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An Adaptive Multi-Level Max-Plus Method for Deterministic Optimal Control Problems

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Abstract: We introduce a new numerical method to approximate the solution of a finite horizon deterministic optimal control problem. We exploit two Hamilton-Jacobi-Bellman PDE, arising by considering the dynamics in forward and backward time. This allows us to compute a neighborhood of the set of optimal trajectories, in order to reduce the search space. The solutions of both PDE are successively approximated by max-plus linear combinations of appropriate basis functions, using a hierarchy of finer and finer grids. We show that the sequence of approximate value functions obtained in this way does converge to the viscosity solution of the HJB equation in a neighborhood of optimal trajectories. Then, under certain regularity assumptions, we show that the number of arithmetic operations needed to compute an approximate optimal solution of a d -dimensional problem, up to a precision ε , is bounded by $O(C^d |\log \varepsilon|)$, for some constant $C > 1$, whereas ordinary grid-based methods have a complexity in $O(1/\varepsilon^{ad})$ for some constant $a > 0$.

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Keywords: Optimal control, dynamic programming, Hamilton-Jacobi-Bellman PDE, max-plus algebra, curse-of-dimensionality

1. INTRODUCTION

We are interested in numerically solving a finite horizon deterministic optimal control problem. Such a problem is associated to a Hamilton-Jacobi-Bellman (HJB) equation via the Bellman dynamic programming principle (see for instance Fleming and Soner (2006)). The value function of this class of optimal control problems has been characterized as the solution of the associated HJB PDE in the viscosity sense (Crandall and Lions (1983)). Several classes of numerical methods have been proposed to solve such PDE. Among them, we mention the finite difference schemes introduced in Crandall and Lions (1984), and the semi-lagrangian schemes, studied in particular in (Falcone (1987), Falcone and Ferretti (2014)).

More recently, max-plus based discretization schemes have been developed by Fleming and McEneaney (2000), Akian et al. (2008), McEneaney (2007), McEneaney (2006), Qu (2014), Yegorov and Dower (2021). These methods take advantage of the max-plus linearity of the evolution semi-group of the HJB PDE, the so called *Lax-Oleinik semi-group*. After a time discretization, this allows one to approximate the value function for a given horizon, by a supremum of appropriate basis functions, for instance quadratic forms. Such suprema are propagated by the action of the Lax-Oleinik semigroup, between two successive time steps.

McEneaney (2007) showed that the max-plus based methods have the advantages to attenuate the *curse-*

of-dimensionality in some structured cases, including switched control problems, see also Sridharan et al. (2010), and Qu (2014) for further complexity results. Other attempts to reduce *curse-of-dimensionality* include the computation of the value function at one given point by constructing the grid from the possible trajectories and reducing the set of trajectories using Lipschitz continuity properties, together with the low dimensionality of the control set, like in Alla et al. (2019), Alla et al. (2020), and Bokanowski et al. (2022).

In this paper, we address the curse-of-dimensionality issues with another approach. The main idea is to consider a hierarchy of finer and finer irregular grids, concentrated around optimal trajectories, thus allowing us to dynamically reduce the search space, while increasing the precision. This is achieved by considering a pair of HJB PDE, associated to two optimal control problems: one with a forward dynamics, fixed initial state and free final state, and a dual one, with a backward dynamics, fixed final state and free initial state. The value functions of these two PDE allow us to compute a family of nested neighborhoods of optimal trajectories. Then, we adaptively add new basis functions, from one grid level to the next one, to refine the approximation. These new basis functions are chosen to be concentrated near the optimal trajectories of the control problem, and the refined neighborhood of optimal trajectories is computed from the solutions of the two HJB PDE in the coarser grid.

We show that using our algorithm, the number of basis functions needed to get a certain error ε is considerably reduced. Indeed, for a d -dimensional problem, under certain regularity assumptions, we get a complexity bound of $C^d |\log \varepsilon|$ arithmetic operations, for some constant $C > 1$. This should be compared with methods based on regular grids, which yield complexity bounds of order $\mathcal{O}(1/\varepsilon^{ad})$ in which $a > 0$ depends on regularity assumptions and on the order of the scheme (see for instance Bardi and Capuzzo-Dolcetta (2008)). With our adaptative method, the curse of dimensionality remains only present in the term C^d , in particular the complexity becomes polynomial in the bit-size of the numerical precision ε .

The present work extends the idea of dynamic grid refinement, originally presented in Akian et al. (2023) to solve semi-Lagrangian discretizations of special, minimal time, problems. Here, we exploit max-plus approximations combined with direct methods, allowing a higher degree of accuracy, and we adress finite horizon problems with more general cost and dynamics structure.

2. OPTIMAL CONTROL PROBLEM, HJB EQUATION, CHARACTERIZATION OF OPTIMAL TRAJECTORIES

We intend to solve the following finite horizon deterministic optimal control problem:

$$\max \left\{ \int_0^T \ell(x(s), u(s)) ds + \phi_0(x(0)) + \phi_T(x(T)) \right\} \quad (1)$$

over the set of trajectories $(x(s), u(s))$ satisfying:

$$\begin{cases} \dot{x}(s) = f(x(s), u(s)) , \\ x(s) \in X, u(s) \in U , \end{cases} \quad (2)$$

for all $s \in [0, T]$. Let us denote v^* the maximum in (1). Here, $X \subset \mathbb{R}^d$, assumed to be bounded, is the state space and $U \subset \mathbb{R}^m$ is the control space. We further assume that the running cost $\ell : X \times U \mapsto \mathbb{R}$, the dynamics $f : X \times U \mapsto \mathbb{R}$, the initial and final cost $\phi_0, \phi_T : X \mapsto \mathbb{R}$ are sufficiently regular: bounded, continuous and Lipschitz w.r.t. all variables.

