



HAL
open science

On a discontinuity capturing Finite Volume method that can solve fully incompressible flows

Esteban Coiffier, Michael Ndjinga

► To cite this version:

Esteban Coiffier, Michael Ndjinga. On a discontinuity capturing Finite Volume method that can solve fully incompressible flows. SMAI 2023 - Congrès de la Société des Mathématiques Appliquées et Industrielles, May 2023, Le Gosier, Guadeloupe, France. hal-04397820

HAL Id: hal-04397820

<https://inria.hal.science/hal-04397820>

Submitted on 16 Jan 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

ON A DISCONTINUITY CAPTURING FINITE VOLUMES METHOD THAT CAN SOLVE FULLY INCOMPRESSIBLE FLOWS

Esteban Coiffier^{1,2}, Michael Ndjinga¹

¹CEA Saclay ISAS/DM2S/STMF/LMEC, France ²Pau University, France



1. Context and Aim

Nuclear thermal hydraulics : simulation for safety in nuclear reactors is based on multiphase flows models (liquid, vapor), **Low-Mach flows**, quasi-incompressible or totally **incompressible** flows (historically approximated by staggered schemes [2]). However, the presence of two phases can generate **discontinuities** (treated classically with Riemann Solvers [5] [1]).

↔ In need of a scheme that can solve incompressible flows and capturing discontinuities.

Multiphase models are very complicated, we started from Euler barotropic equations :

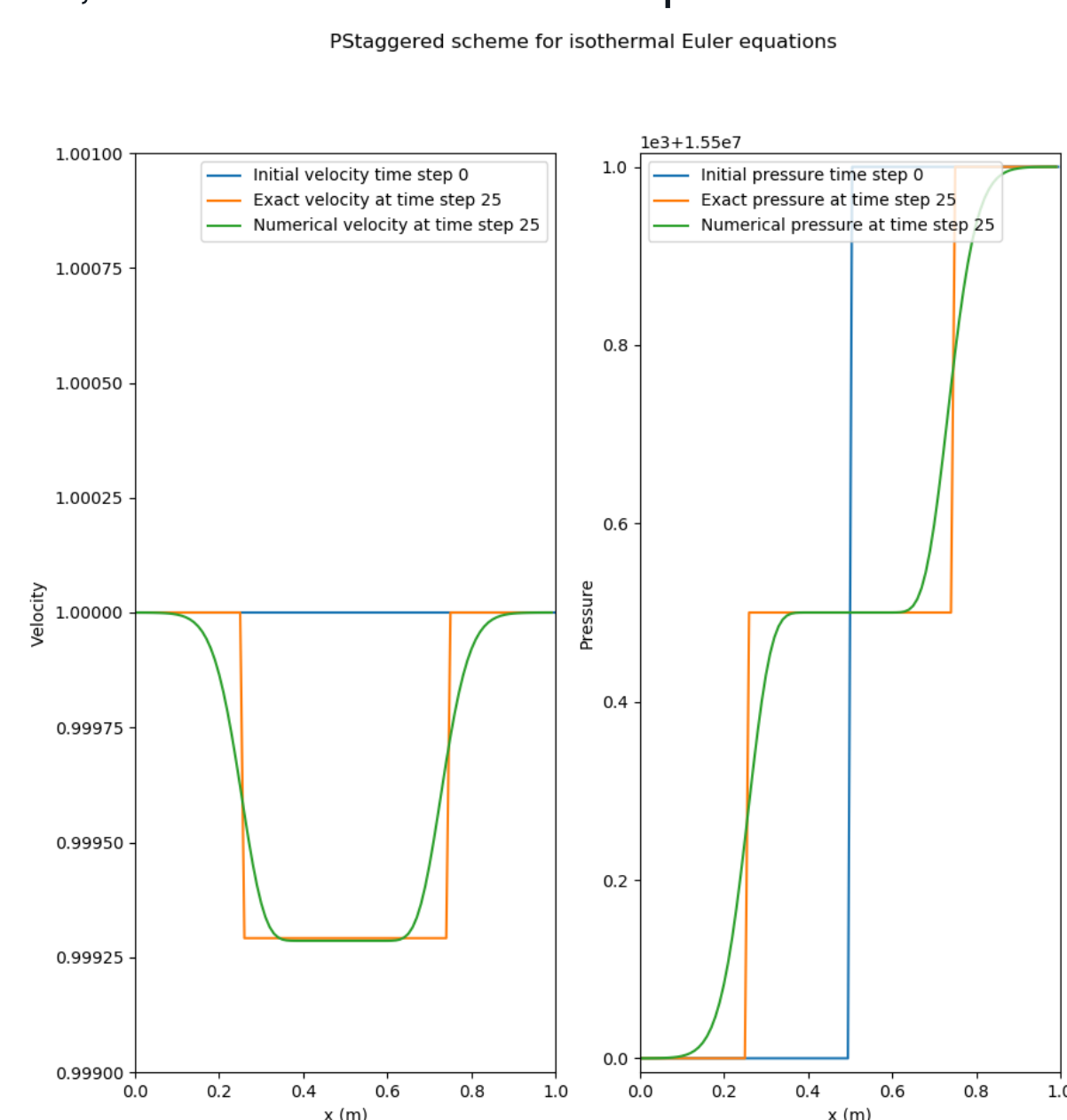
$$\begin{cases} \partial_t \rho + \nabla \cdot \mathbf{q} = 0 \\ \partial_t \mathbf{q} + \nabla \cdot \left(\frac{\mathbf{q} \otimes \mathbf{q}}{\rho} \right) + \nabla p = \mathbf{f} \\ p = g(\rho) \end{cases} \quad (1)$$

