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Transient Analysis of Markov Chains by Controlled Probability Mass Selection

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(Virtual)

April 1-2, 2023

- 1 Motivation
- 2 Background: CTMC Transient Analysis by Uniformization
- 3 Error Description/Analysis
- 4 Our Proposal to Improve the Uniformization Method
- 5 Conclusions

- **Continuous-time Markov chains** (CTMCs) are widely used to model real-world systems and processes in many areas such as (amongst many others)
 - engineering and environmental processes
 - computer and communication networks
 - production and manufacturing systems
 - healthcare systems
 - epidemic diseases
 - biochemically reacting systems
- Often the interest is in **transient (time dependent) system behavior** or performance properties, respectively, and the analysis requires the determination of transient probability distributions

Uniformization, also known as **randomization** or **Jensen's method**, provides a numerically stable computational scheme to approximate the **transient probability distributions** of CTMCs with uniformly bounded transition rates

- Reliability/Availability analysis usually involves systems that were design for high availability: system is working [properly] most of the time.
- The state space could be seen as a partition with "working" states and "failure" states
- In these systems, most of the probability mass is [generally] concentrated in a small subset of states
- The same [usually] occurs in queuing systems

Can we take advantage of this behavior, in order to discard some probability mass during the computation of transient measures, and simultaneously control the introduced error?

Continuous Time Markov Chain (CTMC)

Let $\mathcal{X} = \{X(t), t \geq 0\}$ be a **CTMC** with

- \mathcal{S} : finite state-space
- \mathbf{Q} : infinitesimal generator matrix
- $\pi_i(t) = P[X(t) = i]$
- $\vec{\pi}(t) = [\pi_1(t), \dots, \pi_{|\mathcal{S}|}(t)]$: transient probability vector

System Dynamics:

$$\frac{d}{dt} \vec{\pi}(t) = \vec{\pi}(t) \mathbf{Q},$$

$\vec{\pi}(0)$: initial condition

System Dynamics: Differential Equations

$$\frac{d}{dt} \vec{\pi}(t) = \vec{\pi}(t) \mathbf{Q},$$

$\vec{\pi}(0)$: initial condition

System Dynamics: Solution

- Direct:

$$\vec{\pi}(t) = \vec{\pi}(0)e^{\mathbf{Q}t}$$

- Taylor Series Expansion:

$$\vec{\pi}(t) = \vec{\pi}(0) \sum_{n=0}^{\infty} \frac{(\mathbf{Q}t)^n}{n!}$$

\mathbf{Q} contains negative and unbounded entries, which implies that the evaluation of $e^{\mathbf{Q}t}$ is numerically unstable.

Direct Solution \rightarrow Taylor Series Expansion + Stability

$$\vec{\pi}(t) = \vec{\pi}(0) e^{\mathbf{Q}t}$$

Idea: define the stochastic matrix $\mathbf{P} = \mathbf{Q}/\Lambda + \mathbf{I}$, with $\Lambda \geq \max_i(|Q_{i,i}|)$, then

Uniformization (a.k.a. Randomization or Jensen's Method)

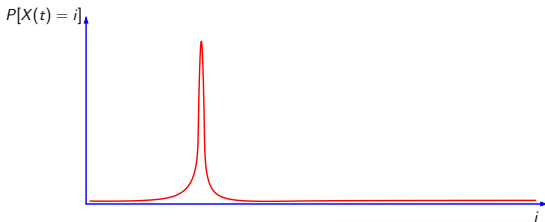
$$\vec{\pi}(t) = \sum_{n=0}^{\infty} e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} \vec{\pi}(0) \mathbf{P}^n$$

which can be computed efficiently by,

$$\vec{\pi}(t) = \sum_{n=0}^{\infty} e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} \vec{\nu}(n), \quad \text{where } \vec{\nu}(n) = \vec{\nu}(n-1) \mathbf{P}, \\ \vec{\nu}(0) = \vec{\pi}(0)$$

$\vec{\nu}(n)$ is the transient probability vector of the *uniformized DTMC*, characterized by \mathbf{P}

In availability/reliability and queueing models is usual to have the probability mass concentrated in a small subset of the state space.



