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# Connecting irreducible and absorbing Markov chains

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AMS Spring Eastern Sectional Meeting

Virtual, April 1st, 2023

# Talk's content

- 1 CTMC  $X$ : a single absorbing state
- 2 Pseudo-aggregation
- 3 Connecting  $X$  with an irreducible  $Y^{(i)}$
- 4 Using cuts
- 5  $X$ : several absorbing states

# Introduction

- This is a follow-up of the talk given at the JMM'23 meeting last January.
- It is about connecting irreducible and absorbing Markov chains. One possible connection is using Sigmund's duality and variants (or extensions) of the same idea.
- Here, we will describe simple ways of building direct relationships between these two classes of stochastic processes, and between some of the available tools for working with them. They concern only the steady-state behavior in the irreducible case, not transients.
- To simplify the presentation, we will work with the finite case only. We will illustrate the talk first with continuous time models, later with discrete time ones; the results are basically the same.
- The corresponding paper will be uploaded soon to arXive. I will send a short email to the participants to this session as well as to the participants at previous one during JMM'23.

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# Initial model $X$

- Let  $X$  be a Continuous Time Markov Chain (CTMC) with finite state space  $\mathcal{X}$ , transition rates  $Q_{x,y}$ , a single state  $a \in \mathcal{X}$  absorbing and the rest of states transient. We denote by  $\mathcal{T}$  the set of transient states. We denote  $\alpha_x = \mathbb{P}(X(0) = x)$ , with  $\alpha_a = 0$ .
- We denote by  $P_{x,y}$  the corresponding probability transition rates from  $x$  to  $y$ , that is,  $P_{x,y} = Q_{x,y}/d_x$  for any state  $x \in \mathcal{T}$  and all  $y$ , where  $d_x =$  departure rate from  $x$ ,  $= \sum_y Q_{x,y}$ . For state  $a$  we set  $P_{a,a} = 1$  ( $P_{a,x} = 0$  for all  $x \neq a$ ).
- In the irreducible case, we will focus on the steady-state distribution of the process. In discrete time, the main object of interest is the absorption time  $T$ :

$$T = \inf\{t \geq 0: X(t) = a\} = \int_0^\infty 1(X(t) \neq a) dt.$$

In words,  $T$  is the time taken by  $X$  to get absorbed.

- As for the irreducible case, we will not look at the distribution of  $T$ , we will focus on its mean,  $\mathbb{E}(T) =: \tau$ , and the conditional  $\mathbb{E}_i(T) = \mathbb{E}(T \mid X(0) = i) =: \tau_i$ . Since  $\tau_a = 0$ , we have  $\mathbb{E}(T) = \sum_{i \in \mathcal{T}} \alpha_i \tau_i$ .
- To compute the  $\tau_i$ s (and  $\tau$ ): we have a **simple linear system** to solve.
- If  $S_x$ : the total time spent by  $X$  in state  $x$ ,  $= \int_0^T \mathbf{1}(X(t) = x) dt$ ,  $\mathbb{E}(S_x) =: \sigma_x$  or  $\mathbb{E}_i(S_x) =: \sigma_{i,x}$  are also **simple linear problems**.
- Many other targets are often of interest in applications:
  - mean total time spent by  $X$  in a subset of transient states;
  - $\mathbb{E}(T^2)$  and  $\mathbb{E}(S_x^2)$ , for example;
  - when there are more than one absorbing state, say  $a_1, a_2, \dots$ ,  $\mathbb{P}_i(X(\infty) = a_1)$ ,  $\mathbb{E}(T \mid X(\infty) = a_1)$  and  $\mathbb{E}(T^2 \mid X(\infty) = a_1)$ , etc.
  - All of them are **easy to solve, mathematically speaking** (numerically, it depends on model's size, stiffness of the involved matrices, etc.).