with ρ density of the fluid, \mathbf{q} its momentum, p the pressure and \mathbf{f} an external force

3. Numerical results

Riemann Problem : computation of discontinuities

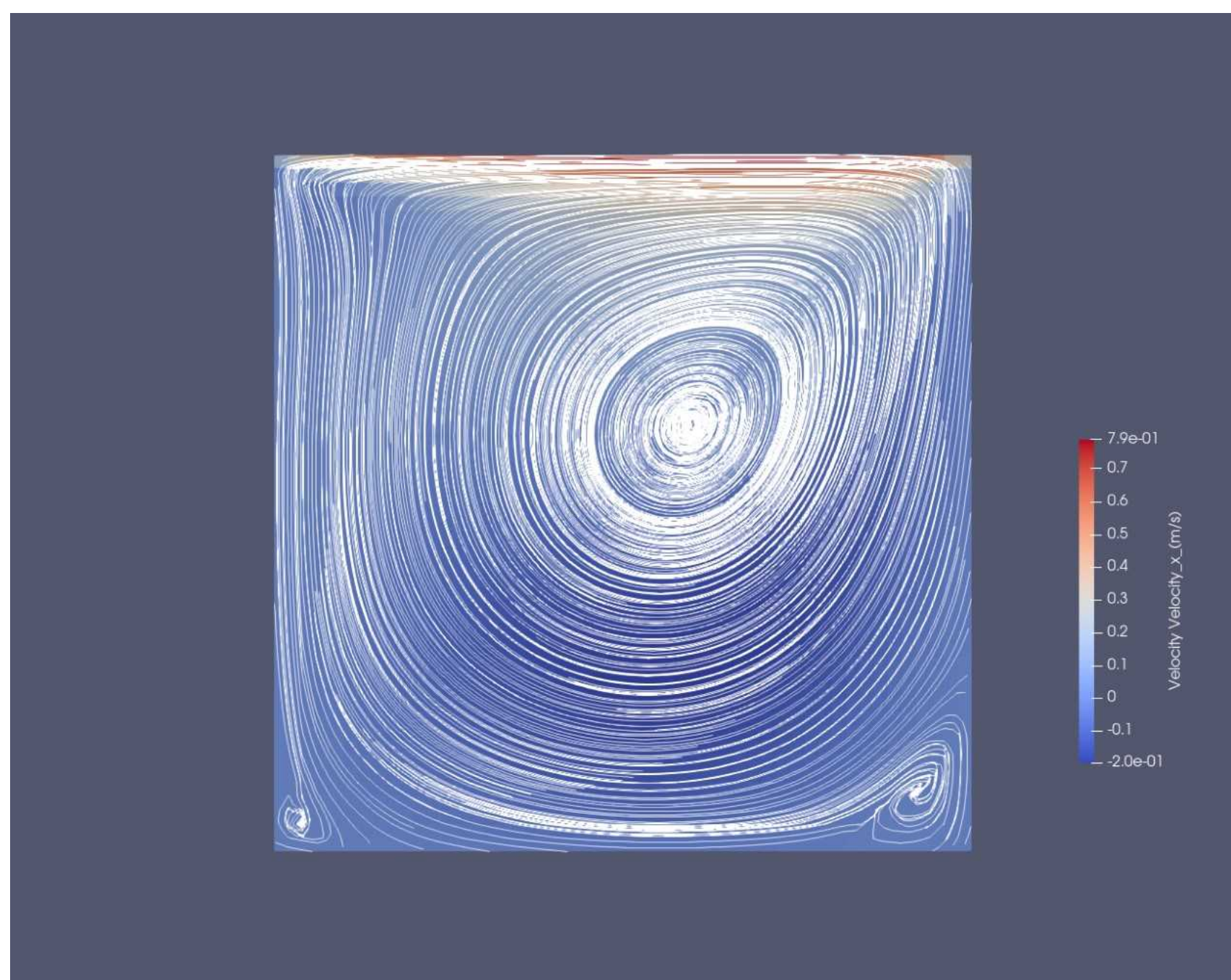
Numerical tests description : 1D problem, initial condition is composed of two constant state separated by a discontinuity, fluid is water.