Improving Uniformization: Strategy

- discarding probability mass that is considered negligible for the overall computation process ($\vec{v}(n) = \vec{v}(n-1)\mathbf{P}$).
- using a data structure to efficiently manage those relevant entries of $\vec{v}(n)$.

Uniformization:

$$\vec{\pi}(t) = \sum_{n=0}^{\infty} e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} \vec{\nu}(n)$$

$$\vec{\nu}(n) = \vec{\nu}(n-1) \mathbf{P},$$

$$\vec{\nu}(0) = \vec{\pi}(0)$$

1) **Truncation** on $n = N$ (uniformization steps): well known technique

$$\vec{\pi}(t) = \sum_{n=0}^N e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} \vec{\nu}(n) + \vec{\epsilon}_t(N)$$

2) **Selection** of probability mass of $\vec{\nu}(n) \leftarrow$ **contribution**

$$\vec{\pi}(t) = \sum_{n=0}^N e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} \hat{\vec{\nu}}(n) + \vec{\epsilon}_t(N) + \vec{\epsilon}_d(N)$$

$\vec{\epsilon}_d(N)$: vector containing to the total discarded probability mass

Approximation: Actual computation

$$\hat{\vec{\pi}}(t) = \sum_{n=0}^N e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} \hat{\vec{\nu}}(n)$$

Error Analysis

$$\begin{aligned} \|\vec{\pi}(t) - \hat{\vec{\pi}}(t)\|_1 &\leq \|\vec{\epsilon}_t(N)\|_1 + \|\vec{\epsilon}_d(N)\|_1 \\ &\leq \varepsilon_t + \varepsilon_d \end{aligned}$$

- Error due to truncation of infinite summation

$$\|\vec{\epsilon}_t(N)\|_1 \leq \varepsilon_t$$

There are several techniques to find out N and to calculate (accurately) $e^{-\Lambda t} \frac{(\Lambda t)^n}{n!}$, by specifying a value of ε_t , beforehand.

- Error due to selection of some entries from the probability vector

$$\|\vec{\epsilon}_d(N)\|_1 \leq \varepsilon_d$$

can be specified beforehand?

Analyzing $\vec{v}(n) = \vec{v}(n-1)\mathbf{P}$

- Let $\vec{\delta}(n)$ be the discarding probability vector of the DTMC at step n , with a total probability mass discarded of δ_n at that step, i.e.,

$$\|\vec{\delta}(n)\|_1 = \delta_n$$

- Let $\hat{\vec{v}}(n)$ be the approximate vector of $\vec{v}(n)$, after discarding the mass $\vec{\delta}(n)$, i.e.,

$$\vec{v}(n) = \hat{\vec{v}}(n) + \vec{\delta}(n)$$

- Therefore, after m iteration steps, we have:

$$\vec{v}(n+m) = \vec{v}(n)\mathbf{P}^m = \hat{\vec{v}}(n)\mathbf{P}^m + \vec{\delta}(n)\mathbf{P}^m$$

- Let $\vec{\epsilon}_d(n, m) = \vec{\delta}(n)\mathbf{P}^m$ be the error after m iteration steps, introduced by discarding $\vec{\delta}(n)$ at step n . Consequently,

$$\|\vec{\epsilon}_d(n, m)\|_1 = \|\vec{\delta}(n)\mathbf{P}^m\|_1 = \|\vec{\delta}(n)\|_1 = \delta_n$$

The error (norm-1) is the total probability mass discarded at step n and does not change with the evolution of the DTMC

Selection of probability mass

$$\vec{\pi}(t) = \sum_{n=0}^N e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} \hat{\vec{v}}(n) + \vec{\epsilon}_t(N) + \vec{\epsilon}_d(N)$$

Selection: Strategy

Let δ_n be the discarded probability mass of $\hat{\vec{v}}(n)$ at step n . Then, we have to comply with:

$$\sum_{n=0}^N \delta_n \leq \|\vec{\epsilon}_d(N)\|_1$$

Discarding δ_n from $\hat{\vec{v}}(n)$, is equivalent to **select** (preserve) a probability mass of:

$$\omega_n = \left\| \hat{\vec{v}}(n) \right\|_1 - \delta_n$$

This is the point of view that we will consider hereafter

Question

How do we select a probability mass of ω_n at each uniformization step?

