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1 Mathematical analysis of an observer for 2 solving inverse source wave problem

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38 Abstract

39 The objective of this work is to propose a method using observers to estimate a source
40 term of a wave equation from internal measurements in a subdomain. The first part of

41 the work consists in proving an identifiability result from classical observability
42 conditions for wave equations. We show that the source reconstruction is an ill-posed
43 inverse problem of degree 1 or 2 depending on the measurements type. This inverse
44 problem is solved using observers – a sequential strategy – that is proven to be
45 equivalent to a minimization of a cost functional with Tikhonov regularization.

46

47 1 Introduction

48 Inverse problems applied to wave-like propagation phenomena has received a lot of
49 attention with numerous applications in geophysics [35, 16], tomography [38, 44, 32],
50 medical imaging [37, 56] with the development of elastography [45, 52], non-destructive
51 testing [23]. In fact, several subcategories of inverse problems can be distinguished:
52 the reconstruction of initial conditions, the reconstruction of sources, and finally the
53 identification of constitutive parameters. First, the reconstruction of initial conditions
54 has been widely studied [39, 52] and is based on the proof of a so-called observability
55 inequality. This inequality, which has been studied extensively by the control community
56 [14, 27, 30] quantifies the initial condition to measurement operator invertibility which
57 in this case is linear, allowing for the analysis of reconstruction strategies [47, 55].
58 At the other end of the spectrum, parameter reconstruction is a nonlinear inversion
59 problem [28, 53], where identifiability and stability results are hard to obtain in the
60 general cases and often rely on Carleman inequalities, see for example [43, 49, 54] and
61 references therein. Finally, source reconstruction is the intermediate step between initial
62 conditions reconstruction and parameter reconstruction. In a first step, we consider a
63 source term with a separate variable, *i.e.*, the source is the product of a time-dependent
64 function multiplying a space-dependent function to be estimated. This source term
65 can take various forms, even being point sources [24, 26]. One of the advantages of
66 estimating such source terms is that one can show identifiability by relying on initial
67 condition observability inequalities [34, 17], while more general source terms may require
68 the use of dedicated Carleman estimates [48, 21]. From a practical point of view, there
69 are two main categories of methods for solving these inverse problems: Least-squares
70 minimization approaches [38, 46, 55] – also known as variational approaches or 4D-var
71 in the data assimilation community [10, 12, 22] – or sequential approaches, where a
72 new evolution problem integrating the measurements as a feedback pursues the target
73 solution. The least squares approach is certainly the oldest strategy [1, 4] and is
74 typically solved using adjoint equations [11]. The goal is to eventually find the optimal
75 trajectory that minimizes the discrepancy between the solution and the observations.
76 The sequential approach, also called the observer approach, is less studied. The
77 principle is to introduce feedback into the dynamics based on measurement to create
78 a new system, called the observer, which converges asymptotically to the observed
79 system over time. Starting from the reconstruction of the initial conditions of wave
80 problems, algorithms were developed [39] combining the observer and time-reversal
81 strategies in a comparable spirit to the back-and-forth nudging introduced in data
82 assimilation [31]. This strategy has been extended to unbounded domains by [52].
83 The proposed feedbacks are often easy to compute, and their efficiency is based on
84 stabilization principles inspired by the so-called Luenberger observer introduced for
85 finite-dimensional control problems [2, 6]. In the same community, it is also known that
86 the principles of dynamic programming [3] can also help to formulate observers that
87 asymptotically solve the least square minimization problem. Such a strategy is often
88 referred to as optimal filtering, since the feedback is also called a filter, and it is derived
89 directly from an optimal criterion. It has been extended to the estimation of initial

90 condition for infinite dimensional systems [5, 7] with recent works on numerical methods
 91 and their analysis, see e.g. [57] and references therein. Observers for reconstructing
 92 sources or identifying parameters have also been introduced in finite dimensional cases
 93 [25] with applications to discretized wave-like problems [33, 56].

94 Our goal is to extend the definition of such an optimal observer to source recon-
 95 struction in the infinite dimensional context of source inversion for wave problem. Our
 96 contribution includes the formulation of the observer, namely its existence, but also its
 97 asymptotic property, here called stability, and its robustness to measurement noise, as
 98 in all inverse problems. Notably, we base our analysis on general field measurements
 99 on a subdomain whereas most observability results for wave equations are classically
 100 based on velocity measurements.

101 This work is organized as follows. In the first section, we present the problem
 102 underlying the nature of the present observations. In the second section, we present
 103 how observability inequality allows us to quantify how poor the reconstruction of source
 104 terms is from the available field measurements. The next section is devoted to the
 105 formulation and analysis of the observer in the infinite dimensional context, developing
 106 an infinite dimensional Kalman filtering reduced to the source space. We then propose
 107 a discretization strategy based on the same principles at the discrete level. Finally, we
 108 illustrate the strategy with numerical experiments.

109 2 Problem setting

110 Let Ω be a bounded, connected open domain of class C^2 in \mathbb{R}^d . We consider the
 111 following wave equation

$$\begin{cases} \partial_{tt}u(x, t) - \Delta u(x, t) = \sigma(t)\theta(x), & (x, t) \in \Omega \times (0, T), \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), & x \in \Omega, \\ \partial_t u(x, 0) = v_0(x), & x \in \Omega, \end{cases} \quad (1)$$

112 where (u_0, v_0) is the initial condition, $\sigma \in L^2(0, T)$ is known and $\theta \in \mathcal{P} \subseteq L^2(\Omega)$, a
 113 Hilbert space, to be recovered from available measurements.

114 2.1 Semigroup approach

115 We rewrite (1) as a first order system in the state-space $\mathcal{Z} := H_0^1(\Omega) \times L^2(\Omega)$ equipped
 116 with the semi-norm H^1 and the L^2 standard scalar product,

$$\begin{cases} \dot{z}(t) = Az(t) + B(t)\theta, & t \in [0, T], \\ z(0) = z_0, \end{cases} \quad (2)$$

117 where

$$z_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \sigma(t)\text{Id} \end{pmatrix}, \quad (3)$$

and the operator A is an unbounded skew-adjoint operator from $\mathcal{D}(A) \subset \mathcal{Z}$ into \mathcal{Z}
 defined by

$$A = \begin{pmatrix} 0 & \text{Id} \\ \Delta_0 & 0 \end{pmatrix}, \quad \mathcal{D}(A) = \mathcal{D}(-\Delta_0) \times H_0^1(\Omega),$$

118 where Δ_0 denotes the Laplacian operator with homogeneous Dirichlet boundary condi-
 119 tions [29, Part II, Section 1.2.10]. The operator A is maximal dissipative, hence it is a
 120 generator of a C^0 -semigroup $(\Phi(t))_{t \geq 0}$ on \mathcal{Z} also denoted $(e^{tA})_{t \geq 0}$. Note that $(B(t))_{t \geq 0}$
 121 is a one parameter family of bounded operator from \mathcal{P} to \mathcal{Z} . We recall the various
 122 notions of solution and associated regularity.

123 **Theorem 2.1.** For all $T > 0$, the evolution equation (2) admits the following solutions:

124 1. For all $z_0 \in \mathcal{D}(A)$, $\sigma \in \mathbf{H}^1([0, T])$ and $\theta \in \mathcal{P}$, there exists a unique strong solution
125 $z \in \mathbf{C}^0([0, T]; \mathcal{D}(A)) \cap \mathbf{C}^1([0, T]; \mathcal{Z})$ of (2) and (3).

126

127 2. For all $z_0 \in \mathcal{Z}$, $\sigma \in \mathbf{L}^1(0, T)$ and $\theta \in \mathcal{P}$, there exists a unique mild-solution
128 $z \in \mathbf{C}^0([0, T]; \mathcal{Z})$ of (2) and (3) given by Duhamel's formula

$$z(t) = e^{tA}z_0 + \int_0^t e^{(t-s)A} \begin{pmatrix} 0 \\ \sigma(s) \end{pmatrix} \theta \, ds, \quad t \in [0, T]. \quad (4)$$

129 Moreover, the solution is a weak solution, namely $z \in \mathbf{L}^p((0, T); \mathcal{Z})$, for all
130 $v \in D(A^*)$, $v, z(\cdot) \in \mathbf{W}^{1,p}(0, T)$ and

$$\begin{cases} \frac{d}{dt}(v, z(t)) = (A^*v, z(t)) + (B(t)\sigma, v), & t \in [0, T] \text{ a.e.}, \\ z(0) = z_0. \end{cases}$$

131 *Proof.* See [29, Part II, Chapter 1, Section 3, Propositions 3.1 and 3.2]. □

132 **Remark 2.2.** From Theorem 2.1, it follows naturally:

133 1. For all $(u_0, v_0) \in \mathcal{D}(-\Delta_0) \times \mathbf{H}_0^1(\Omega)$, $\sigma \in \mathbf{H}^1([0, T])$ and $\theta \in \mathcal{P}$, there exists a
134 unique strong solution u of (1) belonging to

$$\mathbf{C}^0([0, T]; \mathcal{D}(-\Delta_0)) \cap \mathbf{C}^1([0, T]; \mathbf{H}_0^1(\Omega)) \cap \mathbf{C}^2([0, T]; \mathbf{L}^2(\Omega)).$$

135 2. For all $(u_0, v_0) \in \mathbf{H}_0^1(\Omega) \times \mathbf{L}^2(\Omega)$, $\sigma \in \mathbf{L}^1(0, T)$ and $\theta \in \mathcal{P}$, there exists a unique
136 mild-solution u of (1) belonging to

$$\mathbf{C}^0([0, T]; \mathbf{H}_0^1(\Omega)) \cap \mathbf{C}^1([0, T]; \mathbf{L}^2(\Omega)).$$

137 We introduce the so-called *sensitivity* operators, namely the one parameter family
138 $(L(t))_{t \geq 0}$ of bounded operator, such that $t \mapsto L(t)\theta \in \mathbf{C}^0([0, T]; \mathcal{Z})$ is solution to

$$\begin{cases} \dot{z}(t) = Az(t) + B(t)\theta, & t \in [0, T] \\ z(0) = 0, \end{cases} \quad (5)$$

139 then, the Duhamel's formula (4) also reads $z(t) = \Phi(t)z_0 + L(t)\theta$, $t \in [0, T]$. For further
140 use, we define \bar{L} which corresponds to L in the case of $\sigma \equiv 1$ namely the one parameter
141 family $(\bar{L}(t))_{t \geq 0}$ of bounded operator, such that $t \mapsto \bar{L}(t)\theta \in \mathbf{C}^0([0, T]; \mathcal{Z})$ is solution to

$$\begin{cases} \dot{z}(t) = Az(t) + \begin{pmatrix} 0 \\ \theta \end{pmatrix}, & t \in [0, T] \\ z(0) = 0. \end{cases} \quad (6)$$

142 2.2 Observation operator

143 We now need to describe the assumption concerning the available observations. In this
144 work, we suppose that we have at our disposal, at every time t , measurements during
145 the period $(0, T)$ of the restriction of u in $\omega \subset \Omega$ an open and nonempty subset of Ω
146 with Lipchitz boundary $\partial\omega$ and $\partial(\Omega \setminus \omega)$ also being Lipchitz. Moreover, we assume that
147 $\omega \supseteq \omega_{gcc}$ where (ω_{gcc}, T) satisfies the following condition [51]

$$\exists T, \forall \xi \in \Omega, \exists s \in [0, T], \exists \delta > 0; \omega_{gcc} \supset B(x_\xi(s); \delta), \quad (\text{GCC})$$

148 where $B(x_\xi(s); \delta)$ is a ball centered in $x_\xi(s)$ with a radius of δ and where a ray, starting
 149 from ξ and following the Descartes law of refraction in its most general senses [13, 51],
 150 is parametrized by $\{x_\epsilon(s), s \in [0, T]\}$. Note that is well known that such subdomain
 151 should contain a part of the boundary that we denote $\Gamma = \partial\Omega \cap \partial\omega$ with no cups.
 152 Otherwise, some *whispering gallery* rays may prevent the GCC to be fulfilled.
 153



Figure 1. An interior subdomain of observation ω which does satisfy the GCC condition (left) or does not (right)

154 Let us now present the definition of the observation operator C usually introduced
 155 in control and observation theory – see for instance [36] and references therein. We
 156 consider a restriction operator $I_\omega \in \mathcal{L}(H_0^1(\Omega), \mathcal{Y})$, where \mathcal{Y} is the observation space
 157 to be defined. Then formally, and in the absence of noise, at any time $t \in [0, T]$, the
 158 measurement procedure could produce a measurement $y(t) = I_\omega u(t)$. If we assume
 159 that the wave equation solution $u(t) \in H_0^1(\Omega)$ for all t , it is then natural to consider
 160 that $\mathcal{Y} = H_\Gamma^1(\omega)$ – the subspace of functions in $H^1(\omega)$ null on Γ . However, in practice,
 161 the measurements are always polluted by noise, hence in our deterministic context, the
 162 resulting measurement should belong to a less regular space, typically $\mathcal{Y} = L^2(\omega)$.
 163 If we consider regular measurements namely the observation space is $\mathcal{Y} = H_\Gamma^1(\omega)$, it
 164 is rightful to introduce

$$C = (I_\omega \ 0) \in \mathcal{L}(\mathcal{Z}, \mathcal{Y}), \text{ where } I_\omega \in \mathcal{L}(H_0^1(\Omega), H_\Gamma^1(\omega)). \quad (7)$$

165 As in [41], we equipped this space with the norm $\|\cdot\|_{H_\omega^1} = \|E_\omega(\cdot)\|_{H_0^1(\Omega)}$, where
 166 $E_\omega \in \mathcal{L}(H_\Gamma^1(\omega), H_0^1(\Omega))$ is defined by

$$E_\omega \phi := \psi \text{ the solution of } \begin{cases} \Delta \psi = 0, & \text{in } \Omega \setminus \omega, \\ \psi = \phi, & \text{in } \bar{\omega}, \\ \psi = 0, & \text{on } \partial\Omega. \end{cases} \quad (8)$$

167 We recall the following lemma.

168 **Lemma 2.3.** *The norm $\|\cdot\|_{H_\omega^1}$ is equivalent to the norm $\|\cdot\|_{H^1(\omega)}$.*

Proof. As in [41], on the one hand, we show that E_ω is bounded from $H_\Gamma^1(\omega)$ to $H_0^1(\Omega)$. Using a trace property as well as a lifting property, we obtain

$$\begin{aligned} \|\nabla(E_\omega \phi)\|_{L^2(\Omega)}^2 &\leq \|\nabla \phi\|_{L^2(\omega)}^2 + C^{\text{st}} \|\phi\|_{\partial\omega}^2_{H^{1/2}(\partial\omega)} \\ &\leq C^{\text{st}} \|\phi\|_{H^1(\omega)}^2. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \|\phi\|_{H^1(\omega)}^2 &= \|\nabla(E_\omega \phi)\|_{L^2(\omega)}^2 + \|\phi\|_{L^2(\omega)}^2 \\ &\leq \|\nabla(E_\omega \phi)\|_{L^2(\Omega)}^2 + \|E_\omega \phi\|_{L^2(\Omega)}^2. \end{aligned}$$

169 Using Poincaré inequality, we get

$$\|\phi\|_{\mathbf{H}^1(\omega)}^2 \leq (1 + C_p) \|\nabla(E_\omega \phi)\|_{\mathbf{L}^2(\Omega)}^2.$$

170

□

171 For the sake of completeness and since it will be a central tool, for our results, we
172 define the observation operator adjoint. One can show the following orthogonality
173 property using (8)

$$\forall \phi \in \mathbf{H}_\Gamma^1(\omega), \forall \psi \in \mathbf{H}_0^1(\Omega) \text{ such that } \psi|_\omega = 0, (\nabla(E_\omega \phi), \nabla \psi)_{\mathbf{L}^2(\Omega)} = 0.$$

Therefore, we obtain that for all $\phi \in \mathbf{H}_\Gamma^1(\omega)$ and $\psi \in \mathbf{H}_0^1(\Omega)$,

$$\begin{aligned} (\nabla(E_\omega \phi), \nabla \psi)_{\mathbf{L}^2(\Omega)} &= (\nabla(E_\omega \phi), \nabla(E_\omega \psi|_\omega))_{\mathbf{L}^2(\Omega)} - (\nabla(E_\omega \phi), \nabla(E_\omega \psi|_\omega - \psi))_{\mathbf{L}^2(\Omega)} \\ &= (\nabla(E_\omega \phi), \nabla(E_\omega \psi|_\omega))_{\mathbf{L}^2(\Omega)} = (\phi, \psi|_\omega)_{\mathbf{H}_\omega^1}. \end{aligned}$$

174 Then, with the Riesz representation, the adjoint of E_ω is I_ω , and we can directly infer
175 that

$$C^* = (E_\omega \quad 0)^t \in \mathcal{L}(\mathbf{H}_\Gamma^1(\omega); \mathcal{Z}).$$

176 We can see that the choice of norms for the observation operator is important, in
177 particular it directly impacts the definition of its adjoint. With this choice of observation
178 operator C , we restrict the space of observations to $\mathbf{H}_\Gamma^1(\omega)$. In practice, for noisy
179 measurements, the noise could be less regular, *i.e.* it could belong to a larger space
180 such as $\mathbf{L}^2(\omega)$. In this case, we define an observation operator where the image space
181 is larger than in the previous case, namely $\mathbf{L}^2(\omega)$ instead of $\mathbf{H}_\Gamma^1(\omega)$. The observation
182 operator becomes

$$C = (I_\omega \quad 0) : \mathcal{Z} \rightarrow \mathbf{L}^2(\omega) \text{ where } I_\omega \in \mathcal{L}(\mathbf{H}_0^1(\Omega), \mathbf{L}^2(\omega)). \quad (9)$$

183 Then, the observation operator adjoint is given by $C^* = (F_\omega \quad 0)^t$ where $F_\omega \in$
184 $\mathcal{L}(\mathbf{L}^2(\omega), \mathbf{H}_0^1(\Omega))$ is defined by

$$F_\omega \phi := \psi \text{ the solution of } \begin{cases} -\Delta \psi = \chi_\omega \phi, & \text{in } \Omega, \\ \psi = 0, & \text{on } \partial\Omega, \end{cases} \quad (10)$$

185 where χ_ω is the characteristic function of ω .

