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# A Markovian analysis of an IEEE-802.11 station with buffering

Guy Fayolle\*      Paul Muhlethaler †

## Abstract

The purpose of this paper is to analyze the so-called backoff technique of the IEEE 802.11 protocol with buffers. In contrast to existing models, packets arriving when a station (or node) is in backoff state are not discarded, but are stored in a buffer of infinite capacity. As in previous studies, the key point of our analysis hinges on the assumption that the time on the channel is viewed as a random succession of transmission slots (whose duration corresponds to the length of a packet) and mini-slots during which the backoff of the station is decremented. These events occur independently, with given probabilities and there is a Poisson external process. The state of a node is represented by a three-dimensional Markov chain in discrete-time, formed by the triple (backoff counter, number of packets at the station and the transmission attempt number). The stability (ergodicity) conditions are obtained for an arbitrary station and interpreted in terms of maximum throughput. Several approximations related to these models are also discussed.

**Keywords:** Wireless LAN, IEEE 802.11 protocol, Analytical Model, Markov chain, Generating Function, Ergodicity

## 1 Introduction and model definition

This section presents the main assumptions of the study and briefly summarizes how the access pattern is organized in the network.

The channel is supposed to be sampled at discrete time instants  $Z_i$ , so that it is sensed during consecutive time intervals  $Z_i - Z_{i-1}, i \geq 1$ , which are equal either to  $T$  (a *normal slot*) or to  $\sigma$  (a *mini-slot*). In some sense, we have thus introduced two embedded time-scales. Normal slots are used to transmit packets on the channel, whereas mini-slots correspond to time-intervals during which the backoff counters of all stations of the network are decremented.

In general, the selection between these two types of slots is random: a slot is a *normal* with probability  $r$ , and a *mini-slot* with probability  $1 - r$ . Nonetheless an arbitrary node will be said to be *greedy*, as, when the backoff of a station reaches 0, a transmission takes place in a slot which is necessarily a normal one of length  $T$ . This implies in particular that a sequence of slots is not exactly obtained as a pure outcome of repeated Bernoulli trials. We also assume that a collision can occur if at least two packets from two different stations are sent simultaneously, in which case this collision (usually indicated by a lack of acknowledgment) has the duration of the normal slot of length  $T$ . The duration of the acknowledgment (or lack of acknowledgment) is supposed to be part of the normal slot of length  $T$ .

The backoff and the retransmission are really the two main ingredients of the access algorithm of the IEEE 802.11 protocol. When a packet arrives at a station, it receives a backoff which counts the idle time before the transmission of the packet. If a collision occurs then the packet receives another backoff (usually greater than the previous one) for retransmission. This process stops when

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the packet is successfully transmitted. The detailed automaton of the access scheme is presented in the next section.

The main purpose of this article is to analyze the stability of an arbitrary IEEE 802.11 node. In contrast to the seminal model proposed by G. Bianchi [1], we allow infinite buffer at the node. In [2], the broadcast transmission was analyzed, but collision events were not considered by the access protocol.

The paper is organized as follows. Section 2 defines the parameters and the dynamics of the system model. Section 3 is devoted to the forward Kolmogorov's equations. Sections 4, 5, 6 are devoted to the detailed analysis of the steady-state behavior of the process. In particular, ergodicity conditions are explicitly obtained.

## 2 Dynamics

### 2.1 First notations

Let us introduce the following basic parameters.

- $n$  is the number of pending packets in the ego station.
- $m$  is the backoff stage, where  $m \in [0, 1, \dots, M]$ .
- $k$  is an integer representing the backoff time counter in an arbitrary window  $W_m$ , so that  $k \in [0, 1, 2, \dots, W_m - 1]$ . Typically, the size of the backoff window at the  $m$ -th stage will have the form  $W_m = \alpha^m W_0$ , with  $\alpha > 1$ .

An arbitrary station (or node) will be represented by a stochastic process, which is a three-dimensional random walk  $(K_{Z_i}, N_{Z_i}, M_{Z_i})_{i \geq 1}$ , where  $Z_i$  is the embedded increasing sequence of discrete random times at which the chain is observed. Indeed, at time  $Z_i$ ,  $K_{Z_i}$  stands for the value of the backoff counter,  $N_{Z_i}$  is the number of packets in the waiting queue at the station, and  $M_{Z_i}$  is the backoff stage number. The evolution of  $(K_{Z_i}, N_{Z_i}, M_{Z_i})$  will be analyzed in detail in the next section.