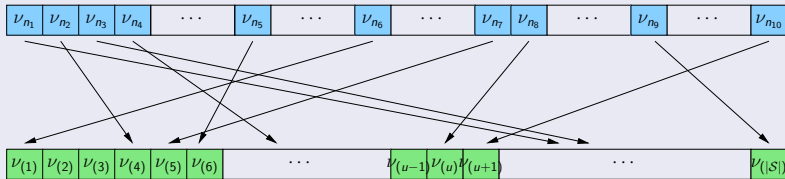
Question

How do we select a probability mass of ω_n at each uniformization step?

Answer

Selecting, in "order", from largest down to the smallest vector component

Sorting and Selection of entries of $\vec{\nu}(n)$

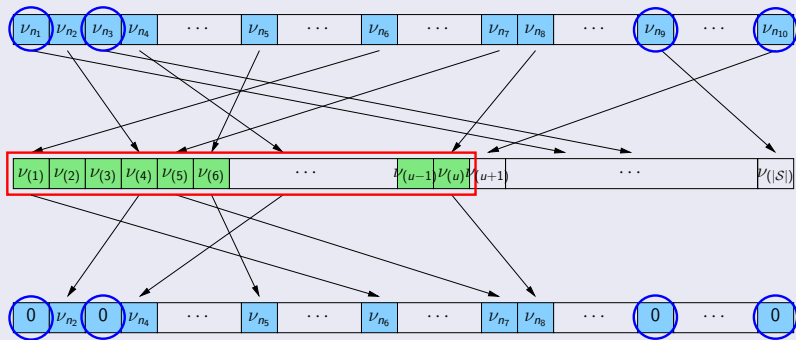


Conditions

- Sorting: $\nu_{(1)} \geq \nu_{(2)} \geq \nu_{(3)} \geq \dots \geq \nu_{(u-1)} \geq \nu_{(u)} \geq \dots \geq \nu_{(|S|)}$

MC Transient Analysis: Selection Approach

Sorting and Selection of entries of $\vec{v}(n)$



Conditions

- **Sorting:** $v_{(1)} \geq v_{(2)} \geq v_{(3)} \geq \dots \geq v_{(u-1)} \geq v_{(u)} \geq \dots \geq v_{(|S|)}$
- **Selection:** $\min_u \left\{ u \mid \sum_{i=1}^u v_{(i)} \geq \omega_n \right\}$

Comments

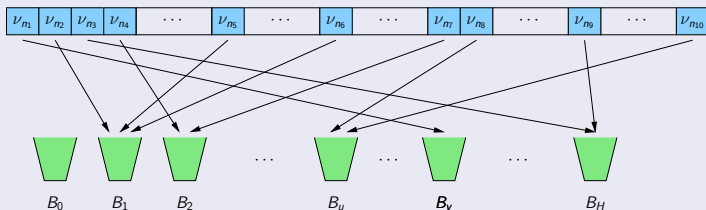
- Sorting all elements of an array of dimension $|\mathcal{S}|$ is a computationally expensive task ($|\mathcal{S}| \log_2 |\mathcal{S}|$, in average)
- The high computational cost of sorting the probability vector (in each step), have discouraged researchers to consider this approach to deal with negligible probabilities
- We propose an efficient method for "sorting" the probability vector

Other approaches

An alternative approach, that has been used recently, consists in just discarding those probabilities that are below a certain threshold, fixed by the user. While it speeds up computation of uniformization, there is no formal way of defining a threshold and the error is not predictable

