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# Transient Analysis of Markov Chains by Controlled Probability Mass Selection

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(Virtual)

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- 1 Motivation
- 2 Background: CTMC Transient Analysis by Uniformization
- 3 Error Description/Analysis
- 4 Our Proposal to Improve the Uniformization Method
- 5 Conclusions

- **Continuous-time Markov chains** (CTMCs) are widely used to model real-world systems and processes in many areas such as (amongst many others)
  - engineering and environmental processes
  - computer and communication networks
  - production and manufacturing systems
  - healthcare systems
  - epidemic diseases
  - biochemically reacting systems
- Often the interest is in **transient (time dependent) system behavior** or performance properties, respectively, and the analysis requires the determination of transient probability distributions

**Uniformization**, also known as **randomization** or **Jensen's method**, provides a numerically stable computational scheme to approximate the **transient probability distributions** of CTMCs with uniformly bounded transition rates

- Reliability/Availability analysis usually involves systems that were design for high availability: system is working [properly] most of the time.
- The state space could be seen as a partition with "working" states and "failure" states
- In these systems, most of the probability mass is [generally] concentrated in a small subset of states
- The same [usually] occurs in queuing systems

Can we take advantage of this behavior, in order to discard some probability mass during the computation of transient measures, and simultaneously control the introduced error?

## Continuous Time Markov Chain (CTMC)

Let  $\mathcal{X} = \{X(t), t \geq 0\}$  be a **CTMC** with

- $\mathcal{S}$ : finite state-space
- $\mathbf{Q}$ : infinitesimal generator matrix
- $\pi_i(t) = P[X(t) = i]$
- $\vec{\pi}(t) = [\pi_1(t), \dots, \pi_{|\mathcal{S}|}(t)]$ : transient probability vector

## System Dynamics:

$$\frac{d}{dt} \vec{\pi}(t) = \vec{\pi}(t) \mathbf{Q},$$

$\vec{\pi}(0)$  : initial condition

## System Dynamics: Differential Equations

$$\frac{d}{dt} \vec{\pi}(t) = \vec{\pi}(t) \mathbf{Q},$$

$\vec{\pi}(0)$  : initial condition

## System Dynamics: Solution

- Direct:

$$\vec{\pi}(t) = \vec{\pi}(0) e^{\mathbf{Q}t}$$

- Taylor Series Expansion:

$$\vec{\pi}(t) = \vec{\pi}(0) \sum_{n=0}^{\infty} \frac{(\mathbf{Q}t)^n}{n!}$$

$\mathbf{Q}$  contains negative and unbounded entries, which implies that the evaluation of  $e^{\mathbf{Q}t}$  is numerically unstable.

## Direct Solution $\rightarrow$ Taylor Series Expansion + Stability

$$\vec{\pi}(t) = \vec{\pi}(0) e^{\mathbf{Q}t}$$

**Idea:** define the stochastic matrix  $\mathbf{P} = \mathbf{Q}/\Lambda + \mathbf{I}$ , with  $\Lambda \geq \max_i(|Q_{i,i}|)$ , then

## Uniformization (a.k.a. Randomization or Jensen's Method)

$$\vec{\pi}(t) = \sum_{n=0}^{\infty} e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} \vec{\pi}(0) \mathbf{P}^n$$

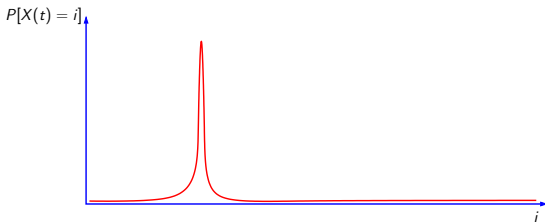
which can be computed efficiently by,

$$\vec{\pi}(t) = \sum_{n=0}^{\infty} e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} \vec{\nu}(n), \quad \text{where } \vec{\nu}(n) = \vec{\nu}(n-1) \mathbf{P}, \\ \vec{\nu}(0) = \vec{\pi}(0)$$

$\vec{\nu}(n)$  is the transient probability vector of the *uniformized DTMC*, characterized by  $\mathbf{P}$



In availability/reliability and queueing models is usual to have the probability mass concentrated in a small subset of the state space.