A well known sufficient and necessary optimality condition for the above problem is given by the Hamilton-Jacobi-Bellman equation, which is deduced from the dynamic programming principle. Indeed, we consider the value function v_+ , defined as follows, for any $(x, t) \in X \times [0, T]$:

$$v_+(x, t) = \sup \left\{ \int_t^T \ell(x(s), u(s)) ds + \phi_T(x(T)) \right\} , \quad (3)$$

under the constraint (2) with the initial state $x(t) = x$. Here, the symbol “ \leftarrow ” indicates that (x, t) is the *source*, so that the corresponding HJB PDE is of a backward nature. Indeed, v_+ is known to be the viscosity solution of the following HJB equation (see for instance Fleming and Soner (2006)):

$$\begin{cases} -\frac{\partial v_+}{\partial t} - H(x, \nabla v_+) = 0, & (x, t) \in X \times [0, T] , \\ v_+(x, T) = \phi_T(x), & x \in X , \end{cases} \quad (4)$$

where $H(x, p) = \sup_{u \in U} \{ p \cdot f(x, u) + \ell(x, u) \}$ is the Hamiltonian of the problem. Once (4) is solved, one can

easily obtain the value of the original problem (1) by further taking the maximum over X , i.e.,

$$v^* = \max_{x \in X} \{ \phi_0(x) + v_+(x, 0) \} . \quad (5)$$

We shall also use another, equivalent, optimality condition for problem (1), obtained by applying the dynamic programming principle in the reverse direction. This leads us to consider the value function v_+ , such that

$$v_+(x, t) = \sup \left\{ \int_0^t \ell(x(s), u(s)) ds + \phi_0(x(0)) \right\} , \quad (6)$$

under the same constraint (2), but with the final state $x(t) = x$. The notation “ \rightarrow ” indicates that (x, t) is now the destination. Then, v_+ is known to be the viscosity solution of the following HJB equation, in forward time:

$$\begin{cases} \frac{\partial v_+}{\partial t} - H(x, -\nabla v_+) = 0, & (x, t) \in X \times [0, T] , \\ v_+(x, 0) = \phi_0(x), & x \in X . \end{cases} \quad (7)$$

Once (7) is solved, we can then get the maximum in (1) by

$$v^* = \max_{x \in X} \{ \phi_T(x) + v_+(x, T) \} . \quad (8)$$

The two value functions v_+ and v_- allow us to determine the points belonging to optimal trajectories:

Definition 2.1. We say that $x^*(\cdot)$ is an optimal trajectory of the optimal control problem (1) if there exists a control $u^*(\cdot)$ such that $(x^*(\cdot), u^*(\cdot))$ achieves the maximum in (1), under the constraint (2). We assume that the set of optimal trajectories is non-empty, and denote, for all $t \in [0, T]$:

$$\Gamma_t^* = \{ x^*(t) \mid x^*(\cdot) \text{ is an optimal trajectory} \} , \quad (9)$$

and $\Gamma^* = \cup_{t \in [0, T]} \Gamma_t^*$.

Then, we have the following result:

Proposition 2.1.

$$v^* = \sup_{x \in X} \{ v_+(x, t) + v_-(x, t) \}, \quad \forall t \in [0, T] . \quad (10)$$

Moreover, for all $t \in [0, T]$, the above supremum is achieved for some $x \in \Gamma_t^*$. Conversely, for all $x \in \Gamma_t^*$, the above supremum is achieved at point x .

Proof. The equality (10) follows in a straightforward way from the definition of the value functions v_+, v_- in (3) and (6). Moreover, since there exists an optimal trajectory $x^*(\cdot)$, then the supremum in (10) is achieved at $x^*(t) \in \Gamma_t^*$, for all $t \in [0, T]$. Conversely, for all $x \in \Gamma_t^*$, there exists an optimal trajectory x^* such that $x^*(t) = x$, and the supremum in (10) is achieved at $x = x^*(t)$. \square

For all $t \in [0, T]$, let us define the map $\mathcal{F}_v^t : X \mapsto \mathbb{R}$ by

$$\mathcal{F}_v^t(x) = \mathcal{F}_v(x, t) = v_+(x, t) + v_-(x, t) . \quad (11)$$

Consider for every $t \in [0, T]$ the subdomain $\mathcal{O}_\eta^t \subset X$, depending on a parameter $\eta > 0$, and defined as follows:

$$\mathcal{O}_\eta^t = \{ x \in X \mid \mathcal{F}_v^t(x) > \sup_{x \in X} \mathcal{F}_v^t(x) - \eta \} . \quad (12)$$

In fact, \mathcal{O}_η^t can be thought of as a η -neighborhood around the geodesic points at time t , Γ_t^* . We set $\mathcal{O}_\eta := \{ (x, t) \mid x \in \mathcal{O}_\eta^t \}$. We intend to reduce the (state,time)-space $X \times [0, T]$ of our optimal control problem to such an η -neighborhood. I.e., we replace the constraint (2) by

$$\begin{cases} \dot{x}(s) = f(x(s), u(s)) , \\ x(s) \in \mathcal{O}_\eta^s, u(s) \in U , \end{cases} \quad (13)$$

for all $s \in [0, T]$. Let us denote v_η^* the maximum of (1) under the new constraint (13). Then we have

Proposition 2.2. $v_\eta^* = v^*$.

Proof. The inequality $v^* \geq v_\eta^*$ is straightforward since $\mathcal{O}_\eta^s \subset X$ for all $s \in [0, T]$. To show the reverse inequality, let us take an optimal trajectory $x^*(\cdot)$ for the original problem. Then, by the result of Proposition 2.1, we have $x^*(s) \in \mathcal{O}_\eta^s, \forall s \in [0, T]$. Thus $v_\eta^* \geq v^*$ since v^* is exactly the value of the integral in (1) following the optimal trajectory $x^*(\cdot)$. \square

Proposition 2.2 indeed tells us that, to solve the problem (1), only the η -neighborhood, \mathcal{O}_η , around the optimal trajectory is relevant. In the following, we will focus on solving the problem (1) using an approximation of such a neighborhood.