Taking into account the definitions given in Section 1, the sequence of transition instants  $Z_i, i \geq 1$ , satisfies the recursive stochastic relationship

$$Z_{i+1} = Z_i + \Delta_i [\mathbb{1}_{\{K_{Z_i} > 0\}} + \mathbb{1}_{\{K_{Z_i} = N_{Z_i} = 0\}}] + T \mathbb{1}_{\{K_{Z_i} = 0, N_{Z_i} > 0\}}, \quad (2.1)$$

with

$$\Delta_i = T \mathbb{1}_{\{B_i\}} + \sigma(1 - \mathbb{1}_{\{B_i\}}), \quad (2.2)$$

where  $\mathbb{1}_{\{\cdot\}}$  denotes the indicator function and  $B_i$  is the event  $\{\text{channel busy}\}$  at time  $Z_i$ . From the assumptions made above,  $\{\mathbb{1}_{\{B_i\}}, i \geq 1\}$  form a sequence of independent identically distributed random variables with  $\mathbb{P}(B_i) = r$ . The listening probability is thus  $1 - \mathbb{P}(B_i) = 1 - r$

### 2.2 Description of the protocol

The Wifi IEEE 802.11 algorithm is based on a backoff scheme and a retransmission if case of a collision on the channel. When a packet arrives in the ego station it receives a random backoff of  $k \in [0, 1, 2, \dots, W_0 - 1]$ , i.e the idle time before transmission will be  $k\sigma$ . The ego station monitors the channel and the backoff is decremented by one unit when an empty slot is sensed on the transmission channel, i.e. the event on the channel is a mini-slot  $\sigma$ . When the backoff counter  $k$  reaches 0, the packet in the ego station is transmitted. If the transmission is successful, then the ego station removes the pending packet from its waiting queue. If the transmission is a collision, then the transmission level is incremented, going from  $m$  to  $m + 1$ , and the value of the backoff counter of the packet is uniformly chosen over the window  $W_{m+1}$ . For  $W_m = \alpha^m W_0$ , the backoff window size is thus increased by the factor  $\alpha > 1$  (in IEEE 802.11,  $\alpha$  is usually 2). Whenever  $m = M$ , the packet remains at the  $M$ -th transmission level, with a backoff drawn uniformly over  $W_M$ . When the transmission is a success (this occurs with probability  $1 - p$ ), the pending packet is removed from the waiting queue of the ego station. If at least one pending packet remains in the queue, then this packet directly receives a backoff  $k$ , uniformly distributed over  $[0, 1, 2, \dots, W_0 - 1]$ ;

