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Learning Market Equilibria Preserving Statistical Privacy Using Performative Prediction

Hélène Le Cadre¹, Mandar Datar², Mathis Guckert¹, and Eitan Altman², Fellow, IEEE

Abstract—We consider a peer-to-peer electricity market modeled as a network game, where End Users (EUs) minimize their cost by computing their demand and generation while satisfying a set of local and coupling constraints. Their nominal demand constitutes sensitive information, that they might want to keep private. We prove that the network game admits a unique Variational Equilibrium, which depends on the private information of all the EUs. A data aggregator (DA) is introduced, which aims to learn the EUs' private information, while remunerating them depending on the quality of the readings they report to the DA. The EUs might have incentives to report biased and noisy readings to preserve their privacy. Relying on performative prediction, we define a decision-dependent game $\mathcal{G}^{\text{stoch}}$, to explicitly take into account the shift caused by the EUs' strategic information on their strategies and market equilibria. To compute market equilibria solutions of $\mathcal{G}^{\text{stoch}}$, two variants of the Repeated Stochastic Gradient Method (RSGM) and a two-time-scale stochastic approximation algorithm are proposed. We prove the convergence of each algorithm. Finally, the algorithms performance is assessed on a numerical example, by comparing the achieved efficiency loss, privacy preservation capabilities, convergence rates, and EUs' utility functions at equilibrium. The results highlight the benefits for the EUs to model performative effects.

Index Terms— *Learning Game, Mechanism Design, Performatively Stable Equilibrium, Statistical Privacy.*

I. INTRODUCTION

A. Problem Statement

In distributed control system literature, there is a usual trade-off between cost minimization and privacy preservation: to obtain a better outcome, each End User (EU) in the system relies on its own information and the information the others have reported. However, the EUs might be reluctant to release sensitive information because of privacy limitations or because, by acting strategically on their readings – therefore providing fake information, they might decrease their cost. The former situation might give rise to moral hazard, which could generate inefficiencies at the system level. To penalize free-rider behaviors and incentivize the EUs to minimize the variance of their readings, we aim to define contracts which remunerate the EUs as increasing functions of the data quality while discriminating between the EUs depending on their local privacy breach.

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In energy systems, data – and the information it contains – is becoming an asset that the agents might want to acquire to learn market equilibria [2], [3]. For example, in many demand-response programs, the aggregator uses the EUs' historical energy consumption to determine their nominal demand profiles, and design incentives to flatten peak times. The nominal demand profiles are then used to compute incentives depending on how much they curtail relative to their nominal profiles. However, EUs can act strategically on the data they share. For example, energy consumers may artificially inflate their nominal demand prior to a demand-response program event, in order to receive larger payments under the program (see [4] and references therein).

In this paper, we want to quantify the EUs' trade-off between statistical privacy preservation, i.e., minimum information release, and indistinguishability, i.e., the inability for the aggregator to determine whether the EUs' readings are biased or not, and the efficiency loss for the market, under different coordination schemes between the DA and EUs.

B. Main Contributions

We consider a peer-to-peer electricity trading problem formulated as a network game, where strategic EUs want to minimize their cost by computing their demand (flexible around a nominal value), controllable generation, and trades with their peers, while satisfying a set of local constraints including the local balance of supply and demand, and trading reciprocity constraints that couple the EUs' optimization problems. Their nominal demand constitutes sensitive information, that is kept private to the EUs. Following [2], [3] and the electricity market literature, the EUs' utility functions are assumed additive and quadratic. We prove that under mild assumptions, the network game admits a unique Variational Equilibrium (VE), which is efficient and stable. In addition, we prove that, at the VE, the EUs' best-responses depend on the private information of *all* the EUs. A Data Aggregator (DA) might then be introduced, which aims to learn the EUs' private information. Indeed the EUs might have incentives to report fake readings which might create shifts in the EUs' strategies. Relying on performative prediction, we define a decision-dependent game to couple the peer-to-peer trading problem with a data market. On the data market, the DA designs a privacy contract incentivizing the EUs to report their private information truthfully. The EUs' readings, demand and controllable generations are then determined as unique solutions of the decision-dependent game, relying on Nash (NE) and Performatively Stable (PSE) Equilibria. We propose two gradient method-based

algorithms to compute PSE, under different coordination schemes between the DA and the EUs, that we test against the Repeated Stochastic Gradient Method (RSGM) from [5] and a classical distributed gradient-descent method for NE computation. We also introduce a two-time-scale stochastic approximation recursive variant, which enables to consider the EUs' decisions taken on the electricity market as best-responses to the readings reported to the DA, leading to the computation of a Stackelberg Equilibrium (SE) of the decision-dependent game. We prove the convergence of each algorithm. Finally, the algorithms performance is assessed on a numerical example, by comparing the resulting efficiency loss, privacy preservation capabilities, convergence rates, and EUs' utility functions at equilibrium. The results highlight the benefits for the EUs to model performative effects.

C. Reviewed Literature

Over the last few years, there has been a growing body of literature on data markets. On the market design side, Agarwal et al. propose an innovative design involving robust real-time matching, to efficiently buy and sell data [6], while Shilov et al. quantify the impact of privacy constraints on the market equilibrium [3]. On the pricing side, Cai et al. in [7] study a data market with many strategic data sources and one DA. The DA aims to build an estimator of a function, thanks to the readings reported by the sources. The quality of the readings depends on the level of effort exerted by the sources, which determines the variance of the sample as a decreasing function of the effort exerted, and the aggregator wants to design incentives to obtain the best possible estimator. Cai et al. prove that there exists a unique NE, which is also a dominant strategy equilibrium. Westenbroek et al. extend this result in [8] by taking into account the non-rivalry nature of data and considering multiple DAs among which the same data can be freely reproduced and shared. They prove that either no equilibrium exists or an infinite number of Generalized Nash Equilibria (GNEs), depending on the contract payment parameters. They also provide necessary and sufficient conditions for GNEs to be socially inefficient. In the same vein, Gast et al. in [9] propose a non-cooperative game model of strategic data sources, which optimize the variance of the noise level of their readings to minimize their cost, which can be split into a disclosure cost and an estimation cost based on linear regression. The impact of strategic information is quantified through explicit bounds on the price of stability. The above works differ from ours in that we explicitly take into account the dependence of the reading distributions in the decision variables of the EUs. Furthermore, our work differs significantly from the framework of ε -differential privacy, because in differential privacy, noise is added by a principal to the output of a computation which is subsequently publicly released [10], [11], and the principal performing the computation is trusted (i.e., it does not act strategically). In contrast, we do not assume that the EU is trusted, which motivates input perturbations that can be related to the more recent notion of local differential privacy, though ignoring the strategic aspect of the input perturbation

[12]. Although strategic information and privacy often lead to system inefficiencies, uncertainty can be beneficial in some cases. For example, in transportation networks, the overall congestion can be decreased when a principal carefully calibrates the level of information available to each user [13]. Much as tolls can push the system to a better outcome, information can similarly affect the system performance at equilibrium [14]. Finally, some papers study the trade-off between the accuracy of equilibrium computation and privacy or fairness guarantees [15], [16]. An extension to online aggregative games for Nash Equilibrium tracking subject to privacy preservation, where the cost functions are time-varying, is provided in [17]. Indeed, most distributed equilibrium computation algorithms have to either sacrifice the accuracy of the computation to gain rigorous privacy guarantees or violate some privacy constraints to achieve accurate computation.

This paper is organized as follows. In Sec. II, we formulate the peer-to-peer electricity trading problem as a non-cooperative network game. Relying on performative prediction, we define a decision-dependent game to couple the network game with a data market, in Sec. III. The DA's contract parameter optimization is formulated as a bilevel optimization problem with induced equilibrium between the EUs, in Sec. IV. We subsequently reformulate it as a more tractable nonlinear program. Algorithms for equilibrium computation are provided in Sec. V, and convergence analysis is performed in Sec. VI. Simulations are run in Sec. VII. We conclude in Sec. VIII.

Notation

$\mathbb{E}_X[f]$ specifies that the expectation of the function $f(\cdot)$ is taken with respect to the random variable X (when X is clear from the context, we abbreviate the notation to $\mathbb{E}[f]$). Vectors and matrices will be represented by bold letters. The stack of the N vectors $\mathbf{x}_1, \dots, \mathbf{x}_N$ is denoted $\mathbf{x} \stackrel{\text{def}}{=} \text{col}(\mathbf{x}_1, \dots, \mathbf{x}_N)$. \mathbf{x}_{-n} is the stack of the vectors containing the decision variables of all the agents in \mathcal{N} except n . The collection of N variables $\mathbf{x}_1, \dots, \mathbf{x}_N$ will be denoted $(\mathbf{x}_n)_{n=1, \dots, N}$. \mathbf{x}^T will denote the transpose of the vector \mathbf{x} . $|\mathcal{X}|$ will refer to the cardinal of the set \mathcal{X} . $A_{(i,j)}$ will denote the element at the intersection of the i^{th} row and j^{th} column of matrix \mathbf{A} . $\mathbf{A}_{(i,:)}$ will stand for the i^{th} row of the matrix \mathbf{A} . For a closed set $\mathcal{C} \subseteq \mathbb{R}^m$, the mapping $\text{proj}_{\mathcal{C}} : \mathbb{R}^m \rightarrow \mathcal{C}$ denotes the projection onto \mathcal{C} , i.e., $\text{proj}_{\mathcal{C}}(\mathbf{x}) \stackrel{\text{def}}{=} \arg\min_{\mathbf{y} \in \mathcal{C}} \|\mathbf{y} - \mathbf{x}\|$. $\text{SOL}(P)$ will denote the set of solutions of Problem P .

II. A NETWORK GAME MODEL

A. Description of the Agents

1) *The End Users:* We let \mathcal{N} be a set of N EUs, forming a directed graph $\Gamma \stackrel{\text{def}}{=} (\mathcal{N}, E)$, where $E \subseteq \mathcal{N} \times \mathcal{N}$ is the set of links between the EUs. Let Γ_n be the set of out-neighbors of EU n . Furthermore, we make the assumption that at node 0, i.e., the root node, $\Gamma_0 = \mathcal{N} \setminus \{0\}$.

Each EU n updates the vector of decision variables $\mathbf{x}_n \stackrel{\text{def}}{=} (D_n, G_n^c, (q_{mn})_{m \in \Gamma_n}) \in \mathcal{X}_n$, which contains its

demand, controllable generation, and bilateral trades with its neighbors, where $\mathcal{X}_n \subseteq \mathbb{R}^{m_n}$, $m_n = |\Gamma_n|+2$, is the strategy set of EU n that captures only local constraints. In addition, we impose the following convention on EU n 's bilateral trades: $q_{mn} \geq 0$ means that n buys q_{mn} from m ; while $q_{mn} < 0$ means that n sells $-q_{mn}$ to m . Coupling constraints are added to capture bilateral trading reciprocity between EU n and its neighbors such that EU n 's strategy set takes the form: $\mathcal{C}_n(\mathbf{x}_{-n}) \stackrel{\text{def}}{=} \left\{ \mathbf{x}_n = (D_n, G_n^c, (q_{mn})_{m \in \Gamma_n}) \in \mathcal{X}_n \mid q_{mn} = -q_{nm}, \forall m \in \Gamma_n \right\}$, with $\mathcal{X}_n \stackrel{\text{def}}{=} \left\{ \mathbf{x}_n = (D_n, G_n^c, (q_{mn})_{m \in \Gamma_n}) \in \mathbb{R}^{m_n} \mid \underline{D}_n \leq D_n \leq \bar{D}_n, \underline{G}_n^c \leq G_n^c \leq \bar{G}_n^c, D_n \leq G_n^c + G_n^{mc} + Q_n \right\}$ where \underline{D}_n and \bar{D}_n are lower and upper-bounds on n 's demand, G_n^{mc} is the uncontrollable generation of EU n (solar PV panels generation), and $Q_n \stackrel{\text{def}}{=} \sum_{m \in \Gamma_n} q_{mn}$ is the net import at node n . The constraint $D_n \leq G_n^c + G_n^{mc} + Q_n$ captures the fact that demand at node n should not exceed the sum of local generation and net import.

Typically, the EUs' utility functions $J_n(\cdot)$ depend on some parameters that are sensitive, and that they might not want to disclose [3], [2]. We define $\theta_n \in \Theta_n \subseteq \mathbb{R}$ as the private information of EU n . Let $J_n : \mathbb{R}^{m_n} \times \Theta_n \rightarrow \mathbb{R}$ be the utility function of EU n . Following [2], we define EU n 's utility (in the cost minimization sense) as the difference between the sum of the total cost incurred from the bilateral trades with its out-neighbors and the controllable generation cost, and its usage benefit $U_n(\cdot)$ ($-U_n(\cdot)$ is the consumption cost of EU n):

$$J_n(\mathbf{x}_n, \theta_n) = \sum_{m \in \Gamma_n} c_{nm} q_{mn} - U_n(D_n, \theta_n) + C_n^G(G_n^c), \quad (1)$$

where $\mathbf{c}_n > \mathbf{0}$ is a known vector of parameters capturing EU n 's preferences (willingness-to-pay or taxes), and $-U_n(\cdot)$ is its consumption cost. The controllable generation cost is assumed to be quadratic in G_n^c , i.e., $C_n^G(G_n^c) = \frac{a_n}{2} G_n^c{}^2 + b_n G_n^c + c_n$ where a_n, b_n, c_n are positive input coefficients.