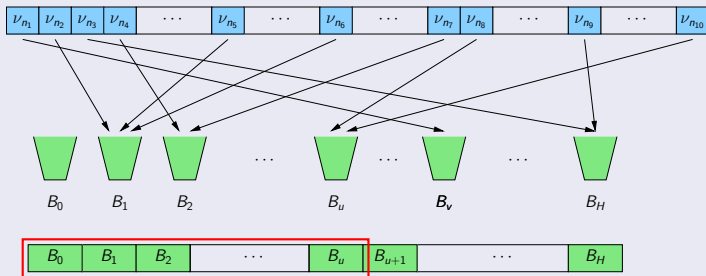
Probability Mass Selection: Strategy (*)

- 1 A number x can be represented in any base as: $x = \pm b^{c_x \cdot m_x}$, where b is the base, c_x is the characteristic, and m_x is the mantissa
- 2 Define a bucket-selection algorithm where:
 - $g(x) = |c_x|$, (for $x > 0$).
 - B_k : bucket (set) to store those entries ν_i , where $g(\nu_i) = k$.



There exist a strict order relation among the buckets $B_j, j \in \{0, \dots, H\}$:
All elements of B_m are smaller than all elements of B_n , for $m > n$

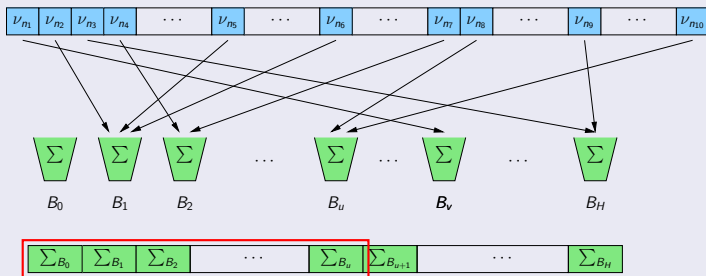
Selection by number representation of entries of $\vec{\nu}(n)$



Comments:

- Selection: $\min_u \left\{ u \mid \sum_{i=0}^u (\Sigma_{B_i}) \geq \omega_n \right\}$, where $\Sigma_{B_i} = \sum_{j: \nu_j \in B_i} \nu_j$
- The optimum value to be selected ($\nu_{(u)}$) is in the bucket B_u

Selection by number representation of entries of $\vec{v}(n)$



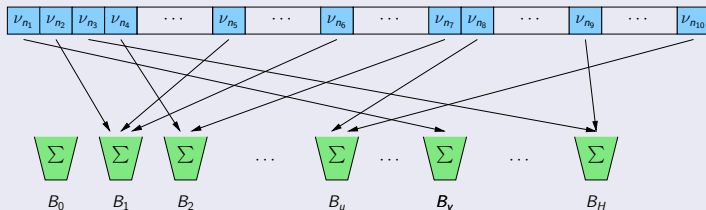
- Once we determine B_u , or equivalently, u , the selection of the components that satisfy the condition given by ω_n , can be done by picking up those entries that comply with:

$$g(v_i) \leq u$$

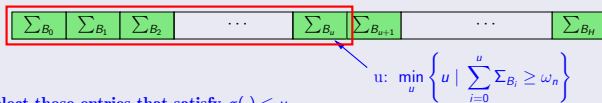
MC Transient Analysis: Selection Approach

Full scheme of selection

1) Select and sum in corresponding bucket the entries of $\vec{v}(n)$



2) Determine u (B_u)



3) Select those entries that satisfy $g(\cdot) \leq u$



Uniformization by Probability Mass Selection

1 Uniformization (approximate)

$$\vec{\pi}(t) = \sum_{n=0}^N e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} \hat{\vec{v}}(n) + \vec{\epsilon}_t(N) + \vec{\epsilon}_d(N)$$

2 Error due to the truncation of infinite summation

$$\|\vec{\epsilon}_t(N)\|_t \leq \epsilon_t: \quad e^{-\lambda t} \frac{(\lambda t)^N}{N!}, N \leftarrow \epsilon_t \text{ (known technique)}$$

3 Error due to the discarding procedure of probability mass

$$\|\vec{\epsilon}_d(N)\|_1 \leq \epsilon_d: \quad \text{choose } \omega_n, \text{ such that } \sum_{n=0}^N \|\hat{\vec{v}}(n)\|_1 - \omega_n \leq \epsilon_d$$