186 2.3 Inverse problem setting

187 We can now formulate the inverse problems. We consider an actual trajectory \check{u} modeled
188 as the first component of a mild solution \check{z} of (2) for a given and known (u_0, v_0) and
189 $\sigma(t)$ but an unknown $\check{\theta}$. Since the initial data are known, we can assume that they are
190 zero thanks to the linearity of the problem. Therefore, \check{u} is a particular solution of the
191 system

$$\begin{cases} \partial_{tt} u(x, t) - \Delta u(x, t) = \sigma(t)\theta(x), & (x, t) \in \Omega \times (0, T), \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = 0, & x \in \Omega, \\ \partial_t u(x, 0) = 0, & x \in \Omega, \end{cases} \quad (11)$$

192 and its first-order formulation

$$\begin{cases} \dot{z}(t) = Az(t) + B(t)\theta, & t \in [0, T], \\ z(0) = 0. \end{cases} \quad (12)$$

193 This actual trajectory is subject to measurements modeled by the bounded observation
 194 operator $C \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$. We denote the available measurements (or observations) y^δ
 195 which, as they are perturbed by some measurements' error – also known as the noise in
 196 our deterministic framework – are assumed to belong to $L^2(0, T; \mathcal{Y})$ equipped with the
 197 standard L^2 norm, hence $y^\delta(t)$ is defined for almost all $t \in (0, T)$. In our deterministic
 198 framework, the measurement *noise* intensity is represented by saying that there exists
 199 $\delta \in \mathbb{R}^+$ such that

$$\|y^\delta - Cz\|_{L^2(0, T; \mathcal{Y})}^2 \leq \delta^2 T.$$

200 Therefore, estimating $\tilde{\theta} \in \mathcal{P} \subset L^2(\Omega)$ from y^δ consists in inverting the linear *input-*
 201 *output* operator

$$\Psi_T : \begin{cases} L^2(\Omega) \rightarrow L^2((0, T); L^2(\omega)), \\ \theta \mapsto (t \mapsto CL(t)\theta). \end{cases} \quad (13)$$

202 We immediately find that Ψ_T is bounded as C is bounded from \mathcal{Z} to \mathcal{Y} and $L(t)$ is
 203 bounded from \mathcal{P} to \mathcal{Z} for all $t \in [0, T]$. Our objective is to study the condition for
 204 which Ψ_T is injective and more importantly when Ψ_T is surjective. In fact, we are
 205 going to show that Ψ_T is not surjective in $L^2(0, T; L^2(\omega))$ but define a *mildly* ill-posed
 206 problem of degree 1 or 2 (see [18]) depending on the measurements space. We will then
 207 propose a regularization strategy to invert Ψ_T using observers' methods.

208 3 From observability to identifiability

209 In this first part, our objective is to quantify the degree of ill-posedness of Ψ_T . In
 210 particular, we prove that we have enough information in the measurements to reconstruct
 211 θ . A fundamental tool for measuring the degree of ill-posedness are observability
 212 inequalities. These inequalities are used to control parameters from observations. The
 213 most important elements are the norms that appear in this type of inequality which
 214 quantifies the ill-posedness [9, 18].

215 3.1 Observability conditions for the wave equation

216 For the homogeneous wave equation with Dirichlet conditions in the case of *field*
 217 measurements in a subdomain, we will first develop a so-called observability inequality.

218 In our framework, we present the proposition which allows us to express an observability
 219 condition with *field* measurements by relying on the geometric control condition.
 220 We adopt the observability condition shown in [41], here revisited for observation
 221 domains that potentially contain a boundary part. In our case, the observation domain
 222 must contain part of the boundary.

223 **Proposition 3.1.** *Let u be an arbitrary solution of (1) with $\theta = 0$. Let $T_0 > 0$.
 224 Assume that there exists $T > T_0$ such that the GCC condition is satisfied for some ω_{gcc}
 225 strict subset of ω , in the time interval $(0, T)$. Then the following observability condition
 226 $\forall (u_0, v_0) \in H_0^1(\Omega) \times L^2(\Omega)$*

$$\int_0^T \|u(\cdot, t)\|_{H^1(\omega)}^2 dt \geq c_{obs}(T) \left(\|u_0\|_{H_0^1(\Omega)}^2 + \|v_0\|_{L^2(\Omega)}^2 \right) \quad (14)$$

227 *holds.*

228 *Proof.* The proof extended the one of [41]. In this proof, we write $a \lesssim b$ if there exists
 229 $c > 0$ independent of T and M such that $a \leq cb$. Since the geometric control condition
 230 of [13] holds, we have the classical observability inequality, with $c'_{obs}(T) > 0$

$$\int_0^T \int_\omega |\partial_t u(x, t)|^2 dx dt \geq c'_{obs}(T) \left(\|u_0\|_{H_0^1(\Omega)}^2 + \|v_0\|_{L^2(\Omega)}^2 \right). \quad (15)$$

We consider several subdomains satisfying $\omega_{gcc} \subseteq \omega_0 \subset \omega_\epsilon \subset \omega$. We define a cutoff function $\psi \in C^2(\Omega)$ satisfying

$$\psi(x) = \begin{cases} 0, & \text{if } x \in \Omega \setminus \omega_\epsilon \\ 1, & \text{if } x \in \omega_0, \end{cases}$$

231 and $0 \leq \psi(x) \leq 1$ for every $x \in \bar{\Omega}$. Particularly, we have

$$\frac{\partial \psi}{\partial n} \Big|_{\partial \omega \setminus \partial \Omega} = 0.$$

We introduce $\varphi(t) = t^2(T-t)^2$. Successively, integrating by parts, we obtain

$$\begin{aligned} 0 &= \int_0^T \int_\omega \varphi \psi (\partial_{tt} u - \Delta u) u \, dx \, dt \\ &= \int_0^T \int_\omega \ddot{\varphi} \psi \frac{|u|^2}{2} \, dx \, dt - \int_0^T \int_\omega \varphi \psi |\partial_t u|^2 \, dx \, dt \\ &\quad + \int_0^T \int_{\partial \omega} \varphi \frac{\partial \psi}{\partial n} \frac{|u|^2}{2} \, d\sigma \, dt - \int_0^T \int_\omega \varphi \Delta \psi \frac{|u|^2}{2} \, dx \, dt \\ &\quad \quad \quad + \int_0^T \int_\omega \varphi \psi |\nabla u|^2 \, dx \, dt. \end{aligned}$$

Using the definition of ψ , we get

$$\begin{aligned} \int_0^T \int_\omega \varphi \psi |\partial_t u|^2 \, dx \, dt + \int_0^T \int_\omega \varphi \Delta \psi \frac{|u|^2}{2} \, dx \, dt \\ = \int_0^T \int_\omega \varphi \psi |\nabla u|^2 \, dx \, dt + \int_0^T \int_\omega \ddot{\varphi} \psi \frac{|u|^2}{2} \, dx \, dt. \end{aligned}$$

232 We recall that for the system (1) with $\theta = 0$, we have classically the conservation energy
233 property namely

$$\mathcal{E}(0) := \frac{1}{2} \|u_0\|_{H_0^1(\Omega)}^2 + \frac{1}{2} \|v_0\|_{L_0^2(\Omega)}^2 = \frac{1}{2} \|u(t)\|_{H_0^1(\Omega)}^2 + \frac{1}{2} \|\partial_t u(t)\|_{L^2(\Omega)}^2 = \mathcal{E}(t). \quad (16)$$

234 Therefore, for a time $T > T_0$ and ϵ such that $T - 2\epsilon > T_0$, we find

$$\begin{aligned} \mathcal{E}(0) = \mathcal{E}(\epsilon) &\lesssim (c'_{\text{obs}}(T))^{-1} \int_\epsilon^{T-\epsilon} \int_{\omega_0} |\partial_t u|^2 \, dx \, dt \\ &\lesssim (c'_{\text{obs}}(T))^{-1} \int_\epsilon^{T-\epsilon} \int_{\omega_0} [\varphi \psi |\partial_t u|^2 + |\nabla u|^2] \, dx \, dt \\ &\lesssim (c'_{\text{obs}}(T))^{-1} \int_0^T \int_\omega [|\nabla u|^2 + |u|^2] \, dx \, dt, \end{aligned}$$

235 hence justifying (14). □

236 **Proposition 3.2.** *The observability constant $c_{\text{obs}}(T)$ of Proposition 3.1 satisfies, for*
237 *T large enough*

$$c_{\text{obs}}(T) \geq cT,$$

238 *where $c > 0$ is independent of T .*

239 *Proof.* Let $t > 0$. Applying Proposition 3.1 for a time $T_0 > 0$, we have

$$\int_t^{T_0+t} \|u(\cdot, t)\|_{\mathbf{H}^1(\omega)}^2 dt \geq 2c_{\text{obs}}(T_0)\mathcal{E}(t). \quad (17)$$

240 Let $T \in [NT_0; (N+1)T_0]$, $N \in \mathbb{N}^*$, we have

$$\int_0^T \|u(\cdot, t)\|_{\mathbf{H}^1(\omega)}^2 dt \geq \sum_{n=0}^{N-1} \int_{nT_0}^{(n+1)T_0} \|u(\cdot, t)\|_{\mathbf{H}^1(\omega)}^2 dt.$$

241 Using (17), we get

$$\int_0^T \|u(\cdot, t)\|_{\mathbf{H}^1(\omega)}^2 dt \geq \sum_{n=0}^{N-1} 2c_{\text{obs}}(T_0)\mathcal{E}(nT_0).$$

For all weak solution of (1) with $\theta = 0$, we have classically by energy conservation (16)

$$\int_0^T \|u(\cdot, t)\|_{\mathbf{H}^1(\omega)}^2 dt \geq 2NT_0 c_{\text{obs}}(T_0)\mathcal{E}(0) \geq 2(T - T_0) c_{\text{obs}}(T_0)\mathcal{E}(0).$$

242

□

243 This proposition provides a rough estimate of the behavior in time T of the
244 observation constant $c_{\text{obs}}(T)$. A more precise estimate can be obtained using micro-
245 local techniques, as in [50].

246 3.2 Regular field measurements in H^1

247 3.2.1 Identifiability with regular field measurements

248 From the observability condition adapted to initial condition reconstruction, we are
249 now going to deduce the parameter identifiability of our source reconstruction problem.
250 As a first step, we start by studying (12) when $\sigma \equiv 1$.

251 **Theorem 3.3.** *Let $\theta \in \mathcal{P}$, when ω satisfies the GCC condition, then for T large enough,*
252 *there exists a constant $c_{\text{idf}}(T) \geq cT$ where $c > 0$ independent of T such that*

$$\int_0^T \|C\bar{L}(t)\theta\|_{\mathbf{H}_\omega^1}^2 dt \geq c_{\text{idf}}(T)\|\theta\|_{\mathbf{H}^{-1}(\Omega)}^2, \quad (18)$$

253 *where the observation operator is the observation operator in $\mathbf{H}^1(\omega)$, namely $C =$
254 $(I_\omega \ 0) \in \mathcal{L}(\mathcal{Z}, \mathbf{H}_\omega^1(\omega))$ and \bar{L} is given by (6).*

255 *Proof.* The idea is to introduce the Riesz representation of θ seen as an element of \mathbf{H}^{-1}
256 that we denote $g \in \mathbf{H}_0^1(\Omega) \cap \mathcal{D}(-\Delta_0)$ and is given by $g = \Delta_0^{-1}\theta$. We introduce the
257 following dynamics which in fact is equivalent to (11) with $\sigma(t) \equiv 1$.

$$\begin{cases} \partial_{tt}u_g(x, t) - \Delta u_g(x, t) = -\Delta_0 g(x), & (x, t) \in \Omega \times (0, T), \\ u_g(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u_g(x, 0) = 0, & x \in \Omega, \\ \partial_t u_g(x, 0) = 0, & x \in \Omega. \end{cases} \quad (19)$$

258 Note that $u_g \in \mathcal{D}(\Delta_0)$ and $\partial_t u_g \in \mathbf{H}^1(\Omega)$. We now proceed by standard energy estimate
259 using the following energy functional

$$\mathcal{E}_g(t) := \frac{1}{2}\|\nabla u_g(t)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\partial_t u_g(t)\|_{L^2(\Omega)}^2.$$

260 Integrating in space and time the first equation of (19) multiplied by $\partial_t u_g$, we get

$$\int_0^T \int_{\Omega} \partial_{tt} u_g \partial_t u_g \, dx \, dt - \int_0^T \int_{\Omega} \Delta u_g \partial_t u_g \, dx \, dt = - \int_0^T \int_{\Omega} \Delta_0 g \partial_t u_g \, dx \, dt.$$

261 Integrating by parts while using that g is time-independent, we obtain

$$\mathcal{E}_g(T) = - \int_0^T \int_{\Omega} \Delta_0 g \partial_t u_g \, dx \, dt = - \int_{\Omega} \Delta_0 g u_g(T) \, dx = \int_{\Omega} \nabla g \nabla u_g(T) \, dx.$$

262 By Cauchy–Schwarz inequality, we get

$$\mathcal{E}_g(T) \leq \|\nabla g\|_{L^2(\Omega)} \|\nabla u_g(T)\|_{L^2(\Omega)} \leq \sqrt{2} \|\nabla g\|_{L^2(\Omega)} \mathcal{E}_g^{\frac{1}{2}}(T).$$

263 This calculation finally leads to

$$\|\partial_t u_g(T)\|_{L^2(\Omega)} \leq 2 \|\nabla g\|_{L^2(\Omega)}. \quad (20)$$

264 We now introduce the system satisfied by $\tilde{u}_g = u_g - g$ in order to obtain an initial
265 value problem for which we can use the observability condition of Proposition 3.1.

$$\begin{cases} \partial_{tt} \tilde{u}_g(x, t) - \Delta \tilde{u}_g(x, t) = 0, & (x, t) \in \Omega \times (0, T), \\ \tilde{u}_g(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ \tilde{u}_g(x, 0) = -g(x), & x \in \Omega, \\ \partial_t \tilde{u}_g(x, 0) = 0, & x \in \Omega. \end{cases} \quad (21)$$

Using the observability condition of Proposition 3.1, we get

$$\int_0^T \|u_g - g\|_{\mathbb{H}^1(\omega)}^2 \, dt = \int_0^T \|\tilde{u}_g\|_{\mathbb{H}^1(\omega)}^2 \, dt \geq C_{\text{obs}}(T) \|g\|_{\mathbb{H}_0^1(\Omega)}^2. \quad (22)$$

266 In order to get rid of the variable g in the left-hand side, we write that

$$\int_0^T \|u_g - g\|_{\mathbb{H}^1(\omega)}^2 \, dt = \int_0^T \|u_g\|_{\mathbb{H}^1(\omega)}^2 \, dt + \int_0^T \|g\|_{\mathbb{H}^1(\omega)}^2 \, dt - 2 \int_0^T (u_g, g)_{\mathbb{H}^1(\omega)} \, dt. \quad (23)$$

267 The cross product can be controlled. Integrating the first equation of (19) in time
268 times Δ_0^{-1} and then multiplying by g in the sense of the scalar product, we have

$$\int_0^T (u_g, g)_{\mathbb{H}^1(\omega)} \, dt = T \|g\|_{\mathbb{H}^1(\omega)}^2 + (\Delta_0^{-1} \partial_t u_g(T), g)_{\mathbb{H}^1(\omega)}.$$

269 Injecting this result in (23), we get

$$\int_0^T \|u_g - g\|_{\mathbb{H}^1(\omega)}^2 \, dt = \int_0^T \|u_g\|_{\mathbb{H}^1(\omega)}^2 \, dt - T \|g\|_{\mathbb{H}^1(\omega)}^2 - 2(\Delta_0^{-1} \partial_t u_g(T), g)_{\mathbb{H}^1(\omega)}.$$

From Cauchy–Schwarz and Poincaré inequalities, we obtain

$$\begin{aligned} \int_0^T \|u_g - g\|_{\mathbb{H}^1(\omega)}^2 \, dt &\leq \int_0^T \|u_g\|_{\mathbb{H}^1(\omega)}^2 \, dt - T \|g\|_{\mathbb{H}^1(\omega)}^2 \\ &\quad + 2(C_p + 1) \|\nabla \Delta_0^{-1} \partial_t u_g(T)\|_{L^2(\Omega)} \|g\|_{\mathbb{H}^1(\omega)}. \end{aligned} \quad (24)$$

270 Since $\nabla \Delta_0^{-1}$ is a bounded operator from $L^2(\Omega)$ to $L^2(\Omega)$, there exists a positive scalar
271 C^{st} such that

$$\|\nabla \Delta_0^{-1} \partial_t u_g(T)\|_{L^2(\Omega)} \leq C^{\text{st}} \|\partial_t u_g(T)\|_{L^2(\Omega)}. \quad (25)$$

Injecting now (25) in (24), we have

$$\begin{aligned} \int_0^T \|u_g - g\|_{\mathbb{H}^1(\omega)}^2 dt &\leq \int_0^T \|u_g\|_{\mathbb{H}^1(\omega)}^2 dt - T\|g\|_{\mathbb{H}^1(\omega)}^2 \\ &\quad + 2C^{\text{st}}(C_p + 1)\|\partial_t u_g(T)\|_{\mathbb{L}^2(\Omega)}\|g\|_{\mathbb{H}^1(\omega)}. \end{aligned}$$

Thanks to (20) and using Young inequality, we obtain

$$\begin{aligned} \int_0^T \|u_g - g\|_{\mathbb{H}^1(\omega)}^2 dt &\leq \int_0^T \|u_g\|_{\mathbb{H}^1(\omega)}^2 dt - T\|g\|_{\mathbb{H}^1(\omega)}^2 \\ &\quad + 2\epsilon C^{\text{st}}(C_p + 1)\|g\|_{\mathbb{H}_0^1(\Omega)}^2 + \frac{2}{\epsilon}C^{\text{st}}(C_p + 1)\|g\|_{\mathbb{H}^1(\omega)}^2. \end{aligned}$$

From the observability condition (22), we finally obtain

$$(C_{\text{obs}}(T) - 2\epsilon C^{\text{st}}(C_p + 1))\|g\|_{\mathbb{H}_0^1(\Omega)}^2 \leq \int_0^T \|u_g\|_{\mathbb{H}^1(\omega)}^2 dt - \left(T - \frac{2}{\epsilon}C^{\text{st}}(C_p + 1)\right)\|g\|_{\mathbb{H}^1(\omega)}^2.$$

272 Choosing $\epsilon = \frac{C_{\text{obs}}(T)}{4C^{\text{st}}(1 + C_p)}$ gives us

$$\frac{C_{\text{obs}}(T)}{2}\|g\|_{\mathbb{H}_0^1(\Omega)}^2 \leq \int_0^T \|u_g\|_{\mathbb{H}^1(\omega)}^2 dt - \left(T - \frac{8C^{\text{st}^2}(1 + C_p)^2}{C_{\text{obs}}(T)}\right)\|g\|_{\mathbb{H}^1(\omega)}^2.$$

273 For T large enough such that $T \geq \frac{8C^{\text{st}^2}(1 + C_p)^2}{C_{\text{obs}}(T)}$, we have

$$\frac{C_{\text{obs}}(T)}{2}\|g\|_{\mathbb{H}_0^1(\Omega)}^2 \leq \int_0^T \|u_g\|_{\mathbb{H}^1(\omega)}^2 dt.$$

274 We conclude by noting that, by definition, $\|g\|_{\mathbb{H}_0^1(\Omega)} = \|\theta\|_{\mathbb{H}^{-1}(\Omega)}$ and by Lemma 2.3,
275 $\|u_g\|_{\mathbb{H}^1(\omega)} \leq C^{\text{st}}\|u_g\|_{\mathbb{H}_\omega^1}$. □

276 From the identifiability property with $\sigma \equiv 1$, we deduce an identifiability property
277 for the source reconstruction in (12) for a class of $\sigma \in \mathbb{H}^1(0, T)$.