We assume $U_n(\cdot)$ to be a quadratic function in EU n 's demand, i.e., we can write:

$$U_n(D_n, \theta_n) = -\tilde{a}_n (D_n - \theta_n)^2 + \tilde{b}_n, \quad (2)$$

where \tilde{a}_n, \tilde{b}_n are positive input coefficients, and θ_n is EU n 's nominal demand. EU n 's usage benefit is a strictly concave function in its demand D_n . It leads to the following interpretation: by being flexible around its nominal demand, e.g., agreeing to slightly reduce its demand around its nominal value, EU n will decrease its usage benefit but it might also decrease its bilateral trading or controllable generation costs.

2) *The Market Operator*: The economic dispatch is classically obtained through a centralized clearing, i.e., the market operator aims to determine the vector \mathbf{x} that minimizes the social cost defined as the sum of the EUs' costs, i.e., $SC(\mathbf{x}, \boldsymbol{\theta}) \stackrel{\text{def}}{=} \sum_n J_n(\mathbf{x}_n, \theta_n)$ under network coupling constraints, i.e., it solves the following optimization problem:

$$\min_{\mathbf{x}} SC(\mathbf{x}, \boldsymbol{\theta}), \quad (3a)$$

$$\text{s.t. } \mathbf{x} \in \mathcal{C} \stackrel{\text{def}}{=} \prod_n \mathcal{C}_n(\mathbf{x}_{-n}). \quad (3b)$$

We prove in [2] that the solution of (3) can be reached

through a distributed clearing, where each EU $n \in \mathcal{N}$ solves the parametrized optimization problem:

$$\min_{\mathbf{x}_n \in \mathcal{C}_n(\mathbf{x}_{-n})} J_n(\mathbf{x}_n, \theta_n). \quad (4)$$

The latter setting gives rise to a non-cooperative game \mathcal{G} between the EUs.

3) *The Data Aggregator*: Data has seen its role becoming more predominant in the new sharing economy, with a potential for being monetized [18]. As the demand for data increases, we have seen the formation of DAs, which collect data for either use or resale [8]. In our paper, the goal of the DA is to collect readings from the EUs to build an estimate \mathbf{x}^a of the EUs' decisions \mathbf{x} at equilibrium and use it to measure the quality of the readings. To this end, it builds contracts with the EUs, and pays $p_n^a(\cdot)$ to each EU n , which is an increasing function in the accuracy of \mathbf{x}_n^a with respect to \mathbf{x}_n .

B. Existence, Uniqueness and Stability of the Variational Equilibrium

To generalize the notion of NE in the presence of coupling constraints, we use the Generalized Nash Equilibrium (GNE). It is defined as a strategy profile vector \mathbf{x} such that \mathbf{x}_n solves the system KKT_n for each n (see Appendix A). The coupling constraint $q_{mn} = -q_{nm}$ written in the problem of n (respectively m , $\forall m \in \Gamma_n$) is associated to the duals ζ_{nm} (respectively ζ_{mn}). In this work, we focus on the special case of a GNE known as a Variational Equilibrium (VE), i.e., a solution $\mathbf{x} = (\mathbf{x}_n)_n$ such that \mathbf{x}_n solves the system KKT_n for each n and, in addition, such that the duals of the coupling constraints are equal, i.e., $\zeta_{nm} = \zeta_{mn}, \forall n \in \mathcal{N}, \forall m \in \Gamma_n$. Set $\mathbf{F}_x(\mathbf{x}) \stackrel{\text{def}}{=} \left(\nabla_{\mathbf{x}_n} J_n(\mathbf{x}_n, \theta_n) \right)_n$.

Definition 1 (Variational Equilibrium, [19]): The Variational Inequality problem $\text{VI}(\mathbf{F}_x, \mathcal{C})$ consists in finding a vector $\mathbf{x}^* \in \mathcal{C}$ such that $(\mathbf{y} - \mathbf{x}^*)^T \mathbf{F}_x(\mathbf{x}^*) \geq 0, \forall \mathbf{y} \in \mathcal{C}$. A solution of $\text{VI}(\mathbf{F}_x, \mathcal{C})$ is called a VE.

Proposition 1: If no energy surplus is available at node n , then EU n 's best-response strategy \mathbf{x}_n^* at equilibrium depends linearly on the nominal demands of *all* the EUs, i.e., $\mathbf{x}_n^* = \mathbf{A}_{n,n} \theta_n + \mathbf{A}_{n,-n} \boldsymbol{\theta}_{-n} + \left[\mathbf{B}_{n,n} G_n^{mc} + \mathbf{B}_{n,-n} \sum_{n' \neq n} G_{n'}^{mc} \right] + \mathbf{C}_n$, with $\mathbf{A}_{n,n}, \mathbf{B}_{n,n}, \mathbf{B}_{n,-n}, \mathbf{C}_n$ vectors and $\mathbf{A}_{n,-n}$ matrix of appropriate dimensions; otherwise \mathbf{x}_n^* depends only on the EU n 's own nominal demand, θ_n .

Proof: See Appendix A. ■

Remark 2.1: The coefficients of the best-responses are not known by the EUs because they rely on coefficients $(\tilde{a}_n, a_n)_n$ that the other EUs might not want to share.

Assumption 1: Demand and supply balance each other at each node, i.e., $D_n = G_n^c + G_n^{mc} + Q_n, \forall n \in \mathcal{N}$.

Assumption 2: We assume that there exists at least one EU $n \in \mathcal{N}$ such that at the VE, $D_n^* \neq 0$ or $G_n^{c*} \neq 0$.

Theorem 1: Under Asm. 2, the game \mathcal{G} admits a unique VE, which is in addition efficient.

Proof: See Appendix B. ■

Equilibrium stability is a sufficient condition for distributed learning methods to converge in [20].

Definition 2 (Stability, [20]): An equilibrium point $\mathbf{x}^* \in \mathcal{C}$ is said to be variationally stable (or simply stable) if there exists a neighborhood \mathcal{U} of \mathbf{x}^* such that $(\mathbf{y} - \mathbf{x}^*)^T \mathbf{F}_x(\mathbf{y}) \geq 0$ for all $\mathbf{y} \in \mathcal{U}$. In particular, if this property holds for all $\mathbf{y} \in \mathcal{C}$, we say that \mathbf{x}^* is globally stable.

Proposition 2: The VE solution of \mathcal{G} is globally stable.

Proof: In Thm. 1 proof, we check that \mathcal{G} belongs to the class of generalized potential games (GPG) [21, Sec. 2] for which the EUs' utility functions are aligned along the game's potential whose minimizers are VEs. From [20, Rk. 1], VEs solutions of convex potential games are automatically stable in the sense of Def. 2. ■

The impact on $(c_{nm})_{n,m}$ on \mathcal{G} equilibria has been analyzed in [2]. In case $(c_{nm})_{n,m}$ capture taxes, it seems natural to take them homogeneous among EUs belonging to the same community.

Assumption 3: We assume that the EUs' preferences for trading are uniform and normalized, i.e., $c_{nm} = 1, \forall n, m$.

We notice that under Asms. 1 and 3, controlling demand and generation allows to control the net import. Therefore, \mathbf{Q} will be treated as a dependent decision variable in the rest of the work. Furthermore, replacing the $(q_{mn})_m$ by Q_n allows to use the linear dependence of $\mathbf{x}_n, \forall n$ in $\boldsymbol{\theta}$ (see Appendix A).

III. COUPLING NETWORK GAME AND DATA MARKET AS A DECISION-DEPENDENT GAME

Decomposition methods are another way to compute equilibrium with minimum information exchange [22], [23]. Nevertheless, works in this category do not distinguish between local and global objectives. In this paper, we focus on the performative prediction setting with decision-dependent distributions which has been introduced in [5] as a game-theoretic model for decision-dependent learning in the presence of competition. It leads to a weaker solution concept than Nash Equilibrium, but allows to endogenously capture the drift (performative effect) caused by the strategic information contained in the EUs' readings on the market equilibrium, and might lead to faster converging algorithms.

A. Balancing Cost and Privacy

We introduce randomization in the reading sent by EU n to the DA, which takes the form $\tilde{\theta}_n = \hat{\theta}_n + \varepsilon_n$. The first part of the reading captures the ability of EU n to add a deterministic bias to its reported information. The second part is a noise following a randomized mechanism $M(\cdot)$, added in order to preserve privacy. We assume that the noise in the readings is independent across the EUs [8].

Assumption 4: For each EU $n \in \mathcal{N}$, ε_n follows a centered Gaussian distribution with variance V_n . Furthermore, the random variables $(\varepsilon_n)_{n \in \mathcal{N}}$ are independent.

To incentivize the EUs to report truthful readings that can later be reused or monetized through targeted advertisements, the DA comes into play. The DA defines a payoff contract to each EU; the payment is decreasing in the distance between the EUs' strategy \mathbf{x}_n and the estimate $\mathbf{x}_n^a \triangleq (D_n^a, G_n^a)$ built

by the DA, defined as follows:

$$\mathbf{x}_n^a = \mathbf{A}_n^a \tilde{\boldsymbol{\theta}} + \mathbf{B}_n^a \mathbf{G}^{nc} + \mathbf{C}_n^a, \quad (5)$$

where $\mathbf{A}_n^a, \mathbf{B}_n^a, \mathbf{C}_n^a$ are estimates of $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n$ introduced in Prop. 1. The form of the DA's estimator is known by all the EUs. It comes from Prop. 1, assuming that Asm. 1 holds. The contract between the EU n and the DA takes the form:

$$p_n^a(\mathbf{x}_n, \mathbf{x}_n^a, \tilde{\boldsymbol{\theta}}) = \gamma_n - \beta_n \left((D_n - D_n^a)^2 + (G_n^c - G_n^{c,a})^2 \right), \quad (6)$$

where γ_n, β_n are non-negative parameters which can be optimized by the DA to check specific market properties [8].

Each EU n chooses the neighboring input $\hat{\theta}_n$ such that $d(\hat{\theta}_n, \theta_n) \triangleq (\hat{\theta}_n - \theta_n)^2 \leq \alpha_n$ [24]. Let $L_{M, \theta_n, \hat{\theta}_n}$ be the privacy loss of $M(\cdot)$, which follows the distribution $\mathcal{N}(\frac{(\theta_n - \hat{\theta}_n)^2}{2V_n}, 2\frac{(\theta_n - \hat{\theta}_n)^2}{2V_n})$ [25]. How much information can be extracted from the reading $\tilde{\theta}_n$ is measured by the expected privacy loss of the mechanism $M(\cdot)$: $I_n(\theta_n, \tilde{\theta}_n) \triangleq -\mathbb{E}_{\varepsilon_n} [L_{M, \theta_n, \hat{\theta}_n}]$, which coincides with the Kullback-Leibler divergence (relative entropy) between $M(\cdot)$'s output distributions on θ_n and $\hat{\theta}_n$. Under Asm. 4, the Kullback-Leibler divergence takes the closed form: $I_n(\theta_n, \hat{\theta}_n, V_n) = \frac{(\hat{\theta}_n - \theta_n)^2}{2V_n}$.

B. Formulation of the Stochastic Game

In the data market, EU n 's extended utility is split in four terms. The first term captures the cost it bears in \mathcal{G} ; the second term measures the cost of information release, while the third term is the contract payment it receives from the DA. A last term is added, which measures the penalty associated with the relaxation of the EUs' coupling constraint:

$$\begin{aligned} \tilde{J}_n(\mathbf{x}_n, \tilde{\boldsymbol{\theta}}, \mathbf{V}, \varepsilon) &= J_n(\mathbf{x}_n, \theta_n) + c_n^I I_n(\theta_n, \hat{\theta}_n, V_n) - p_n^a(\mathbf{x}_n, \mathbf{x}_n^a, \tilde{\boldsymbol{\theta}}) \\ &\quad + \xi \left(\sum_{m=1}^N Q_m \right)^2, \\ &= Q_n + \tilde{a}_n (D_n - \theta_n)^2 - \tilde{b}_n + \frac{a_n}{2} G_n^{c,2} + b_n G_n^c + c_n + c_n^I \frac{(\hat{\theta}_n - \theta_n)^2}{2V_n} \\ &\quad - \gamma_n + \beta_n \left((D_n - D_n^a)^2 + (G_n^c - G_n^{c,a})^2 \right) + \xi \left(\sum_{m=1}^N Q_m \right)^2, \end{aligned} \quad (7)$$

with c_n^I, ξ non-negative parameters. From (5), since $D_n^a, G_n^{c,a}$ depend on $\tilde{\boldsymbol{\theta}}, \varepsilon$, and $I_n(\cdot)$ on $\tilde{\boldsymbol{\theta}}, \mathbf{V}$, $\tilde{J}_n(\cdot)$ also depends on $\tilde{\boldsymbol{\theta}}, \mathbf{V}, \varepsilon$. Under Asms. 1 and 3, the bilateral trading reciprocity constraint at each node can be equivalently reformulated as $\sum_{n=1}^N Q_n = 0$. To avoid coupling between the feasibility sets of the EUs, we put the constraint as a penalty in their utility function, with a weight ξ that is the same for all the EUs. The penalty ξ calibration is discussed in Lem. 1.