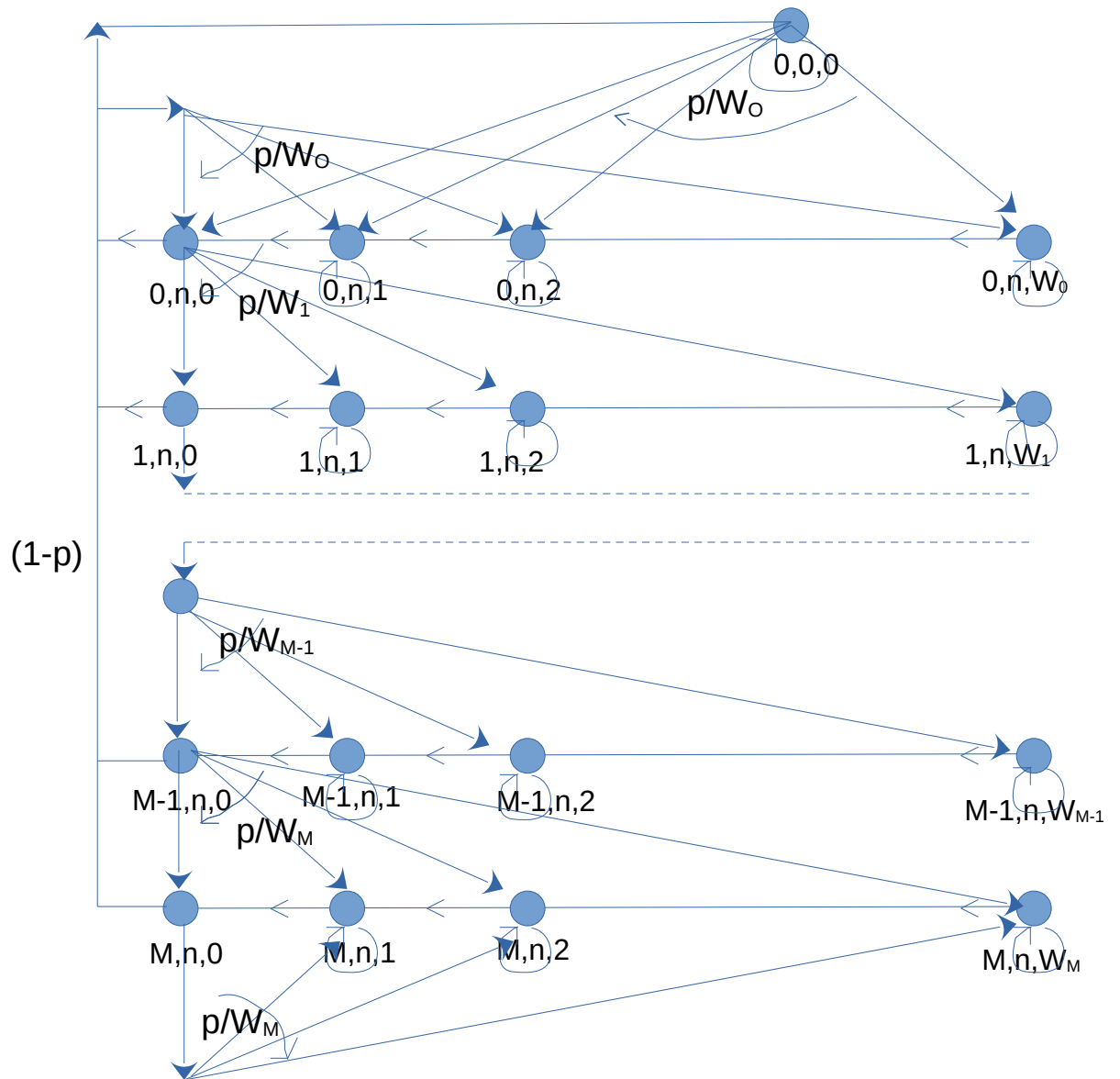


Fig. 2.1: The Wifi Markov chain

otherwise the automaton of the ego station enters an idle state, waiting for a new arrival which will receive a backoff  $k \in [0, 1, 2, \dots, W_0 - 1]$ . The automaton is shown in Figure 2.1.

Consequently, we will study the three-dimensional process  $(K_{Z_i}, N_{Z_i}, M_{Z_i})_{i \geq 1}$  presented in Section 2.1.

### 2.3 Parameters

The *isolated station* which we analyze in the sequel could be in fact viewed as an element of a network made of identical stations. The two following parameters are crucial.

- The probability  $p$  of collision when at least two packets (originating in fact from different stations) are simultaneously transmitted on the channel.
- The probability  $r$  that the channel is busy, whence  $\mathbb{P}(B_i) = r$ .

In our model,  $p$  and  $r$  are essentially viewed as constant exogenous parameters, but in the final Section 7 some suggestions are made pertaining to a network context .

## 3 Forward Kolmogorov's equations

In the sequel, we study the stochastic process  $(K_{Z_i}, N_{Z_i}, M_{Z_i})$ , which as mentioned above forms a three-dimensional Markov chain, with the conditional transition probability

$$p(k, n, m; Z_i) \stackrel{\text{def}}{=} P(K_{Z_i} = k, N_{Z_i} = n, M_{Z_i} = m; Z_i).$$

We are in a position to write the following classical forward Kolmogorov's equations [3].

(1) For  $0 \leq k \leq W_m - 1$  and  $1 \leq m \leq M - 1$ , the following general equilibrium equation holds:

$$\begin{aligned} p(k, n, m; Z_{i+1}) &= \mathbb{1}_{\{k > 0\}} \mathbb{1}_{\{B_i\}} \sum_{j=0}^{n-1} \frac{e^{-\lambda T} (\lambda T)^j}{j!} p(k, n-j, m; Z_i) \\ &+ \frac{p}{W_m} \sum_{j=0}^{n-1} \frac{e^{-\lambda T} (\lambda T)^j}{j!} p(0, n-j, m-1, Z_i) \\ &+ \mathbb{1}_{\{k < W_m - 1\}} (1 - \mathbb{1}_{\{B_i\}}) \sum_{j=0}^{n-1} \frac{e^{-\lambda \sigma} (\lambda \sigma)^j}{j!} p(k+1, n-j, m; Z_i). \end{aligned} \quad (3.1)$$

(2) For  $m = M$  and  $0 \leq k \leq W_M - 1$ , according to the dynamics of the model, we have the special equation

$$\begin{aligned} p(k, n, M; Z_{i+1}) &= \mathbb{1}_{\{k > 0\}} \mathbb{1}_{\{B_i\}} \sum_{j=0}^{n-1} \frac{e^{-\lambda T} (\lambda T)^j}{j!} p(k, n-j, M; Z_i) \\ &+ \frac{p}{W_M} \left( \sum_{j=0}^{n-1} \frac{e^{-\lambda T} (\lambda T)^j}{j!} p(0, n-j, M-1, Z_i) + \sum_{j=0}^{n-1} \frac{e^{-\lambda T} (\lambda T)^j}{j!} p(0, n-j, M, Z_i) \right) \\ &+ \mathbb{1}_{\{k < W_M - 1\}} (1 - \mathbb{1}_{\{B_i\}}) \sum_{j=0}^{n-1} \frac{e^{-\lambda \sigma} (\lambda \sigma)^j}{j!} p(k+1, n-j, M; Z_i). \end{aligned} \quad (3.2)$$