278 **Theorem 3.4.** *Let $\theta \in \mathcal{P}$ and u be a solution of (11). Let $T_0 > 0$. Assuming*
279 *$\sigma(t) \in \mathbb{H}^1(0, T)$ and $\sigma(0) \neq 0$ and (ω, T_0) satisfies the GCC condition, for $T \geq T_0$ large*
280 *enough, then there exists a positive scalar $c'_{\text{idf}}(T)$ depending only on T such that*

$$\int_0^T \|CL\theta\|_{\mathbb{Y}}^2 dt = \int_0^T \|u\|_{\mathbb{H}_\omega^1}^2 dt \geq c'_{\text{idf}}(T)\|\theta\|_{\mathbb{H}^{-1}(\Omega)}^2, \quad (26)$$

281 *where the observation operator is the observation operator in $\mathbb{H}_\Gamma^1(\omega)$, namely $C =$*
282 *$(I_\omega \ 0) \in \mathcal{L}(\mathcal{Z}, \mathbb{H}_\Gamma^1(\omega))$.*

Proof. This proof is inspired by [34]. Given the scalar nature of σ , we use the first-order formulation of the system, so that observations of the solution read, for all $t \in [0, T]$

$$\begin{aligned} u|_\omega(t) &= \int_0^t C e^{(t-s)A} \sigma(s) \begin{pmatrix} 0 \\ \theta \end{pmatrix} ds = CL(t)\theta \\ &= \int_0^t \sigma'(t-s) \left(\int_0^s C e^{\tau A} \begin{pmatrix} 0 \\ \theta \end{pmatrix} d\tau \right) ds + \sigma(0) \left(\int_0^t C e^{\tau A} \begin{pmatrix} 0 \\ \theta \end{pmatrix} d\tau \right) \\ &= \int_0^t \sigma'(t-s) C \bar{L}(s) \theta ds + \sigma(0) C \bar{L}(t) \theta. \end{aligned}$$

283 We define a Volterra operator $S : L^2(0, T; \mathcal{Y}) \rightarrow L^2(0, T; \mathcal{Y})$ ([15, Part 1, Chapter 2]),

$$Sy(t) = y(t) + \int_0^t k_T(t-s)y(s) ds, \text{ where } k_T(t) = \begin{cases} \frac{\sigma'(t)}{\sigma(0)} & t \in [0, T], \\ 0 & t > T. \end{cases} \quad (27)$$

284 Using this Volterra operator, we have

$$u|_\omega(t) = \sigma(0) S \circ C\bar{L}(t)\theta, \quad t \in [0, T]. \quad (28)$$

285 Since $k_T \in L^1(\mathbb{R}^+)$, using [15, Part 1, Chapter 2, Theorem 3.1], it is well-known that S
286 is a linear bijective continuous operator from $L^2(0, T; \mathcal{Y})$ onto $L^2(0, T; \mathcal{Y})$. In particular,
287 there exists a unique $r_T \in L^1(0, T)$ such that

$$y(t) = Sy(t) - \int_0^t r_T(t-s)Sy(s) ds. \quad (29)$$

288 Consequently, we obtain

$$\|y\|_{L^2(0, T; \mathcal{Y})} \leq (1 + \|r_T\|_{L^1(0, T)}) \|Sy\|_{L^2(0, T; \mathcal{Y})}. \quad (30)$$

289 Combining (28) and (30) with $y = C\bar{L}\theta$, we have

$$\|C\bar{L}\theta\|_{L^2(0, T; \mathcal{Y})} \leq (1 + \|r_T\|_{L^1(0, T)}) \frac{1}{|\sigma(0)|} \|u|_\omega\|_{L^2(0, T; \mathcal{Y})}. \quad (31)$$

290 Combining the above equation with the observability condition of Proposition 3.3, we
291 conclude the proof. \square

292 We deduce from the observability inequality (26) that the observations should
293 belong to $H^1(\omega)$, for a stable reconstruction is only in a H^{-1} norm. As a consequence,
294 we face an ill-posed inverse problem of degree 1 for an expected reconstruction in L^2
295 for measurements in H^1 ,

296 **Remark 3.5.** *In some cases, it is possible to affirm that the identifiability constant of*
297 *Theorem 3.4 has the same dependence in T as that of the Theorem 3.3, for instance,*
298 *when σ generates a persistent excitation. Let us take an example of such function*

$$\sigma(t) = \frac{1}{1 + \beta}(e^{-\alpha t} + \beta), \text{ where } \beta, \alpha \text{ are two positive constants.}$$

299 *In fact, this corresponds to a small perturbation of the case where $\sigma(t) \equiv 1$.*

300 *Following the proof of Theorem 3.4 one has to show that $\|r_T\|_{L^1(0, T)}$ is independent of*
301 *T where r_T is defined implicitly by σ (and k_T) in (29). On one hand, we have*

$$\begin{aligned} \left\| \frac{\sigma'(t)}{\sigma(0)} \right\|_{L^1(0, T)} &= \|k_T\|_{L^1(0, T)} = \left\| -\frac{\alpha}{1 + \beta} e^{-\alpha t} \right\|_{L^1(0, T)} \leq \frac{\alpha}{1 + \beta} \int_0^T e^{-\alpha t} dt, \\ &\leq \frac{1}{1 + \beta} (1 - e^{-\alpha T}) \leq \frac{1}{1 + \beta}. \end{aligned}$$

302 *From [15, Part 1, Chapter 2], we have an explicit expression for r_T in terms of k_T ,*

$$r_T = \sum_{j=1}^{\infty} (-1)^{j-1} (k_T)^{*j},$$

303 where $(k_T)^{*j}$ is the $(j - 1)$ -fold convolution of k_T with itself. Hence,

$$\|r_T\|_{L^1(0,T)} \leq \sum_{j=1}^{\infty} \|k_T\|_{L^1(0,T)}^j.$$

304 The common ratio $\|k_T\|_{L^1(0,+\infty)}$ is smaller than one, hence,

$$\|r_T\|_{L^1(0,T)} \leq \left(1 - \|k_T\|_{L^1(0,T)}\right)^{-1} \leq \frac{1}{1 - \frac{1}{1+\beta}} = \frac{1+\beta}{\beta},$$

305 which does not depend on time T .

306 As we have seen in the previous proof, the dependency of the identifiability constant
307 with respect time is not always explicit. When we consider the simpler case $\sigma(t) \equiv 1$,
308 we have a constant $C^{\text{st}}(T) \approx T$ as well as when σ is defined as follows:

$$\forall T > 0, \sigma = \bar{\sigma}|_{[0,T]} \text{ with } \bar{\sigma} \in H^1(\mathbb{R}^+) \text{ and } \left\| \frac{\bar{\sigma}'}{\sigma(0)} \right\|_{L^1(\mathbb{R}^+)} \leq 1,$$

309 as illustrated in the previous example in Remark 3.5. One can compute that any
310 compactly support function will not satisfy this condition. Later, we use the following
311 assumption when explicitly mentioned.

312 **Assumption 1.** *The identifiability constant of Theorem (3.4) satisfies*

$$c'_{\text{idf}}(T) \geq cT, \text{ with } c > 0 \text{ independent of } T.$$

313 3.2.2 From the identifiability result to a generalized Tikhonov regularization
314 strategy

315 We recall that we assume that a target trajectory \check{u} produced by a target source term
316 parameter $\check{\theta}$ has resulted in a set of measurements y^δ where

$$\int_0^T \|y^\delta(x, t) - \check{u}(x, t)\|_{H^1(\omega)}^2 dt \leq \delta^2 T, \quad (32)$$

for a parameter $\delta > 0$. We introduce a cost functional to minimize $\mathcal{J}_T : H_0^1(\Omega) \rightarrow \mathbb{R}$
defined by

$$\begin{aligned} \mathcal{J}_T(\theta) &= \frac{\epsilon}{2} \|\theta\|_{H_0^1(\Omega)}^2 + \frac{1}{2} \int_0^T \|y^\delta(t) - u_{|\theta}\|_{H^1(\omega)}^2 dt \\ &= \frac{\epsilon}{2} \|\theta\|_{H_0^1(\Omega)}^2 + \frac{1}{2} \int_0^T \|y^\delta(t) - Cz_{|\theta}\|_{H^1(\omega)}^2 dt, \end{aligned} \quad (33)$$

317 where $u_{|\theta}$ is the solution of (11) associated with the parameter θ , C is defined by (7)
318 and $\epsilon > 0$. Since the functional is quadratic and the application $\theta \mapsto u_{|\theta}$ is continuous
319 and $\theta \mapsto \|\theta\|_{H_0^1(\Omega)}^2$ is coercive, we have then the existence of a unique minimum of \mathcal{J}_T .
320 We denote the unique minimizer in $H_0^1(\Omega)$ by $\bar{\theta}_T := \underset{\theta \in H_0^1(\Omega)}{\text{argmin}} \mathcal{J}_T$.

321 **Theorem 3.6.** *We assume that there exists $M > 0$ such that*

$$\check{\theta} \in H_0^1(\Omega) \text{ and } \|\check{\theta}\|_{H_0^1(\Omega)} \leq M. \quad (34)$$

322 Furthermore, we also assume that the available measurements are noise-free, i.e. we
323 have for all $t \in [0, T]$

$$y^\delta(t) = \tilde{u}|_\omega(t).$$

324 Let $T_0 > 0$. Assuming $\sigma(t) \in H^1(0, T)$ and $\sigma(0) \neq 0$ and (ω, T_0) satisfies the GCC
325 condition, for $T \geq T_0$ large enough and ϵ small enough, then there exists a positive
326 scalar $C^{st}(T)$ depending only on T such that

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{st}(T) \epsilon^{\frac{1}{4}} M, \quad (35)$$

327 where $\bar{\theta}_T$ minimizes the cost functional (33).

328 Moreover, under Assumption 1, there exists a positive scalar C^{st} independent of T such
329 that

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{st} T^{-\frac{1}{4}} \epsilon^{\frac{1}{4}} M. \quad (36)$$

330 *Proof.* In this proof, we write $a \lesssim b$ if there exists $c > 0$ independent of T, ϵ and M
331 such that $a \leq cb$.

332 Let $\tilde{u} = \bar{u} - \check{u}$ be the solution of the following dynamics

$$\begin{cases} \partial_{tt}\tilde{u}(x, t) - \Delta\tilde{u}(x, t) = \sigma(t)(\bar{\theta}_T(x) - \check{\theta}(x)), & (x, t) \in \Omega \times (0, T), \\ \tilde{u}(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ \tilde{u}(x, 0) = 0, & x \in \Omega, \\ \partial_t\tilde{u}(x, 0) = 0, & x \in \Omega. \end{cases} \quad (37)$$

333 Identifying L^2 to its dual, we have

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 \leq \langle \bar{\theta}_T - \check{\theta}, \bar{\theta}_T - \check{\theta} \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \leq \|\bar{\theta}_T - \check{\theta}\|_{H_0^1(\Omega)} \|\bar{\theta}_T - \check{\theta}\|_{H^{-1}(\Omega)}.$$

334 Using the identifiability property (26) of Theorem 3.4, we get

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 \leq (c'_{\text{idf}}(T))^{-\frac{1}{2}} \|\bar{\theta}_T - \check{\theta}\|_{H_0^1(\Omega)} \left(\int_0^T \|\bar{u} - \check{u}\|_{H_w^1}^2 dt \right)^{\frac{1}{2}}. \quad (38)$$

335 By triangle inequality and since we consider free-noise observations, we have

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 \leq (c'_{\text{idf}}(T))^{-\frac{1}{2}} \left(\|\bar{\theta}_T\|_{H_0^1(\Omega)} + \|\check{\theta}\|_{H_0^1(\Omega)} \right) \left(\int_0^T \|\bar{u} - y^\delta\|_{H_w^1}^2 dt \right)^{\frac{1}{2}}.$$

336 Using the definition of the cost-functional (33) taking into account the norm equivalence
337 of Lemma 2.3, and the prior on the parameter (34), we get

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 \lesssim (c'_{\text{idf}}(T))^{-\frac{1}{2}} \left((2\epsilon^{-1} \mathcal{J}_T(\bar{\theta}_T))^{\frac{1}{2}} + M \right) (2\mathcal{J}_T(\bar{\theta}_T))^{\frac{1}{2}}.$$

338 Remembering that $y^\delta(t) = \tilde{u}|_\omega(t)$ and using again (34), we can bound the cost-functional
339 as follows

$$\mathcal{J}_T(\bar{\theta}_T) \leq \mathcal{J}_T(\check{\theta}) \leq \frac{\epsilon}{2} M^2.$$

340 Eventually, we obtain

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 \lesssim (c'_{\text{idf}}(T))^{-\frac{1}{2}} \left((M^2)^{\frac{1}{2}} + M \right) (\epsilon M^2)^{\frac{1}{2}},$$

341 which corresponds to the desired result (35).

342 In addition, under Assumption 1, we have that there exists $c > 0$ independent of T such
343 that

$$c'_{\text{idf}}(T) \geq cT,$$

344 which leads to (36). □

345 **Theorem 3.7.** *We assume that there exists $M > 0$ such that (34) holds. Let $T_0 > 0$.*
 346 *Assuming $\sigma(t) \in \mathbf{H}^1(0, T)$ and $\sigma(0) \neq 0$ and (ω, T_0) satisfies the GCC condition, for*
 347 *$T \geq T_0$ large enough and ϵ small enough, then there exists a positive scalar $C^{st}(T)$*
 348 *depending only on T such that*

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{st}(T) \epsilon^{-\frac{1}{4}} \delta, \quad (39)$$

349 *where $\bar{\theta}_T$ minimizes the cost functional (33).*

350 *Moreover, under Assumption 1, there exists a positive scalar C^{st} independent of T such*
 351 *that*

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{st} T^{\frac{1}{4}} \epsilon^{-\frac{1}{4}} \delta. \quad (40)$$

Proof. In this proof, we write $a \lesssim b$ if there exists $c > 0$ independent of T, ϵ, δ and M such that $a \leq cb$.

Reasoning exactly as in the proof of Theorem 3.6, we can obtain (38). Inserting $y^\delta - y^\delta = 0$ in (38) and by triangular inequality we find

$$\begin{aligned} \|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 &\lesssim (c'_{\text{idf}}(T))^{-\frac{1}{2}} \left(\|\check{\theta}\|_{\mathbf{H}_0^1(\Omega)} + \|\bar{\theta}_T\|_{\mathbf{H}_0^1(\Omega)} \right) \\ &\quad \times \left[\left(\int_0^T \|u_{|\bar{\theta}_T} - y^\delta\|_{\mathbf{H}_\omega^1}^2 dt \right)^{\frac{1}{2}} + \left(\int_0^T \|\check{u} - y^\delta\|_{\mathbf{H}_\omega^1}^2 dt \right)^{\frac{1}{2}} \right]. \end{aligned}$$

352 Using Lemma 2.3, the estimate on the noise amplitude (32) and the prior on the
 353 parameter (34), we find

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 \lesssim (c'_{\text{idf}}(T))^{-\frac{1}{2}} \left(M + \sqrt{2\epsilon^{-1} \mathcal{J}_T(\bar{\theta}_T)} \right) \left(\sqrt{2 \mathcal{J}_T(\bar{\theta}_T)} + \delta\sqrt{T} \right).$$

354 Using again (32) and (34), we can bound the cost functional as follows

$$\mathcal{J}_T(\bar{\theta}_T) \leq \mathcal{J}_T(\check{\theta}) \leq \frac{\epsilon}{2} M^2 + \frac{1}{2} \delta^2 T.$$

355 Hence, we finally get

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 \lesssim (c'_{\text{idf}}(T))^{-\frac{1}{2}} \left(M + \sqrt{M^2 + \epsilon^{-1} \delta^2 T} \right) \left(\sqrt{\epsilon M^2 + \delta^2 T} + \delta\sqrt{T} \right), \quad (41)$$

356 which implies (39).