The first term in Q in (7) refers to the benefit/cost due to the trades, while the second one encourages the EUs to ensure trading reciprocity among the community. We introduce $\mathcal{V} \triangleq \text{diag}(\mathbf{V})$. Each EU n solves:

$$\min_{\mathbf{x}_n, \tilde{\theta}_n, V_n} \mathbb{E}_{\varepsilon \sim \mathcal{N}(\mathbf{0}, \mathbf{V})} [\tilde{J}_n(\mathbf{x}_n, \tilde{\boldsymbol{\theta}}, \mathbf{V}, \varepsilon)], \quad (8a)$$

$$\text{s.t. } \underline{D}_n \leq D_n \leq \overline{D}_n, \quad (8b)$$

$$\underline{G}_n^c \leq G_n^c \leq \overline{G}_n^c, \quad (8c)$$

$$D_n = G_n^c + G_n^{nc} + Q_n, \quad (8d)$$

$$d(\hat{\theta}_n, \theta_n) \leq \alpha_n, \quad (8e)$$

$$\underline{V}_n \leq V_n \leq \overline{V}_n. \quad (8f)$$

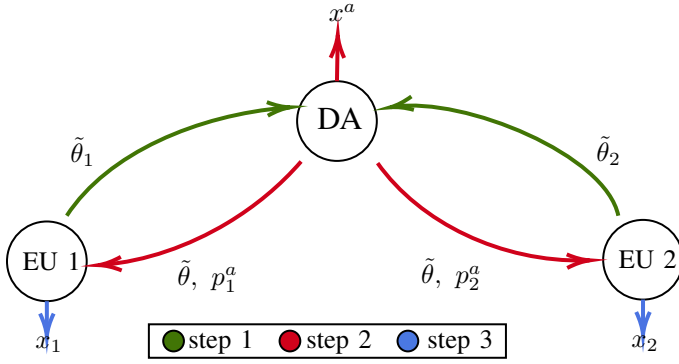


Fig. 1. Sequential interpretation of the game taking place between the DA and two EUs.

This gives rise to the stochastic game $\mathcal{G}^{\text{stoch}}$. Each EU wants to ensure a minimum privacy level, so it imposes a lower bound \underline{V}_n on V_n . To limit the instability of the market, the DA imposes an upper bound \bar{V}_n on V_n . This upper bound does not need to be shared by the EUs, instead it might be encoded by the DA in the communication interface deployed in the EUs' nodes when communicating with the DA.

The game between the EUs, which involves the DA as a coordinator of the data market, takes place as follows:

- Each EU n reports a reading $\tilde{\theta}_n = \hat{\theta}_n + \varepsilon_n$, with $\varepsilon_n \sim \mathcal{N}(0, V_n)$, to the DA by solving (8) with respect to $\hat{\theta}_n, V_n$. The readings $\tilde{\theta}$ are then broadcast to all the EUs by the DA.
- Each EU n computes its x_n by solving (8).

An illustration of the game taking place between the DA and two EUs, interpreted in its sequential form, is given in Fig. 1. Note that depending on the coordination level between the DA and EUs, steps a) and b) can take place *simultaneously* or *sequentially*.

We define \mathcal{Y}_n as the feasible set of Problem (8). We denote $\mathbf{y}_n \triangleq (x_n, \theta_n, V_n) \in \mathcal{Y}_n$ the decision variable of EU n in $\mathcal{G}^{\text{stoch}}$. Due to the penalty in the objective function, it is required that all the EUs have access to the sum of the net imports. It is a reasonable hypothesis because the market operator can broadcast this information without revealing much information about the EUs. In the remaining of the paper, we introduce two equilibrium solution concepts to interpret the outcome of the game $\mathcal{G}^{\text{stoch}}$, that require different levels of information sharing.

C. Equilibrium Analysis

Definition 3: A Nash Equilibrium (NE) is a strategy profile $\mathbf{y}^* = (\mathbf{y}_n^*)_{n \in \mathcal{N}} \in \mathcal{Y} = \prod_{n \in \mathcal{N}} \mathcal{Y}_n$ such that for all $n \in \mathcal{N}$:

$$\mathbf{y}_n^* \in \underset{\mathbf{y}_n \in \mathcal{Y}_n}{\operatorname{argmin}} \mathbb{E}_{\varepsilon \sim \mathcal{N}(\mathbf{0}, \operatorname{diag}(\mathbf{V}_n, \mathbf{V}_{-n}^*))} [\tilde{J}_n(\mathbf{y}_n, \mathbf{y}_{-n}^*, \varepsilon)].$$

There is another solution concept which is more amenable to numerical methods [5], and reduces to Performatively Stable points (PSEs). The idea of the PSE is to decouple the effect of a decision \mathbf{y} on the integrand $\tilde{J}_n(\mathbf{y}_n, \mathbf{y}_{-n}, \varepsilon)$ and on the distribution $\mathcal{N}(\mathbf{0}, \mathcal{V})$. Any vector \mathbf{V}^s induces a static game wherein the distribution for EU n is fixed at $\mathcal{N}(\mathbf{0}, \mathcal{V}^s)$, with $\mathcal{V}^s \stackrel{\text{def}}{=} \operatorname{diag}(\mathbf{V}^s)$, that is:

Definition 4 (Performatively Stable Equilibrium, [5]):

A Performatively Stable Equilibrium (PSE) is a strategy profile $\mathbf{y}^s = (\mathbf{y}_n^s)_{n \in \mathcal{N}} \in \mathcal{Y}$ such that for all $n \in \mathcal{N}$:

$$\mathbf{y}_n^s \in \underset{\mathbf{y}_n \in \mathcal{Y}_n}{\operatorname{argmin}} \mathbb{E}_{\varepsilon \sim \mathcal{N}(\mathbf{0}, \mathcal{V}^s)} [\tilde{J}_n(\mathbf{y}_n, \mathbf{y}_{-n}^s, \varepsilon)]. \quad (9)$$

PSE has a clear intuitive meaning, which makes it more relevant in our setting than NE: each EU has no incentive to deviate from \mathbf{y}^* having access only to readings drawn from $\mathcal{N}(\mathbf{0}, \mathcal{V}^*)$.

Proposition 3: The expectation of the extended utility function $\mathbb{E}_{\varepsilon} [\tilde{J}_n(\mathbf{y}_n, \mathbf{y}_{-n}, \varepsilon)]$ is strongly convex in \mathbf{y}_n , $\forall n$.

Proof: We explicit the expectation of EU n 's extended utility function: $\mathbb{E}_{\varepsilon \sim \mathcal{N}(\mathbf{0}, \mathcal{V})} [\tilde{J}_n(\mathbf{x}_n, \tilde{\theta}, \mathbf{V}, \varepsilon)] = (D_n - G_n^c - G_n^{nc}) + \tilde{a}_n(D_n - \theta_n)^2 - \tilde{b}_n + \frac{\tilde{a}_n}{2} G_n^c + b_n G_n^c + c_n + c_n^I \frac{(\tilde{\theta}_n - \theta_n)^2}{2V_n} - \gamma_n + \beta_n ((D_n - D_n^a(\tilde{\theta}))^2 + (G_n^c - G_n^a(\tilde{\theta}))^2 + \sum_{m=1}^N (A_n^a)_{(1,m)}^2 V_m + \sum_{m=1}^N (A_n^a)_{(2,m)}^2 V_m) + \xi (\sum_{m=1}^N D_m - G_m^c - G_m^{nc})^2$. Using the former expression, we compute the partial derivative $\mathbf{F}_n \stackrel{\text{def}}{=} \nabla_{\mathbf{y}_n}$ and the partial Hessian $\mathbf{H}_n \stackrel{\text{def}}{=} \nabla_{\mathbf{y}_n}^2$ of $\mathbb{E}_{\varepsilon} [\tilde{J}_n(\cdot)]$ with respect to \mathbf{y}_n . We get:

$$\mathbf{H}_n = \begin{bmatrix} \frac{c_n^I}{V_n} + 2\beta_n((A_n^a)_{1,n}^2 + (A_n^a)_{2,n}^2) - \frac{c_n^I(\tilde{\theta}_n - \theta)}{V_n^2} & -2\beta_n(A_n^a)_{1,n} & -2\beta_n(A_n^a)_{2,n} \\ -\frac{c_n^I(\tilde{\theta}_n - \theta)}{V_n^2} & \frac{c_n^I(\tilde{\theta}_n - \theta)}{V_n^3} & 0 & 0 \\ -2\beta_n(A_n^a)_{1,n} & 0 & 2\tilde{a}_n + 2\beta_n + 2\xi & -2\xi \\ -2\beta_n(A_n^a)_{2,n} & 0 & -2\xi & a_n + 2\beta_n + 2\xi \end{bmatrix}.$$

We note that $\mathbf{z}^T \mathbf{H}_n \mathbf{z} > 0, \forall \mathbf{z} \in \mathcal{Y}_n$, i.e., $\mathbb{E}_{\varepsilon} [\tilde{J}_n(\cdot)]$ is strictly convex in \mathbf{y}_n , which will be used later in the proof. In a second time, we prove the strong monotonicity of \mathbf{F}_n , i.e., that $\langle \mathbf{F}_n(\mathbf{y}_n) - \mathbf{F}_n(\mathbf{y}'_n), \mathbf{y}_n - \mathbf{y}'_n \rangle \geq \omega \|\mathbf{y}_n - \mathbf{y}'_n\|^2, \forall \mathbf{y}_n, \mathbf{y}'_n \in \mathcal{Y}_n$. Using the multidimensional Mean Value Theorem on the convex set \mathcal{Y}_n , we obtain the existence of a $\mathbf{z} \in [\mathbf{y}_n, \mathbf{y}'_n]$, i.e., component-wise $z_i \in [(\mathbf{y}_n)_i, (\mathbf{y}'_n)_i] \forall i$, such that $\langle \mathbf{F}_n(\mathbf{y}_n) - \mathbf{F}_n(\mathbf{y}'_n), \mathbf{y}_n - \mathbf{y}'_n \rangle = (\mathbf{y}_n - \mathbf{y}'_n)^T \cdot \nabla_{\mathbf{y}_n} \mathbf{F}_n(\mathbf{z}) \cdot (\mathbf{y}_n - \mathbf{y}'_n)$. We then use the Spectral Theorem on $\nabla_{\mathbf{y}_n} \mathbf{F}_n = \mathbf{H}_n$ to obtain the existence of \mathbf{D} , a diagonal matrix such that $\mathbf{H}_n(\mathbf{z}) = \mathbf{P} \mathbf{D} \mathbf{P}^{-1}$, and \mathbf{P} the orthogonal transition matrix. By replacing \mathbf{H}_n by its expression, identifying the variable in the new base, and using the property that the product of a vector by an orthogonal matrix preserves the Euclidean norm of the vector, we obtain: $(\mathbf{y}_n - \mathbf{y}'_n)^T \cdot \mathbf{P} \mathbf{D} \mathbf{P}^{-1} \cdot (\mathbf{y}_n - \mathbf{y}'_n) \geq \min_i \{\lambda_i(\mathbf{H}_n(\mathbf{z}))\} \|\mathbf{y}_n - \mathbf{y}'_n\|^2$ where the $(\lambda_i(\mathbf{H}_n(\mathbf{z})))_{i \in [1,4]}$ are the eigenvalues of $\mathbf{H}_n(\mathbf{z})$. We obtain the ω -strong monotonicity by taking the minimum over all the feasible set of the eigenvalues of \mathbf{H}_n (which are positive due to the strict convexity), with $\omega = \min_{\mathbf{z} \in \mathcal{Y}_n} \min_{i \in [1,4]} \lambda_i(\mathbf{H}_n(\mathbf{z}))$. Since the conditions listed in [19] are met, we conclude that $\mathbb{E}_{\varepsilon} [\tilde{J}_n(\cdot)]$ is strongly convex. ■

Let $\nabla_{\mathbf{y}_n} h(\mathbf{y})$ be the partial derivative of $h(\cdot)$ with respect to \mathbf{y}_n .

Lemma 1: If the penalty coefficient ξ in $\tilde{J}_n(\cdot)$ is bigger than the Lagrangian multiplier of the constraint $(\sum_{m=1}^N Q_m)^2 = 0$, then the penalized game has the same solutions as $\mathcal{G}^{\text{stoch}}$.

Proof: We adapt the proof of [26, Thm. 2.2]. We call $J_n^{\#}(\cdot)$, the function $\tilde{J}_n(\cdot)$ without the penalty: $J_n^{\#}(\mathbf{x}_n, \tilde{\theta}, \mathbf{V}, \varepsilon) = J_n(\mathbf{x}_n, \theta_n) + c_n^I I_n(\theta_n, \tilde{\theta}_n, V_n) - p_n^a(\mathbf{x}_n, \mathbf{x}_n, \tilde{\theta})$. We set $h(\mathbf{y}) \stackrel{\text{def}}{=} (\sum_{m=1}^N Q_m)^2$ and ϕ is the

Lagrangian multiplier related to the constraint $h(\mathbf{y}) = 0$. We denote $\hat{\mathbf{y}}$ the point that minimizes $J_n^\sharp(\cdot)$ over \mathcal{Y} and satisfies the constraint $h(\hat{\mathbf{y}}) = 0$. For any $\mathbf{y} \in \mathcal{Y}$, we have:

$$J_n^\sharp(\mathbf{y}) + \xi h(\mathbf{y}) \geq J_n^\sharp(\mathbf{y}) + \phi h(\mathbf{y}) \quad (10a)$$

$$\geq J_n^\sharp(\mathbf{y}) + \phi(h(\hat{\mathbf{y}}) + \nabla_n h(\hat{\mathbf{y}})^T(\mathbf{y} - \hat{\mathbf{y}})), \quad (10b)$$

$$= J_n^\sharp(\mathbf{y}) + \phi \nabla_n h(\hat{\mathbf{y}})^T(\mathbf{y} - \hat{\mathbf{y}}), \quad (10c)$$

$$= J_n^\sharp(\mathbf{y}) - \nabla_n J_n^\sharp(\hat{\mathbf{y}})^T(\mathbf{y} - \hat{\mathbf{y}}), \quad (10d)$$

$$\geq J_n^\sharp(\hat{\mathbf{y}}) = J_n^\sharp(\hat{\mathbf{y}}) + \xi h(\hat{\mathbf{y}}). \quad (10e)$$

To go from (10a) to (10b), we use the convexity of $h(\cdot)$. To go from (10c) to (10d), we use the stationarity condition. To go from (10d) to (10e), we use the convexity of $J_n^\sharp(\cdot)$, which can be proved by invoking similar arguments as in the proof of the strict convexity of $\mathbb{E}_\varepsilon[\tilde{J}_n(\cdot)]$ in Prop. 3. Hence, $\hat{\mathbf{y}}$ also minimizes the penalized problem. ■

Proposition 4: There exists a unique NE solution of $\mathcal{G}^{\text{stoch}}$.