(3) For the last boundary case  $m = 0$ ,  $0 \leq k < W_0$  and  $n \geq 1$ , we have

$$\begin{aligned}
p(k, n, 0; Z_{i+1}) &= \mathbb{1}_{\{k>0\}} \mathbb{1}_{\{B_i\}} \sum_{j=0}^{n-1} \frac{e^{-\lambda T} (\lambda T)^j}{j!} p(k, n-j, 0; Z_i) \\
&+ \sum_{m=0}^M \frac{1-p}{W_0} \sum_{j=0}^n \frac{e^{-\lambda T} (\lambda T)^j}{j!} p(0, n+1-j, m; Z_i) \\
&+ \frac{1}{W_0} \left( \mathbb{1}_{\{B_i\}} \frac{e^{-\lambda T} (\lambda T)^n}{n!} + (1 - \mathbb{1}_{\{B_i\}}) \frac{e^{-\lambda \sigma} (\lambda \sigma)^n}{n!} \right) p(0, 0, 0; Z_i) \\
&+ \mathbb{1}_{\{k < W_0 - 1\}} (1 - \mathbb{1}_{\{B_i\}}) \sum_{j=0}^{n-1} \frac{e^{-\lambda \sigma} (\lambda \sigma)^j}{j!} p(k+1, n-j, 0; Z_i). \tag{3.3}
\end{aligned}$$

On the other hand, it is important to note that the dynamic of the algorithm implies that the states  $(k, 0, m)$  do not exist, unless when  $k = m = 0$  which corresponds to the state  $(0, 0, 0)$ . Then one can easily check that the Markov chain is irreducible, in agreement with the following additional equation

$$\begin{aligned}
p(0, 0, 0; Z_{i+1}) &= \mathbb{1}_{\{B_i\}} e^{-\lambda T} p(0, 0, 0; Z_i) + (1 - \mathbb{1}_{\{B_i\}}) e^{-\lambda \sigma} p(0, 0, 0; Z_i) \\
&+ e^{-\lambda T} (1-p) \sum_{m=0}^M p(0, 1, m; Z_i). \tag{3.4}
\end{aligned}$$

Equation 3.1 means that the state  $(k, n, m)$  can be reached from the state  $(k, n-j, m)$  after a normal slot encompassing  $j$  external arrivals. Likewise,  $(k, n, m)$  can be reached from  $(k+1, n-j, m)$  after sensing a mini-slot with  $j$  arrivals. Finally,  $(k, n, m)$  can be reached from  $(0, n-j, m-1)$  after a collision and  $j$  arrivals with probability  $p/W_m$ .

Equation 3.2 means that the state  $(k, n, M)$  can be reached from the state  $(k, n-j, M)$  after a normal slot with  $j$  external arrivals. Similarly,  $(k, n, M)$  can be reached either from  $(0, n-j, M)$  or from  $(0, n-j, M-1)$  after  $j$  arrivals with probability  $p/W_M$ . Finally,  $(k, n, M)$  can be reached from  $(k+1, n-j, M)$  after a mini-slot with  $j$  arrivals.

Equation 3.3 means that  $(k, n, 0)$  can be reached from  $(k, n-j, 0)$  after a normal slot and  $j$  arrivals. Likewise,  $(k, n, 0)$  can be reached from  $(0, n+1-j, m)$  after  $j$  arrivals with probability  $(1-p)/W_0$ , and  $(k, n, 0)$  can be reached from  $(0, 0, 0)$  after  $n$  arrivals during a full slot or a mini-slot with probability  $1/W_0$ . Finally, state  $(k, n, 0)$  can be reached from  $(k+1, n-j, 0)$  after  $j$  arrivals during a mini-slot. Finally state  $(k, n, 0)$  can be reached from state  $(k+1, n-j, 0)$  after sensing a mini-slot with  $j$  arrivals.

Equation 3.4 means that  $(0, 0, 0)$  can be reached either from  $(0, 0, 0)$  (no arrival during a slot or a mini-slot), or from the states  $(0, 1, m)$ , with  $0 \leq m \leq M$ , when no arrival occurs during a normal slot.

In the sequel, we will essentially analyze the steady-state behavior of an arbitrary station.

## 4 Stationary regime

The stationary behavior of the Markov chain  $(K_{Z_i}, N_{Z_i}, M_{Z_i})$  is defined by setting

$$p(k, n, m) \stackrel{\text{def}}{=} \lim_{i \rightarrow \infty} E[P(K_{Z_i} = k, N_{Z_i} = n, M_{Z_i} = m; Z_i)],$$

where the conditional expectation is taken with respect of the sequence of independent Bernoulli trials  $\{B_i, i \geq 1\}$  introduced in Section 1, and noting that  $\lim_{i \rightarrow \infty} Z_i = +\infty$ , almost surely (and even surely!).

