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ON A DISCONTINUITY CAPTURING FINITE VOLUMES METHOD THAT CAN SOLVE FULLY INCOMPRESSIBLE FLOWS

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1. Context and Aim

Nuclear thermal hydraulics : simulation for safety in nuclear reactors is based on multiphase flows models (liquid, vapor), **Low-Mach flows**, quasi-incompressible or totally **incompressible** flows (historically approximated by staggered schemes [2]). However, the presence of two phases can generate **discontinuities** (treated classically with Riemann Solvers [5] [1]).

↔ In need of a scheme that can solve incompressible flows and capturing discontinuities.

Multiphase models are very complicated, we started from Euler barotropic equations :

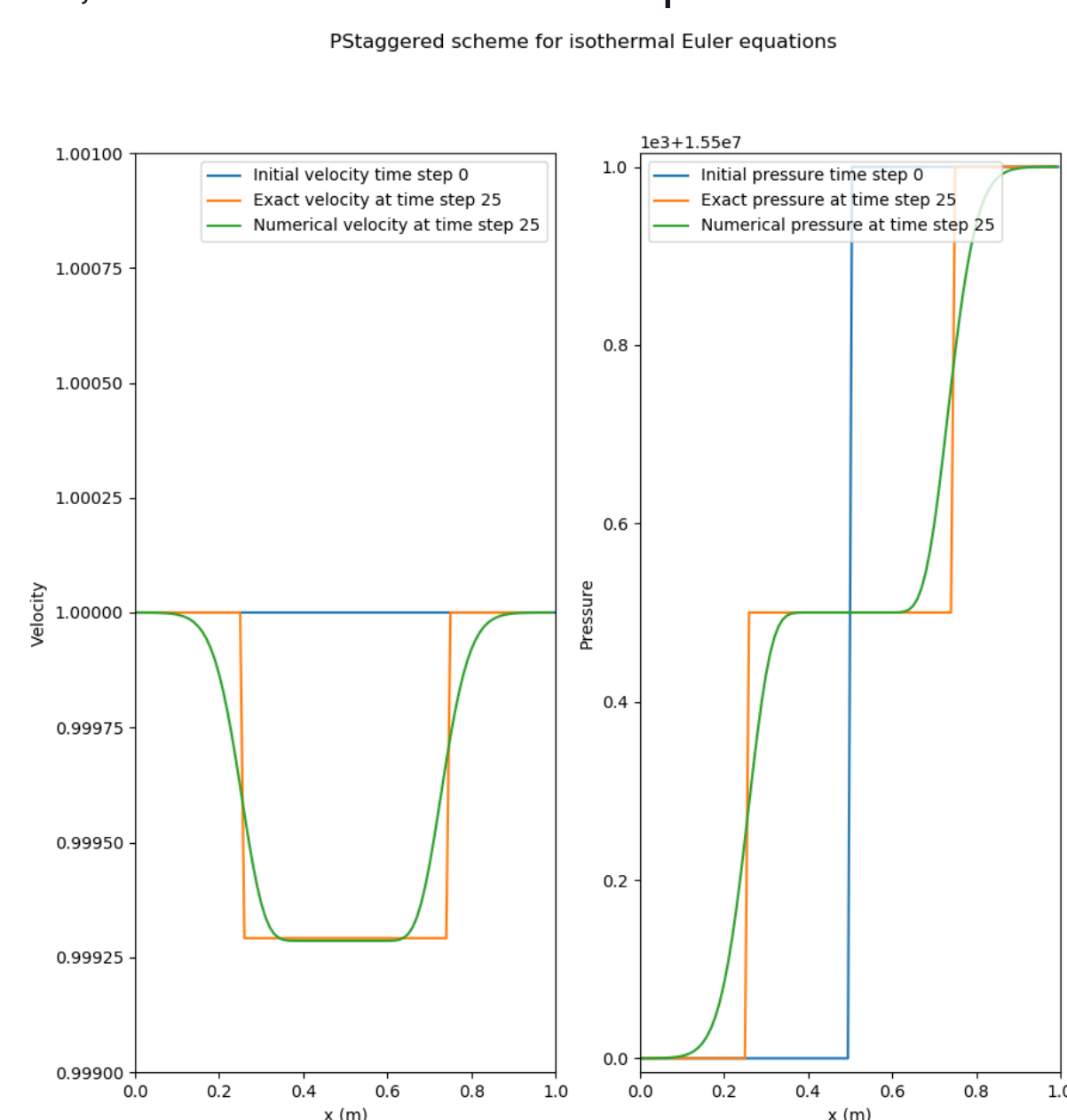
$$\begin{cases} \partial_t \rho + \nabla \cdot \mathbf{q} = 0 \\ \partial_t \mathbf{q} + \nabla \cdot \left(\frac{\mathbf{q} \otimes \mathbf{q}}{\rho} \right) + \nabla p = \mathbf{f} \\ p = g(\rho) \end{cases} \quad (1)$$

with ρ density of the fluid, \mathbf{q} its momentum, p the pressure and \mathbf{f} an external force