Proof: We proved in Prop. 3 proof that for all n , $\mathbb{E}_\varepsilon[\tilde{J}_n(\cdot)]$ is strongly convex. Since for all n , the feasibility set \mathcal{Y}_n is closed and convex, the proposition statement follows. ■

Theorem 2: If the DA's estimate \mathbf{x}^a is perfect for the game \mathcal{G} , i.e., $\mathbf{x}^a(\boldsymbol{\theta}, \mathbf{G}^{nc}) = \mathbf{x}^G$ with $\mathbf{x}^G \in \text{SOL}(\mathcal{G})$, then the NE solution of $\mathcal{G}^{\text{stoch}}$ is efficient.

Proof: We denote \mathbf{x}^G the VE solution of the game \mathcal{G} . We have, $\forall \hat{\boldsymbol{\theta}}, \mathbf{V}, \mathbf{x}_n$: $J_n(\mathbf{x}_n) \leq J_n(\mathbf{x}_n) + c_n^I \frac{(\hat{\theta}_n - \theta_n)^2}{2V_n} + \beta_n((D_n - D_n^a(\hat{\boldsymbol{\theta}}))^2 + (G_n - G_n^a(\hat{\boldsymbol{\theta}}))^2) + \xi(\sum_{m=1}^N Q_m)^2 \implies J_n(\mathbf{x}^G) + \sum_{m=1}^N K_m V_m \leq J_n(\mathbf{x}_n) + c_n^I \frac{(\hat{\theta}_n - \theta_n)^2}{2V_n} + \beta_n((D_n - D_n^a(\hat{\boldsymbol{\theta}}))^2 + (G_n - G_n^a(\hat{\boldsymbol{\theta}}))^2) + \xi(\sum_{m=1}^N Q_m)^2 + \sum_{m=1}^N K_m V_m$ where we set $K_m \stackrel{\text{def}}{=} ((\mathbf{A}_n^a)_{(1,m)}^2 + (\mathbf{A}_n^a)_{(2,m)}^2)$. Note that the term $-\gamma_n$ is constant and does not change the equilibrium, so we omit it. This lower bound is reached in $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}$, $V_n = \underline{V}_n \forall n$ and $\mathbf{x}_n = \mathbf{x}_n^G$. This means that we have proved that assuming \mathbf{x}^a is perfect for \mathcal{G} , $\mathbf{x}^G, \boldsymbol{\theta}, \underline{\mathbf{V}}$ is the NE solution of $\mathcal{G}^{\text{stoch}}$. If we sum the former inequalities over n , we observe that $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}$, $V_n = \underline{V}_n \forall n$ and $\mathbf{x} = \mathbf{x}^G$ also minimizes the EUs' expected social cost. Thus, assuming \mathbf{x}^a is perfect for \mathcal{G} , the NE solution of $\mathcal{G}^{\text{stoch}}$ is efficient. ■

Next, we look at the properties of the PSE.

Proposition 5: There exists a unique PSE solution of $\mathcal{G}^{\text{stoch}}$.

Proof: To prove this result, we rewrite $\mathbb{E}_\varepsilon[\tilde{J}_n(\cdot)]$ as follows, to use [5, Thm. 1]:

$$\mathbb{E}_{\varepsilon \sim \mathcal{N}(\mathbf{0}, \mathbf{V})}[\tilde{J}_n(\mathbf{x}_n, \hat{\boldsymbol{\theta}}, \mathbf{V}, \varepsilon)] = \mathbb{E}_{\varepsilon_n \sim \mathcal{N}(\mathbf{0}, V_n)}[\tilde{J}_n(\mathbf{x}_n, \hat{\boldsymbol{\theta}}, \mathbf{V}, \varepsilon_n)], \quad (11)$$

where $\tilde{J}_n(\mathbf{x}_n, \hat{\boldsymbol{\theta}}, \mathbf{V}, \varepsilon_n) = (D_n - G_n^c - G_n^{nc}) + \tilde{a}_n(D_n - \theta_n)^2 - \tilde{b}_n + \frac{a_n}{2} G_n^c + b_n G_n^c + c_n + c_n^I \frac{(\hat{\theta}_n - \theta_n)^2}{2V_n} - \gamma_n + \beta_n((D_n - D_n^a(\hat{\boldsymbol{\theta}}))^2 + (G_n - G_n^a(\hat{\boldsymbol{\theta}}))^2) + \sum_{m=1, m \neq n}^N (\mathbf{A}_n^a)_{(1,m)}^2 V_m + \sum_{m=1, m \neq n}^N (\mathbf{A}_n^a)_{(2,m)}^2 V_m + (\mathbf{A}_n^a)_{(1,n)}^2 \varepsilon_n^2 + (\mathbf{A}_n^a)_{(2,n)}^2 \varepsilon_n^2 + \xi(\sum_{m=1}^N D_m - G_m^c - G_m^{nc})^2$. To prove the strong convexity of $\tilde{J}_n(\cdot)$, we proceed the same way as to prove the strong convexity of $\tilde{J}_n(\cdot)$, because the partial Hessians of both objective functions are the same. We also check the \mathcal{C}^1 continuity of $\tilde{J}_n(\cdot)$, the Lipschitz-continuity of the

application $\varepsilon \mapsto \tilde{J}_n(\mathbf{y}, \varepsilon)$, and that the distribution of the ε varies in a Lipschitz way according to [5, Asm. 2]. From [5, Thm. 1], we obtain the existence and uniqueness of the PSE. ■

Assuming $\mathcal{G}^{\text{stoch}}$ takes place sequentially, $\hat{\theta}_n, V_n$ and \mathbf{x}_n are computed by solving two distinct parametrized optimization problems, by each EU $n \in \mathcal{N}$. This leads us to introduce the game $\mathcal{G}^\sharp(\hat{\boldsymbol{\theta}}, \mathbf{V})$, where each EU $n \in \mathcal{N}$ solves:

$$\min_{\mathbf{x}_n} \mathbb{E}_{\varepsilon \sim \mathcal{N}(\mathbf{0}, \mathbf{V})} [J_n^\sharp(\mathbf{x}_n, \hat{\boldsymbol{\theta}}, \mathbf{V}, \varepsilon)], \quad (12a)$$

$$\text{s.t. } \underline{D}_n \leq D_n \leq \bar{D}_n, \quad (12b)$$

$$\underline{G}_n^c \leq G_n^c \leq \bar{G}_n^c, \quad (12c)$$

$$D_n = G_n^c + G_n^{nc} + Q_n, \quad (12d)$$

$$\sum Q_m = 0. \quad (12e)$$

Proposition 6: $\mathcal{G}^\sharp(\hat{\boldsymbol{\theta}}, \mathbf{V})$ is a generalized potential game (GPG), $\forall \hat{\boldsymbol{\theta}}, \mathbf{V}$.

Proof: Relying on the GPG definition introduced in [21, Sec. 2], this result follows from the fact that the KKTs of $\mathcal{G}^\sharp(\hat{\boldsymbol{\theta}}, \mathbf{V})$ coincide with the KKTs of the minimization problem with $\sum_n \mathbb{E}_\varepsilon [J_n^\sharp(\mathbf{x}_n, \hat{\boldsymbol{\theta}}, \mathbf{V}, \varepsilon)]$ in the objective, under the concatenation of the constraints (12b)-(12d), $\forall n$, and (12e). ■

From Prop. 6, $\mathcal{G}^\sharp(\hat{\boldsymbol{\theta}}, \mathbf{V})$ can be equivalently formulated as an optimization problem, and its VE computed through exact methods relying on classical optimization solvers. It will be used to compute solutions in steps 1 and 3 of Alg. 1.

IV. FORMULATION OF THE DATA AGGREGATOR PROBLEM

Let $p_n^a(\gamma_n, \beta_n, \mathbf{y}) \stackrel{\text{def}}{=} \mathbb{E}_\varepsilon[p_n^a(\mathbf{x}_n, \mathbf{x}_n^a, \hat{\boldsymbol{\theta}})]$ be the expected value of the payment to EU n . The DA defines payment so as to minimize:

$$L^a(\mathbf{p}^a, \mathbf{y}) = \pi \sum_n p_n^a(\gamma_n, \beta_n, \mathbf{y}) + \mathbb{E}_\varepsilon[l(\mathbf{x}^a, \mathbf{x})], \quad (13)$$

where the parameter $\pi > 0$ denotes a conversion between monetary amounts allocated by the payment functions and the expectation of the loss $l(\mathbf{x}, \mathbf{x}^a) = \sum_n [(D_n - D_n^a)^2 + (G_n^c - G_n^a)^2]$ generated by the estimation gap.

We describe the DA's optimization problem, and specify constraints that the parameter choice γ, β must satisfy.

The first constraint is Individual Rationality (IR). IR requires that each EU prefers contracting with the DA, than with the conventional retailer. This means that ex-ante $\mathbb{E}_\varepsilon[J_n^\sharp(p_n^a(\gamma_n, \beta_n, \mathbf{y}), \mathbf{y}_n)] \leq p^r D_n$, where p^r is the unit price fixed by the conventional retailer.

The second constraint is non-negativity of the payments from the DA, i.e., the payment that the DA offers to each EU n is non-negative ex-ante.

The third constraint is Incentive Compatibility (IC). It is enforced by the DA finding \mathbf{y}^* that minimizes its cost (13), and then designing \mathbf{p}_n^a such that:

$$\mathbf{y}_n^* = \underset{\mathbf{y}_n \in \mathcal{Y}_n}{\text{argmin}} \mathbb{E}_\varepsilon[J_n^\sharp(p_n^a(\gamma_n, \beta_n, \mathbf{y}), \mathbf{y}_n)], \forall \mathbf{y}_{-n} \in \mathcal{Y}_{-n}.$$

Notice that the standard definition of IC assumes that the induced equilibrium between the EUs is a NE. It can

be extended considering that the EUs reach a PSE. In that case, IC is enforced by the DA designing \mathbf{p}_n^a such that $\mathbf{y}_n^s = \operatorname{argmin}_{\mathbf{y}_n \in \mathcal{Y}_n} \mathbb{E}_{\varepsilon \sim \mathcal{N}(\mathbf{0}, \mathbf{V}^s)} [J_n^\#(\mathbf{p}_n^a(\gamma_n, \beta_n, \mathbf{y}_n, \mathbf{y}_{-n}^s), \mathbf{y}_n)]$, $\forall \mathbf{y}_{-n} \in \mathcal{Y}_{-n}$.

With IC, IR, non-negativity constraints, we formulate a bilevel optimization problem for the DA:

$$\begin{aligned} \min_{\gamma, \beta} \quad & L^a(\mathbf{p}^a, \mathbf{y}), \\ \text{s.t.} \quad & \mathbf{y}_n^* = \operatorname{argmin}_{\mathbf{y}_n \in \mathcal{Y}_n} \mathbb{E}_{\varepsilon} [J_n^\#(\mathbf{p}_n^a(\gamma_n, \beta_n, \mathbf{y}), \mathbf{y}_n)], \\ & \forall \mathbf{y}_{-n} \in \mathcal{Y}_{-n}, \forall n \in \mathcal{N}, \sum_{m=1}^N Q_m = 0, \\ & p^r D_n - \mathbb{E}_{\varepsilon} [J_n^\#(\mathbf{p}_n^a(\gamma_n, \beta_n, \mathbf{y}), \mathbf{y}_n)] \geq 0, \forall n \in \mathcal{N}, \\ & \mathbf{p}_n^a(\gamma_n, \beta_n, \mathbf{y}^*) \geq 0, \forall n \in \mathcal{N}, \\ & \gamma_n \geq 0, \beta_n \geq 0, \forall n \in \mathcal{N}. \end{aligned}$$

This problem actually has N coupled optimization problems as constraints, making it a difficult bilevel program. However, we will reformulate the DA's problem to a more manageable nonlinear program in the sequel.

A. Induced Equilibrium between the End Users

To ensure the notion of IC in equilibrium, we show that there is a well-defined mapping $(\gamma, \beta) \mapsto \mathbf{y}^*$.

Definition 5: For fixed contract payments $(\mathbf{p}_n^a)_n$, we say that \mathbf{y}^* is an induced NE if for each EU $n \in \mathcal{N}$,

$$\mathbf{y}_n^* = \operatorname{argmin}_{\mathbf{y}_n \in \mathcal{Y}_n} \mathbb{E}_{\varepsilon} [J_n^\#(\mathbf{p}_n^a(\gamma_n, \beta_n, \mathbf{y}_n, \mathbf{y}_{-n}^*), \mathbf{y}_n)]. \quad (14)$$

Let $D_n^a(\hat{\theta}) = \mathbf{A}_n^a(1,:) \hat{\theta} + \mathbf{B}_n^a(1,:) \mathbf{G}^{nc} + C_n^a(1,1)$ and $G_n^c(\hat{\theta}) = \mathbf{A}_n^a(2,:) \hat{\theta} + \mathbf{B}_n^a(2,:) \mathbf{G}^{nc} + C_n^a(2,1)$. Suppose that we have a set of payments (γ, β) , EU n chooses \mathbf{y}_n^* according to:

$$\begin{aligned} \mathbf{y}_n^* = \operatorname{argmin}_{\mathbf{y}_n \in \mathcal{Y}_n} \quad & \left[(D_n - G_n^c - G_n^{nc}) + \tilde{a}_n(D_n - \theta_n)^2 - \tilde{b}_n \right. \\ & + \frac{a_n}{2} G_n^c{}^2 + b_n G_n^c + c_n + \frac{(\hat{\theta}_n - \theta_n)^2}{2V_n} - \gamma_n + \beta_n \left((D_n - D_n^a(\hat{\theta}))^2 \right. \\ & \left. \left. + (G_n - G_n^a(\hat{\theta}))^2 + \sum_{m=1}^N (A_n^a(1,m))^2 V_m + \sum_{m=1}^N (A_n^a(2,m))^2 V_m \right) \right], \end{aligned} \quad (15)$$

for each choice of $\mathbf{y}_{-n} \in \mathcal{Y}_{-n}$ made by the other EUs. According to Prop. 4, (15) admits a unique optimum for EU n parametrized in \mathbf{y}_{-n} . We notice that each choice of contract parameters β selected by the DA leads to an induced NE.