Throughout this study, for the sake of brevity, we will simply write

$$p(0) \stackrel{\text{def}}{=} p(0, 0, 0).$$

Then, for  $1 \leq m < M$  and  $0 \leq k < W_m$ , equations (3.1) and (3.2) yield

$$\begin{aligned} p(k, n, m) &= \mathbb{1}_{\{k>0\}} r \sum_{j=0}^{n-1} \frac{e^{-\lambda T} (\lambda T)^j}{j!} p(k, n-j, m) + \frac{p}{W_m} \sum_{j=0}^{n-1} \frac{e^{-\lambda T} (\lambda T)^j}{j!} p(0, n-j, m-1) \\ &\quad + \mathbb{1}_{\{k < W_{m-1}\}} (1-r) \sum_{j=0}^{n-1} \frac{e^{-\lambda \sigma} (\lambda \sigma)^j}{j!} p(k+1, n-j, m), \end{aligned} \quad (4.1)$$

together with (the case  $m = M$ )

$$\begin{aligned} p(k, n, M) &= \mathbb{1}_{\{k>0\}} r \sum_{j=0}^{n-1} \frac{e^{-\lambda T} (\lambda T)^j}{j!} p(k, n-j, M) \\ &\quad + \frac{p}{W_M} \left( \sum_{j=0}^{n-1} \frac{e^{-\lambda T} (\lambda T)^j}{j!} p(0, n-j, M-1) + \sum_{j=0}^{n-1} \frac{e^{-\lambda T} (\lambda T)^j}{j!} p(0, n-j, M) \right) \\ &\quad + \mathbb{1}_{\{k < W_{M-1}\}} (1-r) \sum_{j=0}^{n-1} \frac{e^{-\lambda \sigma} (\lambda \sigma)^j}{j!} p(k+1, n-j, M). \end{aligned} \quad (4.2)$$

Similarly, for  $m = 0$ ,  $0 \leq k < W_0$  and  $n \geq 1$ , we get from (3.3) and (3.4)

$$\begin{aligned} p(k, n, 0) &= \mathbb{1}_{\{k>0\}} r \sum_{j=0}^{n-1} \frac{e^{-\lambda T} (\lambda T)^j}{j!} p(k, n-j, 0) + \sum_{m=0}^M \frac{1-p}{W_0} \sum_{j=0}^n \frac{e^{-\lambda T} (\lambda T)^j}{j!} p(0, n+1-j, m) \\ &\quad + \mathbb{1}_{\{k < W_{0-1}\}} (1-r) \sum_{j=0}^{n-1} \frac{e^{-\lambda \sigma} (\lambda \sigma)^j}{j!} p(k+1, n-j, 0) \\ &\quad + \frac{p(0)}{W_0} \left( \frac{r e^{-\lambda T} (\lambda T)^n + (1-r) e^{-\lambda \sigma} (\lambda \sigma)^n}{n!} \right), \end{aligned} \quad (4.3)$$

together with

$$[1 - r e^{-\lambda T} - (1-r) e^{-\lambda \sigma}] p(0) = (1-p) e^{-\lambda T} \sum_{m=0}^M p(0, 1, m). \quad (4.4)$$

## 5 Functional equations

Let us introduce the generating functions

$$G_{k,m}(x) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} p(k, n+1, m) x^n, \quad H_m(x, y) \stackrel{\text{def}}{=} \sum_{k=0}^{W_m-1} G_{k,m}(x) y^k, \quad 0 \leq m \leq M, \quad (5.1)$$

keeping in mind, in particular, that  $G_{0,m}(x) = H_m(x, 0) = 0, \forall m \geq 0$ .

### 5.1 First recurrence relationships

Let

$$R(x, y) \stackrel{\text{def}}{=} 1 - \frac{(1-r) e^{-\lambda \sigma (1-x)}}{y} - r e^{-\lambda T (1-x)}. \quad (5.2)$$

As before, we have to consider three different cases, depending on the values of  $m$ .

(a) For  $1 \leq m < M$ , by multiplying equation (4.1) by  $x^{n-1}$  and summing over  $n \geq 1$ , we obtain the following recurrence.

$$G_{k,m}(x) = r \mathbb{1}_{\{k>0\}} e^{-\lambda T(x-1)} G_{k,m}(x) + \frac{p e^{-\lambda T(1-x)}}{W_m} G_{0,m-1}(x) \\ + \mathbb{1}_{\{k < W_m - 1\}} (1-r) G_{k+1,m}(x) e^{-\lambda \sigma(1-x)},$$

whence, after summing over  $k$ ,  $0 \leq k < W_m$ ,

$$R(x, y) H_m(x, y) = (R(x, y) - 1) H_m(x, 0) + \frac{p(1-y^{W_m}) e^{-\lambda T(1-x)}}{W_m(1-y)} H_{m-1}(x, 0). \quad (5.3)$$