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## Key tool: pseudo-aggregation

- Let  $U$  be an irreducible CTMC with state space  $\mathcal{U}$  and transition rates  $Q$ , ergodic, and  $\mathcal{P} = (B_0, B_1, B_2, \dots)$  a partition of  $\mathcal{U}$ ; the stationary distribution of  $U$  is  $\pi$ .
- Define  $V$ , the pseudo-aggregation of  $U$  with respect to  $\mathcal{P}$ , as the CTMC defined on  $\mathcal{P}$  with rates  $R$  given by

$$R_{B,B'} = \frac{\sum_{x \in B} \pi_x Q_{x,B'}}{\pi_B},$$

plus, if needed,  $\mathbb{P}(V(0) = B) = \pi_B$ .

- Then,  $V$  is ergodic, with stationary distribution  $\pi'$ , where for any  $B \in \mathcal{P}$ ,  $\pi'_B = \sum_{x \in B} \pi_x = \pi_B$ .
- See that  $V$  is Markov **by construction**. The aggregation of  $U$  w.r.t.  $\mathcal{P}$  is, in general, not Markovian (in general, it is “nothing”).

## Some references

Concerning pseudo-aggregation and their context, see

- “*Sojourn times in Finite Markov processes*”, G. Rubino and B. Sericola, Journal of Applied Probability, 26, pages 744-756, 1989.

and then

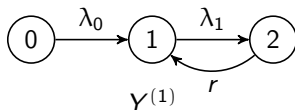
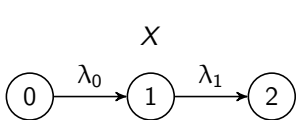
- “*A finite characterization of weak lumpable Markov processes. Part I: the discrete time case*”, G. Rubino and B. Sericola, Stochastic Processes and their Applications, 38, pages 195-204, 1991.
- “*A finite characterization of weak lumpable Markov processes. Part II: the continuous time case*”, G. Rubino and B. Sericola, Stochastic Processes and their Applications, 45, pages 115-125, 1993.

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# Absorbing $\rightarrow$ irreducible

- From  $X$  we build  $Y$  by adding transitions from  $a$  to any  $x \in \mathcal{T}$  with rate  $r\alpha_x$  ( $r > 0$  is arbitrary).
- Fact 1:  $Y$  has a single recurrent class but it may have transient states. Its stationary distribution is denoted by  $\pi$ .
- Fact 2:  $\pi_a > 0$ .
- We also denote by  $Y^{(i)}$  the same type of construction but simply adding a single transition from  $a$  to  $i \in \mathcal{T}$ , with rate  $r$ .
- Facts 1 and 2 hold obviously for  $Y^{(i)}$ , whose stationary distribution is denoted by  $\pi^{(i)}$ ; in particular, we have Fact 3:  $\pi_a^{(i)} > 0$ .



## Relations between $X$ and $Y, Y^{(i)}$

- Consider the pseudo-aggregation of  $Y$  into two macro-states  $a$  and  $b$ , where the latter is the result of collapsing all the transient states.
- The mean (the holding) time of this new 2-state process in state  $b$  is  $\mathbb{E}(T)$  and we have the following balance relation:

$$\pi_b \frac{1}{\mathbb{E}(T)} = \pi_a r,$$

where  $\pi_b = \sum_{i: i \neq a} \pi_i$ .

- Replacing  $\pi_b$  by  $1 - \pi_a$ , we get

$$\mathbb{E}(T) = \frac{1 - \pi_a}{r \pi_a},$$

or equivalently,

$$\pi_a = \frac{1}{1 + r \mathbb{E}(T)}.$$

- If we use  $Y^{(i)}$  instead of  $Y$ , we obtain

$$\tau_i = \frac{1 - \pi_a^{(i)}}{r \pi_a^{(i)}},$$

or equivalently,

$$\pi_a^{(i)} = \frac{1}{1 + r \tau_i}.$$

- If we fix a state  $i$  and use the **regenerative** properties of  $Y^{(i)}$  w.r.t.  $a$ , we obtain

