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# ON A DISCONTINUITY CAPTURING FINITE VOLUMES METHOD THAT CAN SOLVE FULLY INCOMPRESSIBLE FLOWS

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## 1. Context and Aim

**Nuclear thermal hydraulics** : simulation for safety in nuclear reactors is based on multiphase flows models (liquid, vapor), **Low-Mach flows**, quasi-incompressible or totally **incompressible** flows (historically approximated by staggered schemes [2]). However, the presence of two phases can generate **discontinuities** (treated classically with Riemann Solvers [5] [1]).

↔ In need of a scheme that can solve incompressible flows and capturing discontinuities.

Multiphase models are very complicated, we started from Euler barotropic equations :

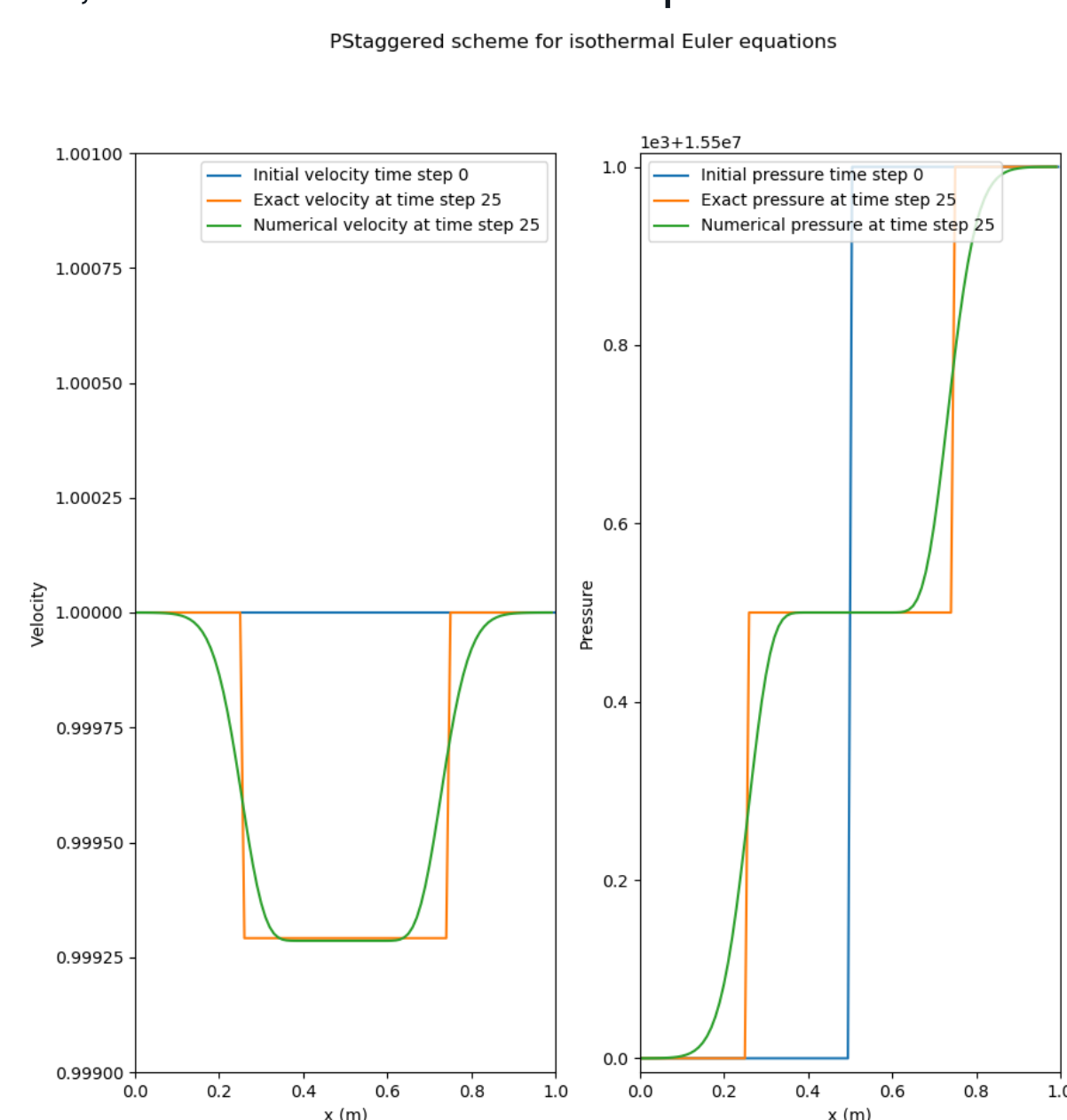
$$\begin{cases} \partial_t \rho + \nabla \cdot \mathbf{q} = 0 \\ \partial_t \mathbf{q} + \nabla \cdot \left( \frac{\mathbf{q} \otimes \mathbf{q}}{\rho} \right) + \nabla p = \mathbf{f} \\ p = g(\rho) \end{cases} \quad (1)$$

with  $\rho$  density of the fluid,  $\mathbf{q}$  its momentum,  $p$  the pressure and  $\mathbf{f}$  an external force