(b) Similarly, for  $m = M$ , we get from (4.2)

$$G_{k,M}(x) = r \mathbb{1}_{\{k>0\}} e^{-\lambda T(x-1)} G_{k,M}(x) + \frac{p e^{-\lambda T(1-x)}}{W_M} [G_{0,M-1}(x) + G_{0,M}(x)] \\ + \mathbb{1}_{\{k < W_M - 1\}} (1-r) G_{k+1,M}(x) e^{-\lambda \sigma(1-x)},$$

which yields in turn

$$R(x, y) H_M(x, y) = (R(x, y) - 1) H_M(x, 0) + \frac{p(1-y^{W_M}) e^{-\lambda T(1-x)}}{W_M(1-y)} [H_{M-1}(x, 0) + H_M(x, 0)]. \quad (5.4)$$

(c) For  $m = 0$ , the derivation of a functional relation from equation (4.3) is slightly more awkward, due to the presence of boundary terms appearing when  $n = 1$ . After a preliminary tedious algebra, we obtain

$$G_{k,0}(x) = r \mathbb{1}_{\{k>0\}} r e^{-\lambda T(1-x)} G_{k,0}(x) + \mathbb{1}_{\{k < W_0 - 1\}} (1-r) G_{k+1,0}(x) e^{-\lambda \sigma(1-x)} \\ + \frac{p(0) [R(0, 1) - R(x, 1)] + (1-p) [e^{-\lambda T(1-x)} S(x) - K]}{x W_0}, \quad (5.5)$$

with

$$S(x) \stackrel{\text{def}}{=} \sum_{m=0}^M G_{0,m}(x) = \sum_{m=0}^M H_m(x, 0), \text{ and } K \stackrel{\text{def}}{=} e^{-\lambda T} \sum_{m=0}^M p(0, 1, m) = e^{-\lambda T} S(0).$$

By equation (4.4), we have the simple relation  $(1-p)K = R(0, 1)p(0, 0, 0)$ , which allows to rewrite (5.5) in the slightly more compact form

$$G_{k,0}(x) = r \mathbb{1}_{\{k>0\}} r e^{-\lambda T(1-x)} G_{k,0}(x) + \mathbb{1}_{\{k < W_0 - 1\}} (1-r) G_{k+1,0}(x) e^{-\lambda \sigma(1-x)} \\ + \frac{(1-p) e^{-\lambda T(1-x)} S(x) - p(0) R(x, 1)}{x W_0}, \quad (5.6)$$

which, after multiplying by  $y^k$  and summing over  $k$ , leads to

$$R(x, y) H_0(x, y) = (R(x, y) - 1) H_0(x, 0) \\ + \frac{1-y^{W_0}}{W_0 x(1-y)} [(1-p) e^{-\lambda T(1-x)} S(x) - p(0) R(x, 1)] \quad (5.7)$$

## 5.2 The basic triangular scheme

The equations obtained in Section 5.1 lead to the following lemma.



**Lemma 5.1.**

$$\begin{cases} H_m(x, 0) = pU_m(x)H_{m-1}(x, 0), & 1 \leq m < M, \\ H_M(x, 0) = \frac{pU_M(x)}{1-pU_M(x)}H_{M-1}(x, 0), \\ H_0(x, 0) = \frac{U_0(x)[(1-p)S(x) - p(0)e^{\lambda T(1-x)}R(x, 1)]}{x}, \end{cases} \quad (5.8)$$

where

$$\begin{cases} \varphi(x) \stackrel{\text{def}}{=} \frac{(1-r)e^{-\lambda\sigma(1-x)}}{1-re^{-\lambda T(1-x)}}, \\ U_m(x) \stackrel{\text{def}}{=} \frac{(1-\varphi(x)W_m)e^{-\lambda T(1-x)}}{W_m(1-\varphi(x))}, & 0 \leq m \leq M. \end{cases} \quad (5.9)$$

*Proof.* Straightforward, noting that,  $\forall 0 \leq m \leq M$ , the function  $H_m(x, y)$  is analytic in  $y$ , as being a polynomial in  $y$ . So, for  $y = \varphi(x)$ , the kernel  $R(x, y)$  in equations (5.3),(5.4),(5.7) does vanish, and system (5.8) follows immediately.  $\blacksquare$

Furthermore, by using the definition of  $S(x)$  given in (5.5), we see that system (5.8) allows to get an explicit form for  $H_m(x, 0)$ ,  $\forall m \geq 0$ . But, before going into more detailed calculations, we will state the following important quantitative result.