357 Furthermore, under Assumption 1, we have that there exists $c >$ independent of T
 358 such that

$$c'_{\text{idf}}(T) \geq cT,$$

359 which proves (40). □

360 Finally, choosing $\epsilon = \delta^2 M^{-2}$ in (41), we obtain the following corollary

361 **Corollary 3.8.** *Under the hypothesis of Theorem 3.7, for $T \geq T_0$ large enough, there*
 362 *exists a positive scalar $C^{st}(T)$ depending only on T such that*

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{st}(T) \sqrt{M} \sqrt{\delta}, \quad (42)$$

363 *where $\bar{\theta}_T$ minimizes the cost functional (33) with $\epsilon = \delta^2 M^{-2}$.*

364 *Moreover, under Assumption 1, there exists a positive scalar C^{st} independent of T such*
 365 *that*

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{st} T^{\frac{1}{4}} \sqrt{M} \sqrt{\delta}. \quad (43)$$

366 3.3 Field measurements with L^2 -noise

367 In the rest of this work, we assume – in order to model a more realistic case in practice –
 368 that the measurements are less regular. Namely, we suppose that the target trajectory
 369 \tilde{u} generated by the target source term parameter $\check{\theta}$ has given rise to a collection of
 370 measurements y^δ such that

$$\int_0^T \|y^\delta(x, t) - \tilde{u}(x, t)\|_{L^2(\omega)}^2 dt \leq \delta^2 T, \quad (44)$$

371 for a parameter $\delta > 0$.

372 We present a method to deal with this additional difficulty of regularity loss. This
 373 approach consists in modifying the definition of the observation operator with second
 374 observation operator C introduced in (9).

375 3.3.1 Identifiability from observability

376 The proof of the desired identifiability result for the identification estimation problem
 377 is based on the classical following initial condition observability inequality – for a
 378 homogeneous wave equation in weakened norms.

379 **Proposition 3.9.** *Let u be an arbitrary solution of (1) with $\theta = 0$. Assume that ω
 380 satisfies the GCC condition, then there exists $cT \leq c_{obs}(T)$ with $c > 0$ independent of
 381 T such that*

$$\int_0^T \|u(\cdot, t)\|_{L^2(\omega)}^2 dt \geq c_{obs}(T) \left(\|u_0\|_{L^2(\Omega)}^2 + \|v_0\|_{H^{-1}(\Omega)}^2 \right). \quad (45)$$

382 **Theorem 3.10.** *Let $\theta \in \mathcal{P}$. Let u be a solution of (11). Assuming $\sigma(t) \in H^1(0, T)$
 383 and $\sigma(0) \neq 0$, and that ω satisfies the GCC condition, for T large enough, there exists
 384 $c'_{idf}(T) > 0$, such that*

$$\|CL\theta\|_{L^2(0, T; L^2(\omega))}^2 = \|u\|_{L^2(0, T; L^2(\omega))}^2 \geq c'_{idf}(T) \|\Delta_0^{-1}\theta\|_{L^2(\Omega)}^2.$$

385 *Proof.* One can show, as for Theorem 3.4, that we have the inequality

$$\|u|_\omega\|_{L^2(0, T; L^2(\omega))} \geq C^{st}(T) \|C\bar{L}\theta\|_{L^2(0, T; L^2(\omega))}. \quad (46)$$

386 As in the proof of Theorem 3.3, we introduce $g \in H_0^1(\Omega) \cap \mathcal{D}(-\Delta_0)$ the Riesz represen-
 387 tation of θ (seen as an element of H^{-1}) given by $g = -\Delta_0^{-1}\theta$, and u_g solution of the
 388 dynamics (19). The difference $\tilde{u}_g = u_g - g$ satisfies the system (21) in the same way.
 389 We apply the observability condition of Proposition 3.9 on this system, and we obtain

$$\|\tilde{u}_g\|_{L^2(0, T; L^2(\omega))}^2 = \|u_g - g\|_{L^2(0, T; L^2(\omega))}^2 \geq c_{obs}(T) \|g\|_{L^2(\Omega)}^2. \quad (47)$$

390 Now we want to discard the variable g in the left-hand of (47). That is why, we develop
 391 the following scalar product

$$\int_0^T \|u_g - g\|_{L^2(\omega)}^2 dt = \int_0^T \|u_g\|_{L^2(\omega)}^2 dt + T \|g\|_{L^2(\omega)}^2 - 2 \int_0^T (u_g, g)_{L^2(\omega)} dt. \quad (48)$$

392 Applying Δ_0^{-1} to the first equation of (19), then integrate in time and finally the
 393 resulting equation given by the scalar product with g is

$$\int_0^T (u_g, g)_{L^2(\omega)} dt = (\Delta_0^{-1} \partial_t u_g(T), g)_{L^2(\omega)} + T \|g\|_{L^2(\omega)}^2.$$

Injecting this result in (48) and using Cauchy-Schwarz and Poincaré inequalities, we obtain

$$\int_0^T \|u_g - g\|_{L^2(\omega)}^2 dt \leq \int_0^T \|u_g\|_{L^2(\omega)}^2 dt - T\|g\|_{L^2(\omega)}^2 + 2C_p \|\nabla \Delta_0^{-1} \partial_t u_g(T)\|_{L^2(\Omega)} \|g\|_{L^2(\omega)}. \quad (49)$$

394 We expect to bound $\|\nabla \Delta_0^{-1} \partial_t u_g(T)\|_{L^2(\Omega)}$ by $\|g\|_{L^2(\Omega)}$. Therefore, we apply Δ_0^{-1} to
395 the first equation of (19). Integrating in time, the scalar product with $-\partial_t u_g$ leads to

$$-\int_{\Omega} (\Delta_0^{-1} \partial_{tt} u_g) \partial_t u_g dx + \int_{\Omega} u_g \partial_t u_g dx = \int_{\Omega} g \partial_t u_g dx.$$

396 Inserting $\Delta_0 \Delta_0^{-1} = \text{Id}$ in the first term we get

$$-\int_{\Omega} (\Delta_0^{-1} \partial_{tt} u_g) \Delta_0 \Delta_0^{-1} \partial_t u_g dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} u_g^2 dx = \int_{\Omega} g \partial_t u_g dx.$$

397 Integrating by parts, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} (\nabla \Delta_0^{-1} \partial_t u_g)^2 dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} u_g^2 dx = \int_{\Omega} g \partial_t u_g dx.$$

Integrating in time and using Cauchy-Schwarz and Young inequalities, we get

$$\begin{aligned} \frac{1}{2} \|\nabla \Delta_0^{-1} \partial_t u_g(T)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|u_g(T)\|_{L^2(\Omega)}^2 &= \int_{\Omega} g u_g(T) dx \\ &\leq \frac{1}{2} \|g\|_{L^2(\Omega)}^2 + \frac{1}{2} \|u_g(T)\|_{L^2(\Omega)}^2, \end{aligned}$$

398 thus,

$$\|\nabla \Delta_0^{-1} \partial_t u_g(T)\|_{L^2(\Omega)}^2 \leq \|g\|_{L^2(\Omega)}^2.$$

399 Injecting this result in (49), we have

$$\int_0^T \|u_g - g\|_{L^2(\omega)}^2 dt \leq \int_0^T \|u_g\|_{L^2(\omega)}^2 dt - T\|g\|_{L^2(\omega)}^2 + 2C_p \|g\|_{L^2(\Omega)} \|g\|_{L^2(\omega)}.$$

400 Using Young inequality, we finally have

$$\int_0^T \|u_g - g\|_{L^2(\omega)}^2 dt \leq \int_0^T \|u_g\|_{L^2(\omega)}^2 dt - T\|g\|_{L^2(\omega)}^2 + \epsilon C_p \|g\|_{L^2(\Omega)}^2 + \frac{C_p}{\epsilon} \|g\|_{L^2(\omega)}^2. \quad (50)$$

401 Gathering (47) and (50), we get

$$\int_0^T \|u_g\|_{L^2(\omega)}^2 dt \geq \left(T - \frac{C_p}{\epsilon}\right) \|g\|_{L^2(\omega)}^2 + (c_{\text{obs}}(T) - \epsilon C_p) \|g\|_{L^2(\Omega)}^2.$$

402 We choose $\epsilon = \frac{c_{\text{obs}}}{2C_p}$, then we have

$$\int_0^T \|u_g\|_{L^2(\omega)}^2 dt \geq \left(T - \frac{2C_p^2}{c_{\text{obs}}(T)}\right) \|g\|_{L^2(\omega)}^2 + \frac{c_{\text{obs}}(T)}{2} \|g\|_{L^2(\Omega)}^2.$$

403 For $T > \frac{2C_p^2}{c_{\text{obs}}(T)}$, and setting $-\Delta_0 g = \theta$, we obtain

$$\|C\bar{L}\theta\|_{L^2((0,T);L^2(\omega))} = \int_0^T \|u_g\|_{L^2(\omega)}^2 dt \geq \frac{c_{\text{obs}}(T)}{2} \|\Delta_0^{-1} \theta\|_{L^2(\Omega)}^2. \quad (51)$$

404 Gathering (46) and (51), we conclude the proof. \square

405 As in the previous section, we assume that the dependence of the observability
406 constant is in $O(T)$.

407 **Assumption 2.** *The identifiability constant of Theorem (3.10) satisfies*

$$c'_{idf}(T) \geq cT, \text{ with } c > 0 \text{ independent of } T.$$

408 3.3.2 Updated convergence rate for the Tikhonov regularization

409 We infer from Theorem 3.10 that we now face an ill-posed problem of degree 2. This will
410 modify the convergence rate of our Tikhonov regularization without further assumption
411 on the prior regularity. Let us first recall the following classical interpolation lemma.

412 **Lemma 3.11.** *Let (μ_i, e_i) be the orthonormal basis for the L^2 scalar product such that
413 the Laplacian eigenvalues $\mu_i > 0$ satisfies $\mu_{i+1} > \mu_i$, and $\mu_i \rightarrow +\infty$. Let $\theta \in \mathbf{H}_0^1(\Omega)$,
414 we have the interpolation property*

$$\|\theta\|_{L^2(\Omega)}^2 \leq \left(\|\Delta_0^{-1}\theta\|_{L^2(\Omega)}^2 \right)^{\frac{1}{3}} \left(\|\theta\|_{\mathbf{H}_0^1(\Omega)}^2 \right)^{\frac{2}{3}},$$

415 where $\|\Delta_0^{-1}\theta\|_{L^2(\Omega)} = \sqrt{\sum_i |(e_i, \theta)|^2 \mu_i^{-2}}$.

416 We recall here the proof for the sake of completeness.

417

418 *Proof.* We express the L^2 -norm of θ in the orthonormal basis.

$$\|\theta\|_{L^2(\Omega)}^2 = \sum_{i=0}^{+\infty} |(e_i, \theta)|^2 = \sum_{i=1}^{+\infty} |(e_i, \theta)|^2 \mu_i^{2q \frac{-s}{s-q}} \mu_i^{2s \frac{-q}{s-q}}.$$

419 Using Hölder inequality, we have

$$\|\theta\|_{L^2(\Omega)}^2 \leq \left(\sum_i |(e_i, \theta)|^2 \mu_i^{2q} \right)^{\frac{s}{s-q}} \left(\sum_i |(e_i, \theta)|^2 \mu_i^{2s} \right)^{\frac{-q}{s-q}}.$$

420 Choosing $s = \frac{1}{2}$ and $q = -1$, we obtain

$$\|\theta\|_{L^2(\Omega)}^2 \leq \left(\sum_i |(e_i, \theta)|^2 \mu_i^{-2} \right)^{\frac{1}{3}} \left(\sum_i |(e_i, \theta)|^2 \mu_i \right)^{\frac{2}{3}} = \left(\|\Delta_0^{-1}\theta\|_{L^2(\Omega)}^2 \right)^{\frac{1}{3}} \left(\|\theta\|_{\mathbf{H}_0^1(\Omega)}^2 \right)^{\frac{2}{3}}.$$

421

□

422 Keeping the same regularization in our criterion, we introduce the cost functional
423 to minimize

$$\mathcal{J}_T(\theta) = \frac{\epsilon}{2} \|\theta\|_{\mathbf{H}_0^1(\Omega)}^2 + \frac{1}{2} \|y^\delta - Cz|_\theta\|_{L^2(0,T;L^2(\omega))}^2, \quad (52)$$

424 where C is defined as operator in L^2 , namely $C = (I_\omega \ 0) \in \mathcal{L}(\mathcal{Z} \rightarrow L^2(\omega))$. We can
425 then estimate the reconstruction error between $\bar{\theta}_T$ the functional minimizer and the
426 target parameter $\check{\theta}$.

427 **Theorem 3.12.** *We assume that there exists $M > 0$ such that (34) holds.*

428 *Furthermore, we also assume that the available measurements are noise-free, i.e. we
429 have for all $t \in [0, T]$*

$$y^\delta(t) = \check{u}|_\omega(t).$$

430 Let $T_0 > 0$. Assuming $\sigma(t) \in H^1(0, T)$ and $\sigma(0) \neq 0$ and (ω, T_0) satisfies the GCC
 431 condition, for $T \geq T_0$ large enough and ϵ small enough, then there exists a positive
 432 scalar $C^{st}(T)$ depending only on T such that

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{st}(T) \epsilon^{\frac{1}{6}} M, \quad (53)$$

433 where $\bar{\theta}_T$ minimizes the cost functional (52).

434 Moreover, under Assumption 2, there exists a positive scalar C^{st} independent of T such
 435 that

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{st} T^{-\frac{1}{6}} \epsilon^{\frac{1}{6}} M. \quad (54)$$

436 *Proof.* In this proof, we write $a \lesssim b$ if there exists $c > 0$ independent of T, ϵ and M
 437 such that $a \leq cb$.

438 We use the interpolation inequality of Lemma 3.11 to obtain

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 \leq \left(\|\bar{\theta}_T - \check{\theta}\|_{H_0^1(\Omega)}^2 \right)^{\frac{2}{3}} \left(\|\Delta_0^{-1}(\bar{\theta}_T - \check{\theta})\|_{L^2(\Omega)}^2 \right)^{\frac{1}{3}}.$$

439 Using the identifiability result given by Theorem 3.10, we have

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 \leq (c'_{\text{idf}}(T))^{-\frac{1}{3}} \left(\|\bar{\theta}_T\|_{H_0^1(\Omega)}^2 + \|\check{\theta}\|_{H_0^1(\Omega)}^2 \right)^{\frac{2}{3}} \left(\int_0^T \|u_{|\bar{\theta}_T} - \check{u}\|_{L^2(\omega)}^2 dt \right)^{\frac{1}{3}}. \quad (55)$$

440 Since we consider noise-free observations, we have $y^\delta = \check{u}_{|\omega}$. Therefore, we obtain

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 \lesssim (c'_{\text{idf}}(T))^{-\frac{1}{3}} \left(\|\bar{\theta}_T\|_{H_0^1(\Omega)}^2 + \|\check{\theta}\|_{H_0^1(\Omega)}^2 \right)^{\frac{2}{3}} \left(\int_0^T \|u_{|\bar{\theta}_T} - y^\delta\|_{L^2(\omega)}^2 dt \right)^{\frac{1}{3}}.$$

441 Using the definition of the cost-functional (52) and our prior on the parameter (34),
 442 we obtain

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 \lesssim (c'_{\text{idf}}(T))^{-\frac{1}{3}} (2\epsilon^{-1} \mathcal{J}_T(\bar{\theta}_T) + M^2)^{\frac{2}{3}} (2\mathcal{J}_T(\bar{\theta}_T))^{\frac{1}{3}}.$$

443 We can bound the cost-functional as follows

$$\mathcal{J}_T(\bar{\theta}_T) \leq \mathcal{J}_T(\check{\theta}) \leq \frac{\epsilon}{2} M^2.$$

444 Finally, we have

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 \lesssim (c'_{\text{idf}}(T))^{-\frac{1}{3}} (M^2 + M^2)^{\frac{2}{3}} (\epsilon M^2)^{\frac{1}{3}},$$

445 which leads to (53). Furthermore, under Assumption 2, we have that there exists $c >$
 446 independent of T such that

$$c'_{\text{idf}}(T) \geq cT,$$

447 which proves (54). □

448 **Theorem 3.13.** We assume that there exists $M > 0$ such that (34) holds.

449 Let $T_0 > 0$. Assuming $\sigma(t) \in H^1(0, T)$ and $\sigma(0) \neq 0$ and (ω, T_0) satisfies the GCC
 450 condition, for $T \geq T_0$ large enough and ϵ small enough, then there exists a positive
 451 scalar $C^{st}(T)$ depending only on T such that

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{st}(T) \epsilon^{-\frac{1}{3}} \delta, \quad (56)$$

452 where $\bar{\theta}_T$ minimizes the cost functional (52).

453 Moreover, under Assumption 2, there exists a positive scalar C^{st} independent of T such
 454 that

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{st} T^{\frac{1}{3}} \epsilon^{-\frac{1}{3}} \delta. \quad (57)$$

455 *Proof.* In this proof, we write $a \lesssim b$ if there exists $c > 0$ independent of T, ϵ, δ and M
 456 such that $a \leq cb$.

457 On one hand, using the estimate on the amplitude noise (44) and the prior (34), we
 458 intend to bound the cost functional.

$$\mathcal{J}_T(\check{\theta}) \leq \frac{\epsilon}{2}M^2 + \frac{1}{2}\delta^2T. \quad (58)$$

On the other hand, as in the proof of Theorem 3.12, one can show that (55) holds. Inserting $y^\delta - y^\delta = 0$ into (55) to reintroduce the cost functional and using the prior on the parameter (34), we get

$$\begin{aligned} \|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 &\lesssim (c'_{\text{idf}}(T))^{-\frac{1}{3}} (2\epsilon^{-1}\mathcal{J}_T(\check{\theta}) + M^2)^{\frac{2}{3}} \\ &\quad \times \left(\int_0^T \|u_{|\bar{\theta}_T} - y^\delta\|_{L^2(\omega)}^2 dt + \int_0^T \|\check{u} - y^\delta\|_{L^2(\omega)}^2 dt \right)^{\frac{1}{3}}. \end{aligned}$$

459 Again, using again (32) and the cost functional bound (58), we have

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)}^2 \lesssim (c'_{\text{idf}}(T))^{-\frac{1}{3}} (2M^2 + \epsilon^{-1}\delta^2T)^{\frac{2}{3}} (2\delta^2T + \epsilon M^2)^{\frac{1}{3}}, \quad (59)$$

460 which corresponds to the desired result (56).