### Improving Uniformization: Strategy

- discarding probability mass that is considered negligible for the overall computation process ( $\vec{\nu}(n) = \vec{\nu}(n-1)\mathbf{P}$ ).
- using a data structure to efficiently manage those relevant entries of  $\vec{\nu}(n)$ .

Uniformization:

$$\vec{\pi}(t) = \sum_{n=0}^{\infty} e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} \vec{\nu}(n)$$

$$\vec{\nu}(n) = \vec{\nu}(n-1) \mathbf{P},$$

$$\vec{\nu}(0) = \vec{\pi}(0)$$

1) **Truncation** on  $n = N$  (uniformization steps): well known technique

$$\vec{\pi}(t) = \sum_{n=0}^N e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} \vec{\nu}(n) + \vec{\epsilon}_t(N)$$

2) **Selection** of probability mass of  $\vec{\nu}(n) \leftarrow$  **contribution**

$$\vec{\pi}(t) = \sum_{n=0}^N e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} \hat{\vec{\nu}}(n) + \vec{\epsilon}_t(N) + \vec{\epsilon}_d(N)$$

$\vec{\epsilon}_d(N)$ : vector containing to the total discarded probability mass

## Approximation: Actual computation

$$\hat{\vec{\pi}}(t) = \sum_{n=0}^N e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} \hat{\vec{\nu}}(n)$$

## Error Analysis

$$\begin{aligned} \|\vec{\pi}(t) - \hat{\vec{\pi}}(t)\|_1 &\leq \|\vec{\epsilon}_t(N)\|_1 + \|\vec{\epsilon}_d(N)\|_1 \\ &\leq \varepsilon_t + \varepsilon_d \end{aligned}$$

- Error due to truncation of infinite summation

$$\|\vec{\epsilon}_t(N)\|_1 \leq \varepsilon_t$$

There are several techniques to find out  $N$  and to calculate (accurately)  $e^{-\Lambda t} \frac{(\Lambda t)^n}{n!}$ , by specifying a value of  $\varepsilon_t$ , beforehand.

- Error due to selection of some entries from the probability vector

$$\|\vec{\epsilon}_d(N)\|_1 \leq \varepsilon_d$$

**can be specified beforehand?**

### Analyzing $\vec{v}(n) = \vec{v}(n-1)\mathbf{P}$

- Let  $\vec{\delta}(n)$  be the discarding probability vector of the DTMC at step  $n$ , with a total probability mass discarded of  $\delta_n$  at that step, i.e.,

$$\|\vec{\delta}(n)\|_1 = \delta_n$$

- Let  $\hat{\vec{v}}(n)$  be the approximate vector of  $\vec{v}(n)$ , after discarding the mass  $\vec{\delta}(n)$ , i.e.,

$$\vec{v}(n) = \hat{\vec{v}}(n) + \vec{\delta}(n)$$

- Therefore, after  $m$  iteration steps, we have:

$$\vec{v}(n+m) = \vec{v}(n)\mathbf{P}^m = \hat{\vec{v}}(n)\mathbf{P}^m + \vec{\delta}(n)\mathbf{P}^m$$

- Let  $\vec{\epsilon}_d(n, m) = \vec{\delta}(n)\mathbf{P}^m$  be the error after  $m$  iteration steps, introduced by discarding  $\vec{\delta}(n)$  at step  $n$ . Consequently,

$$\|\vec{\epsilon}_d(n, m)\|_1 = \|\vec{\delta}(n)\mathbf{P}^m\|_1 = \|\vec{\delta}(n)\|_1 = \delta_n$$

The error (norm-1) is the total probability mass discarded at step  $n$  and does not change with the evolution of the DTMC

Selection of probability mass

$$\vec{\pi}(t) = \sum_{n=0}^N e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} \hat{\vec{v}}(n) + \vec{\epsilon}_t(N) + \vec{\epsilon}_d(N)$$

## Selection: Strategy

Let  $\delta_n$  be the discarded probability mass of  $\hat{\vec{v}}(n)$  at step  $n$ . Then, we have to comply with:

$$\sum_{n=0}^N \delta_n \leq \|\vec{\epsilon}_d(N)\|_1$$

Discarding  $\delta_n$  from  $\hat{\vec{v}}(n)$ , is equivalent to **select** (preserve) a probability mass of:

$$\omega_n = \left\| \hat{\vec{v}}(n) \right\|_1 - \delta_n$$

This is the point of view that we will consider hereafter

## Question

How do we select a probability mass of  $\omega_n$  at each uniformization step?