Define $\Psi_n : \mathbb{R}_+^N \rightarrow \mathcal{Y}_n$ to be the implicitly defined map such that $\Psi_n : \beta \mapsto \mathbf{y}_n^*$ returns the solution of (15) for each EU $n \in \mathcal{N}$ for a given choice of $\beta \in \mathbb{R}_+^N$. By definition, for each EU n , \mathcal{Y}_n is a compact as the product of compact sets, therefore there exists $\underline{\mathbf{y}}_n, \bar{\mathbf{y}}_n$ such that $\underline{\mathbf{y}}_n \leq \mathbf{y}_n \leq \bar{\mathbf{y}}_n, \forall n \in \mathcal{N}$. Let $\underline{\beta} \triangleq \min\{n \in \mathcal{N}, \beta \geq \mathbf{0} | \Psi_n(\beta) = \underline{\mathbf{y}}_n\}$. We define $\bar{\beta} \triangleq [\underline{\beta}, \bar{\beta}]$ where $\bar{\beta} \triangleq \min\{n \in \mathcal{N}, \beta \in \mathbb{R}_+^N | \Psi_n(\beta) = \bar{\mathbf{y}}_n\}$. Thus, when reformulating each EU optimization problem in the following section, we will additionally constraint $\beta \in B$.

B. Reformulation of the End Users' Optimization Problems

To solve (15), we compute the gradient of the right-hand side expression. We obtain:

$$\begin{aligned} \frac{\partial \mathbb{E}_{\varepsilon}[\tilde{J}_n^\#]}{\partial D_n} &= 2\tilde{a}_n(D_n - \theta_n) + 2\beta_n(D_n - D_n^a(\hat{\theta})) + 1, \\ \frac{\partial \mathbb{E}_{\varepsilon}[\tilde{J}_n^\#]}{\partial G_n^c} &= a_n G_n + b_n + 2\beta_n(G_n^c - G_n^a(\hat{\theta})) - 1, \\ \frac{\partial \mathbb{E}_{\varepsilon}[\tilde{J}_n^\#]}{\partial \hat{\theta}_n} &= \frac{(\hat{\theta}_n - \theta_n)}{V_n} + 2\beta_n((A_n^a(1,n)(D_n^a(\hat{\theta}) - D_n) \\ &\quad + (A_n^a(2,n)(G_n^c(\hat{\theta}) - G_n^a(\hat{\theta}))), \\ \frac{\partial \mathbb{E}_{\varepsilon}[\tilde{J}_n^\#]}{\partial V_n} &= \frac{-(\hat{\theta}_n - \theta_n)^2}{2V_n^2} + \beta_n((A_n^a(1,n))^2 + (A_n^a(2,n))^2). \end{aligned}$$

Set $E_n^a \stackrel{\text{def}}{=} C_n^a(1,1) - \mathbf{B}_n^a(1,:) \mathbf{G}^{nc}$ and $F_n^a \stackrel{\text{def}}{=} C_n^a(2,1) - \mathbf{B}_n^a(2,:) \mathbf{G}^{nc}$. By combining the above expressions, we find the following expression for $\mathbf{y}_n(\beta)$:

$$\begin{aligned} & \sum_n \left(\frac{\beta_n}{\tilde{a}_n + \beta_n} \mathbf{A}_n^a(1,:) - \frac{2\beta_n}{a_n + \beta_n} \mathbf{A}_n^a(2,:) \right) \hat{\theta}(\beta) \\ &= \sum_n \left[\frac{1}{\tilde{a}_n + \beta_n} (\tilde{a}_n \theta_n + \beta_n E_n^a + 1) + \frac{1}{a_n + 2\beta_n} (b_n - 2\beta_n F_n^a + 1) \right. \\ &\quad \left. - G_n^{nc} \right], \end{aligned} \quad (17)$$

and

$$G_n^c(\beta) = \frac{1}{a_n + 2\beta_n} [2\beta_n \mathbf{A}_n^a(2,:) \hat{\theta}(\beta) - b_n + 2\beta_n F_n^a + 1], \quad (18a)$$

$$\begin{aligned} D_n(\beta) &= \frac{1}{2\tilde{a}_n \mathbf{A}_n^a(1,n)} \left[-\sqrt{2\beta_n((A_n^a(1,n))^2 + (A_n^a(2,n))^2)} \right. \\ &\quad \left. + A_n^a(1,n)(2\tilde{a}_n \theta_n - 1) + A_n^a(2,n)(1 - b_n) - a_n A_n^a(2,n) \right. \\ &\quad \left. \frac{1}{a_n + 2\beta_n} [2\beta_n \mathbf{A}_n^a(2,:) \hat{\theta}(\beta) - b_n + 2\beta_n F_n^a + 1] \right], \end{aligned} \quad (18b)$$

$$V_n(\beta) = \frac{1}{\sqrt{2\beta_n((A_n^a(1,n))^2 + (A_n^a(2,n))^2)}} (\hat{\theta}_n(\beta) - \theta_n). \quad (18c)$$

We introduce now the following result which is classical in the optimal contract literature.

Lemma 2: Suppose $\mathbf{z} = (\gamma_n, \beta_n)_n$ is an equilibrium solution of the DA's optimization problem. Then, for each EU $n \in \mathcal{N}$:

$$\mathbb{E}_{\varepsilon} [J_n^\#(\mathbf{z})] = p^r D_n(\beta). \quad (19)$$

In other words, the IR constraint is always binding in equilibrium.

Proof: We reason by contradiction. Suppose that there exists an equilibrium \mathbf{z} in which the IR constraint is not binding for some EU n . Then, the DA can unilaterally decrease its expected cost by decreasing γ_n without causing any of the constraints to be violated. Indeed, the latter depend only on β . This contradicts the assumption that \mathbf{z} is an equilibrium. ■

In the following, (19) will become an implicit constraint that will appear both in the DA's objective function and non-negativity constraints. Using (19), the DA's objective

function takes the closed form:

$$L^a(\mathbf{p}^a, \mathbf{y}) = \sum_n \frac{\gamma_n}{\beta_n} + \sum_n \left(\pi - \frac{1}{\beta_n} \right) \left[J_n(\mathbf{x}_n(\boldsymbol{\beta}), \theta_n) + c_n^I \frac{(\hat{\theta}_n(\boldsymbol{\beta}) - \theta_n)^2}{2V_n(\boldsymbol{\beta})} - p^r D_n(\boldsymbol{\beta}) \right].$$

Using (19), the non-negativity constraints become:

$$J_n(\mathbf{x}_n(\boldsymbol{\beta}), \theta_n) + c_n^I \frac{(\hat{\theta}_n(\boldsymbol{\beta}) - \theta_n)^2}{2V_n(\boldsymbol{\beta})} - p^r D_n(\boldsymbol{\beta}) \geq 0, \forall n \in \mathcal{N}.$$

Substituting the previous expressions, the DA's bilevel optimization can then be reformulated as follows:

$$\begin{aligned} \min_{\gamma, \boldsymbol{\beta}} \quad & \sum_n \left(\frac{\gamma_n}{\beta_n} + \left(\pi - \frac{1}{\beta_n} \right) \left[J_n(\mathbf{x}_n(\boldsymbol{\beta}), \theta_n) + c_n^I \frac{(\hat{\theta}_n(\boldsymbol{\beta}) - \theta_n)^2}{2V_n(\boldsymbol{\beta})} - p^r D_n(\boldsymbol{\beta}) \right] \right), \\ \text{s.t.} \quad & \mathbf{y}_n(\boldsymbol{\beta}) \text{ checks (17), (18a)-(18c), } \forall n \in \mathcal{N}, \\ & J_n(\mathbf{x}_n(\boldsymbol{\beta}), \theta_n) + c_n^I \frac{(\hat{\theta}_n(\boldsymbol{\beta}) - \theta_n)^2}{2V_n(\boldsymbol{\beta})} - p^r D_n(\boldsymbol{\beta}) \geq 0, \forall n \in \mathcal{N}, \\ & \sum_n (D_n(\boldsymbol{\beta}) - G_n^c(\boldsymbol{\beta}) - G_n^{nc}) = 0, \\ & \Psi_n(\boldsymbol{\beta}) \in \mathcal{Y}_n, \gamma_n \geq 0, \forall n \in \mathcal{N}. \end{aligned}$$

The DA's problem is nonlinear nonconvex in $\boldsymbol{\beta}$, and linear in γ . From (17), the bias of one EU can be compensated by the bias of another EU because it couples the EUs' biases. This implies that the map $\Psi(\cdot)$ is not necessarily bijective. Therefore, we cannot guarantee that the DA's bilevel optimization problem admits a unique solution.

Suppose the EUs at the lower-level of the DA's bilevel optimization problem reach a PSE, we need to replace (17), (18a)-(18c) with the following equations:

$$\begin{aligned} \hat{\theta}_n &= \theta_n, \\ D_n(\beta_n) &= \frac{\tilde{a}_n \theta_n + \beta_n (A_n^a(1,:) \boldsymbol{\theta} - E_n^a)}{\tilde{a}_n + \beta_n} - \frac{1}{2(\tilde{a}_n + \beta_n)}, \\ G_n^c(\beta_n) &= \frac{2\beta_n (A_n^a(2,:) \boldsymbol{\theta} - F_n^a)}{a_n + 2\beta_n} - \frac{1}{a_n + 2\beta_n}, \\ V_n &= \bar{V}_n. \end{aligned}$$

Proposition 7: Suppose the EUs at the lower-level reach a PSE, then the DA's bilevel optimization problem admits a unique equilibrium \mathbf{z} if and only if $A_n^a(l,:) \boldsymbol{\theta} + B_n^a(l,:) \mathbf{G}^c > C_n^a(l,1), \forall l = 1, 2, \forall n$.

Proof: From Prop. 5, we know that the lower-level of the DA's optimization problem admits a unique PSE. Furthermore, at equilibrium, we observe that $\hat{\theta}_n$ and V_n are constant. By differentiating $D_n(\beta_n)$ and $G_n^c(\beta_n)$ with respect to β_n , we obtain that $\frac{\partial D_n}{\partial \beta_n} > 0$ and $\frac{\partial G_n^c}{\partial \beta_n} > 0$ if and only if $A_n^a(1,:) \boldsymbol{\theta} + B_n^a(1,:) \mathbf{G}^c > C_n^a(1,1)$ and $\frac{\partial G_n^c}{\partial \beta_n} > 0$ if and only if $A_n^a(2,:) \boldsymbol{\theta} + B_n^a(2,:) \mathbf{G}^c > C_n^a(2,1)$. Under these assumptions, D_n and G_n are strictly increasing concave in β_n , therefore $\Psi_n(\cdot)$ is bijective over B_n and the DA's bilevel optimization problem admits a unique solution. ■

V. ALGORITHMS

A. Nash Equilibrium and Social Optimum Computation

To compute the NE, we formulate (8) as a deterministic optimization problem, by making the expectation in the objective function explicit. This is possible because the $(\varepsilon_n)_n$ are independent and $\varepsilon_n \sim \mathcal{N}(0, V_n), \forall n$. For the social optimum (SO), we sum these deterministic objective functions over all the EUs. To compute the NE and the SO, we apply a classic Gradient Descent Method (GDM) in a distributed fashion to compute the NE or in a centralized fashion, considering the social cost, to compute the SO.

The drawback of this approach is that the information decision variables of the other EUs appear in each EU's objective function. When we explicit the expectation, the $\hat{\theta}_n$ and V_n of the other players appear. Thus, it means that when implementing distributed GDM, the EUs need to share their decision variables with each other to reach the NE.

B. Performatively Stable Equilibrium Computation

To compute the PSE, we use the Repeated Stochastic Gradient Method (RSGM) [5]. In RSGM, having initialized \mathbf{y}^0 , at each step t , each EU samples its ε_n^t , and updates its variable \mathbf{y}_n^t using the formula:

$$\mathbf{y}_n^{t+1} = \text{proj}_{\mathcal{Y}_n} \left(\mathbf{y}_n^t - \eta \nabla_{\mathbf{y}_n} \bar{J}_n(\mathbf{y}^t, \varepsilon_n^t) \right), \quad (22)$$

where $\bar{J}_n(\cdot)$ is defined in Prop. 5 proof. We used a backtracking line search for the choice of the step size η .