rated by a discontinuity, fluid is water.

Lid Driven Cavity : Low Mach 2D

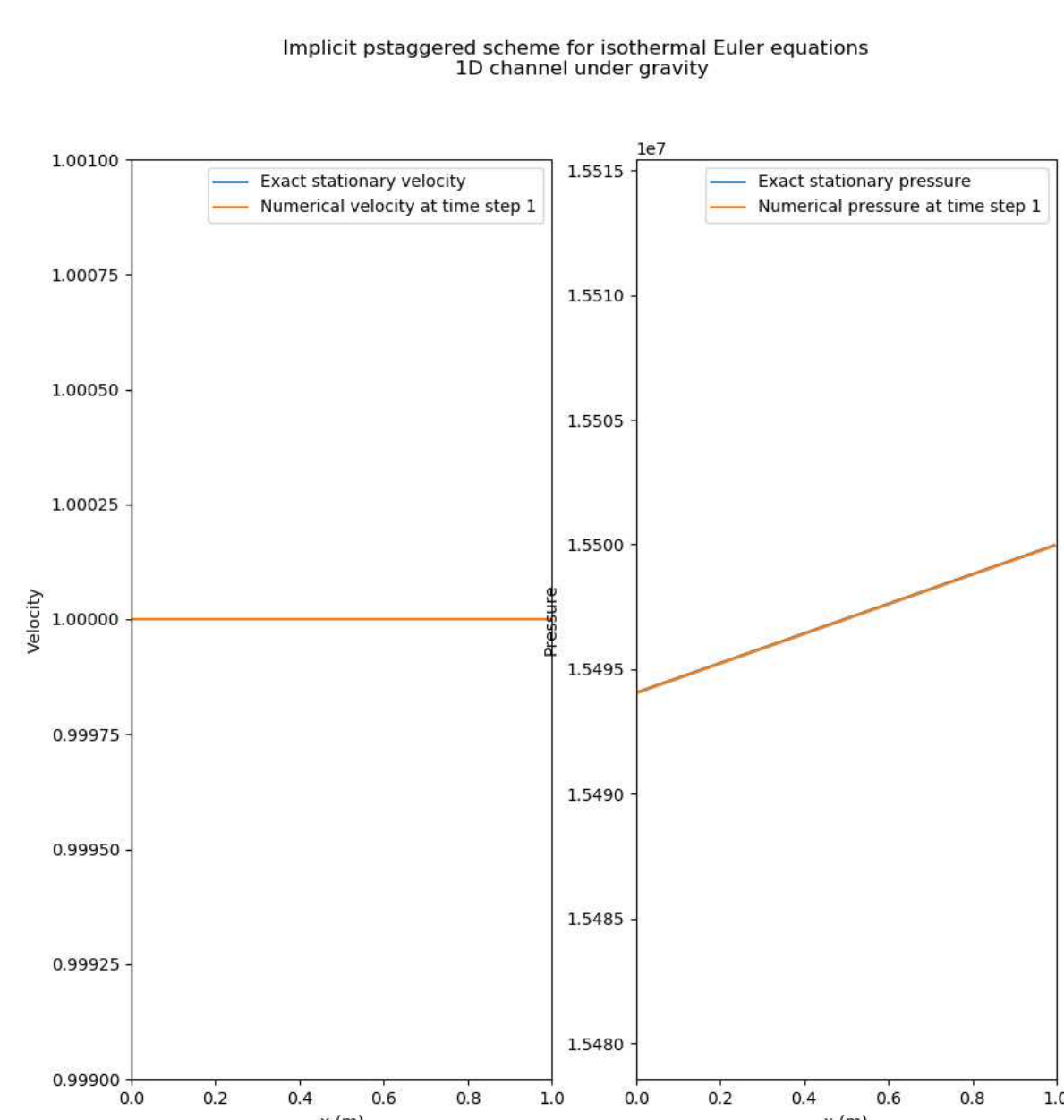
Numerical tests description : $\mathbf{u} = 1\text{m/s}$ on the top, and equal to 0 on the other sides on a 50 x 50 cells cartesian mesh. Fluid is water again.