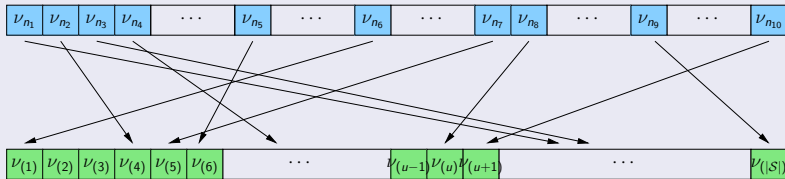
### Question

How do we select a probability mass of  $\omega_n$  at each uniformization step?

### Answer

Selecting, in "order", from largest down to the smallest vector component

## Sorting and Selection of entries of $\vec{\nu}(n)$



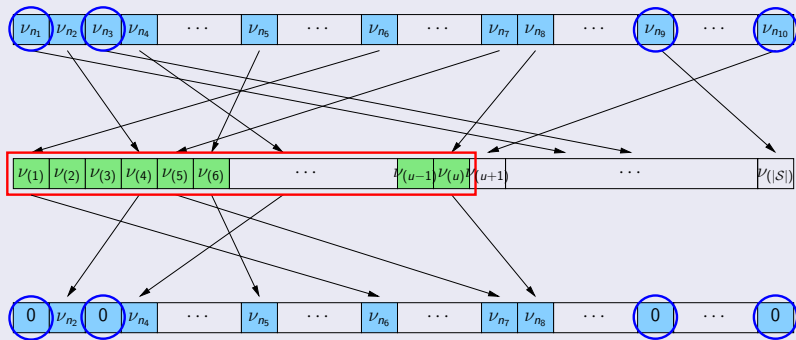
## Conditions

- Sorting:  $\nu_{(1)} \geq \nu_{(2)} \geq \nu_{(3)} \geq \dots \geq \nu_{(u-1)} \geq \nu_{(u)} \geq \dots \geq \nu_{(|S|)}$



# MC Transient Analysis: Selection Approach

## Sorting and Selection of entries of $\vec{v}(n)$



## Conditions

- **Sorting:**  $v_{(1)} \geq v_{(2)} \geq v_{(3)} \geq \dots \geq v_{(u-1)} \geq v_{(u)} \geq \dots \geq v_{(|S|)}$
- **Selection:**  $\min_u \left\{ u \mid \sum_{i=1}^u v_{(i)} \geq \omega_n \right\}$

### Comments

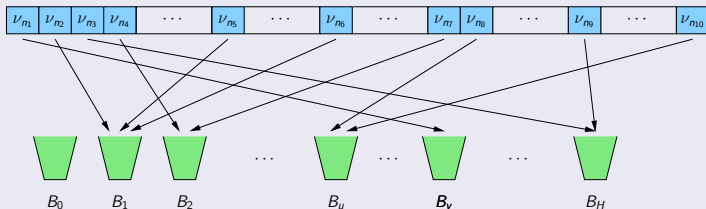
- Sorting all elements of an array of dimension  $|\mathcal{S}|$  is a computationally expensive task ( $|\mathcal{S}| \log_2 |\mathcal{S}|$ , in average)
- The high computational cost of sorting the probability vector (in each step), have discouraged researchers to consider this approach to deal with negligible probabilities
- We propose an efficient method for "sorting" the probability vector

### Other approaches

An alternative approach, that has been used recently, consists in just discarding those probabilities that are below a certain threshold, fixed by the user. While it speeds up computation of uniformization, there is no formal way of defining a threshold and the error is not predictable

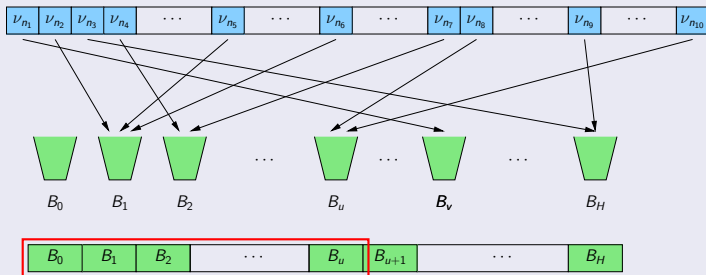
## Probability Mass Selection: Strategy (\*)