3. PROPAGATION BY LAX-OLEINIK SEMI-GROUPS AND MAX-PLUS APPROXIMATION

We denote by S_τ^τ the *Lax Oleinik semigroup* of (4) with $\tau = T - t$, i.e., the evolution semigroup of this PDE, meaning that, for all $0 \leq t \leq T$, S_τ^τ is the map sending the final cost function $\phi_T(\cdot)$ to the value function $v_\tau(\cdot, \tau)$, so that the semi-group property $S^{\tau_1 + \tau_2} = S^{\tau_1} \circ S^{\tau_2}$ is satisfied. In addition, the map S_τ^τ is *max-plus linear*, meaning that for all $\lambda \in \mathbb{R}$ and for all functions ϕ_T^1 and $\phi_T^2 : X \rightarrow \mathbb{R}$, we have:

$$\begin{aligned} S_\tau^\tau[\sup(\phi_T^1, \phi_T^2)] &= \sup(S_\tau^\tau[\phi_T^1], S_\tau^\tau[\phi_T^2]), \\ S_\tau^\tau[\lambda + \phi_T^1] &= \lambda + S_\tau^\tau[\phi_T^1], \end{aligned} \quad (14)$$

where for any function ϕ on X , $\lambda + \phi$ is the function $x \in X \mapsto \lambda + \phi(x)$ (see for instance Fleming and McEneaney (2000), Akian et al. (2008), Yegorov and Dower (2021)). Indeed, the property (14) can be interpreted as the linearity in the sense of the max-plus semiring, which is the set $\mathbb{R}_{\max} := \mathbb{R} \cup \{-\infty\}$ equipped with the addition $a \oplus b := \max(a, b)$ and the multiplication $a \odot b := a + b$, with $-\infty$ as the zero and 0 as the unit. Notice that the above properties hold, mutatis mutandis, for the evolution operator of the dual equation (7). We will then briefly describe the approximation method based on the max-plus linearity introduced in Akian et al. (2008), which may be thought of as a max-plus analogue of the finite element methods.

Let us discretize the time horizon by $N = \frac{T}{\delta}$ steps. Denote $v_\tau^t = v_\tau(\cdot, t)$. By the semigroup property we have:

$$v_\tau^{t-\delta} = S_\tau^\delta[v_\tau^t], \quad \forall t = \delta, 2\delta, \dots, T, \quad v_\tau^T = \phi_T. \quad (15)$$

Denote $\overline{\mathbb{R}}_{\max} := \mathbb{R}_{\max} \cup \{+\infty\}$ the complete semiring extending \mathbb{R}_{\max} , and let \mathcal{W} be a complete $\overline{\mathbb{R}}_{\max}$ -semimodule of functions $w : X \rightarrow \overline{\mathbb{R}}_{\max}$, meaning that \mathcal{W} is stable under taking the supremum of an arbitrary family of functions, and by the addition of a constant, see McEneaney (2006); Cohen et al. (2004) for background. We choose this semimodule \mathcal{W} in such a way that $v_\tau^t \in \mathcal{W}$ for all $t \geq 0$. In many applications, the value function v^t is known to be c -semiconcave for all $t \in [0, T]$, and then \mathcal{W} can be taken to be the set of c -semiconcave functions, which is a complete module, see McEneaney (2006); Akian et al. (2008). We also choose \mathcal{Z} , a complete $\overline{\mathbb{R}}_{\max}$ -semimodule of

test functions $z : X \mapsto \overline{\mathbb{R}}_{\max}$. If the space of test functions \mathcal{Z} is large enough, (15) is equivalent to:

$$\langle z, v_\tau^{t-\delta} \rangle = \langle z, S_\tau^\delta[v_\tau^t] \rangle \quad \forall t, \quad \langle z, v^T \rangle = \langle z, \phi_T \rangle \quad \forall z \in \mathcal{Z}, \quad (16)$$

where the max-plus scalar product of $u \in \mathcal{W}$ and $v \in \mathcal{Z}$ is defined by $\langle u, v \rangle = \sup_{x \in X} (u(x) + v(x)) \in \overline{\mathbb{R}}_{\max}$.

Note that in the system (16), the unknown value functions are elements of \mathcal{W} , therefore having an infinite number of degrees of freedom, and that there are infinitely many equations (one for each element $z \in \mathcal{Z}$). Hence, we need to discretize this system. To do so, we consider $\mathcal{W}^h \subset \mathcal{W}$, a semimodule generated by a finite family of basis functions $\{w_i\}_{1 \leq i \leq p}$. The value function v_τ^t at time t is approximated by $v_\tau^{t,h} \in \mathcal{W}^h$, that is:

$$v_\tau^{t,h} := \sup_{1 \leq i \leq p} \{\lambda_i^t + w_i\} : x \mapsto \max_{1 \leq i \leq p} \{\lambda_i^t + w_i(x)\}, \quad (17)$$

where $\{\lambda_i^t\}_{1 \leq i \leq p}$ is a family of scalars. We then consider $\mathcal{Z}^h \subset \mathcal{Z}$, a semimodule generated by a finite family of test functions $\{z_j\}_{1 \leq j \leq q}$, and, instead of requiring (16) to hold for all $z \in \mathcal{Z}$, we only require that it holds for generators, leading to a finite system of equations. Therefore, the approximation $v_\tau^{t-\delta,h}$ and v^T should satisfy:

$$\langle z_j, v_\tau^{t-\delta,h} \rangle = \langle z_j, S_\tau^\delta[v_\tau^{t,h}] \rangle, \quad \langle z_j, v^T \rangle = \langle z_j, \phi_T \rangle \quad \forall j. \quad (18)$$

It is a key property of max-plus algebra that a system of linear equations, even when the number of equations coincides with the number of degrees of freedom, and when the system is “nonsingular”, may have no solution, so that the notion of solution must be replaced by a notion of maximal subsolution, which is always well posed. In particular, (18) may not have a solution. Hence, we define $v_\tau^{t-\delta,h}$ to be the maximal solution of the following system of inequalities:

$$\langle z_j, v_\tau^{t-\delta,h} \rangle \leq \langle z_j, S_\tau^\delta[v_\tau^{t,h}] \rangle, \quad \langle z_j, v^T \rangle \leq \langle z_j, \phi_T \rangle, \quad \forall j. \quad (19)$$