3. Numerical results

Riemann Problem : computation of discontinuities

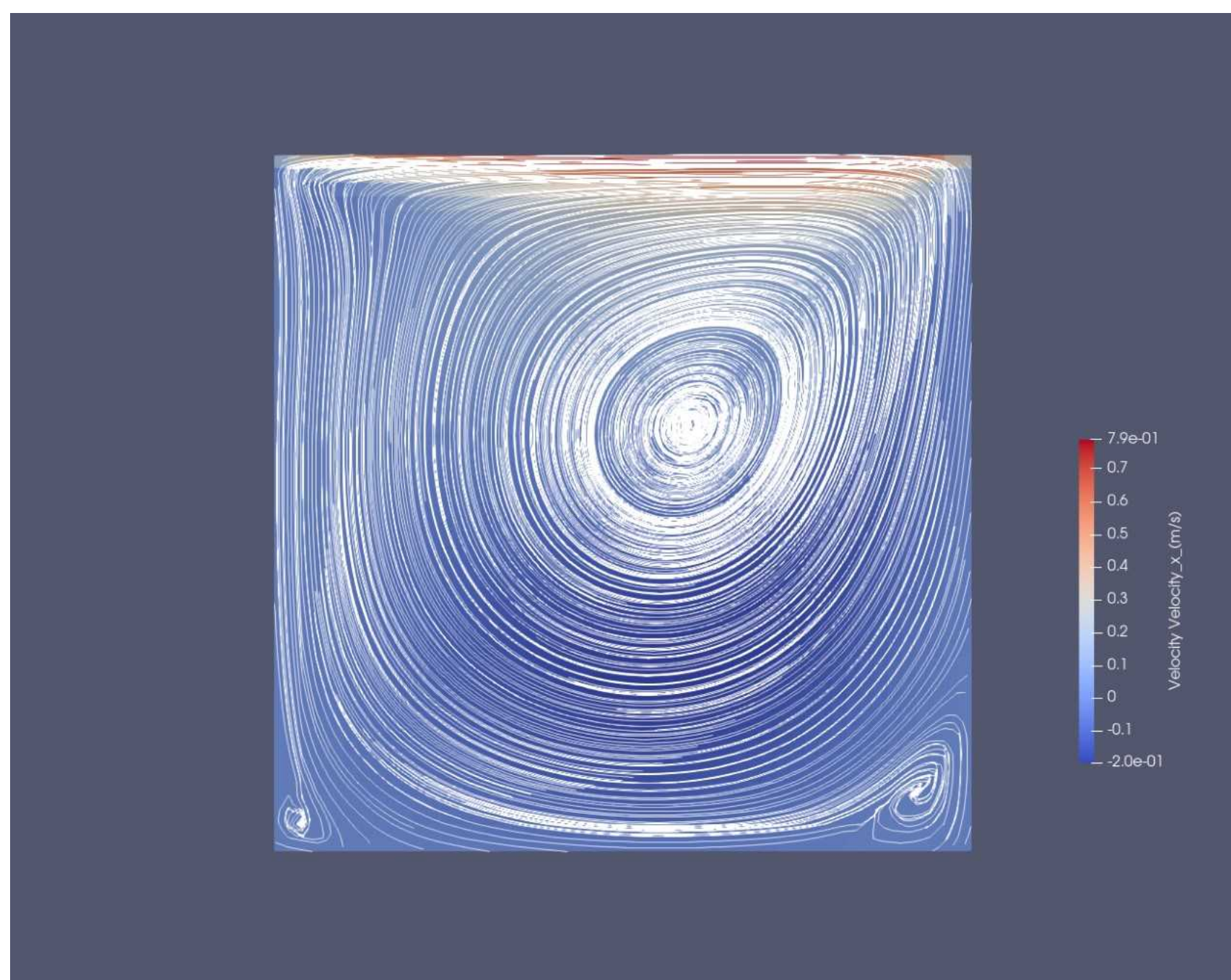
Numerical tests description : 1D problem, initial condition is composed of two constant state separated by a discontinuity, fluid is water.



rated by a discontinuity, fluid is water.