- 1 A number  $x$  can be represented in any base as:  $x = \pm b^{c_x \cdot m_x}$ , where  $b$  is the base,  $c_x$  is the characteristic, and  $m_x$  is the mantissa
- 2 Define a bucket-selection algorithm where:
  - $g(x) = |c_x|$ , (for  $x > 0$ ).
  - $B_k$ : bucket (set) to store those entries  $\nu_i$ , where  $g(\nu_i) = k$ .



There exist a strict order relation among the buckets  $B_j, j \in \{0, \dots, H\}$ :  
All elements of  $B_m$  are smaller than all elements of  $B_n$ , for  $m > n$

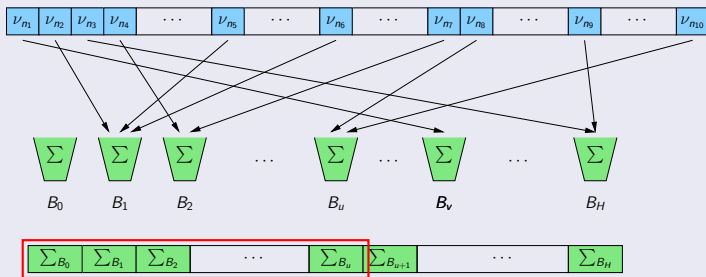
## Selection by number representation of entries of $\vec{\nu}(n)$



### Comments:

- Selection:  $\min_u \left\{ u \mid \sum_{i=0}^u (\Sigma_{B_i}) \geq \omega_n \right\}$ , where  $\Sigma_{B_i} = \sum_{j: \nu_j \in B_i} \nu_j$
- The optimum value to be selected ( $\nu_{(u)}$ ) is in the bucket  $B_u$

## Selection by number representation of entries of $\vec{v}(n)$



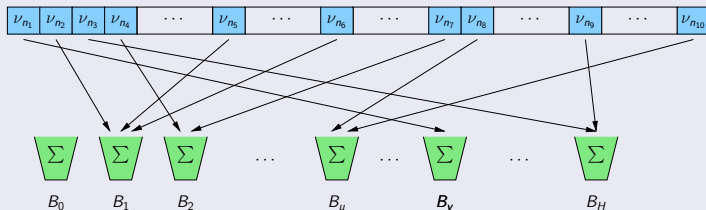
- Once we determine  $B_u$ , or equivalently,  $u$ , the selection of the components that satisfy the condition given by  $\omega_n$ , can be done by picking up those entries that comply with:

$$g(v_i) \leq u$$

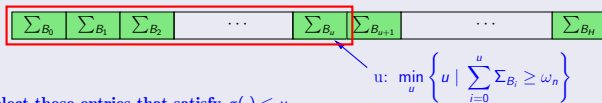
# MC Transient Analysis: Selection Approach

## Full scheme of selection

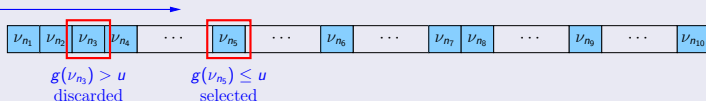
1) Select and sum in corresponding bucket the entries of  $\vec{v}(n)$



2) Determine  $u$  ( $B_u$ )



3) Select those entries that satisfy  $g(\cdot) \leq u$



## Uniformization by Probability Mass Selection

### 1 Uniformization (approximate)

$$\vec{\pi}(t) = \sum_{n=0}^N e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} \hat{\vec{v}}(n) + \vec{\epsilon}_t(N) + \vec{\epsilon}_d(N)$$

### 2 Error due to the truncation of infinite summation

$$\|\vec{\epsilon}_t(N)\|_t \leq \epsilon_t: \quad e^{-\lambda t} \frac{(\lambda t)^N}{N!}, N \leftarrow \epsilon_t \text{ (known technique)}$$