Let us denote $W_h : \mathbb{R}_{\max}^p \mapsto \mathcal{W}$ the max-plus linear operator such that $W_h(\lambda) = \bigoplus_{1 \leq i \leq p} \{\lambda_i \odot w_i\}$, and $Z_h^* : \mathcal{W} \mapsto \mathbb{R}_{\max}^q$ with $(Z_h^*(w))_j = \langle z_j, w \rangle, \forall 1 \leq j \leq q$. Recall that, for every ordered sets \mathcal{S}, \mathcal{T} and order preserving map $g : \mathcal{S} \mapsto \mathcal{T}$, the residuated map $g^\#$ is defined as $g^\#(t) = \max\{s \in \mathcal{S} \mid g(s) \leq t\}$, when it exists. Max-plus linear operators have a residuated map. Moreover, by (Cohen et al., 1996, Th. 1), for all max-plus linear operators $B : \mathcal{U} \mapsto \mathcal{X}, C : \mathcal{X} \mapsto \mathcal{Y}$ over complete semimodules $\mathcal{X}, \mathcal{Y}, \mathcal{U}$, the operator $\Pi_B^C := B \circ (C \circ B)^\# \circ C$ is a projector, and we have, for all $x \in \mathcal{X}$:

$$\Pi_B^C(x) = \max\{y \in \text{im} B \mid Cy \leq Cx\}. \quad (20)$$

Then, the approximations $v_\tau^{t,h}$ can be expressed as follows.

Proposition 3.1. (Akian et al. (2008)). Consider the maximal $\lambda^t \in \mathbb{R}_{\max}^p$ and $v_\tau^{t,h} \in \mathcal{W}_h, t = 0, \delta, \dots, T$, such that $v_\tau^{t,h} = W_h \lambda^t$, with $v_\tau^{t-\delta,h}, t \geq \delta$, and v^T solutions of (19). We have,

$$v_\tau^{t-\delta,h} = S_\tau^{\delta,h}[v_\tau^{t,h}], \quad \text{where } S_\tau^{\delta,h} = \Pi_{W_h}^{Z_h^*} \circ S_\tau^\delta,$$

and

$$\begin{cases} \lambda^{t-\delta} = (Z_h^* W_h)^\#(Z_h^* S_\tau^\delta W_h \lambda^t), \quad \forall t = \delta, 2\delta, \dots, T, \\ \lambda^T = W_h^\# \phi_T. \end{cases}$$

The above formula can be expressed using the linear operators $M_h := Z_h^* W_h$ and $K_h := Z_h^* S_\tau^\delta W_h$, with entries:

$$(M_h)_{j,i} = \langle z_j, w_i \rangle, \quad (K_h)_{j,i} = \langle z_j, S_\tau^\delta w_i \rangle. \quad (21)$$

The matrices M_h and K_h may be thought of as max-plus analogues of the *mass* and *stiffness* matrices arising in the finite element method, see Akian et al. (2008). Computing $(M_h)_{j,i}$ is a convex programming problem, which can be solved by standard optimization methods (sometimes the solution can even be computed analytically). Computing $(K_h)_{j,i}$ is equivalent to solve the associated control problem in a small time horizon δ . An approximation method proposed in Akian et al. (2008) is to use the Hamiltonian of the problem. Alternatively, a direct method can be used, see e.g. Bonnans et al. (2017) for background on direct methods in optimal control. After M_h, K_h are computed (or approximated), the max-plus method works as follows:

Algorithm 1 Max-Plus Approximation Method

- 1: Discretize time horizon by $N = \frac{T}{\delta}$ steps.
 - 2: Choose basis functions $\{w_i\}_{1 \leq i \leq p}$ and $\{z_j\}_{1 \leq j \leq q}$.
 - 3: Compute (or approximate) M_h and K_h .
 - 4: Initialize $\lambda^T = W_h^\# \phi_T$, $v_\leftarrow^{T,h} = W_h \lambda^T$.
 - 5: **for** $t = T, T-\delta, \dots, \delta$ **do**
 - 6: $\lambda^{t-\delta} = M_h^\# K_h \lambda^t$.
 - 7: $v_\leftarrow^{t-\delta,h} = W_h \lambda^{t-\delta}$
 - 8: **end for**
-

4. ADAPTIVE MAX-PLUS APPROXIMATION METHOD

In Section 2, we observed that to solve the optimal control problem (1), we need only to focus on a neighborhood of the optimal trajectory—if we could approximately know it in advance. This also works when we intend to find an approximation of the value function. In this section, we will propose an adaptive max-plus approximation method to solve problem (1). The general idea is to start with a small set of basis functions, then adaptively add more basis functions to better approximate the two value functions v_\rightarrow and v_\leftarrow in a suitable neighborhood of the optimal trajectories, obtained from the approximate value functions.

We discretize the time horizon by $N = \frac{T}{\delta}$ steps. Then, our algorithm consists of three main steps:

Step 1. Coarse Approximation. Let us start with a regular grid $X^H = \{x_1, x_2, \dots, x_{p^H}\}$, with step size H , and let $I^H := \{1, \dots, p^H\}$ be the index set of X^H . Natural choices of basis functions and test functions, are the Lipschitz functions of the form $w_{x_i}(x) := -c\|x - x_i\|_1$, and the quadratic functions of the form $w_{x_i}(x) := -c\|x - x_i\|_2^2$, for every $x_i \in X^H$. We fix some sets of basis functions $\{w_{x_i}\}_{x_i \in X^H}$ and test functions $\{z_{x_i}\}_{x_i \in X^H}$, and apply Algorithm 1. This leads to an approximation of the two value functions v_\rightarrow and v_\leftarrow , by the maps $v_\rightarrow^{t,H}$ and $v_\leftarrow^{t,H}$, with $t \in \{0, \delta, \dots, T\}$, respectively.