Lid Driven Cavity : Low Mach 2D

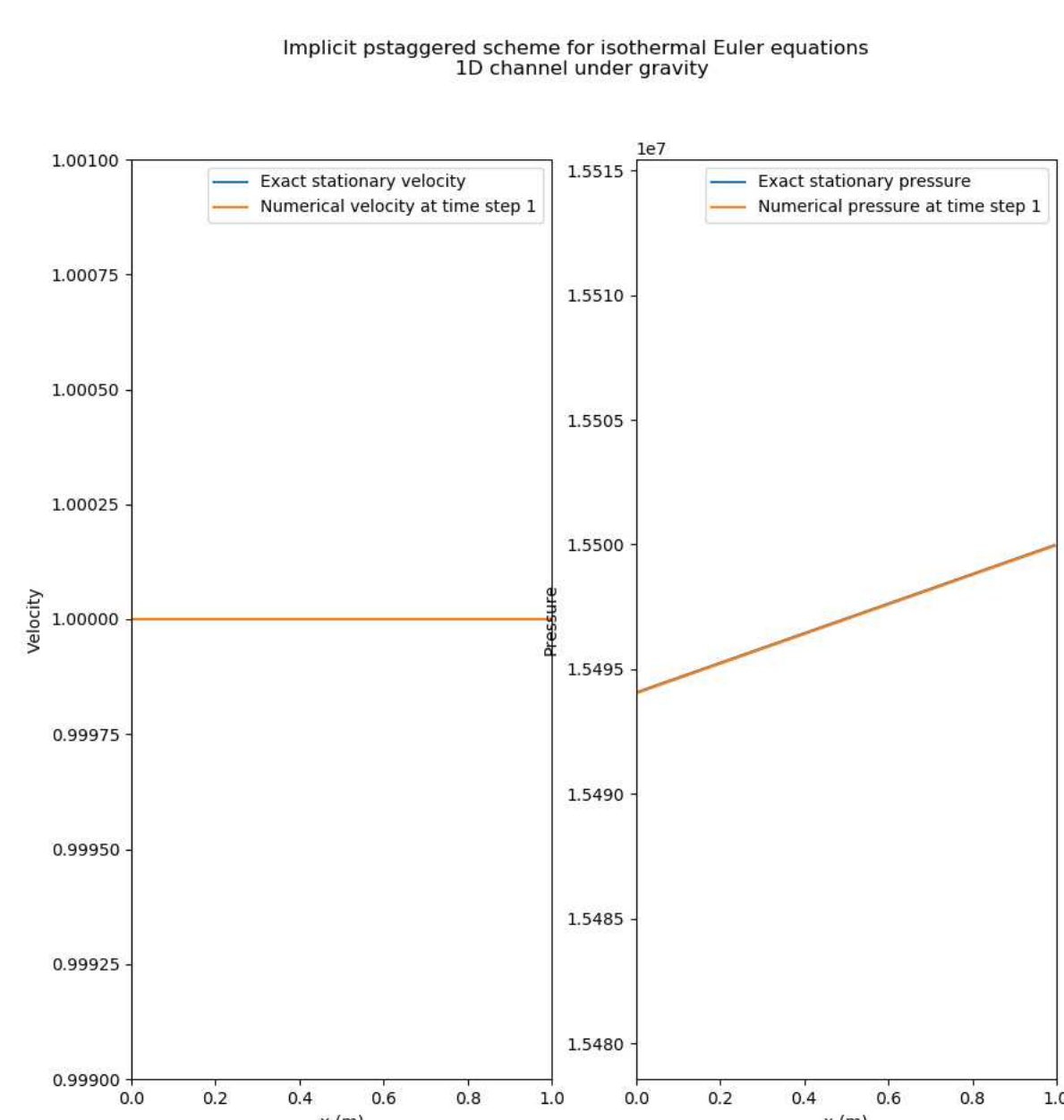
Numerical tests description : $\mathbf{u} = 1\text{m/s}$ on the top, and equal to 0 on the other sides on a 50 x 50 cells cartesian mesh. Fluid is water again.



We converged to a stationary state with **viscosity** that was also implemented in the manner of the previous paragraph, we can observe the formation of vortices, which is not guaranteed by classical schemes at Low Mach numbers

Stationary with gravity : Incompressible 1D

Numerical tests description : incompressible flow, and stationary test case, with the external force $\mathbf{f} = \rho\mathbf{g}$ being the gravity.



2. Process

The two main ingredients are :

Numerical Dissipation of staggered schemes

We recall where the expression "numerical dissipation" comes from. For a two point semi-discretization

$$\partial_t \mathbf{U} + \partial_x \mathcal{F}(\mathbf{U}) = 0 \quad \text{becomes} \quad \mathbf{U}'_i(t) + \frac{F_{i,i+1} - F_{i-1,i}}{\Delta x} = 0 \quad (2)$$

where the discrete flux is defined as

$$F_{i,j} = \frac{\mathcal{F}(\mathbf{U}_i) + \mathcal{F}(\mathbf{U}_j)}{2} - \mathcal{D} \frac{\mathbf{U}_i - \mathbf{U}_j}{2}$$

so that, when $\Delta x \rightarrow 0$: $\partial_t \mathbf{U} + \partial_x \mathcal{F}(\mathbf{U}) = \Delta x \mathcal{D} \partial_{xx} \mathbf{U} + \mathcal{O}((\Delta x)^2)$

\mathcal{D} is the **numerical dissipation operator**.