However, RSGM requires that each EU n shares its decision variables \mathbf{y}_n with the other EUs at each iteration. But, in $\mathcal{G}^{\text{stoch}}$, the EUs are only aware of their own decision variables \mathbf{y}_n , and of $\hat{\boldsymbol{\theta}}$. Also, depending on the structure of the game, it might not be possible to reformulate the objective function $\mathbb{E}_{\varepsilon \sim \mathcal{D}(\mathbf{y})} [\tilde{J}_n(\mathbf{x}_n, \hat{\boldsymbol{\theta}}, \mathbf{V}, \varepsilon)]$, in the form $\mathbb{E}_{\varepsilon_n \sim \mathcal{D}_n(\mathbf{y})} [\bar{J}_n(\mathbf{x}_n, \hat{\boldsymbol{\theta}}, \mathbf{V}, \varepsilon_n)]$, $\forall n$ – for example, in case the $(\varepsilon_n)_n$ are correlated. For this reason, we propose an alternative algorithm, Coupled Privacy Repeated Stochastic Gradient Method (CP-RSGM), in which at each time step t , each EU n samples its ε_n^t and shares its $\hat{\theta}_n^t$ with the DA, which then broadcasts it to the other EUs. In fact, in our game $\tilde{J}_n(\mathbf{y}^t, \varepsilon^t) \equiv \tilde{J}_n(\mathbf{x}_n^t, \hat{\theta}_n^t, V_n^t, \hat{\boldsymbol{\theta}}^t)$. So, each EU can update its decision variables following the rule:

$$\mathbf{y}_n^{t+1} = \text{proj}_{\mathcal{Y}_n} \left(\mathbf{y}_n^t - \eta \nabla_{\mathbf{y}_n} \tilde{J}_n(\mathbf{y}^t, \varepsilon^t) \right). \quad (23)$$

In CP-RSGM, the update of the readings parameters, demand and generation variables is done simultaneously. In case where the readings parameters on the one side, demand and generation variables on the other side need to be updated sequentially, we introduce another algorithm, Privacy Split Repeated Stochastic Gradient Method (PS-RSGM), which is described in Alg. 1.

Algorithm 1 Privacy Split Repeated Stochastic Gradient Method (PS-RSGM)

- 1: Find $\mathbf{x}^G \in \text{SOL}(\mathcal{G})$.
 - 2: Find $\hat{\theta}_n^\#, V_n^\# = \underset{\hat{\theta}_n, V_n}{\text{argmin}} \mathbb{E}_\varepsilon \left[\tilde{J}_n(\mathbf{x}_n^G, \hat{\theta}, \mathbf{V}, \varepsilon) \right]$ under constraints (8e), (8f) $\forall n$. Set $\mathbf{x} = \mathbf{x}^G$ and compute a solution with CP-RSGM.
 - 3: Find $\mathbf{x}_n^\# = \underset{\mathbf{x}_n}{\text{argmin}} \mathbb{E}_{\varepsilon \sim \mathcal{N}(\mathbf{0}, \mathbf{V}^\#)} \left[J_n^\#(\mathbf{x}_n, \hat{\theta}^\#, \mathbf{V}^\#, \varepsilon) \right] \forall n$ under constraints (8b), (8c) and $\sum_{n=1}^N (D_n - G_n^c - G_n^{nc}) = 0$. $J_n^\#(\cdot)$ is define above (10).
-

A drawback of Alg. 1 is that $\hat{\theta}^\#, \mathbf{V}^\#$ are not best-responses to $\mathbf{x}^\#$. Taking the paradigm of best-response learning methods, we rely on stochastic approximation which is a simulation-based approach that can be used for finding the fixed point of some unknown operator $\Xi(\cdot)$ represented by the form of an expectation, i.e., $\Xi(\mathbf{y}) \triangleq \text{col} \left(\mathbb{E}_\varepsilon \left[\nabla_{\mathbf{y}_n} \tilde{J}_n(\mathbf{x}_n, \hat{\theta}, \mathbf{V}, \varepsilon) \right] \right)_n$. Specifically, this method seeks a solution $\mathbf{y}^\#$ based on noisy observations $\nabla_{\mathbf{y}_n} \tilde{J}_n(\mathbf{x}_n, \hat{\theta}, \mathbf{V}, \varepsilon)$. The iterate \mathbf{y} is updated by moving along the direction of $\nabla_{\mathbf{y}_n} \tilde{J}_n(\mathbf{x}_n, \hat{\theta}, \mathbf{V}, \varepsilon)$ scaled by some step size. Set $\Upsilon_n \triangleq \{(\hat{\theta}_n, V_n) \in \mathbb{R}_+^2 | (8e), (8f) \text{ hold}\}$ and $\mathbf{v}_n^t \triangleq (\hat{\theta}_n^t, V_n^t)$. We consider the so-called two-time-scale stochastic approximation [27], [28], a variant of the classic stochastic approximation, which allows to decouple the updating steps in \mathbf{v} and \mathbf{x} and to replace the exact computation step of \mathbf{x} from PS-RSGM (step 3 of Alg. 1) by a gradient-play update. At each time step t , we first apply a gradient-play update to compute \mathbf{v}_n^{t+1} , and $\tilde{\theta}^{t+1}$ is broadcast to all the EUs. Then, we apply a second time gradient-play to compute \mathbf{x}^{t+1} taking as input \mathbf{v}^{t+1} and $\tilde{\theta}^{t+1}$. The convergence of this algorithm requires a proper choice of the two step sizes η_1^t, η_2^t , which is discussed in Sec. VI B. This two-time-scale variant is described in Alg. 2.

Algorithm 2 Two-Time-Scale Stochastic Approximation Method (PS-SAM)

- 1: Find $\mathbf{x}^G \in \text{SOL}(\mathcal{G})$. For $t = 1$, set $\mathbf{x}^1 = \mathbf{x}^G$.
 - 2: At each time step $t = 1, \dots, T$
 - a: Find $\mathbf{v}_n^{t+1} = \text{proj}_{\Upsilon_n}(\mathbf{v}_n^t - \eta_1^t \nabla_{\mathbf{v}_n} \tilde{J}_n(\mathbf{x}_n^t, \mathbf{v}_n^t, \tilde{\theta}^t)) \forall n$, broadcast $\tilde{\theta}^{t+1}$ to all the EUs.
 - b: Find
$$\mathbf{x}_n^{t+1} = \text{proj}_{\mathcal{X}_n}(\mathbf{x}_n^t - \eta_2^t \nabla_{\mathbf{x}_n} \tilde{J}_n(\mathbf{x}_n^t, \mathbf{v}_n^{t+1}, \tilde{\theta}^{t+1})) \forall n.$$
-

Remark 5.1: Alg. 2 can be interpreted as a recursive way to solve a game in which each EU n first chooses their readings parameters $\mathbf{v}_n, \forall n$ and report its sampled reading $\tilde{\theta}_n$ to the DA. Then, each EU n updates its demand and controllable generation in reaction to the information broadcast by the DA, $\tilde{\theta}$. The outcome of Alg. 2, reached after convergence, coincides with a Stackelberg Equilibrium (SE) [29]. From game theory, we know that except in special cases (e.g., finite zero-sum games), NE and SE do not necessarily coincide [30].

The convergence of CP-RSGM, PS-RSGM, and PS-SAM is analyzed in the next section.

VI. CONVERGENCE ANALYSIS

Before starting the proofs, we introduce notation. We denote: $\mathbb{E}_t \stackrel{\text{def}}{=} \mathbb{E}[\cdot | \mathcal{F}_t]$, where $(\mathcal{F}_t)_t$ is a filtration such that ε^t is \mathcal{F}_{t+1} -measurable. We also introduce: $g(\mathbf{y}, \varepsilon) \stackrel{\text{def}}{=} (\nabla_n \tilde{J}_n(\mathbf{y}, \varepsilon))_{n \in \mathcal{N}}$ and $G_{\mathbf{y}'}(\mathbf{y}) \stackrel{\text{def}}{=} \mathbb{E}_{\varepsilon \sim \mathcal{N}(\mathbf{0}, \mathbf{V}')} [g(\mathbf{y}, \varepsilon)]$. Finally, we introduce Lip_1 as the set of 1-Lipschitz continuous functions.

A. Convergence Proofs for CP-RSGM and PS-RSGM

Theorem 3: CP-RSGM converges in expectation to a neighborhood of the PSE. The size of the neighborhood is determined by the step size η of the gradient descent.

Proof: We observe that $g_n(\cdot)$ is linear in ε , hence it is κ_n -Lipschitz in ε . Now, we want to obtain the L -Lipschitz continuity of $G_z(\cdot)$ in \mathbf{y} . To do so, we observe that $G_{z,n}(\mathbf{y}) - G_{z,n}(\mathbf{y}') = F_{y_n}(\mathbf{y}) - F_{y_n}(\mathbf{y}')$. We proceed the same way as we did to prove the strong monotonicity of the objective function, and sum over all the EUs. We obtain $L = N \max_{w \in \mathcal{Y}_n} \max_{i \in [1, 4]} \lambda_i(H(w))$.

We check a property on the regularity of the distribution used. Using [31, Lem. 2.4], we obtain an upper bound on the Wasserstein-1 distance between two Gaussian distributions:

$$W_1(\mathcal{N}(\mathbf{0}, V), \mathcal{N}(\mathbf{0}, V')) \leq \sqrt{\sum_{i=1}^N (\sqrt{V_i} - \sqrt{V'_i})^2} \quad (24a)$$

$$\leq \sqrt{\sum_{i=1}^N \tau_i^2 (V_i - V'_i)^2} \leq \tau \|V - V'\| \quad (24b)$$

with τ_i obtained by the Mean Value Theorem and $\tau = \max \tau_i$. Let $\rho \stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^N (\frac{\kappa_i \tau}{\omega})^2}$. We aim to prove the convergence of CP-RSGM taking inspiration from the proof of convergence of RSGM in [5]. First, we need to obtain the following property on the variance of the gradient of $\tilde{\Pi}_n(\cdot)$:

$$\mathbb{E}_t \|g(\mathbf{y}, \varepsilon) - \mathbb{E}_t[g(\mathbf{y}, \varepsilon)]\|^2 \leq \sigma^2. \quad (25)$$

To do so, we use the explicit expression of the gradient $g(\cdot)$ and its linearity in ε . This leads us to the following bound:

$$\sigma^2 = 4 \sum_{n=1}^N \beta_n^2 S_n \bar{V}_n, \quad (26)$$

with S_n a scalar being quadratic in the elements of A_n^a defined as follows:

$$\begin{aligned} S_n = & \left(\sum_{m=1}^N (A_{n(1,m)}^a)^2 \right) \left((A_{n(1,n)}^a)^2 + 1 \right) + \\ & \left(\sum_{m=1}^N (A_{n(2,m)}^a)^2 \right) \left((A_{n(2,n)}^a)^2 + 1 \right) + \\ & A_{n(1,n)}^a A_{n(2,n)}^a \left(\sum_{m=1}^N A_{n(1,m)}^a A_{n(2,m)}^a \right). \end{aligned}$$

Let \mathbf{y}^s be the PSE. Then, we need to check the next condition on the bias of our estimator of the gradient:

$$\|\mathbb{E}_t g(\mathbf{y}, \varepsilon) - G_{\mathbf{y}^s}(\mathbf{y})\| \leq P \|\mathbf{y} - \mathbf{y}^s\|. \quad (28)$$

To do so, we decompose the norm in (28) over each EU n . We use the κ_n -Lipschitz continuity of $g_n(\cdot)$ in ε and the bound on the Wasserstein-1 distance between two Gaussian density functions. This gives us:

$$\|\mathbb{E}_t g_n(\mathbf{y}, \varepsilon) - G_{\mathbf{y}^s, n}(\mathbf{y})\| \quad (29a)$$

$$\leq \kappa_n \sup_{h \in \text{Lip}_1} \left[\mathbb{E}_{\varepsilon \sim \mathcal{N}(0, \mathbf{V})} [h(\mathbf{y}, \varepsilon)] - \mathbb{E}_{\varepsilon' \sim \mathcal{N}(0, \mathbf{V}_s)} [h(\mathbf{y}, \varepsilon')] \right], \quad (29b)$$

$$\leq \kappa_n \tau \|\mathbf{V} - \mathbf{V}^s\|. \quad (29c)$$

To go from equation (29b) to (29c), we use the conditions obtained in (24a)-(24b), because this sup is the definition of the Wasserstein-1 distance.

Now that we have this inequality, we square it, sum it over all the EUs, major $\|\mathbf{V} - \mathbf{V}^s\|$ by $\|\mathbf{y} - \mathbf{y}^s\|$, and apply a square root on the inequality. We obtain:

$$\|\mathbb{E}_t g(\mathbf{y}, \varepsilon) - G_{\mathbf{y}^s}(\mathbf{y})\| \leq \sqrt{\sum_{n=1}^N (\kappa_n \tau)^2} \|\mathbf{y} - \mathbf{y}^s\| \quad (30)$$

We obtain $P = \omega \rho$. Now that we have proved those two inequalities, we can apply [5, Thm. 8]. We obtain that, if $\rho \in (0, 1)$, for any step $\eta < \frac{\omega(1-\rho)}{8L^2}$ the CP-RSGM generates a sequence \mathbf{y}^t satisfying:

$$\mathbb{E}_t \|\mathbf{y}^{t+1} - \mathbf{y}^s\|^2 \leq \frac{1+2\eta\omega\rho+2\eta^2\omega^2\rho^2}{1+2\eta\omega(\frac{1+\rho}{2})} \|\mathbf{y}^t - \mathbf{y}^s\|^2 + \frac{4\eta^2\sigma^2}{1+2\eta\omega(\frac{1+\rho}{2})} \quad (31)$$

Thus, we observe that we converge in expectation to a neighborhood of the PSE. Notice that we can reduce the size of the neighborhood by reducing the step size η of the gradient descent, but the latter will converge slower. So, there is a trade-off to find between the accuracy of the PSE computation and the convergence speed. ■

Proposition 8: PS-RSGM converges in expectation to a neighborhood of the PSE.