461 Furthermore, under Assumption 2, we have that there exists $c >$ independent of T
 462 such that

$$c'_{\text{idf}}(T) \geq cT,$$

463 which leads to (57). □

464 Eventually, choosing $\epsilon = \delta^2M^{-2}$ in (59), we obtain the following corollary.

465 **Corollary 3.14.** *Under the hypothesis of Theorem 3.13, for $T \geq T_0$ large enough, then*
 466 *there exists a positive scalar $C^{st}(T)$ depending only on T such that*

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{st}(T)M^{\frac{2}{3}}\delta^{\frac{1}{3}}, \quad (60)$$

467 where $\bar{\theta}_T$ minimizes the cost functional (52) with $\epsilon = \delta^2M^{-2}$.

468 Moreover, under Assumption 2, there exists a positive scalar C^{st} independent of T such
 469 that

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{st}T^{\frac{1}{3}}M^{\frac{2}{3}}\delta^{\frac{1}{3}}. \quad (61)$$

470 4 Observer design

471 Our goal in this section is to propose a sequential method for computing $\bar{\theta}_T$ and the
 472 associated wave equation trajectory \bar{z}_T . By sequential, we mean that we want to update
 473 the estimator when measurements are available, without recomputing the estimator
 474 each time a new measurement is available. This leads to the definition of an observer
 475 in the sense of [20], which is formally defined in a general framework of an observed
 476 system of a dynamics of the form

$$\begin{cases} \dot{z} = F(z, t) \\ z(0) = z_0 \end{cases} \quad (62)$$

477 where we observed a target trajectory \check{z} with unknown initial state \check{z}_0 by measuring
 478 $(y^\delta(t))_{t \geq 0}$. We here propose to slightly simplify this definition.

479 **Definition 4.1.** An asymptotic observer of (62) is a function $\mathbb{R}_+ \ni t \mapsto \hat{z}(t; (y^\delta(s))_{s \leq t})$
 480 such that

481 1. **Robustness to Noise:** for all $0 \leq \tau$, and for all $\varepsilon > 0$, there exist $\delta, \alpha > 0$,
 482 such that

$$\left. \begin{array}{l} \|\hat{z}(0) - \check{z}(0)\|_{\mathcal{Z}} \leq \alpha \\ \|\check{y} - y^\delta\|_{L^2((0, \tau); \mathcal{Y})} \leq \delta \end{array} \right\} \Rightarrow \|\check{z}(t) - \hat{z}(t)\|_{\mathcal{Z}} \leq \varepsilon, \quad t \geq \tau.$$

483

484 2. **Convergence:** for all $0 \leq \tau$, and for all $\varepsilon, \alpha, \delta > 0$, there exists $T_0 > 0$ such that

$$\left. \begin{array}{l} \|\hat{z}(0) - \check{z}(0)\|_{\mathcal{Z}} \leq \alpha \\ \|\check{y} - y^\delta\|_{L^2((0, \tau); \mathcal{Y})} \leq \delta \\ \text{supp}(\check{y} - y^\delta) \subset (0, \tau) \end{array} \right\} \Rightarrow \|\check{z}(t) - \hat{z}(t)\|_{\mathcal{Z}} \leq \varepsilon, \quad t \geq T_0 + \tau.$$

485

486 Note that the two above properties correspond respectively to an asymptotic
 487 convergence property in the absence of noise and property of robustness to noise. As
 488 the observer can be computed at time t from measurements available at time $s \leq t$, we
 489 say that it is a causal function of the measurements. Note also that, as a consequence
 490 of (i), an asymptotic observer satisfies for all $s \geq 0$

$$\hat{z}(s) = \check{z}(s), \quad (y^\delta(t) = \check{y}(t), t \geq s) \Rightarrow (\hat{z}(t) = \check{z}(t), t \geq s). \quad (63)$$

491 In our case, we try to define such an observer for the joint dynamics of $z = (z, \theta)$,
 492 where the solution z of the wave equation is extended by the zero dynamics of the
 493 parameter θ . Our method is based on a dynamic programming strategy that relate
 494 functional minimization to Kalman observer, which we will apply to the dynamics of
 495 the state and parameters. But let us first recall some results about Kalman filtering [5,
 496 57], adapted here to our infinite-dimensional system.

497 4.1 Reduced-order Kalman filters for quasi-evolution operators

498 To present our strategy, we first need to review some results in the analysis of evolution
 499 problems with non-homogeneous generators.

500 4.1.1 Quasi-evolution equation

501 Let us consider the following abstract dynamics defined in the state-space \mathcal{Z}

$$\begin{cases} \dot{z}(t) = \mathbf{A}(t)z(t) + \beta(t), & \text{in } (0, T), \\ z(0) = z_0. \end{cases} \quad (64)$$

502 from given $z_0 \in \mathcal{Z}$, $\beta \in L^2((0, T); \mathcal{Z})$. Here, the operator

$$\mathbf{A}(t) = \mathbf{A} + \mathbf{G}(t)$$

503 where $(\mathbf{A}, \mathcal{D}(\mathbf{A}))$ is maximal dissipative and for all $t > 0$, $\mathbf{G}(t)$ is a strongly continuous
 504 bounded operator, namely for all $z \in \mathcal{Z}$, $\mathbf{G}(\cdot)z \in C^0([0, T], \mathcal{Z})$. Note that by uniform
 505 boundedness theorem, there exists $c_{\text{st}} > 0$ such that $\|\mathbf{G}(t)\|_{\mathcal{L}(\mathcal{Z})} \leq c_{\text{st}}$. Classical results
 506 for evolution equation give the existence of a solution of (64) knowing z_0 and $\beta(t)$ and
 507 denoted $z|_{z_0, \beta}$. Indeed, from [29, II-1 Propostion 3.4], if $z_0 \in \mathcal{Z}$ and $\beta \in L^2((0, T); \mathcal{Z})$
 508 then $z|_{z_0, \beta} \in C^0([0, T]; \mathcal{Z})$ is a mild solution

$$z(t) = \Phi(t, 0)z_0 + \int_0^t \Phi(t, s)\beta(s) ds, \quad (65)$$

509 where $\Phi : \Delta_T \rightarrow \mathcal{L}(\mathcal{Z})$ with $\Delta_T = \{(t, s) : s \in [0, T], t \in [s, T]\}$ is a quasi-evolution
 510 operator in the sense of [8, Definition 1.2].

511 **Remark 4.1.** Note that it is possible to give an implicit definition of Φ . Indeed,
512 consider the system,

$$\begin{cases} \dot{z}(t) = \mathbf{A}z(t) + \mathbf{G}(t)z(t), & \text{in } (s, T), \\ z(s) = z_s \in \mathcal{Z}. \end{cases}$$

513 By definition of the quasi-evolution operator Φ , we have, using Duhamel's formula

$$z(t) = \Phi(t, s)z_s = e^{(t-s)\mathbf{A}}z_s + \int_s^t e^{(t-\tau)\mathbf{A}}\mathbf{G}(\tau)z(\tau) d\tau, \quad (66)$$

514 which, using (66), finally becomes

$$\Phi(t, s)z_s = e^{(t-s)\mathbf{A}}z_s + \int_s^t e^{(t-\tau)\mathbf{A}}\mathbf{G}(\tau)\Phi(\tau, s)z_s d\tau.$$

515 The mild solution (65) is also a weak solution in the sense that $z|_{z_0, \beta} \in L^2((0, T); \mathcal{Z}) \cap$
516 $H^1((0, T); \mathcal{D}(\mathbf{A}^*))$, namely for all $q \in \mathcal{D}(\mathbf{A}^*)$, $(q, z(\cdot))_{\mathcal{Z}}$ belongs to $H^1(0, T)$ and for
517 almost all $t \in (0, T)$,

$$\forall q \in \mathcal{D}(\mathbf{A}^*), \quad \frac{d}{dt}(q, z(t))_{\mathcal{Z}} + (\mathbf{A}(t)^*q, z(t))_{\mathcal{Z}} = (q, \beta(t))_{\mathcal{Z}}. \quad (67)$$

518 Moreover, from [29, II-1 Propostion 3.5], if we further assume that for all $z \in$
519 $\mathcal{D}(\mathbf{A})$, $\mathbf{G}(\cdot)z \in C^0([0, T], \mathcal{D}(\mathbf{A}))$, $z_0 \in \mathcal{D}(\mathbf{A})$ and $\beta \in L^2((0, T); \mathcal{D}(\mathbf{A}))$ then $z|_{z_0, \beta} \in$
520 $C^0([0, T]; \mathcal{D}(\mathbf{A})) \cap H^1((0, T); \mathcal{Z})$. Finally, when for all $z \in \mathcal{D}(\mathbf{A})$, $\mathbf{G}(\cdot)z \in C^1([0, T]; \mathcal{Z})$,
521 $z_0 \in \mathcal{D}(\mathbf{A})$ and $\beta \in H^1((0, T); \mathcal{D}(\mathbf{A}))$ then $z|_{z_0, \beta} \in C^0([0, T], \mathcal{D}(\mathbf{A})) \cap C^1([0, T]; \mathcal{Z})$.

522 4.1.2 Least-square estimation

523 We introduce a space $\mathcal{V} \subset \mathcal{Z}$ with continuous injection and a symmetric and coercive
524 bilinear form $m : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$. We want to minimize a criterion $\mathcal{J}_T : \mathcal{V} \rightarrow \mathbb{R}$ defined by

$$\mathcal{J}_T(\zeta) = \frac{1}{2}m(\zeta, \zeta) + \frac{1}{2} \int_0^T \|y^\delta(t) - \mathbf{C}(t)z|_{\zeta}(t)\|_{\mathcal{Y}}^2 dt, \quad (68)$$

525 where $z|_{\zeta}$ is the solution of (64) for an initial condition $z_0 = \hat{z}_0 + \zeta$, where \hat{z}_0 is a
526 known prior. In this section, the observation operator is a time-dependent operator
527 $\mathbf{C} \in C^0([0, T], \mathcal{L}(\mathcal{Z}, \mathcal{Y}))$. Note that depending on the choice of \mathcal{V} , the bilinear form m
528 can be seen as Tikhonov regularization or a generalized Tikhonov regularization [18].

The functional \mathcal{J}_T is continuous, differentiable, and strictly convex, so it admits one and only one minimizer denoted $\bar{\zeta}|_T$ where we underline the dependency with respect to T . Computing the derivative of \mathcal{J}_T , we have for all $\xi \in \mathcal{V}$,

$$\langle D\mathcal{J}_T(\zeta), \xi \rangle_{\mathcal{V}, \mathcal{V}} = m(\zeta, \xi) + \int_0^T (\xi, \Phi(t, 0)^* \mathbf{C}(t)^* (\mathbf{C}(t)z|_{\zeta} - y^\delta))_{\mathcal{Y}} dt.$$

529 Therefore, introducing the adjoint dynamics for any trajectory $z|_{\zeta} \in L^2((0, T); \mathcal{Z})$ and
530 $y^\delta \in L^2((0, T); \mathcal{Y})$,

$$\begin{cases} \dot{q}(t) + \mathbf{A}(t)^*q(t) = -\mathbf{C}(t)^*(y^\delta(t) - \mathbf{C}(t)z|_{\zeta}(t)), & t \in (0, T) \\ q(T) = 0, \end{cases} \quad (69)$$

531 whose mild solution in $C^0([0, T], \mathcal{Z})$ is given by the Duhamel formulae – adapted to
532 this backward formulation –

$$q(t) = \int_t^T \Phi(t, s)^* \mathbf{C}(t)^* (y^\delta(s) - \mathbf{C}(t)z|_{\zeta}(s)) ds, \quad t \in [0, T], \quad (70)$$

533 we find

$$\forall \xi \in \mathcal{V}, \quad \langle \mathbb{D} \mathcal{J}_T(\zeta), \xi \rangle_{\mathcal{V}, \mathcal{V}} = m(\zeta, \xi) - (\mathbf{q}(0), \xi)_{\mathcal{Z}}. \quad (71)$$

534 If we now introduce the operator $\mathbf{\Pi}_0 \in \mathcal{L}(\mathcal{Z}, \mathcal{V})$ such that for all $\mathbf{q} \in \mathcal{Z}$, $\mathbf{\Pi}_0 \mathbf{q}$ is the
535 solution of $m(\mathbf{\Pi}_0 \mathbf{q}, \xi) = (\mathbf{q}, \xi)_{\mathcal{Z}}$ for all $\xi \in \mathcal{V}$, we deduce from (71) the optimality
536 condition

$$\bar{\zeta}|_T = \mathbf{\Pi}_0 \bar{\mathbf{q}}_T(0),$$

537 where $\bar{\mathbf{q}}_T$ is associated to the optimal trajectory $\bar{\mathbf{z}}_T = \mathbf{z}|_{\bar{\zeta}|_T}$ through the two-ends
538 optimality system – as called in [14, 5]

$$\begin{cases} \dot{\bar{\mathbf{z}}}_T(t) = \mathbf{A}(t)\bar{\mathbf{z}}_T + \boldsymbol{\beta}(t), & t \in (0, T), \\ \dot{\bar{\mathbf{q}}}_T(t) + \mathbf{A}(t)^* \bar{\mathbf{q}}_T = -\mathbf{C}(t)^*(y^\delta(t) - \mathbf{C}(t)\bar{\mathbf{z}}_T(t)), & t \in (0, T), \\ \bar{\mathbf{z}}_T(0) = \hat{\mathbf{z}}_0 + \mathbf{\Pi}_0 \bar{\mathbf{q}}_T(0), \\ \bar{\mathbf{q}}_T(T) = 0. \end{cases} \quad (72)$$

539 Note finally that if $\mathcal{V} \subset \mathcal{D}(\mathbf{A})$, then $\bar{\mathbf{z}}$ and $\bar{\mathbf{q}}$ belong to $C^1([0, T], \mathcal{Z}) \cap C^0([0, T], \mathcal{D}(\mathbf{A}))$.

540 4.2 Kalman estimator definition

541 The objective of this section is to present briefly the Kalman estimator strategy to
542 solve sequentially the two-ends problem (72). First, we introduce the Riccati dynamics

$$\begin{cases} \dot{\mathbf{\Pi}}(t) = \mathbf{A}(t)\mathbf{\Pi}(t) + \mathbf{\Pi}(t)\mathbf{A}(t)^* - \mathbf{\Pi}(t)\mathbf{C}(t)^*\mathbf{C}(t)\mathbf{\Pi}(t), & \text{in } (0, T), \\ \mathbf{\Pi}(0) = \mathbf{\Pi}_0, \end{cases} \quad (73)$$

543 where $\mathbf{C} \in C^0([0, T], \mathcal{L}(\mathcal{Z}, \mathcal{Y}))$. The initial condition $\mathbf{\Pi}_0 \in \mathcal{S}^+(\mathcal{Z})$ the space of bounded
544 symmetric operators. The following theorem gives the existence of solution of (73) in
545 $\mathcal{S}^+(\mathcal{Z})$. The resulting operator $\mathbf{\Pi} \in \mathcal{S}^+(\mathcal{Z})$ is called Riccati operator or *covariance*
546 *operator* for its interpretation in the context of stochastic filtering, see for instance [5,
547 7].

548 **Theorem 4.2** (Existence of Riccati solution).

549 *The Riccati dynamics (73) admits one and only one mild solution $\mathbf{\Pi} \in C^0([0, T], \mathcal{S}^+(\mathcal{Z}))$*
550 *given by*

$$\mathbf{\Pi}(t) = \boldsymbol{\Phi}(t, 0)\mathbf{\Pi}_0\boldsymbol{\Phi}(t, 0)^* - \int_0^t \boldsymbol{\Phi}(t, s)\mathbf{\Pi}(s)\mathbf{C}(t)^*\mathbf{C}(t)\mathbf{\Pi}(s)\boldsymbol{\Phi}(t, s)^* ds. \quad (74)$$

Moreover, for all $(\mathbf{z}_1, \mathbf{z}_2) \in \mathcal{D}(\mathbf{A}^*)$, $(\mathbf{\Pi}(\cdot)\mathbf{z}_1, \mathbf{z}_2)_{\mathcal{Z}}$ is differentiable and satisfies the weak formulation

$$\begin{aligned} \frac{d}{dt}(\mathbf{\Pi}(t)\mathbf{z}_1, \mathbf{z}_2)_{\mathcal{Z}} &= (\mathbf{\Pi}(t)\mathbf{z}_1, \mathbf{A}(t)^*\mathbf{z}_2)_{\mathcal{Z}} + (\mathbf{\Pi}(t)\mathbf{A}(t)^*\mathbf{z}_1, \mathbf{z}_2)_{\mathcal{Z}} \\ &\quad - (\mathbf{C}(t)\mathbf{\Pi}(t)\mathbf{z}_1, \mathbf{C}(t)\mathbf{\Pi}(t)\mathbf{z}_2)_{\mathcal{Y}} = 0, \quad t \in [0, T]. \end{aligned} \quad (75)$$

551 *Proof.* We refer to [8, Corollary 2.2], for a proof of existence of a mild solution of
552 a Riccati equation where $\mathbf{A}(t) = \mathbf{A} + \mathbf{G}(t)$ with $(\mathbf{A}, \mathcal{D}(\mathbf{A}))$ unbounded and $\mathbf{G}(t)$ a
553 bounded perturbation. The existence of a weak solution is a direct consequence, see
554 for instance [29, IV-1 Proposition 2.1]. \square

555 From the Riccati operator definition, we can introduce the so-called *Kalman observer*
556 – also called *Kalman estimator* in the context of stochastic filtering.