We converged to a stationary state with **viscosity** that was also implemented in the manner of the previous paragraph, we can observe the formation of vortices, which is not guaranteed by classical schemes at Low Mach numbers

Stationary with gravity : Incompressible 1D

Numerical tests description : incompressible flow, and stationary test case, with the external force $\mathbf{f} = \rho g$ being the gravity.



2. Process

The two main ingredients are :

Numerical Dissipation of staggered schemes

We recall where the expression "numerical dissipation" comes from. For a two point semi-discretization

$$\partial_t \mathbf{U} + \partial_x \mathcal{F}(\mathbf{U}) = 0 \quad \text{becomes} \quad \mathbf{U}'_i(t) + \frac{F_{i,i+1} - F_{i-1,i}}{\Delta x} = 0 \quad (2)$$

where the discrete flux is defined as

$$F_{i,j} = \frac{\mathcal{F}(\mathbf{U}_i) + \mathcal{F}(\mathbf{U}_j)}{2} - \mathcal{D} \frac{\mathbf{U}_i - \mathbf{U}_j}{2}$$

so that, when $\Delta x \rightarrow 0$: $\partial_t \mathbf{U} + \partial_x \mathcal{F}(\mathbf{U}) = \Delta x \mathcal{D} \partial_{xx} \mathbf{U} + \mathcal{O}((\Delta x)^2)$

\mathcal{D} is the **numerical dissipation operator**.

Example on wave system : Godunov scheme

$$\begin{cases} \frac{d}{dt} \rho_i + \frac{(q_{i+1} - q_{i-1})}{2\Delta x} = \frac{c(\rho_{i+1} - 2\rho_i + \rho_{i-1})}{2\Delta x} \\ \frac{d}{dt} q_i + \frac{c^2(\rho_{i+1} - \rho_{i-1})}{2\Delta x} = \frac{c(q_{i+1} - 2q_i + q_{i-1})}{2\Delta x} \end{cases}, \quad \mathbf{U}_i := (\rho_i, q_i)^t \text{ and } \mathcal{D} := \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \quad (3)$$

Example on wave equation : Staggered scheme

$$\begin{cases} \Delta x \frac{d}{dt} \rho_i + (q_{i+1/2} - q_{i-1/2}) = 0 \\ \Delta x \frac{d}{dt} q_{i+1/2} + c^2(\rho_{i+1} - \rho_i) = 0 \end{cases}, \quad \mathbf{U}_i := (\rho_i, q_{i+1/2})^t \text{ and } \mathcal{D} := \begin{pmatrix} 0 & 1 \\ -c^2 & 0 \end{pmatrix}$$