Proof: In PS-RSGM, the first step is an exact method to find the solution \mathbf{x}^G of the game \mathcal{G} . Then, we find $\hat{\boldsymbol{\theta}}^\#, \mathbf{V}^\#$ using CP-RSGM with fixed $\mathbf{x} = \mathbf{x}^G$. Finally, using Prop. 6, it is obvious that $\mathcal{G}^\#(\hat{\boldsymbol{\theta}}, \mathbf{V})$ can be solved as an optimization problem, and we can find $\mathbf{x}^\#$ with an exact method relying on classical optimization solvers. Thus, the proof of convergence of the CP-RSGM also proves the convergence of the PS-RSGM. ■

B. Convergence Proof for PS-SAM

Let $\mathbf{H}_{\mathbf{x}_n} \stackrel{\text{def}}{=} \mathbf{H}_{n(3:4,3:4)}$, $\mathbf{H}_{\mathbf{v}_n} \stackrel{\text{def}}{=} \mathbf{H}_{n(1:2,1:2)}$, and $\tilde{L} \stackrel{\text{def}}{=} N \max_i \lambda_i(\mathbf{H}_{\mathbf{x}_n})$. In addition, we set $\mu_1 \stackrel{\text{def}}{=} \min_{\mathbf{v}_n \in \Upsilon_n} \min_i \lambda_i(\mathbf{H}_{\mathbf{v}_n}(\mathbf{v}_n))$, and $\mu_2 \stackrel{\text{def}}{=} \min_i \lambda_i(\mathbf{H}_{\mathbf{x}_n})$.

Proposition 9: If the two-time-scales of PS-SAM are chosen such that $\frac{\eta_2^0}{\eta_1^0} \leq \frac{\mu_1 \mu_2}{2L}$, $\sum_{t=0}^{+\infty} \eta_1^t = \sum_{t=0}^{+\infty} \eta_2^t = +\infty$, $\max\{\sum_{t=0}^{+\infty} (\eta_1^t)^2, \sum_{t=0}^{+\infty} (\eta_2^t)^2, \sum_{t=0}^{+\infty} \frac{(\eta_2^t)^2}{(\eta_1^t)^2}\} < +\infty$, then $(\mathbf{y}^t)_t$ converges almost surely to a SE solution of $\mathcal{G}^{\text{stoch}}$.

Proof: The proof can be adapted from [27, Thm. 1]. ■

Remark 6.1: Following [27, Rk. 2], we choose $\eta_1^t = \frac{\eta_1^0}{(t+2)^a}$, $\eta_2^t = \frac{\eta_2^0}{(t+2)^b}$, with $\frac{1}{2} < a < b \leq 1$, $2b-a > 1$ in Sec. VII.

VII. SIMULATIONS

Numerical results are presented on a three-node network, with $\boldsymbol{\theta} = [0.23, 0.32, 0.55]$ and $\mathbf{G}^{nc} = [0.1, 0.2, 0.4]$. $\mathcal{G}^{\text{stoch}}$ parameters are described in Tab. I. In addition, we choose $\xi = 10^3$ and $\pi = 1.0$.

TABLE I
STOCHASTIC GAME $\mathcal{G}^{\text{stoch}}$ PARAMETERS.

Player	\hat{a}_n	\hat{b}_n	a_n	b_n	c_n	D_n	\bar{D}_n	G_n^c	\bar{G}_n^c	V_n	\bar{V}_n	c_n^t	α_n
EU 0	15.0	0.0	1.0	2.0	1.0	0.2	10.0	0.0	3.0	0.05	10.0	1.0	3.0
EU 1	15.0	0.0	1.5	1.5	1.0	0.0	10.0	0.0	3.0	0.05	10.0	1.0	3.0
EU 2	10.0	0.0	2.0	1.0	1.0	0.4	10.0	0.0	3.0	0.05	10.0	1.0	3.0

A. Indistinguishability and Convergence Rates

In Fig. 2, we plot the final value of $\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}$ reached by GDM, RSGM, CP-RSGM, PS-RSGM and PS-SAM, considering 50 different initializations of \mathbf{y}^0 . We proved that the NE and the PSE solutions of $\mathcal{G}^{\text{stoch}}$ are unique. However, CP-RSGM and PS-RSGM only enable to approximate the PSE. This is due to the sampling on ε and to the fact that we impose an upper bound on the number of iterations. We depict the resulting neighborhoods as the convex hulls of the set of approximate PSEs that can be reached by CP and PS-RSGM. We also notice that despite the upper bound on the number of iterations, PS-SAM reaches a SE, with smaller biases than RSGM and variants.

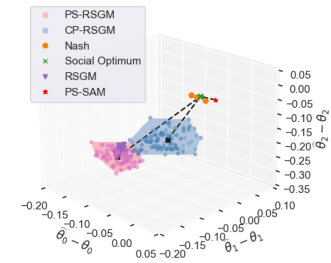


Fig. 2. Final value $\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}$ reached by GDM, RSGM, CP-RSGM, PS-RSGM, PS-SAM for 50 different starting points.

In Fig. 2, we also observe that the NE is very close to the SO, but still a gap remains. This gap can be explained from Thm. 2 as resulting from the estimation bias of the DA's estimator.

We compare the convergence rates in number of iterations, of RSGM, CP-RSGM and PS-RSGM in $\hat{\boldsymbol{\theta}}$, in Fig. 3. To build this plot, we ran the algorithm for 50 random starting points. Due to the sampling on ε at each step, the algorithms only converge in expectation. A first observation is that CP-RSGM and PS-RSGM provide upper and lower bounds on the approximate PSE computed by RSGM. Convergence rates in time are additionally provided in Tab. II. To compute a PSE, we observe that PS-RSGM converges faster in time than RSGM. To compute a SE, PS-SAM achieves a better convergence rate in time than RSGM. But, the fastest in time

remains GDM for NE computation. As mentioned in Sec. V A, the good performance of GDM should be mitigated by the full coordination it requires between the DA and the EUs.

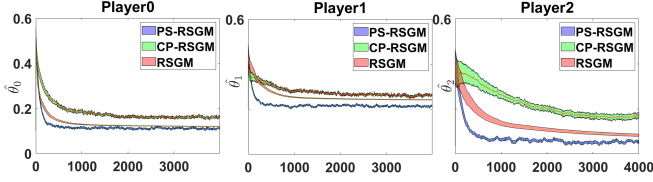


Fig. 3. 95% confidence interval for $\hat{\theta}$ while learning the PSE.

B. Balancing Efficiency and Privacy

Let y^{so} be the SO solution of the optimization problem with objective the sum over n of the expected utilities (12a), denoted $SC(\cdot)$, under the concatenation of the constraints (8b)-(8f) for all the EUs. In an equilibrium y , we evaluate the efficiency gap as the difference between the social cost in the equilibrium and the optimal social cost: $SC(y) - SC(y^{so})$. We plot the distribution of the efficiency gap for the GDM, RSGM, CP and PS-RSM in Fig. 4. The efficiency gap of the NE is close to 0, i.e., the NE is a desirable equilibrium to reach for the coupled markets. The PS-RSGM and the CP-RSGM have a 5% lower efficiency gap than the RSGM. The total information release by the EUs on the market, illustrated in Fig. 5, is defined as $\sum_n I_n(\theta_n, \hat{\theta}_n, V_n)$. The smaller it is, the less information is released. We summarize the performance of each algorithm in Tab. II.

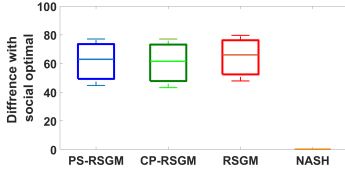


Fig. 4. Box plots of the efficiency gaps for each algorithm.

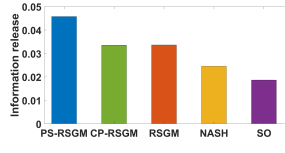


Fig. 5. Total information released by the EUs for each algorithm.

TABLE II

COMPARISON OF THE ALGORITHMS FOR EQUILIBRIUM LEARNING.

Algorithm	Coordination	Eff. Gap	Info. Release	$\sum_n d(\theta_n, \hat{\theta}_n)$	Iter. Compl. [iter]	Time Compl. [s]	Eq.
GDM	simultaneous	0.26	0.025	0.077	180	2.35	NE
RSGM	simultaneous	66.06	0.034	0.452	94	2.89	PSE
CP-RSGM	simultaneous	61.61	0.034	0.378	730	13.52	PSE
PS-RSGM	sequential	62.99	0.046	0.490	217	2.86	PSE
PS-SAM	recursive	74.14	0.203	0.086	35580	2.83	SE

To synthesize the results: taking the perspective of the coupled markets, GDM leads to the smallest efficiency gap, information release and bias. However, it requires full coordination between the DA and the EUs, which is very hard to impose in practice. CP-RSGM generates a smaller efficiency gap than RSGM, that we interpret as the benchmark, for almost the same information release, but with smaller bias, i.e., higher indistinguishability for the EUs, and requires less coordination among the EUs. Allowing sequentiality in the DA and EUs' decisions through PS-RSGM leads to a smaller variance (more information release) and larger bias (less indistinguishability) than RSGM and CP-RSGM. Finally, PS-RSGM converges faster in time than RSGM, which is itself

far faster than CP-RSGM. PS-SAM achieves even faster convergence rates in time than PS-RSGM, smaller biases (more indistinguishability) but this good performance is at the cost of the highest efficiency gap and information release.

These results, which consider the problem from a market operator's perspective, have to be mitigated by what is observed at the EUs' levels.

C. Benefits of Performative Effects Modeling

In Fig. 6-8, all the bar plots are represented in log scale. We observe that the costs borne by the EUs on the electricity market are 10 times higher under PS-SAM and RSGM, than under GDM. Indeed, while the NE computed with GDM might be close to efficient, SE and PSE lead to larger efficiency losses on the electricity market. Furthermore, depending on the EUs' consumption and generation profiles, the information release under PS-SAM and RSGM can be decreased for some of the EUs by comparison with the GDM outcome. Finally, PS-SAM and RSGM lead to 10 times higher contract payments than under GDM. These plots demonstrate that performative effects modeling is beneficial to the EUs when coupling the electricity market with a data market, because it leads to far higher contract payments than when such effects are ignored and can compensate the inefficiencies of the electricity market equilibrium. Furthermore, the information release can even be decreased for some of the EUs under PS-SAM and RSGM. Finally, RSGM tends to provide more homogeneous information release than GDM, while PS-SAM introduces disparities among the EUs.

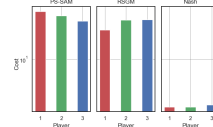


Fig. 6. Cost $(J_n)_n$.

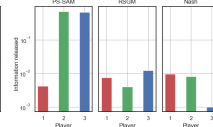


Fig. 7. Information released $(I_n)_n$.

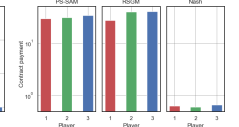


Fig. 8. Contract payment $(p_n^a)_n$.

D. Solving the DA's Bilevel Optimization Problem

The DA's bilevel optimization problem being linear increasing in each component of γ and unconstrained on \mathbb{R}_+^N , the DA's objective is minimized in $\gamma = 0$. In β , the problem is more difficult as it is nonlinear nonconvex. To compute a solution, we rely on simulated annealing. In Fig. 9, we plot the frequencies of occurrence of all the algorithm sampled solutions, evaluated in the DA's objective, that were visited (current BL objective), selected (last BL objective), optimal (optimal BL objective) in the sense better than the previous one. We observe in Fig. 10 that the simulated annealing algorithm converges in less than 2.10^3 iterations to an equilibrium in β .

VIII. CONCLUSION

We formulate a peer-to-peer electricity market as a network game, and characterize its Variational Equilibrium in terms of existence, uniqueness, and stability. In addition, we prove that the Variational Equilibrium depends on the nominal demands of all the EUs, which is seen as sensitive

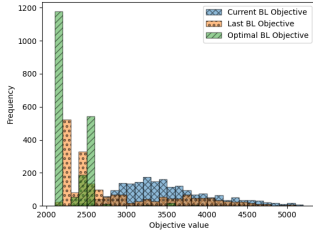


Fig. 9. Frequencies of occurrence of the sampled solutions.

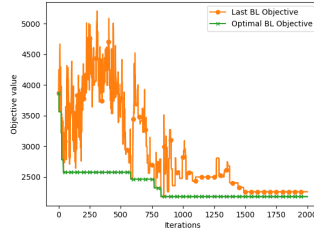


Fig. 10. Convergence of the simulated annealing algorithm.

information that they might be reluctant to disclose. To protect their privacy, the EUs rely on strategic information, therefore communicating biased and noisy readings to the others. A DA is introduced to coordinate the information exchanges between the EUs, with the aim to learn the EUs' private information, by providing them with contracts which remunerate them depending on the accuracy of their readings. The originality of our work relies on the framework of performative prediction that we introduce to explicitly take into account the shift caused by the EUs' strategic information on their strategies and market equilibria. To analyse the outcome of the resulting decision-dependent game, various solution concepts are introduced such as Nash, Performatively Stable, and Stackelberg Equilibria. Two variants of RSGM and a two-time-scale stochastic approximation algorithm are proposed to compute these equilibria. We prove the convergence of each algorithm. Their performance is subsequently compared on a numerical example by measuring the achieved efficiency loss, privacy preservation capabilities, convergence rates, and the EUs' utility functions at equilibrium, highlighting the benefits of performative effects modeling for the EUs.