$$\frac{\sigma_{i,x}}{\frac{1}{r} + \tau_i} = \pi_x^{(i)}.$$

Similarly, reasoning in  $Y$  now, we have

$$\frac{\sigma_x}{\frac{1}{r} + \mathbb{E}(T)} = \pi_x.$$

# Conclusion

- Given an absorbing Markov chain  $X$  as before, we can compute the quantities  $\mathbb{E}(T)$  and  $\sigma_x$  for all state  $x$  by working with the irreducible model  $Y$ .
- We can also go the other way, computing in an absorbing model by working with an irreducible one.

# Applications

- (i) Analyzing an absorbing chain  $X$  using  $Y$  or the  $Y^{(i)}$ s, and viceversa.
- (ii) New numerical techniques to compute the stationary distribution of irreducible chains.
- (iii) New tools to compute bounds of quantities calculated using the stationary distribution of irreducible chains (for instance, the stationary probability of being in a subset of states).
- (iv) New results for absorbing models similar to the balance relations through cuts in irreducible ones.

In the rest of the talk, we will only illustrate some aspects of (iv) and then, of (i).



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## Cuts in irreducible chains

- The point here is the equilibrium in the “probability flow” through the border of a cut.
- Typical example: if  $U$  is an irreducible and ergodic chain with transition rates  $Q$  and stationary distribution  $\pi$ , and  $(A, B)$  a partition of  $U$ 's state space, then we have the balance equation

$$\sum_{x \in A} \pi_x Q_{x,B} = \sum_{y \in B} \pi_y Q_{y,A},$$

which can be read **“in equilibrium, the probabilistic flow from  $A$  to  $B$  equals the probabilistic flow from  $B$  to  $A$ ”**.

- We will explore here equivalent relations, in some sense, defined on absorbing chains.

## First, two examples in the irreducible case

- In the stable  $M/M/1$  (or in the  $M/M/1/H$ , out of the borders), the equilibrium equation at state  $n \geq 1$  is

$$(\lambda + \mu)\pi_n = \lambda\pi_{n-1} + \mu\pi_{n+1}.$$

Using the cut ( $\{0, 1, \dots, n\}, \{n+1, n+2, \dots\}$ ) (I call it a “vertical” cut), we directly obtain

$$\lambda\pi_n = \mu\pi_{n+1}.$$

- Now, consider a stable  $M/H_K/1$  queue with arrival rate  $\lambda$  and service time c.d.f.  $F(x) = \sum_{k=1}^K \alpha_k (1 - e^{-\mu_k x})$  (we say that this is a Hyperexponential distribution with  $K$  phases, implicitly assuming that if  $k \neq j$ ,  $\mu_k \neq \mu_j$ ).

If we define  $N(t) = \#$  of customers at time  $t$ , and  $Ph(t) =$  phase of the server at time  $t$ , the vector  $(N(t), Ph(t))$  is Markov.

Let us denote  $G(z) = \sum_{n \geq 0} \mathbb{P}(N(\infty) = n) z^n$ .

- Using the equilibrium equations, or the PK-formula in terms of transforms, we obtain the expression

$$G(z) = \frac{(1 - \rho)(1 - z) \sum_{k=1}^K \alpha_k (\varphi_k - 1) E_k(z)}{\sum_{k=1}^K \alpha_k (\varphi_k - 1) E_k(z) - z E_0(z)},$$

where

$$\varphi_k = \frac{\lambda + \mu_k}{\mu_k}, \quad E_0(z) = \prod_{k=1}^K (\varphi_k - z), \quad E_k(z) = \prod_{j=1, j \neq k}^K (\varphi_j - z)$$

- Observe that in previous expression, there must be a pole at  $z = 1$ , which means that we must find the result of dividing the denominator by  $1 - z$ .
- Using the standard “vertical” cut as for the  $M/M/1$ , we directly obtain

$$G(z) = \frac{(1 - \rho) \sum_{k=1}^K \alpha_k (\varphi_k - 1) E_k(z)}{\sum_{k=1}^K \alpha_k E_k(z) - E_0(z)}.$$

## Central idea in this section

- Start with some  $X$  having an absorbing state.
- Build the associated irreducible chain  $Y$ , use cuts in  $Y$  and equilibrium through them, and see if they correspond to new relations in  $X$ .
- The answer is positive, and we find in that way new relations concerning specific time expectations in  $X$ .