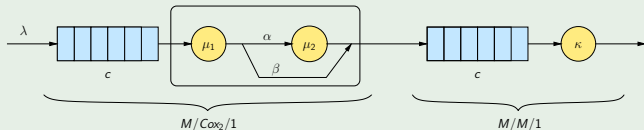
4 Evaluation of $\hat{\vec{v}}(n) = \hat{\vec{v}}(n-1)\mathbf{P}$

$$\hat{\vec{v}}(n) = \sum_{i: g(\hat{v}_i(n-1)) \leq u} \hat{v}_i(n-1) \vec{P}_i$$

$$g(b^{c_x \cdot m_x}) = |c_x|, \quad u: \min_u \left\{ u \mid \sum_{i=0}^u \Sigma_{B_i} \geq \omega_n \right\}, \quad \Sigma_{B_i} = \sum_{j: g(\hat{v}_j(n-1))=i} \hat{v}_j(n-1)$$

MC Transient Analysis: Example

Two Tandem Queues



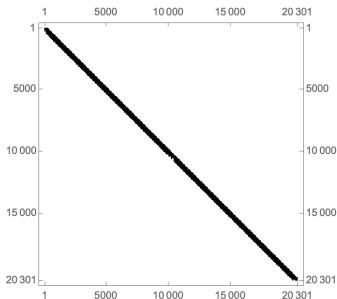
source: H. Hermanns, J. Meyer-Kayser and M. Siegle. "Multi-terminal binary decision diagrams to represent and analyse continuous-time Markov chains". In Proc. 3rd Int. Workshop on the Num. Sol. of Markov Chains, pp. 188-207, 1999

Parameters:

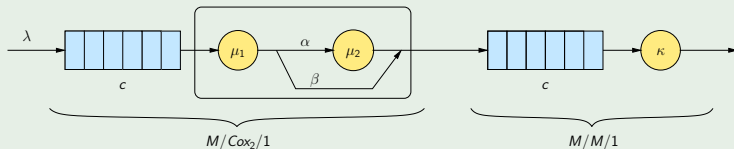
- $c = 100$
- $\lambda = 4c$
- $\mu_1 = 2$
- $\mu_2 = 2$
- $\alpha = 0.1$
- $\beta = 0.9$
- $\kappa = 4$

starting condition: empty system (state 0)

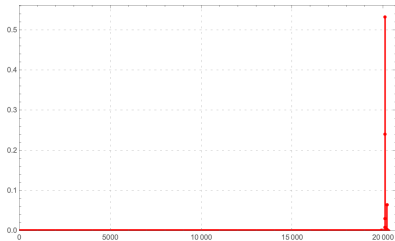
Matrix Q



Two Tandem Queues



State probabilities after $t = 1$



probability vs. state number

Number of states (probability mass)



number of states vs. uniformization step

- This technique allows us to take advantage of the concentration of probability mass in few states
- We provide a bound (beforehand) on the numerical error, and the procedure guarantees that its answer complies with that specification
- If the probability mass is not concentrated, our method could perform worse than the standard uniformization (i.e., without discarding probabilities)
- A hybrid approach can be used (standard uniformization + our approach) in order to avoid the under-performance of the discarding procedure

MC Transient Analysis: Note on Implementation

Saving in the evaluation of $\vec{\nu}(n)$

$$\vec{\nu}(n) = \vec{\nu}(n-1) \mathbf{P}, \quad \vec{\nu}(0) = \vec{\pi}(0)$$

Consider \mathbf{P} in terms of row-vectors:

$$\mathbf{P} = \begin{bmatrix} \vec{P}_1 \\ \vec{P}_2 \\ \vdots \\ \vec{P}_{|S|} \end{bmatrix}, \quad \vec{P}_i = [P_{i,1}, P_{i,2}, \dots, P_{i,|S|}]$$

Evaluation of $\vec{\nu}(n)$ can be written:

$$\vec{\nu}(n) = \sum_{i \in S} \nu_i(n-1) \vec{P}_i$$