## 3. Numerical results

### Riemann Problem : computation of discontinuities

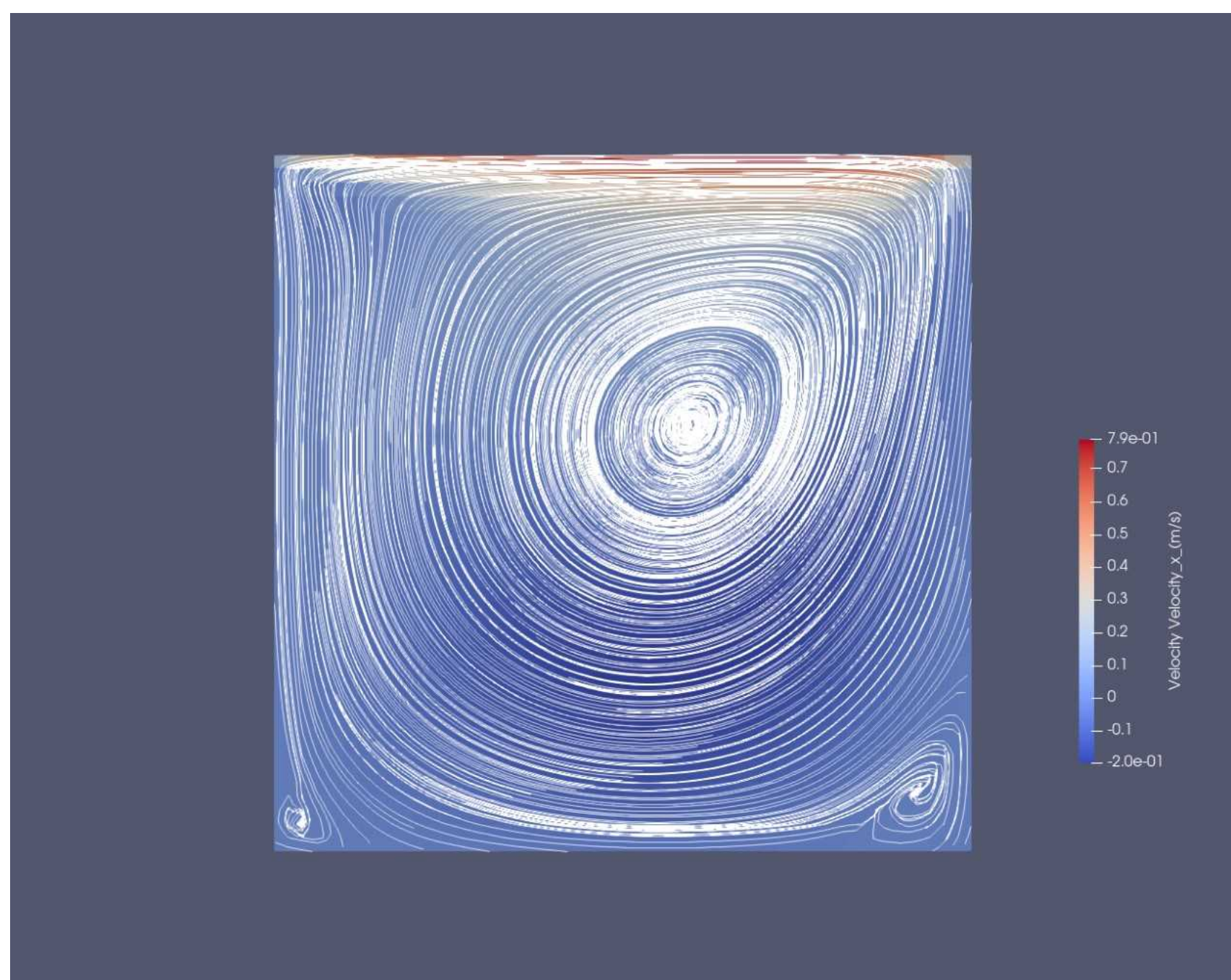
Numerical tests description : 1D problem, initial condition is composed of two constant state separated by a discontinuity, fluid is water.



rated by a discontinuity, fluid is water.

