(x_n, x_{-n}), \quad \forall x \in \mathcal{U}, \quad \forall n, m \in \mathcal{N}.$$

$$H_{n,m}^{\text{conv}} = \begin{cases} -C_n^{\text{conv}} & \text{if } m = n, \\ \frac{\alpha^{\text{conv}}}{\xi_n} \phi' \left(\frac{x_{\sigma(n)}^{\text{conv}}}{\xi_n} \right) \frac{\partial x_{\sigma(n)}^{\text{conv}}}{\partial x_m^{\text{conv}}} & \text{if } m \neq n. \end{cases} \quad (39)$$

$$H_{n,m}^{\text{nuc}} = \begin{cases} -\lambda_n^2 \sum_{\tau} \frac{\alpha^{\tau}}{x_{\sigma(n)}^{\tau}} \exp \left(-C_n^{\text{nuc}} - \lambda_n \frac{x_n^{\text{nuc}}}{x_{\sigma(n)}^{\tau}} \right) & \text{if } m = n, \\ \left[\frac{\alpha^{\text{nuc}}}{\xi_n} \phi' \left(\frac{x_{\sigma(n)}^{\text{conv}} + x_{\sigma(n)}^{\text{nuc}}}{\xi_n} \right) + \lambda_n^2 \frac{\alpha^{\text{nuc}} x_n^{\text{nuc}}}{(x_{\sigma(n)}^{\text{nuc}})^2} \exp \left(-\lambda_n \frac{x_n^{\text{nuc}}}{x_{\sigma(n)}^{\text{nuc}}} \right) \right] \frac{\partial x_{\sigma(n)}^{\text{nuc}}}{\partial x_m^{\text{nuc}}} & \text{if } m \neq n. \end{cases} \quad (40)$$

$$H_{n,n}(x) = \begin{pmatrix} -C_n^{\text{conv}} & 0 \\ 0 & -C_n^{\text{nuc}} - \lambda_n^2 \sum_{\tau} \frac{\alpha^{\tau}}{x_{\sigma(n)}^{\tau}} \exp(-\lambda_n x_n^{\text{nuc}}) \end{pmatrix}. \quad (41)$$

B Security Functions Closed-Form Expressions

Below, we recall the analytical expressions of the security functions considering conventional or nuclear armament only.

Security Function for Conventional Armaments

$$f_n^{\text{conv}}(x_n, x_{-n}) = x_n^{\text{conv}} \phi\left(\frac{x_{\sigma(n)}^{\text{conv}}}{\xi_n}\right) - x_{\sigma(n)}^{\text{conv}} \exp\left(-\lambda_n \frac{x_n^{\text{nuc}}}{x_{\sigma(n)}^{\text{conv}}}\right).$$

$$\begin{cases} \frac{\partial f_n^{\text{conv}}}{\partial x_n^{\text{conv}}}(x_n, x_{-n}) = \phi\left(\frac{x_{\sigma(n)}^{\text{conv}}}{\xi_n}\right) \\ \frac{\partial f_n^{\text{conv}}}{\partial x_n^{\text{nuc}}}(x_n, x_{-n}) = -\lambda_n \mathbf{1}_{\{x_{\sigma(n)}^{\text{conv}} > 0\}} \exp\left(-\lambda_n \frac{x_n^{\text{nuc}}}{x_{\sigma(n)}^{\text{conv}}}\right). \end{cases}$$

Security Function for Nuclear Armaments

$$f_n^{\text{nuc}}(x_n, x_{-n}) = x_n^{\text{nuc}} \phi\left(\frac{x_{\sigma(n)}^{\text{conv}} + x_{\sigma(n)}^{\text{nuc}}}{\xi_n}\right) - x_{\sigma(n)}^{\text{nuc}} \exp\left(-\lambda_n \frac{x_n^{\text{nuc}}}{x_{\sigma(n)}^{\text{nuc}}}\right),$$

$$\begin{cases} \frac{\partial f_n^{\text{nuc}}}{\partial x_n^{\text{conv}}}(x_n, x_{-n}) = 0 \\ \frac{\partial f_n^{\text{nuc}}}{\partial x_n^{\text{nuc}}}(x_n, x_{-n}) = \phi\left(\frac{x_{\sigma(n)}^{\text{conv}}}{\xi_n}\right) - \lambda_n \mathbf{1}_{\{x_{\sigma(n)}^{\text{nuc}} > 0\}} \exp\left(-\lambda_n \frac{x_n^{\text{nuc}}}{x_{\sigma(n)}^{\text{nuc}}}\right). \end{cases}$$