**Proposition 5.2.** *Under ergodicity conditions, the probability  $p(0)$  that the system is idle is given by the formula*

$$p(0) = \frac{1}{1 + Ce^{\lambda T}(1 - re^{-\lambda T} - (1-r)e^{-\lambda\sigma})}, \quad (5.10)$$

where

$$C \stackrel{\text{def}}{=} \sum_{m=0}^M C_m p^m, \quad C_m = 1 + \frac{W_m - 1}{2(1-r)}, \quad 0 \leq m < M, \quad C_M = \frac{1}{1-p} \left[ 1 + \frac{W_M - 1}{2(1-r)} \right].$$

*Proof.* The only remaining unknown  $p(0, 0, 0)$  will be obtained from the normalization condition

$$\sum_{k \geq 0} \sum_{n \geq 1} \sum_{m=0}^M p(k, n, m) = \sum_{m=0}^M H_m(1, 1) = 1 - p(0). \quad (5.11)$$

From (5.3),(5.4),(5.7) and (5.8), after a straightforward algebra, we get

$$\begin{cases} R(x, 1)H_m(x, 1) = p[(R(x, 1) - 1)U_m(x) + e^{-\lambda T(1-x)}]H_{m-1}(x, 0), & 1 \leq m < M, \\ R(x, 1)H_M(x, 1) = \left[ R(x, 1) - 1 + \frac{e^{-\lambda T(1-x)}}{U_M(x)} \right] H_M(x, 0), \\ R(x, 1)H_0(x, 1) = \frac{[(1-p)S(x) - p(0)e^{\lambda T(1-x)}R(x, 1)][(R(x, 1) - 1)U_0(x) + e^{-\lambda T(1-x)}]}{x}. \end{cases} \quad (5.12)$$

Letting  $x = 1$  in system (5.8), we obtain in addition, since  $U_m(1) = 1 \forall m \geq 0$ , the following useful relations

$$\begin{cases} H_m(1, 0) = p H_{m-1}(1, 0), & 1 \leq m < M, \\ H_M(1, 0) = \frac{p}{1-p} H_{M-1}(1, 0), \\ H_0(1, 0) = (1-p)S(1). \end{cases} \quad (5.13)$$

**Remark 5.3.** It is worth noting that the 3 equations of system (5.13) are not independent. Indeed, the third equation is a direct consequence of the two first ones.

The idea will be now to divide all equations of (5.12) by  $R(x, 1)$ , to let  $x \rightarrow 1$ , and to use the normalizing condition (5.11). This gives

$$\begin{cases} H_m(1, 1) = p \left[ 1 + \frac{W_m - 1}{2(1-r)} \right] H_{m-1}(1, 0), & 1 \leq m < M, \\ H_M(1, 1) = \left[ 1 + \frac{W_M - 1}{2(1-r)} \right] H_M(1, 0), \\ H_0(1, 1) = \left[ 1 + \frac{W_0 - 1}{2(1-r)} \right] H_0(1, 0), \end{cases} \quad (5.14)$$

where in the last equation of (5.14) we have replaced  $(1-p)S(1)$  by  $H_0(1, 0)$ . Then, by summing all the equations in system (5.14), using (4.4) and (5.13), we obtain (5.10), and proof is terminated.  $\blacksquare$

## 6 Stability conditions for an arbitrary station

To ensure the ergodicity of the Markov chain  $(K_{Z_i}, N_{Z_i}, M_{Z_i})_{i \geq 1}$ , it suffices to exhibit a solution of system (5.8) analytic in the unit disk  $\mathcal{D}$ . With this in mind, we state the following lemma.

**Lemma 6.1.** *We have*

$$H_0(x, 0) = \frac{p(0)e^{\lambda T(1-x)}U_0(x)R(x, 1)}{T(x)}, \quad (6.1)$$

where

$$\begin{cases} F_m(x) \stackrel{\text{def}}{=} \prod_{i=1}^m U_i(x), & \forall m \geq 1, \quad \text{and } F_0(x) \stackrel{\text{def}}{=} 1, \\ T(x) \stackrel{\text{def}}{=} (1-p)U_0(x) \left[ \sum_{m=0}^{M-1} p^m F_m(x) + \frac{p^M F_M(x)}{1-pU_M(x)} \right] - x. \end{cases}$$

*Proof.* The result is obtained by a simple algebra from system (5.8), using in particular the intermediate relationship

$$H_m(x, 0) = p^m F_m(x) H_0(x, 0), \quad 1 \leq m < M. \quad (6.2)$$

**Theorem 6.2.** *The system is ergodic if and only if*

$$T'(1) < 0. \quad (6.3)$$

*Proof.* From equation (6.2), we note straight away that the analyticity of  $H_0(x, 0)$  in  $\mathcal{D}$  yields immediately the analyticity of  $H_m(x, 0)$ ,  $m \geq 1$  in  $\mathcal{D}$ . Hence, the ergodicity of the system is equivalent to the analyticity of  $H_0(x, 0)$  in  $\mathcal{D}$ .