Step 2. Optimal Trajectory Approximation. For every $t \in \{0, \delta, \dots, T\}$ and every $x \in X$, we have:

$$\begin{aligned} v_\rightarrow^{t,H}(x) &= \max_{1 \leq i \leq p^H} \{\lambda_i^{\rightarrow,t} + w_{x_i}(x)\}, \\ v_\leftarrow^{t,H}(x) &= \max_{1 \leq i \leq p^H} \{\lambda_i^{\leftarrow,t} + w_{x_i}(x)\}. \end{aligned} \quad (22)$$

Then, an approximation for \mathcal{F}_v^t is given by:

$$\mathcal{F}_{v^H}^t = \sup_{1 \leq i, i' \leq p^H} \{\lambda_i^{\rightarrow,t} + \lambda_{i'}^{\leftarrow,t} + w_{x_i} + w_{x_{i'}}\}. \quad (23)$$

For a given η^H , let us denote by $\mathcal{O}_{\eta^H, H}^t \subseteq X$ the approximation of $\mathcal{O}_{\eta^H}^t$ defined as follows:

$$\mathcal{O}_{\eta^H, H}^t = \{x \in X \mid \mathcal{F}_{v^H}^t(x) > \max_{y \in X} \{\mathcal{F}_{v^H}^t(y) - \eta^H\}\}. \quad (24)$$

Denote $\mathcal{M}_{i,i'} = w_{x_i} + w_{x_{i'}}$ for all $i, i' \in I^H$. Using (23), the r.h.s. in (24) can be computed as a function of the scalars $\mathcal{M}_{i,i'}^* := \max_{y \in X} \mathcal{M}_{i,i'}(y) = \langle w_{x_i}, w_{x_{i'}} \rangle$:

$$\max_{y \in X} \mathcal{F}_{v^H}^t(y) = \max_{i, i' \in I^H} \{\lambda_i^{\rightarrow,t} + \lambda_{i'}^{\leftarrow,t} + \mathcal{M}_{i,i'}^*\}. \quad (25)$$

Moreover, for the above basis functions w_{x_i} and $w_{x_{i'}}$, the scalars $\mathcal{M}_{i,i'}^*$ can be computed analytically. Denote $\mathcal{N}_{v^H}^t(i, i') = \lambda_i^{\rightarrow,t} + \lambda_{i'}^{\leftarrow,t} + \mathcal{M}_{i,i'}^*$ and let $\mathcal{N}_{v^H}^{t,*} = \max_{i, i' \in I^H} \mathcal{N}_{v^H}^t(i, i')$. In this step, we first select the couples (i, i') as follows:

$$\mathcal{I}_{\eta^H, H}^t := \{(i, i') \in (I^H)^2 \mid \mathcal{N}_{v^H}^t(i, i') > \mathcal{N}_{v^H}^{t,*} - \eta^H\}. \quad (26)$$

Then, based on $\mathcal{I}_{\eta^H, H}^t$, we select $\mathcal{A}_{\eta^H, H}^t \subset X$ as follows:

$$\mathcal{A}_{\eta^H, H}^t = \{x \in X \mid \exists (i, i') \in \mathcal{I}_{\eta^H, H}^t \mathcal{M}_{i,i'}(x) > \mathcal{M}_{i,i'}^* - \eta^H\}. \quad (27)$$

The set $\mathcal{A}_{\eta^H, H}^t$ can be compared with $\mathcal{O}_{\eta^H, H}^t$, and seen as an approximation of $\mathcal{O}_{\eta^H}^t$. At the end of this step, we obtain an *active region*:

$$X_f = \bigcup_{t \in \{0, \delta, \dots, T\}} \{\mathcal{A}_{\eta^H, H}^t\}. \quad (28)$$

Step 3. Fine Approximation. In this step, we consider the discretization of X by a regular grid X^h with step size $h < H$, and set $X_f^h := X^h \cap X_f = \{x_1, x_2, \dots, x_{p^h}\}$. For the purpose of efficiency, we shall directly compute

$$\mathcal{A}_{\eta^H, H}^{t,h} = X^h \cap \mathcal{A}_{\eta^H, H}^t, \quad (29)$$

then $X_f^h = \bigcup_{t \in \{0, \delta, \dots, T\}} \{\mathcal{A}_{\eta^H, H}^{t,h}\}$. Given X_f^h , we add more basis functions and test functions by using the points in X_f^h . We then use the new set of basis functions: $\{w_{x_i}\}_{x_i \in (X^H \cup X_f^h)}$ and the new set of test functions: $\{z_{x_i}\}_{x_i \in (X^H \cup X_f^h)}$ to approximate the two value functions at each time step t .

The above approximation steps can be easily repeated, for instance, m times. Moreover, the discretization grids need not be regular, and in fact, the general error estimate established in Akian et al. (2008) applies to an irregular grid. The error is expressed in terms of an abstract mesh parameter, defined as the maximal diameter of a cell of the Voronoi tessellation induced by the grid points of the *active region*.

To define the repeated steps, we need a family of parameters $\{\eta_l\}_{l=1,2,\dots,m}$ selecting the *active regions* based on the previous two directions' approximations. We also need a family of mesh steps $H_1 > H_2 > \dots > H_{m+1}$ and the corresponding discretization grids $X^{H_1}, \dots, X^{H_{m+1}}$ of X , for constructing the space discretization of the *active regions*.

We assume these parameters are fixed in advance. Then, we get the following algorithm:

Algorithm 2 Adaptive Max-Plus Approximation Method

Discretize time horizon by $N = \frac{T}{\delta}$ steps.

2: Set *Base* and *Test* to empty sets, set $X_f^{H_1}$ to X^{H_1} ;

for $l = 1$ to $m + 1$ **do**

4: $Base = Base \cup \{w_{x_i}\}_{x_i \in X_f^l}$;
 $Test = Test \cup \{z_{x_i}\}_{x_i \in X_f^l}$;

6: Approximate $v_{\rightarrow}, v_{\leftarrow}$ using Algorithm 1 with *Base* and *Test*;

if $l \leq m$ **then**

8: Set $H = H_l, \eta^H = \eta_l$;
Set I^H as an index set for *Base*;

10: Set $h = H_{l+1}$, and $X_f^{H_{l+1}}$ to empty set;

for $t = 0, \delta, \dots, T$ **do**

12: Compute $\mathcal{M}_{i,i'}^*$ for all $i, i' \in I^H$;
Compute $\mathcal{I}_{\eta^H, H}^t$ as in (26);

14: Compute $\mathcal{A}_{\eta^H, H}^{t, h}$ by (27,29);
 $X_f^{H_{l+1}} = X_f^{H_{l+1}} \cup \mathcal{A}_{\eta^H, H}^{t, h}$;

16: **end for**

end if

18: **end for**

We count, in Algorithm 2, each time's computation of one level l , that is the first two main steps above when $H = H_l$ and the discretization part of the last one when $h = H_{l+1}$.