APPENDIX

A. Proof of Proposition 1

For any EU $n \in \mathcal{N}$, Problem (4) can be formulated as follows:

$$\min_{D_n, G_n^c, q_{mn}} J_n(\mathbf{x}_n, \theta_n), \quad (32a)$$

$$s.t. \quad \underline{D}_n \leq D_n \leq \bar{D}_n, \quad (\underline{\mu}_n, \bar{\mu}_n) \quad (32b)$$

$$\underline{G}_n^c \leq G_n^c \leq \bar{G}_n^c, \quad (\underline{\nu}_n, \bar{\nu}_n) \quad (32c)$$

$$q_{mn} = -q_{nm}, \quad (\zeta_{nm}) \quad (32d)$$

$$D_n \leq G_n^c + G_n^{nc} + Q_n. \quad (\lambda_n) \quad (32e)$$

To compute an equilibrium solution of \mathcal{G} , we derive the KKT conditions of the Lagrangian function $\mathcal{L}_n(\cdot)$ with respect to $D_n, G_n^c, q_{mn}, \forall m \in \Gamma_n, m \neq n$, the first order stationarity conditions write down as follows:

$$\frac{\partial \mathcal{L}_n}{\partial D_n} = 2\tilde{a}_n(D_n - \theta_n) + \lambda_n - \underline{\mu}_n + \bar{\mu}_n = 0, \quad (33a)$$

$$\frac{\partial \mathcal{L}_n}{\partial G_n^c} = a_n G_n^c + b_n - \underline{\nu}_n + \bar{\nu}_n - \lambda_n = 0, \quad (33b)$$

$$\frac{\partial \mathcal{L}_n}{\partial q_{mn}} = c_{nm} + \zeta_{nm} - \lambda_n = 0, \forall m \in \Gamma_n, m \neq n. \quad (33c)$$

From (33c), we obtain:

$$\lambda_n = c_{nm} + \zeta_{nm}, \forall m \in \Gamma_n. \quad (34)$$

Then, from (34), we infer that at a VE:

$$\zeta_{nm} = \lambda_n - c_{nm} = \lambda_m - c_{mn}, \forall m \in \Gamma_n. \quad (35)$$

From (33a), we infer that at equilibrium:

$$D_n^* = \theta_n - \frac{1}{2\tilde{a}_n} (\lambda_n + (\bar{\mu}_n - \underline{\mu}_n)). \quad (36)$$

From (33b), we infer that at equilibrium:

$$G_n^{c*} = -\frac{b_n}{a_n} + \frac{1}{a_n} (\lambda_n - (\bar{\nu}_n - \underline{\nu}_n) + (\bar{\mu}_n - \underline{\mu}_n)). \quad (37)$$

Complementarity constraints are:

$$\underline{\mu}_n \geq 0 \perp D_n \leq D_n, \quad (38a)$$

$$\bar{\mu}_n \geq 0 \perp D_n \leq \bar{D}_n, \quad (38b)$$

$$\underline{\nu}_n \geq 0 \perp G_n^c \leq G_n^c, \quad (38c)$$

$$\bar{\nu}_n \geq 0 \perp G_n^c \leq \bar{G}_n^c, \quad (38d)$$

$$\lambda_n \geq 0 \perp D_n \leq G_n^c + G_n^{nc} + Q_n. \quad (38e)$$

Complementarity constraint (38e) implies that $\lambda_n(D_n - G_n^c - G_n^{nc} - Q_n) = 0$, which leads us to distinguish between two cases depending on whether constraint (32e) is binding.

Case 1 [No surplus at node n] In case $\lambda > 0$ then

$$Q_n^* = D_n^* - G_n^{c*} - G_n^{nc}, \quad (39)$$

which is equivalent to $\Delta_n = 0$, i.e., there is no surplus at node n . By substitution of (36), (37) in the net import at node n , Q_n , and using the fact that $\sum_n Q_n = 0$, we obtain:

$$\sum_n \left[\theta_n - \frac{1}{2\tilde{a}_n} (\bar{\mu}_n - \underline{\mu}_n) + \frac{b_n}{a_n} + \frac{1}{a_n} (\bar{\nu}_n - \underline{\nu}_n) - G_n^{nc} \right] = \sum_n \left(\frac{1}{2\tilde{a}_n} + \frac{1}{a_n} \right) \lambda_n. \quad (40)$$

But, from (35), we have the following relationship:

$$\lambda_n = c_{n0} - c_{0n} + \lambda_0, \forall n \in \Gamma_0. \quad (41)$$

By substitution in (40), we obtain a closed form expression for λ_0 :

$$\lambda_0 = \frac{\sum_n \left[\theta_n - \frac{1}{2\tilde{a}_n} (\bar{\mu}_n - \underline{\mu}_n) + \frac{b_n}{a_n} + \frac{1}{a_n} (\bar{\nu}_n - \underline{\nu}_n) - G_n^{nc} \right]}{\sum_n \left(\frac{1}{2\tilde{a}_n} + \frac{1}{a_n} \right)} + \frac{\sum_n \left(\frac{1}{2\tilde{a}_n} + \frac{1}{a_n} \right) (c_{0n} - c_{n0})}{\sum_n \left(\frac{1}{2\tilde{a}_n} + \frac{1}{a_n} \right)}. \quad (42)$$

To simplify the notations, let

$$K \triangleq \frac{\sum_n \left[-\frac{1}{2\tilde{a}_n} (\bar{\mu}_n - \underline{\mu}_n) + \frac{b_n}{a_n} + \frac{1}{a_n} (\bar{\nu}_n - \underline{\nu}_n) \right]}{\sum_n \left(\frac{1}{2\tilde{a}_n} + \frac{1}{a_n} \right)} + \frac{\sum_n \left(\frac{1}{2\tilde{a}_n} + \frac{1}{a_n} \right) (c_{0n} - c_{n0})}{\sum_n \left(\frac{1}{2\tilde{a}_n} + \frac{1}{a_n} \right)}.$$

Substituting λ_0 closed form (42) in (41), and then later in (36), (37), (39), we get:

$$\lambda_n = \frac{\sum_m \theta_m}{\sum_n \left(\frac{1}{2\tilde{a}_n} + \frac{1}{a_n} \right)} - \frac{\sum_m G_m^{nc}}{\sum_n \left(\frac{1}{2\tilde{a}_n} + \frac{1}{a_n} \right)} + K + (c_{n0} - c_{0n}),$$

$$D_n^* = \left(1 - \frac{1}{2\tilde{a}_n \sum_n \left(\frac{1}{2\tilde{a}_n} + \frac{1}{a_n} \right)} \right) \theta_n - \frac{1}{2\tilde{a}_n} \left(\frac{\sum_{m \neq n} \theta_m}{\sum_n \left(\frac{1}{2\tilde{a}_n} + \frac{1}{a_n} \right)} \right) - \frac{\sum_m G_m^{nc}}{\sum_n \left(\frac{1}{2\tilde{a}_n} + \frac{1}{a_n} \right)} + K + (c_{n0} - c_{0n}) + (\bar{\mu}_n - \underline{\mu}_n),$$

$G_n^{c*} = \frac{1}{a_n} \left(\frac{\sum_n \theta_n}{\sum_n (\frac{1}{2\tilde{a}_n} + \frac{1}{a_n})} - \frac{\sum_n G_n^{nc}}{\sum_n (\frac{1}{2\tilde{a}_n} + \frac{1}{a_n})} \right) + K$
 $+ (c_{n0} - c_{0n}) - (\bar{\nu}_n - \underline{\nu}_n) + (\bar{\mu}_n - \underline{\mu}_n) - \frac{b_n}{a_n}$, and
 $Q_n^* = \left(1 - \frac{1}{2\tilde{a}_n \sum_n (\frac{1}{2\tilde{a}_n} + \frac{1}{a_n})} - \frac{1}{a_n \sum_n (\frac{1}{2\tilde{a}_n} + \frac{1}{a_n})} \right) \theta_n$
 $- \left(\frac{1}{2\tilde{a}_n} + \frac{1}{a_n} \right) \frac{\sum_{m \neq n} \theta_m}{\sum_n (\frac{1}{2\tilde{a}_n} + \frac{1}{a_n})} - \left(1 - \frac{1}{2\tilde{a}_n \sum_n (\frac{1}{2\tilde{a}_n} + \frac{1}{a_n})} \right)$
 $- \frac{1}{a_n \sum_n (\frac{1}{2\tilde{a}_n} + \frac{1}{a_n})} \left(G_n^{nc} - \left(\frac{1}{2\tilde{a}_n} + \frac{1}{a_n} \right) \left(- \frac{\sum_{m \neq n} G_m^{nc}}{\sum_n (\frac{1}{2\tilde{a}_n} + \frac{1}{a_n})} \right) \right)$
 $+ K + (c_{n0} - c_{0n}) + (\bar{\mu}_n - \underline{\mu}_n) + \frac{(\bar{\nu}_n - \underline{\nu}_n)}{a_n} + \frac{b_n}{a_n}$. We observe that
 D_n^*, G_n^{c*}, Q_n^* at equilibrium depend linearly on the target
demands of *all* the EUs.

Case 2 [Possible surplus at node n] In case $\lambda_n = 0$, then by substitution in (36), (37), (39) we obtain that
 $D_n^* = \theta_n - \frac{1}{2\tilde{a}_n} (-\tilde{b}_n + (\underline{\mu}_n - \underline{\mu}_n))$, $G_n^{c*} = -\frac{b_n}{a_n} - \frac{1}{a_n} (\bar{\nu}_n - \underline{\nu}_n)$,
and $Q_n^* = \theta_n - \frac{1}{2\tilde{a}_n} (-\tilde{b}_n + (\underline{\mu}_n - \underline{\mu}_n)) + \frac{b_n}{a_n} + \frac{1}{a_n} (\bar{\nu}_n - \underline{\nu}_n)$
 $- G_n^{nc}$.

In both cases, Prop. 1 statement follows from the closed form expressions of $D_n^*, G_n^{c*}, Q_n^*, \forall n \in \mathcal{N}$ at equilibrium.

B. Proof of Theorem 1

In the non-cooperative game \mathcal{G} , the strategy set of each EU is convex and compact; thus, the existence of a NE follows from [32, Thm. 3]. Now let us consider that there exists two VEs \hat{x} and \bar{x} solutions of the game \mathcal{G} . As both \hat{x} and \bar{x} points are equilibrium they must satisfy the KKT conditions, multiplying first order condition with $[\bar{x}_n - \hat{x}_n]$ at point \hat{x}_n and \bar{x}_n for each n , we get:
 $\left[(2\tilde{a}_n - (\hat{D}_n - \theta_n)) - (2\tilde{a}_n - (\bar{D}_n - \theta_n)) \right] (\hat{D}_n - \bar{D}_n) = 0$,
 $+ \left[(\hat{\lambda}_n - \hat{\mu}_n + \hat{\mu}_n) - (\bar{\lambda}_n - \bar{\mu}_n + \bar{\mu}_n) \right] (\hat{D}_n - \bar{D}_n) = 0$,
 $\left[(a_n \hat{G}_n^c + b_n) - (a_n \bar{G}_n^c + b_n) \right] (\hat{G}_n^c - \bar{G}_n^c) -$
 $\left[(\hat{\nu}_n + \hat{\nu}_n - \hat{\lambda}_n) - (\bar{\nu}_n + \bar{\nu}_n - \bar{\lambda}_n) \right] (\hat{G}_n^c - \bar{G}_n^c) = 0$,
 $[c_{nm} - c_{nm}] (\hat{q}_{nm} - \bar{q}_{nm}) + \left[(\hat{\zeta}_{nm} - \hat{\lambda}_n) - (\bar{\zeta}_{nm} - \bar{\lambda}_n) \right]$
 $(\hat{q}_{nm} - \bar{q}_{nm}) = 0, \forall m \in \Gamma_n, m \neq n$. Summing over all the n gives us:

$$(\hat{x} - \bar{x})^T [\mathbf{F}_x(\hat{x}) - \mathbf{F}_x(\bar{x})] + (\hat{x} - \bar{x})^T [\hat{\Lambda} - \bar{\Lambda}] = 0, \quad (43)$$

where $\hat{\Lambda}$ and $\bar{\Lambda}$ represent matrices with Lagrangian multipliers from KKT at point \hat{x} and \bar{x} respectively. We observe that:

$$(\hat{x} - \bar{x})^T [\mathbf{F}_x(\hat{x}) - \mathbf{F}_x(\bar{x})] + (\hat{x} - \bar{x})^T [\hat{\Lambda} - \bar{\Lambda}] > 0, \quad (44)$$

which contradicts the assumption that there exist two VEs. In the above equation to prove that first term is positive, we show that game \mathcal{G} is strongly monotone, i.e.:

$$(\mathbf{y} - \mathbf{x})^T [\mathbf{F}_x(\mathbf{y}) - \mathbf{F}_x(\mathbf{x})] > 0, \forall x \neq y, \quad (45)$$

under Asm. 2.

Consider the pseudo Hessian matrix \mathbf{H} of utilities of the EUs where each n^{th} row and m^{th} column block matrix is given as $H_{n,m} = \left[\frac{\partial \mathbf{F}_{x_n}}{\partial \mathbf{x}_m} \right]$, i.e., in details:

$$H_{n,m} = \begin{cases} \begin{bmatrix} 2\tilde{a}_n & 0 & 0 \\ 0 & a_n & 0 \\ 0 & 0 & 0 \end{bmatrix} & \text{if } n = m, \\ \mathbf{0}_{3 \times 3} & \text{if } n \neq m. \end{cases} \quad (46)$$

However, we observe that the matrix \mathbf{H} is a block diagonal matrix with each diagonal matrix positive semi-definite, which implies that matrix \mathbf{H} is positive semi-definite and in our case $\frac{\partial \mathbf{F}_{x_n}}{\partial \mathbf{x}_n}$ is constant for all $n \in \mathcal{N}$, thus inequality (45) follows from [32], Thm. 6, and the game \mathcal{G} is strongly monotone. The efficiency of the VE follows from the generalized potential game structure of \mathcal{G} , which results from the coincidence of the KKTs of (3) and \mathcal{G} .

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