### Lid Driven Cavity : Low Mach 2D

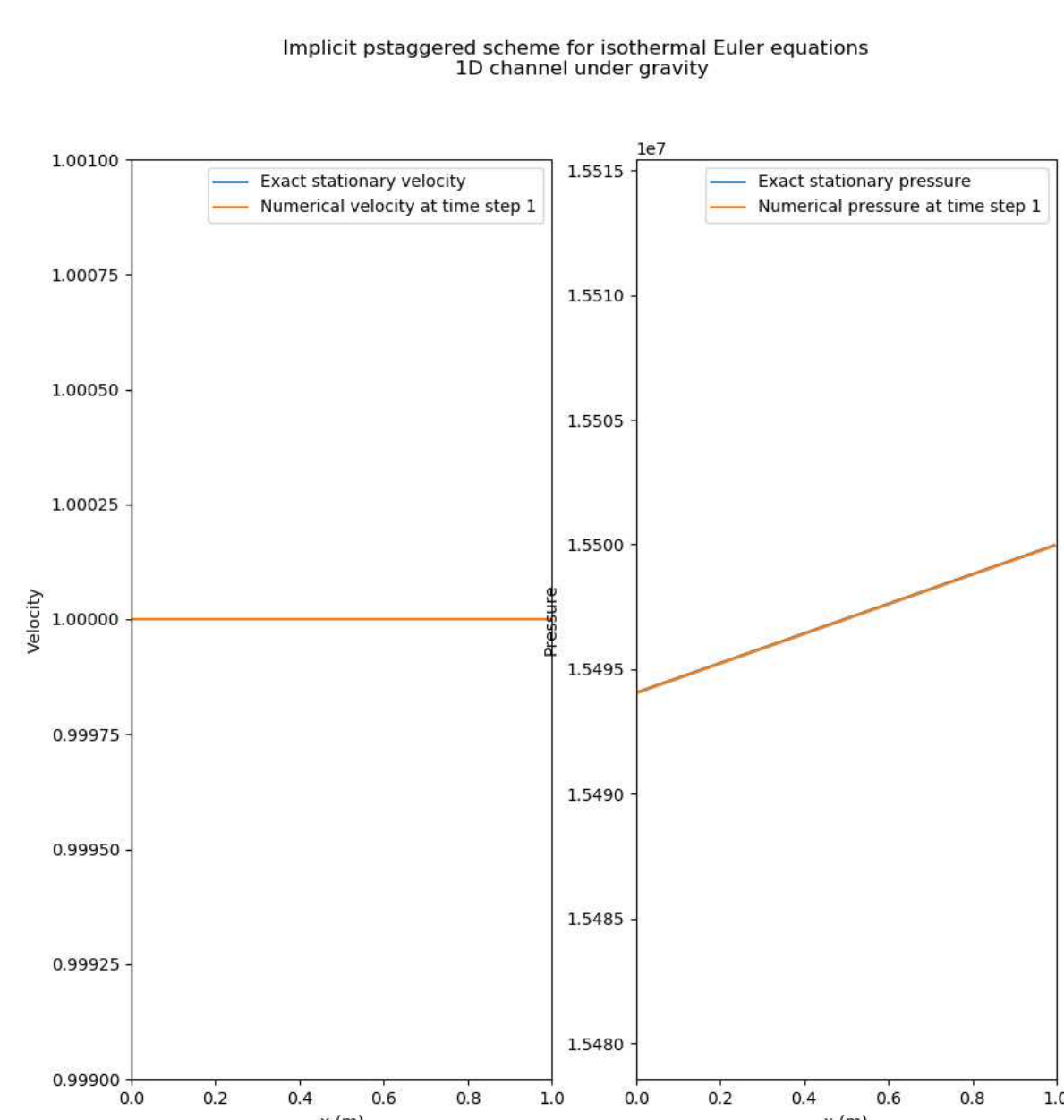
Numerical tests description :  $\mathbf{u} = 1\text{m/s}$  on the top, and equal to 0 on the other sides on a 50 x 50 cells cartesian mesh. Fluid is water again.



We converged to a stationary state with **viscosity** that was also implemented in the manner of the previous paragraph, we can observe the formation of vortices, which is not guaranteed by classical schemes at Low Mach numbers

### Stationary with gravity : Incompressible 1D

Numerical tests description : incompressible flow, and stationary test case, with the external force  $\mathbf{f} = \rho g$  being the gravity.



## 2. Process

The two main ingredients are :

### Numerical Dissipation of staggered schemes

We recall where the expression "numerical dissipation" comes from. For a two point semi-discretization

$$\partial_t \mathbf{U} + \partial_x \mathcal{F}(\mathbf{U}) = 0 \quad \text{becomes} \quad \mathbf{U}'_i(t) + \frac{F_{i,i+1} - F_{i-1,i}}{\Delta x} = 0 \quad (2)$$

where the discrete flux is defined as

$$F_{i,j} = \frac{\mathcal{F}(\mathbf{U}_i) + \mathcal{F}(\mathbf{U}_j)}{2} - \mathcal{D} \frac{\mathbf{U}_i - \mathbf{U}_j}{2}$$

so that, when  $\Delta x \rightarrow 0$  :  $\partial_t \mathbf{U} + \partial_x \mathcal{F}(\mathbf{U}) = \Delta x \mathcal{D} \partial_{xx} \mathbf{U} + \mathcal{O}((\Delta x)^2)$

$\mathcal{D}$  is the **numerical dissipation operator**.