For each level $l \in \{1, 2, \dots, m + 1\}$ of Algorithm 2, let $v_{\rightarrow}^{t, H_l}, v_{\leftarrow}^{t, H_l}$ with $t \in \{0, \delta, \dots, T\}$, be the approximations of v_{\rightarrow}^t and v_{\leftarrow}^t computed using the discrete *active region* $X_f^{H_l}$, and for $l \leq m$, let us denote by $X_f^{(l+1)}$ the union of the *active regions* $\mathcal{A}_{\eta_l, H_l}^t$ at time t , selected by (26,27), using $v_{\rightarrow}^{H_l}, v_{\leftarrow}^{H_l}$ and η_l . We also set $X_f^{(1)} = X$. Then, for all l , the set $X_f^{H_l}$ is the discretization of $X_f^{(l)}$. For all $l \in \{1, 2, \dots, m + 1\}$ and $t \in \{0, \delta, \dots, T\}$, let us denote by $\tilde{v}_{\rightarrow}^{t, H_l}$ and $\tilde{v}_{\leftarrow}^{t, H_l}$ the approximations of v_{\rightarrow}^t and v_{\leftarrow}^t using Algorithm 1 with the sets of basis functions and test functions obtained from the discretization grids $X_f^{H_l}$ of X with mesh step H_l . Due to the initialization, the functions $\tilde{v}_{\rightarrow}^{t, H_1}, \tilde{v}_{\leftarrow}^{t, H_1}$ coincide with $v_{\rightarrow}^{t, H_1}, v_{\leftarrow}^{t, H_1}$ for $l = 1$. We have the following result:

Theorem 4.1.

- (i) For every $l \in \{1, 2, \dots, m\}$, there exists an $\bar{\eta}_l$ depending on H_l and δ such that for all $\eta_l \geq \bar{\eta}_l$, and $t \in \{0, \delta, \dots, T\}$, $X_f^{(l+1)}$ contains Γ_t^* , that is the set of geodesic points for problem (1) at time t .
- (ii) Take η_l as proposed in (i), then for every $l \in \{2, \dots, m + 1\}$, $t \in \{0, \delta, \dots, T\}$ and $x \in \Gamma_t^*$, we have $v_{\rightarrow}^{t, H_l}(x) = \tilde{v}_{\rightarrow}^{t, H_l}(x), v_{\leftarrow}^{t, H_l}(x) = \tilde{v}_{\leftarrow}^{t, H_l}(x)$. Thus, $\{v_{\rightarrow}^{t, H_m}\}, \{v_{\leftarrow}^{t, H_m}\}$ converge to $v_{\rightarrow}^t, v_{\leftarrow}^t$ respectively as $H_m \rightarrow 0$.

Proof. We give the proof in the two-level case (the extension to the multi-level case follows along the same lines). Fix a time step δ , and a time $t \in \{0, \delta, \dots, T\}$. We first notice that $\mathcal{O}_{\eta^H, H}^t \subset \mathcal{A}_{\eta^H, H}^t$. As shown in Proposition 2.1, the value function in a geodesic point $x \in \Gamma_t^*$ satisfies $\mathcal{F}_v^t(x) = \sup_{y \in X} \mathcal{F}_v^t(y)$. We know that the approximations $\tilde{v}_{\rightarrow}^{t, H}$ and $\tilde{v}_{\leftarrow}^{t, H}$ have certain error bounds (for the sup-norm)

$\varepsilon_{\rightarrow}^H, \varepsilon_{\leftarrow}^H$ resp., depending on H and δ , but not on t :

$$\|v_{\rightarrow}^{t, H} - v_{\rightarrow}^t\| \leq \varepsilon_{\rightarrow}^H, \|v_{\leftarrow}^{t, H} - v_{\leftarrow}^t\| \leq \varepsilon_{\leftarrow}^H.$$

Denote $\varepsilon^H = \varepsilon_{\rightarrow}^H + \varepsilon_{\leftarrow}^H$, we have for every $y \in X$:

$$(\mathcal{F}_v^t(y) - \varepsilon^H) \leq \mathcal{F}_{v^H}^t(y) \leq (\mathcal{F}_v^t(y) + \varepsilon^H).$$

Consider now $x' \notin X_f := \cup_{t \in \{0, \delta, \dots, T\}} \mathcal{A}_{\eta^H, H}^t$, so that $x' \notin \mathcal{A}_{\eta^H, H}^t$, and so $x' \notin \mathcal{O}_{\eta^H, H}^t$. Then

$$\begin{aligned} \mathcal{F}_v^t(x') &\leq \mathcal{F}_{v^H}^t(x') + \varepsilon^H \leq \sup_{y \in X} \{(\mathcal{F}_{v^H}^t(y) - \eta^H) + \varepsilon^H\} \\ &\leq \sup_{y \in X} \{\mathcal{F}_v^t(y) + (2\varepsilon^H - \eta^H)\}. \end{aligned}$$

Thus, if we take η^H big enough such that $(2\varepsilon^H - \eta^H) < 0$, we have $x' \notin \Gamma_t^*$, and the result of (i) follows.

The result in (ii) is then straightforward, using Proposition 2.2. \square

5. ERROR ANALYSIS AND COMPUTATIONAL COMPLEXITY

In this section, we will analyze the computational complexity of our algorithm, and give the optimal parameters to turn the algorithm.

Let us start with evaluating the neighborhood of the optimal trajectory:

Proposition 5.1. For every $t \in [0, T]$ and for every $x \in \mathcal{O}_{\eta}^t$, there exists a $x^* \in \Gamma_t^*$ and such that:

$$\|x - x^*\| \leq C(\eta)^\beta,$$

where $C > 0$ and $\beta > 0$ are constants independent of x, t and η .

In Proposition 5.1, the exponent β determines the growth of the neighborhood \mathcal{O}_{η} of the optimal trajectories, as a function of η . This exponent depends on the geometry of the value function. We shall see in Proposition 5.3 that for typical instances, taking $\beta = 1/2$ is admissible.