557 **Theorem 4.3.** *There exists one and only one mild solution in $C^0([0, T]; \mathcal{Z})$ of the*
 558 *dynamics*

$$\begin{cases} \dot{\hat{z}} = \mathbf{A}(t)\hat{z} + \mathbf{\Pi}(t)\mathbf{C}(t)^*(y^\delta(t) - \mathbf{C}(t)\hat{z}(t)), & t \in (0, T), \\ \hat{z}(0) = \hat{z}_0, \end{cases} \quad (76)$$

559 *in the sense that*

$$\hat{z}(t) = \mathbf{\Phi}(t, 0)\hat{z}_0 + \int_0^t \mathbf{\Phi}(t, s)\mathbf{\Pi}(s)\mathbf{C}(s)^*[y^\delta(s) - \mathbf{C}(s)\hat{z}(s)] ds. \quad (77)$$

560 *Moreover, this solution is the unique weak solution in the sense that, (1.) $\mathbf{z} \in$*
 561 *$L^2((0, T); \mathcal{Z})$, (2.) for all $\mathbf{q} \in \mathcal{D}(\mathbf{A}^*)$, $(\mathbf{q}, \hat{z}(\cdot))_{\mathcal{Z}}$ belongs to $H^1(0, T)$ and (3.) for almost*
 562 *all $t \in (0, T)$,*

$$\forall \mathbf{q} \in \mathcal{D}(\mathbf{A}^*), \quad \frac{d}{dt}(\hat{z}(t), \mathbf{q})_{\mathcal{Z}} + (\hat{z}, \mathbf{A}(t)^*\mathbf{q})_{\mathcal{Z}} = (y^\delta(t) - \mathbf{C}(t)\hat{z}, \mathbf{C}(t)\mathbf{\Pi}(t)\mathbf{q})_{\mathcal{Y}}. \quad (78)$$

563 *Proof.* We know that $\mathbf{\Pi} \in C^0([0, T]; \mathcal{S}_+^*(\mathcal{Z}))$. Therefore, for all $t \in [0, T]$, $\mathbf{A}(t) -$
 564 $\mathbf{\Pi}(t)\mathbf{C}(t)^*\mathbf{C}(t)$ is a uniform bounded perturbation of \mathbf{A} , the existence enters exactly
 565 the framework of quasi-evolution operator. \square

566 4.2.1 The optimal filtering property

567 We can now move to the understanding on how the Riccati operator and the associated
 568 Kalman observer decouples the two ends problems (72). The following proof extend
 569 the one proposed in [57] to quasi evolution operator.

570 **Theorem 4.4.** *The Kalman estimator defined by the dynamics (76) satisfies the*
 571 *fundamental identity*

$$\forall t \in [0, T], \quad \bar{z}_T(t) = \hat{z}(t) + \mathbf{\Pi}(t)\bar{\mathbf{q}}_T(t). \quad (79)$$

Proof. Let us introduce $\eta = \hat{z} - \bar{z}_T + \mathbf{\Pi}\bar{\mathbf{q}}_T$ and $v \in \mathcal{D}(\mathbf{A}^*)$, and compute

$$\begin{aligned} \frac{d}{dt}(\eta(t), v)_{\mathcal{Z}} &= (\hat{z}(t), \mathbf{A}(t)^*v)_{\mathcal{Z}} + (y^\delta(t) - \mathbf{C}(t)\hat{z}(t), \mathbf{C}(t)\mathbf{\Pi}(t)v)_{\mathcal{Y}} \\ &\quad - (\bar{z}_T(t), \mathbf{A}(t)^*v)_{\mathcal{Z}} + \frac{d}{dt}(\mathbf{\Pi}(t)\bar{\mathbf{q}}_T(t), v)_{\mathcal{Z}}, \end{aligned} \quad (80)$$

where we need to properly define the last-term time derivative. In fact, as $\bar{\mathbf{q}}$ can be defined from a Duhamel formula of the form (70), and $\mathbf{\Pi}$ is mild solution of the Riccati equation (73), hence is absolutely continuous, we have from [8, Lemme 2.5] that

$$\frac{d}{dt}(\mathbf{\Pi}(t)\bar{\mathbf{q}}_T(t), v)_{\mathcal{Z}} = \frac{d}{dt}(\mathbf{\Pi}(t)\bar{\mathbf{q}}_T(s), v)_{\mathcal{Z}} \Big|_{s=t} + \frac{d}{dt}(\bar{\mathbf{q}}_T(t), \mathbf{\Pi}(s)v)_{\mathcal{Z}} \Big|_{s=t}.$$

Moreover, as $\mathbf{\Pi}$ is a weak solution of (73), we have from (75),

$$\begin{aligned} \frac{d}{dt}(\mathbf{\Pi}(t)\bar{\mathbf{q}}_T(s), v) \Big|_{s=t} &= (\mathbf{\Pi}(t)\bar{\mathbf{q}}_T(t), \mathbf{A}(t)^*v)_{\mathcal{Z}} + (\bar{\mathbf{q}}_T(t), \mathbf{A}(t)^*\mathbf{\Pi}(t)v)_{\mathcal{Z}} \\ &\quad - (\mathbf{C}(t)\mathbf{\Pi}(t)\bar{\mathbf{q}}_T(t), \mathbf{C}(t)\mathbf{\Pi}(t)v)_{\mathcal{Y}}, \end{aligned} \quad (81)$$

and, as $\bar{\mathbf{q}}_T$ is a weak solution of (72),

$$\begin{aligned} \frac{d}{dt}(\bar{\mathbf{q}}_T(t), \mathbf{\Pi}(s)v) \Big|_{s=t} &= -(\bar{\mathbf{q}}_T(t), \mathbf{A}(t)^*\mathbf{\Pi}(t)v)_{\mathcal{Z}} \\ &\quad - (y^\delta(t) - \mathbf{C}(t)\bar{z}_T(t), \mathbf{C}(t)\mathbf{\Pi}(t)v)_{\mathcal{Y}}. \end{aligned} \quad (82)$$

572 Gathering (80), (81) and (82), we finally obtain

$$\begin{cases} \frac{d}{dt}(\eta(t), v)_{\mathbf{Z}} = (\eta(t), \mathbf{A}(t)^* - \mathbf{C}(t)^* \mathbf{C}(t) \mathbf{\Pi}(t)v)_{\mathbf{Z}}, & t \in [0, T] \\ \eta(0) = 0 \end{cases}$$

573 whence, by Theorem 4.3, $\eta = 0$ in $[0, T]$, which concludes the proof. \square

574 In particular, we see that by taking $t = T$ that for all $t > 0$, $\hat{\mathbf{z}}(t) = \bar{\mathbf{z}}_t(t)$. This
575 why the Kalman observer is also called optimal filtering estimator as it consists of
576 sequentially computing the trajectory associated with least-square estimator at final
577 time.

578 4.2.2 The reduced order case

579 As we saw in the previous section, the inverse problem can be solved using the Kalman
580 filter, but this requires us to calculate a Riccati operator in the augmented space
581 \mathbf{Z} . In order to reduce the computational cost, we now show that we can instead
582 compute another Riccati operator on a reduced space \mathbf{Z}_r that takes into account a
583 priori informations on the inverse problem: in our case, described specifically later,
584 only the source terms is not known. The reduced space \mathbf{Z}_r is assumed to be a closed
585 subspace of \mathbf{Z} (equipped with the same scalar product) hence we can assume given a
586 projection operator $P_r \in \mathcal{L}(\mathbf{Z}, \mathbf{Z}_r)$ in this space. Moreover, we also assumed given a
587 compact subspace \mathbf{V}_r of \mathbf{Z}_r that is used to define a regularized version of the functional
588 to minimize. The minimization problem reads

$$\min_{\zeta_r \in \mathbf{V}_r} \left\{ \mathcal{J}_T(\zeta_r) = \frac{1}{2} m_r(\zeta_r, \zeta_r) + \frac{1}{2} \int_0^T \|y^\delta - \mathbf{C}z_{|P_r^* \zeta_r}\|_y^2 dt \right\},$$

589 where $z_{|P_r^* \zeta_r}$ is solution of (64) for an initial condition $\mathbf{z}_0 = \hat{\mathbf{z}}_0 + P_r^* \zeta_r$ and m_r is a
590 continuous and coercive bilinear form on \mathbf{V}_r . The associated two-ends problem reads

$$\begin{cases} \dot{\bar{\mathbf{z}}}_T = \mathbf{A}(t)\bar{\mathbf{z}}_T + \boldsymbol{\beta}, & t \in (0, T), \\ \dot{\bar{\mathbf{q}}}_T + \mathbf{A}(t)^* \bar{\mathbf{q}}_T = -\mathbf{C}^*(y^\delta(t) - \mathbf{C}\bar{\mathbf{z}}_T), & t \in (0, T), \\ \bar{\mathbf{z}}_T(0) = \hat{\mathbf{z}}_0 + P_r^* \Lambda_0 P_r \bar{\mathbf{q}}_T(0), \\ \bar{\mathbf{q}}_T(T) = 0, \end{cases} \quad (83)$$

591 with $\Lambda_0 \in \mathcal{L}(\mathbf{V}_r)$ defined such that

$$\forall (\zeta_r, \xi_r) \in \mathbf{V}_r \times \mathbf{V}_r, \quad m_r(\Lambda_0 \zeta_r, \xi_r) = (\zeta_r, \xi_r)_{\mathbf{Z}}.$$

592 The resulting Kalman observer can still be defined from the Riccati dynamics

$$\begin{cases} \dot{\mathbf{\Pi}}(t) = \mathbf{A}(t)\mathbf{\Pi}(t) + \mathbf{\Pi}(t)\mathbf{A}(t)^* - \mathbf{\Pi}(t)\mathbf{C}^* \mathbf{C} \mathbf{\Pi}(t), & t > 0, \\ \mathbf{\Pi}(0) = P_r^* \Lambda_0 P_r, \end{cases} \quad (84)$$

593 so that we continue to have $\bar{\mathbf{z}}_T(t) = \hat{\mathbf{z}}(t) + \mathbf{\Pi}(t)\bar{\mathbf{q}}_T(t)$ for all $t \in [0, T]$. But moreover,
594 the Riccati solution of (84) can be deduced from a reduced covariance operator Λ ,
595 solution of a Riccati dynamics in the reduced space \mathbf{Z}_r , namely

$$\begin{cases} \dot{\Lambda}(t) = -\Lambda(t)P_r\Phi(t, 0)^* \mathbf{C}^* \mathbf{C} \Phi(t, 0)P_r^* \Lambda(t), & t > 0, \\ \Lambda(0) = \Lambda_0. \end{cases} \quad (85)$$

596 This dynamics satisfies the condition of existence since for all $t \in [0, T]$, $\mathbf{C}\Phi(t, 0)P_r^*$
597 can be considered as a time-dependent bounded observation operator. Moreover, we
598 have the following decomposition.

599 **Theorem 4.5.** Let $\mathbf{\Pi}(t)$ the mild solution of the Riccati dynamic (84) and $\Lambda(t)$ the
600 mild solution of the Riccati dynamic (85), then we have

$$\mathbf{\Pi}(t) = \Phi(t, 0)P_r^* \Lambda(t)P_r \Phi(t, 0)^*. \quad (86)$$

601 *Proof.* The mild solution of the Riccati dynamics (85) reads

$$\Lambda(t) = \Lambda(0) - \int_0^t \Lambda(s)P_r \Phi(s, 0)^* C^* C \Phi(s, 0)P_r^* \Lambda(s) ds.$$

Using this mild solution, we compute the following quantity $\Phi(t, 0)P_r^* \Lambda(t)P_r \Phi(t, 0)^*$.

$$\begin{aligned} \Phi(t, 0)P_r^* \Lambda(t)P_r \Phi(t, 0)^* &= \Phi(t, 0)P_r^* \Lambda(0)P_r \Phi(t, 0)^* \\ &\quad - \int_0^t \Phi(t, 0)P_r^* \Lambda(s)P_r \Phi(s, 0)^* C^* C \Phi(s, 0)P_r^* \Lambda(s)P_r \Phi(t, 0)^* ds. \end{aligned}$$

Note that $\Phi(t, 0) = \Phi(t, s)\Phi(s, 0)$, therefore we obtain

$$\begin{aligned} \Phi(t, 0)P_r^* \Lambda(t)P_r \Phi(t, 0)^* &= \Phi(t, 0)P_r^* \Lambda(0)P_r \Phi(t, 0)^* \\ &\quad - \int_0^t \Phi(t, s)\Phi(s, 0)P_r^* \Lambda(s)P_r \Phi(s, 0)^* C^* C \Phi(s, 0)P_r^* \Lambda(s)P_r \Phi(s, 0)^* \Phi(t, s)^* ds. \end{aligned}$$

602 Setting $\tilde{\mathbf{\Pi}}(t) = \Phi(t, 0)P_r^* \Lambda(t)P_r \Phi(t, 0)^*$, we get

$$\tilde{\mathbf{\Pi}}(t) = \tilde{\mathbf{\Pi}}(0) - \int_0^t \Phi(t, s)\tilde{\mathbf{\Pi}}(s)C^* C \tilde{\mathbf{\Pi}}(s)\Phi(t, s)^* ds.$$

603 Hence, $\tilde{\mathbf{\Pi}}$ is a mild solution of (84) and by uniqueness of the solution of the Riccati
604 dynamics, we have

$$\tilde{\mathbf{\Pi}}(t) = \mathbf{\Pi}(t) = \Phi(t, 0)P_r^* \Lambda(t)P_r \Phi(t, 0)^*,$$

605 which concludes the proof. □

606 Therefore, a computational strategy can be based only on computations restricted
607 to the reduced-space \mathcal{Z}_r .

608 4.2.3 Application to source estimation

609 We will now define an observer, which is equivalent to the calculation of the optimum
610 $\bar{\theta}_T$ with increasing time. To that purpose, we rely on the Kalman estimation theory
611 presented above and more precisely the Reduced-Order Kalman filtering approach of
612 Section 4.2.2, where the uncertainty space is, here, the parameter space, a subspace
613 of the augmented space $\mathcal{Z} = \mathcal{Z} \times \mathcal{P}$. In particular, we set $\mathcal{Z}_r = \{0\} \times L^2(\Omega)$ and
614 $\mathcal{V}_r = \{0\} \times H_0^1(\Omega)$. The dynamics in the augmented space reads

$$\underbrace{\frac{d}{dt} \begin{pmatrix} z \\ \theta \end{pmatrix}}_{\dot{z}} = \underbrace{\begin{pmatrix} A & B(t) \\ 0 & 0 \end{pmatrix}}_{A(t)z} \begin{pmatrix} z \\ \theta \end{pmatrix} \quad \text{with} \quad \underbrace{\begin{pmatrix} z(0) \\ \theta(0) \end{pmatrix}}_{z(0)} = \underbrace{\begin{pmatrix} 0 \\ \theta \end{pmatrix}}_{\zeta}. \quad (87)$$

615 The least square minimization (33) rewritten with respect to the augmented variable
616 reads

$$\mathcal{J}_T(\theta) = \frac{\epsilon}{2} \|\theta\|_{H_0^1(\Omega)}^2 + \frac{1}{2} \int_0^T \|y^\delta(t) - Cz|_\theta\|_Y^2 dt. \quad (88)$$

617 Introducing the projection operator

$$P_\theta = \begin{pmatrix} 0 & 0 \\ 0 & \text{Id}_{\mathcal{P}} \end{pmatrix}$$

618 from the augmented space \mathcal{Z} to the reduced space \mathcal{Z}_r , we have, using Theorem 4.5,

$$\begin{cases} \mathbf{\Pi}(t) = \Phi(t, 0) P_\theta^* \Lambda(t) P_\theta \Phi(t, 0)^*, \\ \mathbf{\Pi}(0) = P_\theta^* \Lambda(0) P_\theta, \end{cases} \quad (89)$$

619 where $\Lambda(t)$ is given by (85) with $P_r = P_\theta$. Then, we can specify the evolution operator
620 Φ associated with the following system

$$\mathbf{A}(t) = \begin{pmatrix} A & B(t) \\ 0 & 0 \end{pmatrix}, \quad \begin{cases} \dot{z}(t) = \mathbf{A}(t)z(t), \\ z(0) = z_0, \end{cases} \quad \Rightarrow \quad \begin{cases} \dot{z}(t) = Az(t) + B(t)\theta, \\ \dot{\theta}(t) = 0, \\ (z, \theta)^t(0) = (z_0, \theta_0)^t, \end{cases}$$

621 Therefore, for initial data of the form $z_0 = P_\theta^*(0, \theta_0)^t = (0, \theta_0)^t$ the solution reads

$$\begin{pmatrix} z(t) \\ \theta(t) \end{pmatrix} = \Phi(t, 0) \begin{pmatrix} 0 \\ \theta_0 \end{pmatrix},$$

622 and, using the definition of L given by (5), $z(t) = L(t)\theta_0$ is the mild solution of

$$\begin{cases} \dot{z} = Az + B\theta_0, & t > 0, \\ z(0) = 0, \end{cases}$$

623 Hence, we obtain that

$$\begin{pmatrix} z(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} L(t) \\ \text{Id}_{\mathcal{P}} \end{pmatrix} \theta_0.$$

624 Eventually, we have shown that for all θ_0 , $\Phi(t, 0)P_\theta^*(z_0, \theta_0)^t$ is given by

$$\begin{pmatrix} z(t) \\ \theta(t) \end{pmatrix} = \Phi(t, 0)P_\theta^* \begin{pmatrix} z_0 \\ \theta_0 \end{pmatrix} = \Phi(t, 0) \begin{pmatrix} 0 \\ \theta_0 \end{pmatrix} = \begin{pmatrix} L(t) \\ \text{Id}_{\mathcal{P}} \end{pmatrix} \theta_0.$$

625 Therefore, we can simplify the Riccati dynamics (85) into

$$\begin{cases} \dot{\Lambda} = -\Lambda L^* C^* C L \Lambda, & t > 0, \\ \Lambda(0) = \Lambda_0, \end{cases} \quad (90)$$

626 and (89) into

$$\mathbf{\Pi}(t) = \begin{pmatrix} L(t) \\ \text{Id}_{\mathcal{P}} \end{pmatrix} \Lambda(t) (L^*(t) \text{Id}_{\mathcal{P}}). \quad (91)$$

627 Note that, in our case $m_r(\cdot, \cdot) = \frac{\epsilon}{2}(\cdot, \cdot)_{\mathbb{H}_0^1(\Omega)}$, therefore we can obtain an explicit
628 expression of Λ_0 given by

$$\Lambda_0 = -\epsilon^{-1} \Delta_0^{-1}.$$

629 Finally, we obtain the equivalent sequential estimator with the following result that is
630 a straightforward application of Theorem 4.3.