To proceed further, we will employ the so-called Rouché's theorem, which we recall below for the sake of completeness.

**Theorem 6.3** (Rouché, see e.g. [4]). *For any two complex-valued functions  $f$  and  $g$  holomorphic inside some region  $\mathcal{H}$  with simple closed contour  $\partial\mathcal{H}$ , if  $|g(z)| < |f(z)|$  on  $\partial\mathcal{H}$ , then  $f$  and  $f + g$  have the same number of zeros inside  $\mathcal{H}$ , where each zero is counted with its multiplicity.*

On the unit circle  $\mathcal{C}$ , when  $|x| = 1, x \neq 1$ , we have  $|x| > |T(x) + x|$ , since  $|U_m(x)| < 1, \forall m$ , and  $T(1) = 0$ .

Assume  $T'(1) < 0$ . We construct a contour  $\mathcal{C}_\varepsilon$  consisting essentially of the unit circle  $\mathcal{C}$  continuously distorted around  $x = 1$  by a small notch (say an arc of circle of radius  $\varepsilon$  centered at 1), keeping the point 1 inside the domain  $\mathcal{D}_\varepsilon$  bounded by  $\mathcal{C}_\varepsilon$ . Then, choosing  $\varepsilon$  sufficiently small, we have

$$|x| > |T(x) + x|, \quad \forall x \in \mathcal{C}_\varepsilon.$$

Then Rouché's theorem says that the functions  $x$  and  $T(x)$  have the same number of zeros inside  $\mathcal{D}_\varepsilon$ , in this case exactly one. Since  $T(1) = 0$ , it follows that  $T(x)$  has no zero inside  $\mathcal{D}$ . Consequently, when  $T'(1) < 0$ , the function  $H_0(x, 0)$  is analytic in  $\mathcal{D}$  and the system is ergodic and the proof is concluded.  $\blacksquare$

## 6.1 Explicit maximum arrival rate $\lambda$ for stability

We will show below that the ergodicity condition (6.3) can be expressed as a linear inequality in  $\lambda$ . Introduce the quantities

$$a_i \stackrel{\text{def}}{=} \frac{W_i - 1}{2}, \quad i \geq 0, \quad \text{and} \quad A_m \stackrel{\text{def}}{=} \sum_{i=1}^m a_i.$$

Then after direct (albeit slightly painful!) calculations, we can write the following equality.

$$\begin{aligned} T'(1) &= \left[ (1-p) \sum_{m=1}^{M-1} p^m A_m + \frac{p^M (A_M - p A_{M-1})}{1-p} + a_0 \right] \varphi'(1) \\ &\quad + \lambda T \left[ (1-p) \sum_{m=1}^{M-1} m p^m + M p^M + \frac{p^{M+1}}{1-p} + 1 \right] - 1. \end{aligned}$$

From the definition (5.9), we have

$$\varphi'(1) = \frac{\lambda [(1-r)\sigma + rT]}{1-r},$$

and all these elements combined together lead to the following theorem.

**Theorem 6.4.** *The system is ergodic if and only if the arrival rate  $\lambda$  satisfies the inequality*

$$\lambda < \frac{1}{\frac{(1-r)\sigma + rT}{1-r} \left[ (1-p) \sum_{m=1}^{M-1} p^m A_m + \frac{p^M (A_M - p A_{M-1})}{1-p} + a_0 \right] + \frac{T}{1-p}}.$$

## 7 About possible extensions to a network context

Theorem 6.4 gives a necessary and sufficient condition for one arbitrary ego station to be stable, when  $p$  and  $r$  are considered as *exogenous parameters*. However, in a network with  $N$  identical stations, one can think of linking  $p$  and  $r$  to the behavior of an arbitrary ego station at steady state, as mentioned at the end of Section 2.3.

Let

$$\tau \stackrel{\text{def}}{=} \sum_{m=0}^M \sum_{n=1}^{\infty} p(0, n, m),$$

so that  $\tau$  is just the sum of the probabilities of being in the ego station with a backoff counter 0 at all the backoff stages, and it can indeed be viewed as a transmission rate.

Then a reasonable heuristic choice for  $(p, r)$  could be as follows.

- $p = 1 - (1 - \tau)^{N-1}$ , the probability of collision when at least two packets (sent by different stations) are simultaneously transmitted on the channel.
- $1 - (1 - \tau)^N$  is the probability that the channel is busy, whence  $\mathbb{P}(B_i) = 1 - (1 - \tau)^N = r$ .
- Hence,  $1 - r = (1 - p)(1 - \tau)$ .

In this context, preliminary numerical experiments show in fact that the stability condition seems also to depend on the existence of a solution to the fixed-point equation (in  $\tau$ )

$$H_0(1, 0) = (1 - p)S(1) = (1 - p)\tau,$$

derived in (5.13). This problem is the subject of an ongoing work.

## References

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