### 3 Error due to the discarding procedure of probability mass

$$\|\vec{\epsilon}_d(N)\|_1 \leq \epsilon_d: \quad \text{choose } \omega_n, \text{ such that } \sum_{n=0}^N \|\hat{\vec{v}}(n)\|_1 - \omega_n \leq \epsilon_d$$

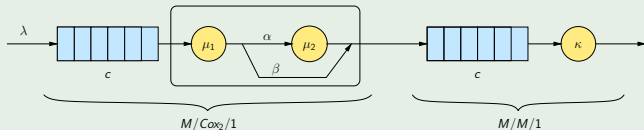
### 4 Evaluation of $\hat{\vec{v}}(n) = \hat{\vec{v}}(n-1)\mathbf{P}$

$$\hat{\vec{v}}(n) = \sum_{i: g(\hat{v}_i(n-1)) \leq u} \hat{v}_i(n-1) \vec{P}_i$$

$$g(b^{c_x \cdot m_x}) = |c_x|, \quad u: \min_u \left\{ u \mid \sum_{i=0}^u \Sigma_{B_i} \geq \omega_n \right\}, \quad \Sigma_{B_i} = \sum_{j: g(\hat{v}_j(n-1))=i} \hat{v}_j(n-1)$$

# MC Transient Analysis: Example

## Two Tandem Queues



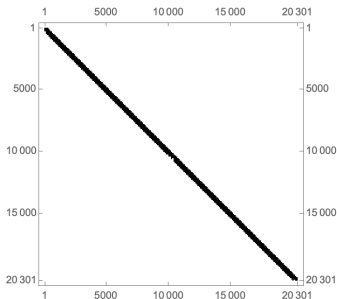
source: H. Hermanns, J. Meyer-Kayser and M. Siegle. "Multi-terminal binary decision diagrams to represent and analyse continuous-time Markov chains". In Proc. 3rd Int. Workshop on the Num. Sol. of Markov Chains, pp. 188-207, 1999

### Parameters:

- $c = 100$
- $\lambda = 4c$
- $\mu_1 = 2$
- $\mu_2 = 2$
- $\alpha = 0.1$
- $\beta = 0.9$
- $\kappa = 4$

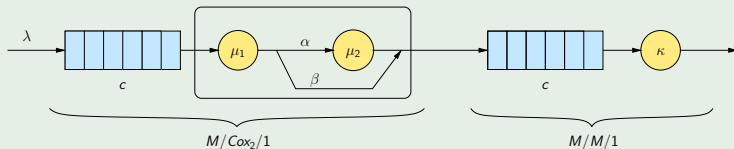
starting condition: empty system (state 0)

### Matrix Q

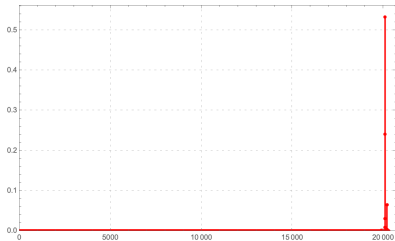




## Two Tandem Queues



State probabilities after  $t = 1$



*probability vs. state number*

Number of states (probability mass)



*number of states vs. uniformization step*

- This technique allows us to take advantage of the concentration of probability mass in few states
- We provide a bound (beforehand) on the numerical error, and the procedure guarantees that its answer complies with that specification
- If the probability mass is not concentrated, our method could perform worse than the standard uniformization (i.e., without discarding probabilities)
- A hybrid approach can be used (standard uniformization + our approach) in order to avoid the under-performance of the discarding procedure

# MC Transient Analysis: Note on Implementation

Saving in the evaluation of  $\vec{\nu}(n)$

$$\vec{\nu}(n) = \vec{\nu}(n-1) \mathbf{P}, \quad \vec{\nu}(0) = \vec{\pi}(0)$$

Consider  $\mathbf{P}$  in terms of row-vectors:

$$\mathbf{P} = \begin{bmatrix} \vec{P}_1 \\ \vec{P}_2 \\ \vdots \\ \vec{P}_{|S|} \end{bmatrix}, \quad \vec{P}_i = [P_{i,1}, P_{i,2}, \dots, P_{i,|S|}]$$

Evaluation of  $\vec{\nu}(n)$  can be written:

$$\vec{\nu}(n) = \sum_{i \in S} \nu_i(n-1) \vec{P}_i$$