631 **Corollary 4.6.** *The minimizer $\hat{\theta} = \bar{\theta}_t$ is solution of the following dynamics*

$$\begin{cases} \dot{\hat{\theta}} = \Lambda L^* C^* (y^\delta - C\hat{z}), & t > 0, \\ \hat{\theta}(0) = \hat{\theta}_0, \\ \dot{\hat{z}} = A\hat{z} + B\hat{\theta} + L\hat{\theta}, & t > 0, \\ z(0) = \hat{z}_0. \end{cases} \quad (92)$$

632 Note that the present estimator can be defined for observations in H^1 or L^2 with an
633 adapted definition of the observation operator C and its adjoint C^* . Using the results
634 obtained in Sections 3.2.2 and 3.3.2, we show that $(\hat{z}, \hat{\theta})$ cannot be entirely defined
635 as observers in the sense of Definition 4.1, but that they satisfy the second property
636 (robustness to noise) and a necessary condition of the first (convergence).

637 **Proposition 4.7.** *Assuming that we have at our disposal the perfect measurements*
638 *$y^\delta = \check{y}$, then, under Assumption 1, the estimators $(\hat{z}, \hat{\theta})$ asymptotically converges to the*
639 *target parameter $(\check{z}, \check{\theta})$, namely*

$$\lim_{t \rightarrow +\infty} \|\hat{\theta}(t) - \check{\theta}(t)\|_{L^2(\Omega)} = 0 \text{ and } \lim_{t \rightarrow +\infty} \|\hat{z}(t) - \check{z}(t)\|_{\mathcal{Z}} = 0.$$

640 *Proof.* We here use the link between the observer and the functional minimization.
641 Under Assumptions 1 and 2, we know that (Theorem 3.6 and 3.12), for both types of
642 observation operator, there exists $a > 0$ and C^{st} such that

$$\|\bar{\theta}_T - \check{\theta}\|_{L^2(\Omega)} \leq C^{\text{st}} T^{-a},$$

643 which, combined with $\hat{\theta} = \bar{\theta}_t$ gives the convergence of the parameter estimator

$$\lim_{t \rightarrow +\infty} \|\hat{\theta}(t) - \check{\theta}\|_{L^2(\Omega)} = 0.$$

644 Moreover, we have that

$$\hat{z} = \bar{z}_t(t) = z|_{\bar{\theta}_t}.$$

645 Therefore,

$$\lim_{t \rightarrow +\infty} \|\hat{z}(t) - \check{z}(t)\|_{\mathcal{Z}} = \lim_{t \rightarrow +\infty} \|z|_{\bar{\theta}_t}(t) - z|_{\check{\theta}_t}\|_{\mathcal{Z}} = 0,$$

646 which concludes the proof. □

647 **Proposition 4.8.** *Assuming that we have at our disposal the noisy measurements*
648 *satisfying (32) or (44), the estimator $(\hat{z}, \hat{\theta})$ is stable with respect to noise, namely there*
649 *exist two positive scalars C_1^{st} and C_2^{st} independent of δ and dependent of t such that*

$$\|\hat{\theta}(t) - \check{\theta}\|_{L^2(\Omega)} \leq C_1^{\text{st}}(t) \delta \text{ and } \|\hat{z} - \check{z}\|_{\mathcal{Z}} \leq C_2^{\text{st}}(t) \delta.$$

650 *Proof.* We take advantage of the link between the observer and the functional mini-
651 mization. From Theorem 3.7 and 3.13, for both types of observation operator, there
652 exist C^{st} such that

$$\|\bar{\theta}_t - \check{\theta}\|_{L^2(\Omega)} \leq C^{\text{st}} \delta,$$

653 which, combined with $\hat{\theta}(t) = \bar{\theta}_t$ gives the stability of the parameter estimator

$$\|\hat{\theta}(t) - \check{\theta}\|_{L^2(\Omega)} \leq C^{\text{st}} \delta.$$

654 Furthermore, we have that

$$\hat{z}(t) = \bar{z}_t(t) = z|_{\bar{\theta}_t}.$$

655 Therefore, we obtain

$$\|\hat{z}(t) - \tilde{z}\|_{\mathcal{Z}} = \|z_{|\bar{\theta}_t}(t) - z_{|\tilde{\theta}}\|_{\mathcal{Z}}.$$

656 Classically using the Grönwall lemma, one can show that there exists a positive scalar
657 C^{st} depending only on t and σ such that

$$\|z_{|\bar{\theta}_t}(t) - z_{|\tilde{\theta}}\|_{\mathcal{Z}} \leq C^{\text{st}} \|\bar{\theta}_t(t) - \tilde{\theta}\|_{L^2(\Omega)},$$

658 which concludes the proof. \square

659 **Remark 4.9.** *Proposition 4.8 allows us to demonstrate the robustness to noise property*
660 *of Definition 4.1. On the other hand, Proposition 4.7 is a necessary condition for the*
661 *convergence property of Definition 4.1. Indeed, the second property of the definition is*
662 *obtained only in this case $\tau = 0$. We would have to take into account the possibility of*
663 *uncertainties in the initial condition for dynamics (92) to fall within the asymptotic*
664 *observer framework of definition 4.1.*

665 4.3 Observer discretization

666 We can now study the observer discretization, where we face two possible options. First
667 we could directly propose a time and space discretization of (92), or we could first
668 discretize the wave equation and then reapply the Tikhonov regularization strategy
669 and observer design to the discretized system. Following [27], we are going to follow
670 this second strategy.

671 4.3.1 Reduced-order Kalman discretization

672 As in the time-continuous case, let us first recall the Kalman discretization for a linear
673 discrete-time dynamics. Let $\mathcal{Z}^h \subset \mathcal{V} \subset \mathcal{Z}$ a finite dimensional space equipped with the
674 scalar product of \mathcal{Z} . Assuming a dynamics of the following form, with a given source
675 term $\beta^h \in \mathcal{Z}^h$

$$\begin{cases} z_{n+1|\zeta}^h = \Phi^{h,\tau} z_{n|\zeta}^h + \beta_{n+1}^h, & n \geq 0, \\ z_{0|\zeta}^h = \hat{z}_0^h + \zeta \end{cases}$$

676 associated with measurements $y_n^{\delta,h}$ and an observation operator $C^h \in \mathcal{L}(\mathcal{Z}^h, \mathcal{Y}^h)$ where
677 \mathcal{Y}^h is the finite dimensional space in which the observations belong. We first consider
678 the minimization of a discrete version of the criterion (68) (see [57])

$$\min_{\zeta \in \mathcal{Z}^h} \left\{ \mathcal{J}_N(\zeta^h) = \frac{1}{2} m(\zeta^h, \zeta^h) + \sum_{n=1}^N \frac{\tau}{2} \|y_n^{\delta,h} - C^h z_{n|\zeta}^h\|_{\mathcal{Y}^h}^2 \right\}, \quad (93)$$

679 where the time integral is replaced by a simple quadrature formula. We easily find the
680 corresponding two-ends problem associated with the minimization of this criterion,

$$\begin{cases} \bar{z}_{n+1|N}^h = \Phi^{h,\tau} \bar{z}_{n|N}^h + \beta_{n+1}^h, & 0 \leq n \leq N-1, \\ \bar{z}_{0|N}^h = \hat{z}_0^h + \Pi_0^h \bar{q}_{0|N}^h, \\ \bar{q}_{n|N}^h = \Phi^{h,\tau*} \bar{q}_{n+1|N}^h + \tau C^{h*} (y_n^{\delta,h} - C^h \bar{z}_{n|N}^h), & 0 \leq n \leq N-1, \\ \bar{q}_{N|N}^h = 0, \end{cases} \quad (94)$$

681 where the notation $|N$ stresses the fact that the criterion is using the N first observations
682 and $\Pi_0^h \in \mathcal{L}(\mathcal{Z}^h)$ is the operator defined by

$$\forall (\mathbf{q}_h, \xi_h) \in \mathcal{Z}^h \times \mathcal{Z}^h, \quad m(\Pi_0^h \mathbf{q}_h, \xi_h) = (\mathbf{q}_h, \xi_h)_{\mathcal{Z}}.$$

683 Then, we can uncouple these systems by solving the following Riccati equation,

$$\begin{cases} \mathbf{\Pi}_{n+1-}^h = \mathbf{\Phi}^{h,\tau} \mathbf{\Pi}_n^h \mathbf{\Phi}^{h,\tau*}, & n \in \mathbb{N}, \\ \mathbf{\Pi}_{n+1}^h = \mathbf{\Pi}_{n+1-}^h - \tau \mathbf{\Pi}_{n+1-}^h \mathbf{C}^{h*} [\text{Id}_{y^h} + \tau \mathbf{C}^h \mathbf{\Pi}_{n+1-}^h \mathbf{C}^{h*}]^{-1} \mathbf{C}^h \mathbf{\Pi}_{n+1-}^h, & n \in \mathbb{N}, \end{cases} \quad (95)$$

684 which defines a sequence of operators $\mathbf{\Pi}_{n-}^h$ and $\mathbf{\Pi}_n^h$. The sequence of $\mathbf{\Pi}_{n-}^h$ and $\mathbf{\Pi}_n^h$ can
685 be interpreted as a splitting time-scheme for the Riccati dynamics (73) [57]. Then, we
686 introduce the estimator

$$\begin{cases} \hat{\mathbf{z}}_{n+1-}^h = \mathbf{\Phi}^{h,\tau} \hat{\mathbf{z}}_n^h + \boldsymbol{\beta}_{n+1}^h, & n \in \mathbb{N}, \\ \hat{\mathbf{z}}_{n+1}^h = \hat{\mathbf{z}}_{n+1-}^h + \tau \mathbf{\Pi}_{n+1}^h \mathbf{C}^{h*} \left(y_{n+1}^{\delta,h} - \mathbf{C}^h \hat{\mathbf{z}}_{n+1-}^h \right), & n \in \mathbb{N}. \end{cases} \quad (96)$$

687 Identically, (96) can be interpreted as a splitting time-scheme for the estimator dynamics
688 (76), see [57]. Then, we have the following equivalence results proved in [57].

689 **Theorem 4.10.** *We have the following decoupling principle*

$$\bar{\mathbf{z}}_{n|N}^h = \hat{\mathbf{z}}_{n-}^h + \mathbf{\Pi}_{n-}^h \bar{\mathbf{q}}_{n|N}^h, \quad \forall 0 \leq n \leq N+1, \quad (97)$$

690 and in particular for all $n \in \mathbb{N}$

$$\hat{\mathbf{z}}_{n-}^h = \bar{\mathbf{z}}_{n|n}^h. \quad (98)$$

691 Let us now apply such equivalence to a reduced space configuration. For this
692 purpose, we define $\mathcal{Z}_r^h \subset \mathcal{Z}^h$ a finite dimensional space such that $\mathcal{Z}_r^h \subset \mathcal{V}_r \subset \mathcal{Z}_r$. To
693 simplify the notation, we assume that $P_r(\mathcal{Z}^h) \subset \mathcal{Z}_r^h$ – which is satisfied for our source
694 identification problem – hence P_r can be seen as an operator in $\mathcal{L}(\mathcal{Z}^h, \mathcal{Z}_r^h)$. The discrete
695 criterion here reads

$$\min_{\zeta_r^h \in \mathcal{Z}_r^h} \left\{ \mathcal{J}_N(\zeta_r^h) = \frac{1}{2} m_r(\zeta_r^h, \zeta_r^h) + \sum_{k=1}^N \frac{\tau}{2} \|y_n^{\delta,h} - \mathbf{C} z_{n|P_r^* \zeta_r^h}^h\|_{y^h}^2 \right\}. \quad (99)$$

696 The two ends problem remains as in (94) except for the initial condition that is changed
697 into

$$\bar{\mathbf{z}}_{0|N}^h = \hat{\mathbf{z}}_{0-}^h + P_r^*(U_0^h)^{-1} P_r \bar{\mathbf{q}}_{0|N}^h, \quad (100)$$

698 Moreover, we have the discrete counterpart of Theorem 4.5.

699 **Theorem 4.11.** *The solution of (95) with initial condition defined by*

$$\mathbf{\Pi}_0^h = P_r^* \Lambda_0^h P_r$$

700 satisfies

$$\mathbf{\Pi}_n^h = \left(\mathbf{\Phi}^{h,\tau} \right)^n P_r^* \Lambda_n^h P_r \left(\mathbf{\Phi}^{h,\tau*} \right)^n \quad (101)$$

701 where Λ_n^h satisfies $\Lambda_n^h = (U_n^h)^{-1}$ with

$$U_{n+1}^h = U_n^h + \tau P_r \left(\mathbf{\Phi}^{h,\tau*} \right)^{n+1} \mathbf{C}^{h*} \mathbf{C}^h \left(\mathbf{\Phi}^{h,\tau} \right)^{n+1} P_r^*, \quad n \geq 0, \quad (102)$$

702 *Proof.* We proceed by induction. We have at iteration 0

$$\mathbf{\Pi}_0^h = P_r^*(U_0^h)^{-1} P_r.$$

703 Let us assume (101) is satisfied at iteration n . First, we directly have

$$\mathbf{\Pi}_{n+1-}^h = \mathbf{\Phi}^{h,\tau} \mathbf{\Pi}_n^h \mathbf{\Phi}^{h,\tau*} = \left(\mathbf{\Phi}^{h,\tau} \right)^{n+1} P_r^* \Lambda_n^h P_r \left(\mathbf{\Phi}^{h,\tau*} \right)^{n+1},$$

Then, using (95) and the expression above of $\mathbf{\Pi}_{n+1}^h$, we have

$$\begin{aligned} \mathbf{\Pi}_{n+1}^h &= \left(\Phi^{h,\tau}\right)^{n+1} P_r^* \left[\Lambda_n^h - \tau \Lambda_n^h P_r \left(\Phi^{h,\tau^*}\right)^{n+1} C^{h*} \left[\text{Id}_{\mathcal{Y}^h} + \tau C^h \left(\Phi^{h,\tau}\right)^{n+1} \right. \right. \\ &\quad \left. \left. \times P_r^* \Lambda_n^h P_r \left(\Phi^{h,\tau^*}\right)^{n+1} C^{h*} \right]^{-1} C^h \left(\Phi^{h,\tau}\right)^{n+1} P_r^* \Lambda_n^h \right] P_r \left(\Phi^{h,\tau}\right)^{n+1}. \end{aligned}$$

704 We therefore have

$$\mathbf{\Pi}_{n+1}^h = \left(\Phi^{h,\tau}\right)^{n+1} P_r^* \Lambda_{n+1}^h P_r \left(\Phi^{h,\tau}\right)^{n+1},$$

as soon as

$$\Lambda_{n+1}^h = \Lambda_n^h - \Lambda_n^h D_n^{h,*} \left(\text{Id}_{\mathcal{Y}^h} + D_n^h \Lambda_n^h D_n^{h,*} \right)^{-1} D_n^h \Lambda_n^h$$

705 with $D_n^h = \sqrt{\tau} C^h \left(\Phi^{h,\tau}\right)^{n+1} P_r^*$. By using Sherman–Morrison–Woodbury formula for
706 operators – see for instance [40, Theorem 1.1] – we obtain

$$\left(\Lambda_{n+1}^h\right)^{-1} = \left(\Lambda_n^h\right)^{-1} + D_n^{h,*} D_n^h$$

707 which ends the proof since it is precisely (102). □

708 4.3.2 Application to source identification

709 In this work, we discretize the wave equation using Lagrange finite element. Let $\theta^h \in \mathcal{P}^h$
710 the finite dimensional space corresponding to \mathcal{P} . We use a conservative mid-point
711 scheme, that ultimately reads

$$\begin{cases} \frac{u_{n+1}^h - u_n^h}{\tau} = \frac{v_{n+1}^h + v_n^h}{2} \\ \frac{v_{n+1}^h - v_n^h}{\tau} - \Delta_0^h \frac{u_{n+1}^h + u_n^h}{2} = \sigma \left(\left(n + \frac{1}{2} \right) \tau \right) \theta^h, \end{cases} \quad (103)$$

712 where Δ_0^h is the bounded linear operator in $\mathcal{V}^h \subset H^1(\Omega)$ associated with the conform
713 finite element discretization of the unbounded operator Δ_0 of $L^2(\Omega)$. Note that this
714 system can be rewritten in a first order form by considering $z_n^h \in \mathcal{Z}^h = \mathcal{V}^h \times \mathcal{V}^h$ solution
715 of

$$z_{n+1}^h = \Phi^{h,\tau} z_n^h + B_{n+1}^{h,\tau} \theta^h,$$

716 where

$$\Phi^{h,\tau} = \begin{pmatrix} \text{Id}^h & -\frac{\tau}{2} \text{Id}^h \\ -\frac{\tau}{2} \Delta_0^h & \text{Id}^h \end{pmatrix}^{-1} \begin{pmatrix} \text{Id}^h & \frac{\tau}{2} \text{Id}^h \\ \frac{\tau}{2} \Delta_0^h & \text{Id}^h \end{pmatrix}$$

717 and

$$B_{n+1}^{h,\tau} = \tau \begin{pmatrix} \text{Id}^h & -\frac{\tau}{2} \text{Id}^h \\ -\frac{\tau}{2} \Delta_0^h & \text{Id}^h \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \sigma \left(\left(n + \frac{1}{2} \right) \tau \right) \end{pmatrix}.$$

718 Note that \mathcal{Z}^h is equipped with the scalar product on \mathcal{Z} that is the natural scalar
719 product on $H_0^1(\Omega) \times L^2(\Omega)$.