## Preliminary relations

- We are in the same setting as at the beginning, with chains  $X$  (absorbing),  $Y$  and  $Y^{(i)}$  (both irreducible, or having a recurrent class that always includes the state absorbing in  $X$ ).
- We fix  $i \in \mathcal{T}$  and, in  $Y^{(i)}$ , we write the equilibrium relation for every state  $x \neq a$ :

$$\pi_x^{(i)} d_x = \sum_{y: y \neq a} \pi_y^{(i)} Q_{y,x} + \pi_a^{(i)} r \mathbf{1}(x = i).$$

- We know that  $\pi_z^{(i)} = r \sigma_{i,z} \pi_a^{(i)}$ , for  $z \neq a$ . Putting this in previous relation, we get, for every state  $x \neq a$ ,

$$\sigma_{i,x} d_x = \sum_{y: y \neq a} \sigma_{i,y} Q_{y,x} + \mathbf{1}(x = i).$$

- Multiplying by  $\alpha_i$  and summing on  $i \in \mathcal{T}$ , we obtain

for all  $x \neq a$ ,  $\sigma_x d_x = \sum_{y: y \neq a} \sigma_y Q_{y,x} + \alpha_x$ .

- As a side effect, with a little algebra, we have

$$\sum_{x: x \neq a} \sigma_{i,x} Q_{x,a} = 1$$

and multiplying by  $\alpha_i$  and summing on  $i \in \mathcal{T}$ ,

$$\sum_{y: y \neq a} \sigma_y Q_{y,a} = 1.$$

# First result

- Let us decompose the transient states of  $X$  into two non-empty classes  $B$  and  $C$ . Choose some state  $i \in B$  and consider, in process  $Y^{(i)}$ , the equilibrium relation w.r.t. the partition  $\{B, C \cup \{a\}\}$ . Use then the bridges between  $X, Y, Y^{(i)}$ , and obtain

$$\sum_{j \in B} \sigma_{i,j} (Q_{j,C} + Q_{j,a}) = \sum_{k \in C} \sigma_{i,k} Q_{k,B} + 1.$$

- Different formulation: let  $N_{B,C}$  = the r.v. “# of transitions from  $B$  to  $C$  in process  $X$  before absorption”, and analogously for  $N_{C,B}, N_{B,a}$  and  $N_{C,a}$ . Observe that  $N_{B,a} + N_{C,a} = 1$ , so,  $N_{B,a}$  and  $N_{C,a}$  are two (dependent) Binary random variables.



- Choosing  $i \in B$ , we have  $\sum_{j \in B} \sigma_{i,j} Q_{j,C} = \mathbb{E}_i(N_{B,C})$ ,  
 $\sum_{k \in C} \sigma_{i,k} Q_{k,B} = \mathbb{E}_i(N_{C,B})$  and  $\sum_{j \in B} \sigma_{i,j} Q_{j,a} = \mathbb{E}_i(N_{B,a})$ .
- We also have  $\mathbb{E}_i(N_{B,a}) = \mathbb{P}_i(X \text{ gets absorbed from } B)$ .
- From these, we obtain

$$\mathbb{E}_i(N_{B,C}) = \mathbb{E}_i(N_{C,B}) + \mathbb{P}_i(X \text{ gets absorbed from } C) \quad (i \in B).$$