### Example on wave system : Godunov scheme

$$\begin{cases} \frac{d}{dt} \rho_i + \frac{(q_{i+1} - q_{i-1})}{2\Delta x} = \frac{c(\rho_{i+1} - 2\rho_i + \rho_{i-1})}{2\Delta x} \\ \frac{d}{dt} q_i + \frac{c^2(\rho_{i+1} - \rho_{i-1})}{2\Delta x} = \frac{c(q_{i+1} - 2q_i + q_{i-1})}{2\Delta x}, \end{cases} \quad \mathbf{U}_i := (\rho_i, q_i)^t \text{ and } \mathcal{D} := \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \quad (3)$$

### Example on wave equation : Staggered scheme

$$\begin{cases} \Delta x \frac{d}{dt} \rho_i + (q_{i+1/2} - q_{i-1/2}) = 0 \\ \Delta x \frac{d}{dt} q_{i+1/2} + c^2(\rho_{i+1} - \rho_i) = 0 \end{cases} \quad \mathbf{U}_i := (\rho_i, q_{i+1/2})^t \text{ and } \mathcal{D} := \begin{pmatrix} 0 & 1 \\ -c^2 & 0 \end{pmatrix}$$

### Primitive variables

In the incompressible regime the density is constant by definition. **It cannot be an unknown of the problem**. We need to replace the density by the pressure. Instead of solving (1) we solve the following system :

$$\begin{cases} \partial_t (\rho(p)) + \nabla \cdot (\rho(p) \mathbf{u}) = 0 \\ \partial_t (\rho(p) \mathbf{u}) + \nabla \cdot (\rho(p) \mathbf{u} \otimes \mathbf{u}) + \nabla p = \mathbf{f} \end{cases} \quad (4)$$

### Putting it together

Putting (4) into the form of (2) we get the following discretization,

$$\mathbf{U}(\mathbf{V}_i)'(t) + \frac{G_{i,i+1} - G_{i-1,i}}{\Delta x} = 0, \quad \mathbf{U}(\mathbf{V}_i) := (\rho(p_i), \mathbf{q}_i)^t, \quad \mathbf{V}_i := (p_i, \mathbf{u}_i)^t$$

where the discrete flux is define as

$$G_{i,j} = \frac{\mathcal{F}(\mathbf{U}(\mathbf{V}_i)) + \mathcal{F}(\mathbf{U}(\mathbf{V}_j))}{2} - \mathcal{D} \frac{\mathbf{V}_i - \mathbf{V}_j}{2}, \quad \mathcal{F}(\mathbf{U}(\mathbf{V}_i)) := \begin{pmatrix} \rho(p_i) \mathbf{u}_i \\ \rho(p_i) \mathbf{u}_i \otimes \mathbf{u}_i + p_i \mathbb{I}_d \end{pmatrix},$$

with the numerical dissipation extracted from classical staggered schemes [4] (in multi-D) (we differ from [3] for example)

$$\mathcal{D} = \text{sign}(\mathbf{u}_{ij} \cdot \mathbf{n}) \begin{pmatrix} \frac{\mathbf{u}_{ij} \cdot \mathbf{n}}{c_{ij}^2} & \rho(p)^t \mathbf{n} \\ -\mathbf{n} - \frac{(\mathbf{u}_{ij} \cdot \mathbf{n})}{c_{ij}^2} \mathbf{u}_{ij} & \rho(p)(\mathbf{u}_{ij} \otimes \mathbf{n} + (\mathbf{u}_{ij} \cdot \mathbf{n}) \mathbb{I}_d) \end{pmatrix}$$

with  $\mathbf{u}_{ij} := \frac{\mathbf{u}_i \sqrt{\rho(p_i)} + \mathbf{u}_j \sqrt{\rho(p_j)}}{\sqrt{\rho(p_i)} + \sqrt{\rho(p_j)}}$  and  $c_{ij} := \frac{p_i - p_j}{\rho(p_i) - \rho(p_j)}$  if  $\rho(p_i) \neq \rho(p_j)$  and  $p'(\rho(p_i))$  else.

Finally, we integrate the time **implicitly** we solve the non-linear system with a Newton scheme

## 4. Conclusions

- Computations of discontinuities are very good
- Low-mach regime is handled effectively
- Incompressible regime can be solved with the same scheme but the linear solver has trouble when the system is singular (Newmann and Wall only boundary conditions)

## 5. References

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