Primitive variables

In the incompressible regime the density is constant by definition. **It cannot be an unknown of the problem**. We need to replace the density by the pressure. Instead of solving (1) we solve the following system :

$$\begin{cases} \partial_t (\rho(p)) + \nabla \cdot (\rho(p) \mathbf{u}) = 0 \\ \partial_t (\rho(p) \mathbf{u}) + \nabla \cdot (\rho(p) \mathbf{u} \otimes \mathbf{u}) + \nabla p = \mathbf{f} \end{cases} \quad (4)$$

Putting it together

Putting (4) into the form of (2) we get the following discretization,

$$\mathbf{U}(\mathbf{V}_i)'(t) + \frac{G_{i,i+1} - G_{i-1,i}}{\Delta x} = 0, \quad \mathbf{U}(\mathbf{V}_i) := (\rho(p_i), \mathbf{q}_i)^t, \quad \mathbf{V}_i := (p_i, \mathbf{u}_i)^t$$

where the discrete flux is define as

$$G_{i,j} = \frac{\mathcal{F}(\mathbf{U}(\mathbf{V}_i)) + \mathcal{F}(\mathbf{U}(\mathbf{V}_j))}{2} - \mathcal{D} \frac{\mathbf{V}_i - \mathbf{V}_j}{2}, \quad \mathcal{F}(\mathbf{U}(\mathbf{V}_i)) := \begin{pmatrix} \rho(p_i) \mathbf{u}_i \\ \rho(p_i) \mathbf{u}_i \otimes \mathbf{u}_i + p_i \mathbb{I}_d \end{pmatrix},$$

with the numerical dissipation extracted from classical staggered schemes [4] (in multi-D) (we differ from [3] for example)

$$\mathcal{D} = \text{sign}(\mathbf{u}_{ij} \cdot \mathbf{n}) \begin{pmatrix} \frac{\mathbf{u}_{ij} \cdot \mathbf{n}}{c_{ij}^2} & \rho(p)^t \mathbf{n} \\ -\mathbf{n} - \frac{(\mathbf{u}_{ij} \cdot \mathbf{n})}{c_{ij}^2} \mathbf{u}_{ij} & \rho(p)(\mathbf{u}_{ij} \otimes \mathbf{n} + (\mathbf{u}_{ij} \cdot \mathbf{n}) \mathbb{I}_d) \end{pmatrix}$$

with $\mathbf{u}_{ij} := \frac{\mathbf{u}_i \sqrt{\rho(p_i)} + \mathbf{u}_j \sqrt{\rho(p_j)}}{\sqrt{\rho(p_i)} + \sqrt{\rho(p_j)}}$ and $c_{ij} := \frac{p_i - p_j}{\rho(p_i) - \rho(p_j)}$ if $\rho(p_i) \neq \rho(p_j)$ and $p'(\rho(p_i))$ else.

Finally, we integrate the time **implicitly** we solve the non-linear system with a Newton scheme

4. Conclusions

- Computations of discontinuities are very good
- Low-mach regime is handled effectively
- Incompressible regime can be solved with the same scheme but the linear solver has trouble when the system is singular (Newmann and Wall only boundary conditions)

5. References

- [1] Sergej K Godunov and I Bohachevsky. "Finite difference method for numerical computation of discontinuous solutions of the equations of fluid dynamics". In: *Matematičeskij sbornik* 47.3 (1959), pp. 271–306.
- [2] Francis H Harlow and J Eddie Welch. "Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface". In: *The physics of fluids* 8.12 (1965), pp. 2182–2189.
- [3] Raphael Herbin, Walid Kheriji, and J-C Latché. "On some implicit and semi-implicit staggered schemes for the shallow water and Euler equations". In: *ESAIM: Mathematical Modelling and Numerical Analysis* 48.6 (2014), pp. 1807–1857.
- [4] Andrea Prosperetti and Grétar Tryggvason. *Computational methods for multiphase flow*. Cambridge university press, 2009.
- [5] Philip L Roe. "Approximate Riemann solvers, parameter vectors, and difference schemes". In: *Journal of computational physics* 43.2 (1981), pp. 357–372.