## Second result

- Similarly as before, and deconditioning,

$$\mathbb{E}(N_{B,C}) + \mathbb{P}(X \text{ gets absorbed from } B) = \mathbb{E}(N_{C,B}) + \alpha(B).$$

- By symmetry, we also have

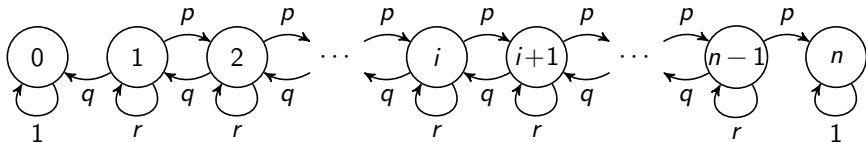
$$\mathbb{E}(N_{B,C}) + \alpha(C) = \mathbb{E}(N_{C,B}) + \mathbb{P}(X \text{ gets absorbed from } C).$$

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# Example

- We will limit the presentation to a discussion around an example.
- To change, let us consider now the gambler ruin problem in discrete time, in the following homogeneous case:



- Goal: to compute the probabilities of absorption

$$\delta_j = \mathbb{P}_j(X(\infty) = 0),$$

for  $j = 1, 2, \dots, n-1$ .

## Direct computation of the $\delta_j$ s

Linear system to solve:

- $\delta_1 = q + r\delta_1 + p\delta_2$
- for  $j = 2, 3, \dots, n-2$ :  $\delta_j = q\delta_{j-1} + r\delta_j + p\delta_{j+1}$
- $\delta_{n-1} = q\delta_{n-2} + r\delta_{n-1}$

Replacing  $r$  by  $1 - p - q$ , dividing by  $q$  and using  $\psi = p/q$ , we obtain

- $(1 + \psi)\delta_1 = 1 + \psi\delta_2$
- for  $j = 2, 3, \dots, n-2$ :  $(1 + \psi)\delta_j = \delta_{j-1} + \psi\delta_{j+1}$
- $(1 + \psi)\delta_{n-1} = \delta_{n-2}$

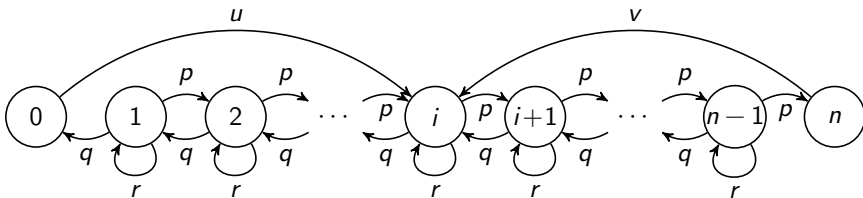
We observe that this is basically an order 2 linear recurrence. Once solved, we obtain  $j = 1, 2, \dots, n-1$ ,

$$\delta_j = \frac{\psi^{n-j} - 1}{\psi^n - 1}, \quad \text{if } \psi \neq 1;$$

if  $\psi = 1$  (if  $p = q$ ), then  $\delta_j = 1 - j/n$ .

# Using the construction introduced before

- Extending the previous results to the case of any number of absorbing states, choosing one of the transient states, say  $i$ , we build  $Y^{(i)}$  depicted here:



Linear system to solve for  $\pi = \pi^{(i)}$  here, using “vertical” cuts:

- $u\pi_0 = q\pi_1$
- for  $j = 1, 2, \dots, i-1$ :  $p\pi_j + u\pi_0 = q\pi_{j+1}$
- for  $k = i, i+1, \dots, n-2$ :  $p\pi_k = q\pi_{k+1} + v\pi_n$
- $p\pi_{n-1} = v\pi_n$

Now, we directly have a basically order 1 recurrence problem to solve.

- Following a similar approach as previously discussed in previous 2 sections, we obtain that

$$\delta_i = \frac{u \pi_0}{u \pi_0 + v \pi_n}.$$

- This can be further simplified, but no room here to go into the details.