Based on Proposition 5.1, and the property that $\mathcal{O}_{\eta^H, H}^t \subset \mathcal{A}_{\eta^H, H}^t \subset \mathcal{O}_{2\eta^H, H}^t$ are approximations of $\mathcal{O}_{\eta^H}^t$, we obtain the following general space complexity result:

Proposition 5.2. Given the sets of parameters $\{\eta_l\}_{l=1,2,\dots,m}$ and $\{H_l\}_{l=1,2,\dots,m+1}$, the number of discretization points generated by the adaptative max-plus approximation method can be bounded as follows:

$$C_{spa}(\{\eta_l, H_l\}) = O\left(\left(\frac{1}{H_1}\right)^d + \sum_{l=2}^{m+1} \left(\frac{(\eta_{l-1})^{\beta(d-1)}}{(H_l)^d}\right)\right). \quad (30)$$

Sketch of Proof. The summand $(\frac{1}{H_1})^d$ is the number of discretization points needed in the first level's grid, for which we discretized using mesh step H_1 . Each summand $(\frac{(\eta_{l-1})^{\beta(d-1)}}{(H_l)^d})$ corresponds to the number of points in the level- l 's grid, which is a "tubular" neighborhood around the optimal trajectory: at each time step, we only approximate the value functions using the points in a ball with radius $(\eta_{l-1})^\beta$ around the optimal trajectory. (This idea of using tubular neighborhoods of optimal paths to obtain complexity estimates originates from our recent work Akian et al. (2023), dealing with a minimal time optimal control problem.) \square

To obtain a complexity bound showing an attenuation of the curse of dimensionality, we certainly do not want the value function to be too “flat” near optimal trajectories. Indeed, this would result in a large neighborhood \mathcal{O}_η , and since this neighborhood is used to reduce the search space and define the new grid in Algorithm 2, the size of the new grid would not be so much reduced. Therefore, we make the following convexity assumption, around the optimal trajectories.

Assumption 5.1. The functions v_\rightarrow^t and v_\leftarrow^t are μ -strongly concave and X is a convex set.

Proposition 5.3. Under Assumption 5.1, we can take $\beta = \frac{1}{2}$ in Proposition 5.1.

Proof. For all $t \in [0, T]$, the function \mathcal{F}_v^t is 2μ -strongly concave on X . Let $x \in \mathcal{O}_\eta^t$ and let $x^* \in \Gamma_t^*$. For all $s \in [0, 1]$, the point $sx + (1-s)x^* \in X$. Then, by the strong concavity property, we have

$$\begin{aligned} & \mathcal{F}_v^t(sx + (1-s)x^*) + \mu(sx + (1-s)x^*)^2 \\ & \geq \frac{1}{2} \{s(\mathcal{F}_v^t(x) + \mu x^2) + (1-s)(\mathcal{F}_v^t(x^*) + \mu(x^*)^2)\}. \end{aligned}$$

By a simple computation we obtain that if $s > 0$, then $\|x - x^*\| \leq (\frac{\eta}{2\mu(1-s)})^{\frac{1}{2}}$, and passing to the limit in s , we deduce that $\|x - x^*\| \leq (\frac{\eta}{2\mu})^{\frac{1}{2}}$. \square

To make sure our *active region* X_f does contain all Γ_t^* , with $t = 0, \delta, \dots, T$, we need to take η_l big enough, as discussed in Theorem 4.1.

Let us first focus on the approximation error, that is the approximation of $S^\delta[w]$. For every basis function w_i and test function z_i , we have:

$$\begin{aligned} & \langle z_i, S^\delta[w_i] \rangle = \\ & \max \left\{ z_i(x(0)) + \int_0^\delta \ell(x(s), u(s)) ds + w_i(x(\delta)) \right\}, \end{aligned} \quad (31)$$

over the set of trajectories $(x(s), u(s))$ satisfying (2). This is an optimal control problem similar to the original one, but with two new essential properties: first, the time horizon δ is *small*, and second, the initial and final costs, z_i and w_i are “nice” concave functions, e.g., strongly concave quadratic forms. Then, the strong convexity of the initial or terminal cost “propagates” over a small horizon, which entails that (31) is actually a *convex* infinite dimensional optimization problem, which, after an appropriate discretization, using a so-called *direct method* in optimal control, can be reduced to a convex finite dimensional optimization problem, which can be solved *globally* by convex optimization methods method. in which the authors used a gradient descent to compute $\langle z_i, S^\delta[w_i] \rangle$. Alternatively, in Akian et al. (2008), the authors approximate (31) using the Hamiltonian, which results in an error $\mathcal{O}(\delta^2)$ or $\mathcal{O}(\delta^{\frac{3}{2}})$, depending the properties of z_i and w_i . However, to get the best complexity bounds, we need to assume that (31) is approximated with a high degree of accuracy. Thus, we shall make the following assumption:

Assumption 5.2. The functions z_i, w_i are strongly concave, and there exists a $\bar{\delta}$ such that, for every $\delta \leq \bar{\delta}$, $\langle z_i, S^\delta[w_i] \rangle$ can be computed exactly, or with an error negligible compared with the projection error, by a direct method.

This will allow us to obtain an *ideal* complexity bound, in an *oracle* Turing machine model, in which the time to solve a convex optimal control problem, in a small horizon, by calling a direct method (calling the oracle), is counted as one unit. This ideal complexity bound can be subsequently refined to get an effective bound in the ordinary Turing model of computation, recalling that ε -approximate solutions of *well conditioned* convex programming problems can be obtained in polynomial time by the ellipsoid or interior point methods. Using such an ideal model of computation is justified, since the only source of curse of dimensionality is the growth of the grid size, and since the execution time in this model is essentially the size of the largest grid.

To bound the projection error, we need to make the following assumption:

Assumption 5.3. The functions $v_\rightarrow^t, v_\leftarrow^t$ are L_v -Lipschitz continuous, α_1 -semiconvex, α_2 -semiconcave w.r.t. x for every $t \in [0, T]$.

By the result of (Lakhoua, 2007, Th. 83), using quadratic basis functions and Lipschitz test functions, we get a projection error $\mathcal{O}(\frac{\Delta x^2}{\delta})$, where Δx is the mesh step of the grid. Then, combining with the result of Theorem 4.1, we have the following result for the total error:

Theorem 5.1. Make Assumptions 5.2 and 5.3, choose quadratic basis functions and Lipschitz test functions, choose $\delta \leq \bar{\delta}$. Then, there exists a constant $C > 0$ depending on δ such that, for a given set of mesh steps $\{H_l\}_{l \in \{1, 2, \dots, m+1\}}$, set $\eta_l = C(H_l)^2$, for every $l \in \{1, 2, \dots, m\}$, we have:

$$\begin{aligned} & \|v_\rightarrow^{H_l, t} - v_\rightarrow^t\|_\infty \leq C(H_l)^2, \\ & \|v_\leftarrow^{H_l, t} - v_\leftarrow^t\|_\infty \leq C(H_l)^2, \end{aligned}$$

for every $l \in \{1, 2, \dots, m+1\}$.