720 We now apply this result to our source identification problem. The joint state-
721 parameter discrete dynamics in $\mathcal{Z}^h = \mathcal{Z}^h \times \mathcal{P}^h$ reads

$$\underbrace{\begin{pmatrix} z_{n+1}^h \\ \theta_{n+1}^h \end{pmatrix}}_{z_{n+1}^h} = \underbrace{\begin{pmatrix} \Phi^{h,\tau} & B_{n+1}^{h,\tau} \\ 0 & \text{Id}^h \end{pmatrix}}_{\Phi^{h,\tau} z_n^h} \underbrace{\begin{pmatrix} z_n^h \\ \theta_n^h \end{pmatrix}}_{z_n^h}, \quad \text{with} \quad \underbrace{\begin{pmatrix} z_0^h \\ \theta_0^h \end{pmatrix}}_{z_0^h} = \underbrace{\begin{pmatrix} 0 \\ \theta^h \end{pmatrix}}_{\zeta}. \quad (104)$$

722 From Theorem 4.5, we know that

$$\begin{cases} \mathbf{\Pi}_n^h = (\Phi^{h,\tau})^n P_r^* \Lambda_n^h P_r (\Phi^{h,\tau^*})^n \text{ and } \mathbf{\Pi}_{n+1}^h = (\Phi^{h,\tau})^{n+1} P_r^* \Lambda_n^h P_r (\Phi^{h,\tau^*})^{n+1}, \\ \mathbf{\Pi}_0 = P_\theta^* (U_0^h)^{-1} P_\theta, \end{cases} \quad (105)$$

723 where $U_n^h = (\Lambda_n^h)^{-1}$ is solution of

$$U_{n+1}^h = U_n^h + \tau P_\theta (\Phi^{h,\tau^*})^{n+1} C^{h*} C^h (\Phi^{h,\tau})^{n+1} P_\theta^*, \quad n \geq 0. \quad (106)$$

724 Moreover, we have

$$\begin{pmatrix} z_n^h \\ \theta_n^h \end{pmatrix} = (\Phi^{h,\tau})^n P_\theta^* \begin{pmatrix} 0 \\ \theta_0^h \end{pmatrix} = (\Phi^{h,\tau})^n \begin{pmatrix} 0 \\ \theta_0^h \end{pmatrix} = \begin{pmatrix} L_n^h \\ \text{Id}^h \end{pmatrix} \theta_0^h$$

725 where the operator $(L_n^h)_{n \in \mathbb{N}}$ is defined by

$$L_n^h \theta = z_{n|\theta}^h \quad \text{with} \quad \begin{cases} z_{n+1|\theta}^h = \Phi^{h,\tau} z_{n|\theta}^h + B_{n+1}^{h,\tau} \theta, & n \in \mathbb{N}, \\ z_{0|\theta}^h = 0. \end{cases} \quad (107)$$

726 Therefore, we can simplify (106) into

$$U_{n+1}^h = U_n^h + \tau L_{n+1}^{h*} C^{h*} C^h L_{n+1}^h, \quad n \geq 0, \quad (108)$$

727 and (105) into

$$\mathbf{\Pi}_n^h = \begin{pmatrix} L_n^h \\ \text{Id} \end{pmatrix} \Lambda_n^h (L_n^{h*} \quad \text{Id}), \text{ and } \mathbf{\Pi}_{n+1}^h = \begin{pmatrix} L_{n+1}^h \\ \text{Id}^h \end{pmatrix} \Lambda_n^h (L_{n+1}^{h*} \quad \text{Id}^h). \quad (109)$$

728 Finally, we obtain the equivalent sequential estimator.

729 **Theorem 4.12.** *The minimizer $\hat{\theta}_n^h = \bar{\theta}_n^h$ is solution from $(\hat{z}_0^h, \hat{\theta}_0^h)$ of the recursive*
730 *dynamics*

$$\begin{cases} \hat{z}_{n+1}^h = \Phi^{h,\tau} \hat{z}_n^h + B_{n+1}^{h,\tau} \hat{\theta}_n^h, \\ \hat{\theta}_{n+1}^h = \hat{\theta}_n^h + \tau (U_{n+1}^h)^{-1} L_{n+1}^{h*} C^{h*} (y_{n+1}^{\delta,h} - C^h \hat{z}_{n+1}^h), \\ \hat{z}_{n+1}^h = \hat{z}_{n+1}^h + L_{n+1}^h (\hat{\theta}_{n+1}^h - \hat{\theta}_n^h). \end{cases} \quad (110)$$

731 We already know that (110) is consistent. A direct convergence study of this
732 splitting scheme could be intricate. Here, by relying on the fundamental relation
733 $\hat{\theta}_n^h = \bar{\theta}_n^h$ we can justify the stability from the minimization. This should lead us to the
734 convergence of the time-scheme (110).

735 5 Numerical illustrations

736 We consider a domain Ω , and assume we have access to observations in the observation
737 subdomain ω illustrated in Figure 2. As an example, we consider the following wave
738 equation

$$\begin{cases} \partial_{tt} \check{u}(x, t) - \Delta \check{u}(x, t) = \sigma(t) \check{\theta}(x), & (x, t) \in \Omega \times (0, T), \\ \check{u}(x, t) = 0, & (x, t) \in \partial\Omega \times (0, T), \\ \check{u}(x, 0) = \check{u}_0(x), & x \in \Omega, \\ \partial_t \check{u}(x, 0) = 0, & x \in \Omega, \end{cases} \quad (111)$$

739 where

$$\sigma(t) = \frac{1}{2} + \frac{1}{2}e^{-0.5t},$$

740 and \tilde{u}_0 is solution of

$$\begin{cases} -\Delta \tilde{u}_0(x) = 10, & x \in \Omega, \\ \tilde{u}_0(x) = 0, & x \in \partial\Omega, \end{cases} \quad (112)$$

741 and $\check{\theta}$ is solution of

$$\begin{cases} -\Delta \check{\theta}(x) = 0 & x \in \Omega \setminus \Omega_{\text{hole}} \\ -\Delta \check{\theta}(x) = 10, & x \in \Omega_{\text{hole}}, \\ \check{\theta}(x) = 0, & x \in \partial\Omega, \end{cases} \quad (113)$$

with Ω_{hole} represented in Figure 2.

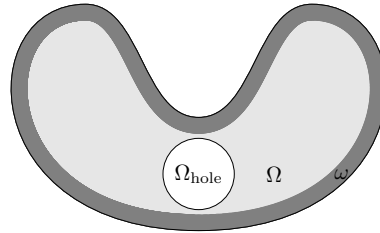


Figure 2. Representation of the domain Ω and the subdomain ω

742

743 The initial conditions and the target source term defined respectively by (112) and the
744 (113) are illustrated in Figure 3 (left) and 3 (right). Note that the choice of σ is the
745 same as that of Remark 3.5 with $\beta = 1$ and $\alpha = 0.5$.

746

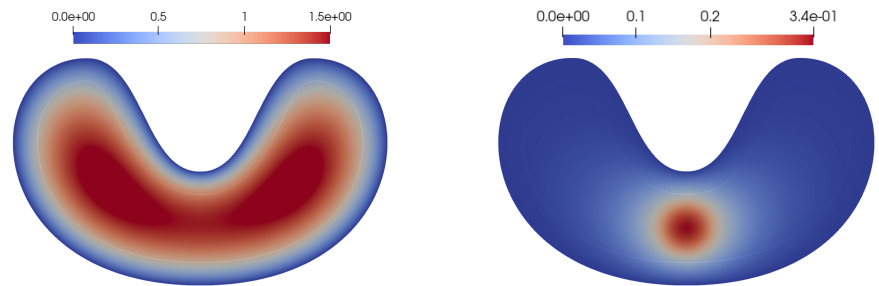


Figure 3. Representation of \tilde{u}_0 (left) and θ (right)

747 We decompose the target source term

$$\tilde{\theta}^h := \sum_{i=0}^{N_{\text{mod}}} \tilde{\theta}_i^h \phi_i^h, \quad (114)$$

748 where $(\phi_i^h)_{i \in [0, N_{\text{mod}}]}$ are the eigenvectors of Δ_0^h . In the following numerical illustrations,
749 we choose $N_{\text{mod}} = 27$ so that the relative error between $\tilde{\theta}$ and $\tilde{\theta}^h$ is less than 10%, see
Figure 4.

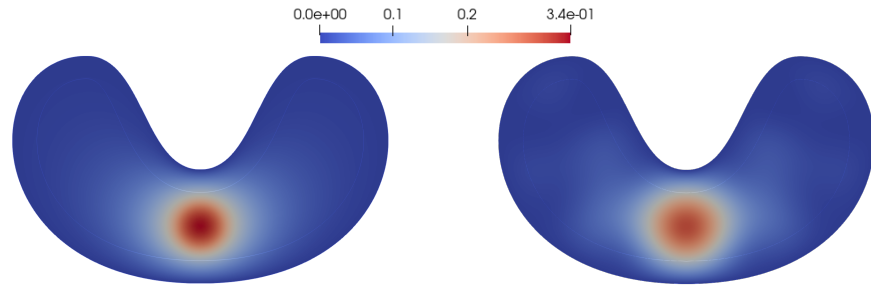


Figure 4. Source term $\tilde{\theta}$ (left) and its decomposition $\tilde{\theta}^h$ on the Laplacian basis with 27 modes (right)

750

751 In the following, we will compare the two methods presented in this work. The
752 method presented is denoted H^1 -method (the noisy measurements belong to $H^1(\omega)$).
753 Similarly, the method presented in Section 3.3 is denoted L^2 -method : the noisy
754 measurements belong to $L^2(\omega)$.

755 5.1 Numerical illustrations for noise-free data

756 First, to generate the measurements, we implement the direct problem on a time
757 window from 0 to 1 with a time step of $\tau = 0.01$ and with \mathbb{P}_1 finite elements and where
758 the space function $\tilde{\theta}^h$ to reconstruct in the source term is defined (114). We give an
759 illustration of a measurement in Figure 5.

760

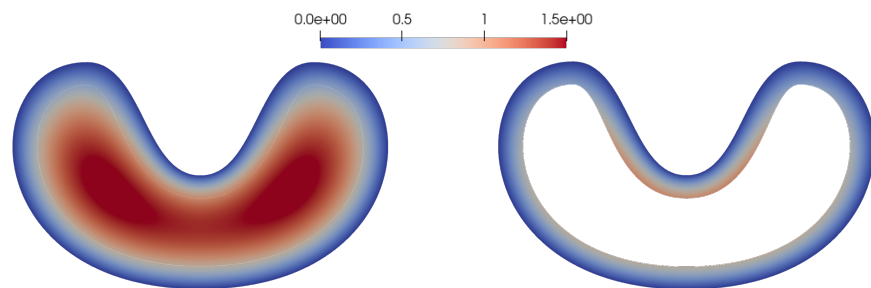


Figure 5. Solution \tilde{u}_n^h of the direct problem (left) and its corresponding measurements $(\tilde{u}_n^h)|_\omega$ (right) at time $t^n = 0.1$

761 We study the inverse problem for the H^1 -method in a noise-free case. We implement
762 the observers scheme (110) (where C is defined by (7)) in Freefem++ [42] over a time
763 window $[0, 1]$ with a time step of 0.01 and with \mathbb{P}_1 finite elements. Moreover, we start
764 from a null initial guess $\hat{\theta}_0^h = 0$ but with the correct initial conditions \tilde{u}_0 . Furthermore,
765 we remove the Tikhonov regularization term since we suppose that we have the perfect
766 measurements at our disposal. The same simulation for the L^2 -method is performed.
767 In other words, scheme 110 is implemented, but this time the observation operator C
768 is defined by (9).

769 Figure 6 shows the relative error between the target source term decomposed on
 770 the Laplacian modes and the observer defined by

$$\|\check{\theta}^h - \hat{\theta}_n^h\|_{L^2(\Omega)} / \|\check{\theta}^h\|_{L^2(\Omega)}$$

771 over the time.

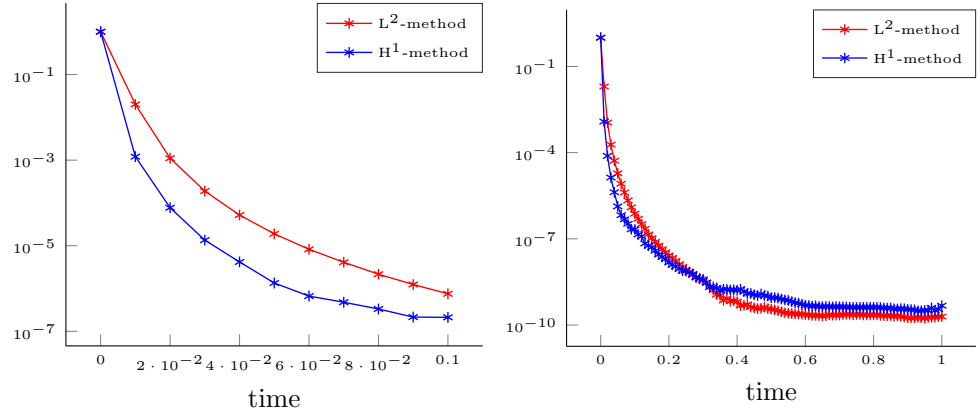


Figure 6. Evolution of the relative error with $\delta = 1$ and without Tikhonov term for $T = 0.1$ (left) and $T = 1$ (right)

772 For the H^1 -method, we can see that the error falls below $10^{-5}\%$ from $t^n = 0.1$. At
 773 the final time $t^n = 1$, the relative error is about $10^{-8}\%$. For the L^2 -method, the results
 774 are very similar. We can see that both methods converge for long time with the same
 775 final error level. However, in the short time range (Figure 6 left), the H^1 -method is
 776 more efficient than the L^2 -method. This indicates that the H^1 -method benefits from
 777 improved observability.

778

779 We will now deal with noisy measurements belonging to $L^2(\omega)$.

780 5.2 Numerical illustrations with noisy measurements

781 We represent the mesh in Figure 7 (left) used to solve the inverse problem. For
 782 visualization purposes, we show a coarse mesh (Figure 7 right), the used mesh is a
 783 refined version of the coarse one (Figure 7 left).



Figure 7. Representation of the mesh of Ω with 56193 degrees of freedom (left) and with 14169 degrees of freedom

784

785 We corrupt the synthetic data $(y_n^{\delta,h})_{0 \leq n \leq N}$ of sampling rate T/N with a random
 786 artificial noise $\chi_n^h \in L^2(\omega)$. To do this, we perturb the measurements at each interpola-
 787 tion point with a random value ξ that follows the Gaussian law $\mathcal{N}(0, 1)$ and scale with
 788 the noise level δ , *i.e.*, at each interpolation point we have

$$(y_n^{\delta,h})_i = (\tilde{y}_n^h)_i + \delta \xi_i / |\omega|, \quad (115)$$

789 where the scaling by $|\omega|$ ensures that the noise is normalized. We represent in Figure 8
 an example of a noisy measurement and the noise associated.

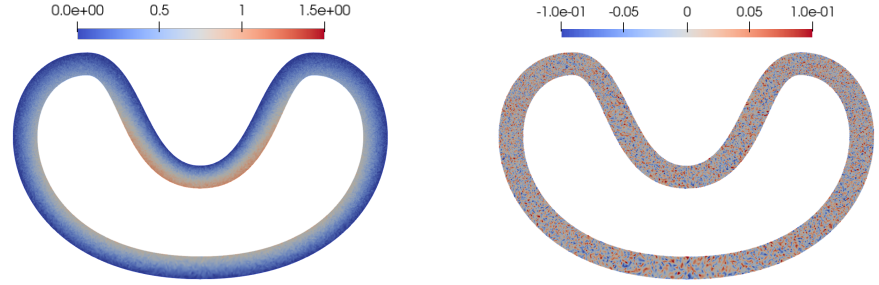


Figure 8. Noisy measurements $y_n^{\delta,h}$ (left) and the corresponding noise $\chi_n^h = y_n^{\delta,h} - \tilde{y}_n^h$ (right) at time $t^n = 0.1$

790

791 We solve the inverse problem with the *prediction-correction* scheme (110) with a
 792 time step of $\tau = 0.01$ and $\delta = 1$. Figure 9 shows the relative error

$$\frac{\|\check{\theta}^h - \hat{\theta}_n^h\|_{L^2(\Omega)}}{\|\check{\theta}^h\|_{L^2(\Omega)}}$$

793 between the target source term decomposed on the basis and the observer at the final
 794 time $t^n = 1$ as a function of the noise level δ in (115) for different regularization levels
 795 M .

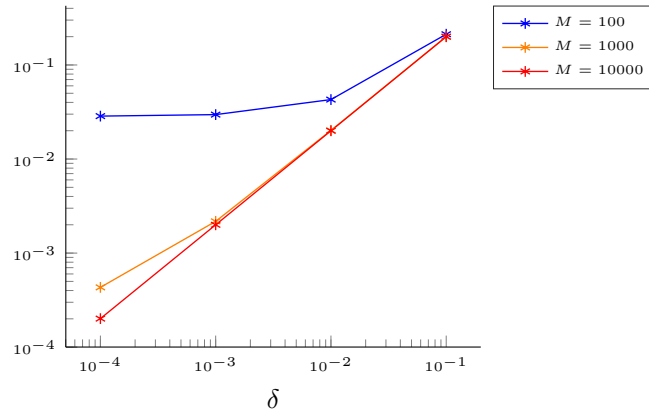


Figure 9. Representation of the relative error according the values of δ in loglog scale for different values of M

796

797 First, we note that the larger the value of M , the lower the weight of the regulariza-
 798 tion. In particular, we can see that the regularization with $M = 10^4$ (in red in Figure
 799 9) offers better results. In particular, for the different noise levels considered – from
 800 $\delta = 0.01\%$ up to $\delta = 10\%$ in (115) – we reach a reconstruction of the order of the noise
 801 level.

802 6 Conclusion

803 We have proposed a strategy to estimate a source term in a wave equation using
804 observers from internal *field* measurements. We have divided the study into two parts
805 depending on whether we have access to regular or less regular measurements. The
806 regularity of the measurements leads to major changes in the definition of the inverse
807 problem through the definition of the observation operator. The proposed approach
808 follows three steps. First we prove identifiability results which rely on observability
809 inequalities. These identifiability results give us information about the degree of the
810 ill-posedness of the inverse problem. We then propose the minimization problem
811 associated with the reconstruction of the source term with a well-adapted Tikhonov
812 regularization. This inverse problem is solved using a sequential strategy that is proven
813 to be equivalent to the minimization of the cost functional. We have assumed that we
814 have not only internal observations on the system, but also perfect knowledge of the
815 initial conditions, which can be a strong assumption in practice. A possible perspective
816 would be to assume that there are uncertainties on the initial conditions. In this case,
817 it would be necessary to further analyze the joint state-source estimation initially
818 proposed in [19].

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