Example on wave system : Godunov scheme

$$\begin{cases} \frac{d}{dt} \rho_i + \frac{(q_{i+1} - q_{i-1})}{2\Delta x} = \frac{c(\rho_{i+1} - 2\rho_i + \rho_{i-1})}{2\Delta x} \\ \frac{d}{dt} q_i + \frac{c^2(\rho_{i+1} - \rho_{i-1})}{2\Delta x} = \frac{c(q_{i+1} - 2q_i + q_{i-1})}{2\Delta x} \end{cases}, \quad \mathbf{U}_i := (\rho_i, q_i)^t \text{ and } \mathcal{D} := \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \quad (3)$$

Example on wave equation : Staggered scheme

$$\begin{cases} \Delta x \frac{d}{dt} \rho_i + (q_{i+1/2} - q_{i-1/2}) = 0 \\ \Delta x \frac{d}{dt} q_{i+1/2} + c^2(\rho_{i+1} - \rho_i) = 0 \end{cases}, \quad \mathbf{U}_i := (\rho_i, q_{i+1/2})^t \text{ and } \mathcal{D} := \begin{pmatrix} 0 & 1 \\ -c^2 & 0 \end{pmatrix}$$

Primitive variables

In the incompressible regime the density is constant by definition. **It cannot be an unknown of the problem**. We need to replace the density by the pressure. Instead of solving (1) we solve the following system :

$$\begin{cases} \partial_t (\rho(p)) + \nabla \cdot (\rho(p)\mathbf{u}) = 0 \\ \partial_t (\rho(p)\mathbf{u}) + \nabla \cdot (\rho(p)\mathbf{u} \otimes \mathbf{u}) + \nabla p = \mathbf{f} \end{cases} \quad (4)$$

Putting it together

Putting (4) into the form of (2) we get the following discretization,

$$\mathbf{U}(\mathbf{V}_i)'(t) + \frac{G_{i,i+1} - G_{i-1,i}}{\Delta x} = 0, \quad \mathbf{U}(\mathbf{V}_i) := (\rho(p_i), \mathbf{q}_i)^t, \quad \mathbf{V}_i := (p_i, \mathbf{u}_i)^t$$

where the discrete flux is define as

$$G_{i,j} = \frac{\mathcal{F}(\mathbf{U}(\mathbf{V}_i)) + \mathcal{F}(\mathbf{U}(\mathbf{V}_j))}{2} - \mathcal{D} \frac{\mathbf{V}_i - \mathbf{V}_j}{2}, \quad \mathcal{F}(\mathbf{U}(\mathbf{V}_i)) := \begin{pmatrix} \rho(p_i)\mathbf{u}_i \\ \rho(p_i)\mathbf{u}_i \otimes \mathbf{u}_i + p_i \mathbb{I}_d \end{pmatrix},$$

with the numerical dissipation extracted from classical staggered schemes [4] (in multi-D) (we differ from [3] for example)

$$\mathcal{D} = \text{sign}(\mathbf{u}_{ij} \cdot \mathbf{n}) \begin{pmatrix} \frac{\mathbf{u}_{ij} \cdot \mathbf{n}}{c_{ij}^2} & \rho(p)^t \mathbf{n} \\ -\mathbf{n} - \frac{(\mathbf{u}_{ij} \cdot \mathbf{n})}{c_{ij}^2} \mathbf{u}_{ij} & \rho(p)(\mathbf{u}_{ij} \otimes \mathbf{n} + (\mathbf{u}_{ij} \cdot \mathbf{n}) \mathbb{I}_d) \end{pmatrix}$$

with $\mathbf{u}_{ij} := \frac{\mathbf{u}_i \sqrt{\rho(p_i)} + \mathbf{u}_j \sqrt{\rho(p_j)}}{\sqrt{\rho(p_i)} + \sqrt{\rho(p_j)}}$ and $c_{ij} := \frac{p_i - p_j}{\rho(p_i) - \rho(p_j)}$ if $\rho(p_i) \neq \rho(p_j)$ and $p'(\rho(p_i))$ else.

Finally, we integrate the time **implicitly** we solve the non-linear system with a Newton scheme

4. Conclusions

- Computations of discontinuities are very good
- Low-mach regime is handled effectively
- Incompressible regime can be solved with the same scheme but the linear solver has trouble when the system is singular (Newmann and Wall only boundary conditions)

5. References

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