Proof. Under Assumption 5.2, we are allowed to ignore the propagation error, so that the total error is only the projection error, by Akian et al. (2008), which, by (Lakhoua, 2007, Coro. 69), is of order $(H_l)^2$ for each level- l , under assumption 5.3. \square

Theorem 5.1 indeed give us an upper bound for choosing the parameters η_l , depending on the parameters H_l . Let us plug this relationship between η_l and H_l into (30), and use the result of proposition 5.3 under the Assumption 5.1, we have

$$\begin{aligned} & \mathcal{C}_{spa}(\{H_l\}_{l=1, \dots, m+1}) \\ & \leq O\left((H_1)^{-d} + C^{d-1} \sum_{l=2}^m ((H_{l-1})^{d-1} (H_l)^{-d})\right). \end{aligned}$$

Suppose now we want to have a final error in the order of ε , then we need to take $H_{m+1} = \mathcal{O}(\varepsilon^{\frac{1}{2}})$. Once H_{m+1} is fixed, \mathcal{C}_{spa} is a convex function w.r.t. $\{H_l\}_{l=1, \dots, m}$. We also notice that, up to a multiplicative factor, the computational complexity, in our oracle model, is the same as space complexity. Then, we have the following main result for the computational complexity of our algorithm:

Theorem 5.2. Assume Assumption 5.1, and take the same condition as in Theorem 5.1, in order to get an error $\mathcal{O}(\varepsilon)$:

- i. We shall take $H_m = C(\frac{1}{\varepsilon})^{\frac{1}{2}}$, and $H_l = C(H_m)^{\frac{1}{m}}$ for all $l \in \{1, 2, \dots, m-1\}$. In this case, the total computational complexity of our m -level method, expressed in the oracle model, is bounded by $(\frac{1}{\varepsilon})^{\frac{d}{2m}}$.
- ii. Set $N = \lceil \frac{1}{2} |d \log(\varepsilon)| \rceil$, and take $\{H_l\}_{l \in \{1, 2, \dots, m+1\}}$ as proposed in i., then the total computational complexity reduces to $\mathcal{O}(C^d |d \log(\varepsilon)|)$.

Sketch of Proof. To get a final error ε , by the result of Theorem 5.1, we need to take $H_m = (\frac{1}{\varepsilon})^{\frac{1}{2}}$. We notice that when H_m is fixed, C_{spa} is a convex function w.r.t. each H_l . Then, by taking it's minimum w.r.t. each H_l we obtain the result of i. Substituting these values into C_{spa} , further taking the minimum of C_{spa} w.r.t. m , we obtain the result of ii. \square

6. NUMERICAL EXPERIMENTS

We applied our algorithm to a simple example, in which the value function is known, so that the final approximation error can be computed exactly: the linear-quadratic control problem.

Consider the problem (1) with $U = \mathbb{R}^d$ and $X = [-5, 5]^d$, the running cost $\ell(x, u) = -\|x\|^2 - \frac{1}{2}\|u\|^2$, dynamics $f(x, u) = u$, initial and final cost functions $\phi_0(x) = -\frac{1}{2}\|x - x_0\|^2$, $\phi_T(x) = -\|x - x_T\|^2$ with $x_0 = (-3, \dots, -3)$ and $x_T = (3, \dots, 3)$. The time horizon is $T = 5$ and is discretized with the time step $\delta = 0.5$.

For our algorithm, we choose quadratic basis functions and test functions with $c = 10$, centered at the points of regular grids and we do two tests. In both, we count the number of discretization points, and so the number of basis functions, for our algorithm. These results have to be compared with the number of basis functions necessary for max-plus method of Akian et al. (2008), or the number of grid points of the grid-based methods.

For the first test, we fix the final grid mesh h to 0.2, so that the final precision is in $\mathcal{O}(0.04)$, we first show the number of max-plus basis functions, when the dimension varies from 2 to 4, we give for comparison the number of grid points for an ordinary finite-difference based method:

dimension d	2	3	4
# basis functions	678	5280	46500
# ordinary grid points	$\mathcal{O}(10^5)$	$\mathcal{O}(10^8)$	$\mathcal{O}(10^{10})$

For the second test, we fix the dimension d to 3, and make the final grid mesh vary from 0.5 to 0.02:

mesh step h	0.5	0.2	0.05	0.02
# basis functions	3170	5280	22490	38970
# ordinary grid points	$\mathcal{O}(10^6)$	$\mathcal{O}(10^8)$	$\mathcal{O}(10^{10})$	$\mathcal{O}(10^{11})$

The algorithm has been implemented in MATLAB with some functions written in C++, and is executed on a single core of a IntelCore I7 at 2.3Gh with 16Gb RAM. We tried different kinds of linear-quadratic control problems, changing the costs and dynamics, and the numerical tests showed results similar to the above tables. We also observed a similar growth rate of the CPU time. In all cases, the CPU time for a 4-dimensional problem, with a final grid mesh size 0.2, discretizing the space $[-5, 5]^4$, was approximately of 80 seconds. Whereas the code is not fully optimized, the computational speed already outper-

forms standard grid-based methods. The tables show in particular that the number of basis function grows moderately with the precision, consistently with the estimate of Theorem 5.2, ii.

7. CONCLUSION

We introduced a new approximation method in optimal control, combining max-plus techniques (approximation of the value function by suprema of elementary functions) and dynamic grid refinements around optimal trajectories. This enable us to reduce the search space. In fact, under regularity assumptions, the grid size needed to obtain a ε -approximation grows linearly in $|\log \varepsilon|$, for a fixed dimension. We presented a first implementation, on a toy example, which already shows a considerable speedup by comparison with grid-based methods. We plan to refine the implementation and provide more systematic tests in future work.

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