Universal First-Order Quantification over Automata

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Abstract. Deciding formulas that mix arithmetic and uninterpreted predicates is of practical interest, notably for applications in verification. Some decision procedures consist in building by structural induction an automaton that recognizes the set of models of the formula under analysis, and then testing whether this automaton accepts a non-empty language. A drawback is that universal quantification is usually handled by a reduction to existential quantification and complementation. For logical formalisms in which models are encoded as infinite words, this hinders the practical use of this method due to the difficulty of complementing infinite-word automata. The contribution of this paper is to introduce an algorithm for directly computing the effect of universal first-order quantifiers on automata recognizing sets of models, for formulas involving natural numbers encoded in unary notation. This paves the way to implementable decision procedures for various arithmetic theories.

Keywords: Infinite-word automata, first-order logic, quantifier elimination, satisfiability

1 Introduction

Automated reasoning with arithmetic theories is of primary importance, notably for verification, where Satisfiability Modulo Theories (SMT) solvers are regularly used to discharge proof obligations. It is well known however that mixing arithmetic with uninterpreted symbols quickly leads to undecidable languages. For instance, extending Presburger arithmetic, i.e., the first-order additive theory of integer numbers, with just one uninterpreted unary predicate makes it undecidable [7, 8, 15]. There exist decidable fragments mixing arithmetic and uninterpreted symbols that are expressive enough to be interesting, for instance, the monadic second-order theory of $\mathbb{N}$ under one successor (S1S).

The decidability of S1S has been established thanks to the concept of infinite-word automaton [6]. In order to decide whether a formula φ is satisfiable, the

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approach consists in building an automaton that recognizes the set of its models, encoded in a suitable way, and then checking that this automaton accepts a non-empty language. Such an automaton has one separate input tape for each first-order and second-order free variable of φ. It is constructed by starting from elementary automata representing the atoms of φ, and then translating the effect of Boolean connectives and quantifiers into corresponding operations over automata. For instance, applying an existential quantifier simply amounts to removing from the automaton the input tape associated to the quantified variable. Universal quantification reduces to existential quantification thanks to the equivalence $\forall x\, \varphi \equiv \neg\exists x\, \neg\varphi$.

Even though this approach has originally been introduced as a purely theoretical tool, it is applied in practice to obtain usable decision procedures for various logics. In particular, the tool MONA [9] uses this method to decide a restricted version of S1S, and tools such as LASH [3] and Shasta [14] use a similar technique to decide Presburger arithmetic. The LASH tool also generalizes this result by providing an implemented decision procedure for the first-order additive theory of mixed integer and real variables [2].

A major issue in practice is that the elimination of universal quantifiers relies on complementation, which is an operation that is not easily implemented for infinite-word automata [13, 17]. Actual implementations of automata-based decision procedures elude this problem by restricting the language of interest or the class of automata that need to be manipulated. For instance, the tool MONA only handles *Weak* S1S (WS1S) which is, schematically, a restriction of S1S to finite subsets of natural numbers [5]. The tool LASH handles the mixed integer and real additive arithmetic by working with *weak deterministic automata*, which are a restricted form of infinite-word automata admitting an easy complementation algorithm [2].

The contribution of this paper is to introduce a direct algorithm for computing the effect of universal first-order quantification over infinite-word automata. This is an essential step towards practical decision procedures for more expressive fragments mixing arithmetic with uninterpreted symbols. The considered automata are those that recognize models of formulas over natural numbers encoded in unary notation. This algorithm does not rely on complementation, and can be implemented straightforwardly on unrestricted infinite-word automata. As an example of its potential applications, this algorithm leads to a practically implementable decision procedure for the first-order theory of natural numbers with the order relation and uninterpreted unary predicates. It also paves the way to a decision procedure for SMT solvers for the UFIDL (Uninterpreted Functions and Integer Difference Logic) logic with only unary predicates.

2 Basic notions

2.1 Logic

We address the problem of deciding satisfiability for formulas expressed in first-order structures of the form $(\mathbb{N}, R_1, R_2, \ldots)$, where $\mathbb{N}$ is the *domain* of natural

numbers, and R_1, R_2, ... are (interpreted) *relations* over tuples of values in $\mathbb{N}$. More precisely, each R_i is defined as a relation $R_i \subseteq \mathbb{N}^{\alpha_i}$ for some $\alpha_i > 0$ called the *arity* of R_i.

The formulas in such a structure involve *first-order variables* x_1, x_2, ..., and *second-order variables* X_1, X_2, ... Formulas are recursively defined as

- $\top$, $\bot$, $x_i = x_j$, $X_i = X_j$, $X_i(x_j)$ or $R_i(x_{j_1}, \ldots, x_{j_{\alpha_i}})$, where $i, j, j_1, j_2, \ldots \in \mathbb{N}_{>0}$ (*atomic formulas*),
- $\varphi_1 \wedge \varphi_2$, $\varphi_1 \vee \varphi_2$ or $\neg\varphi$, where φ_1, φ_2 and φ are formulas, or
- $\exists x_i\, \varphi$ or $\forall x_i\, \varphi$, where φ is a formula.

We write $\varphi(x_1, \ldots, x_k, X_1, \ldots, X_\ell)$ to express that $x_1, \ldots, x_k, X_1, \ldots, X_\ell$ are the free variables of φ, i.e., that φ does not involve other unquantified variables.

An *interpretation* I for a formula $\varphi(x_1, \ldots, x_k, X_1, \ldots, X_\ell)$ is an assignment of values $I(x_i) \in \mathbb{N}$ for all $i \in [1, k]$ and $I(X_j) \subseteq \mathbb{N}$ for all $j \in [1, \ell]$ to its free variables. An interpretation I that makes φ true, which is denoted by $I \models \varphi$, is called a *model* of φ.

The semantics is defined in the usual way. One has

- $I \models \top$ and $I \not\models \bot$ for every I.
- $I \models x_i = x_j$ and $I \models X_i = X_j$ iff (respectively) $I(x_i) = I(x_j)$ and $I(X_i) = I(X_j)$.
- $I \models X_i(x_j)$ iff $I(x_j) \in I(X_i)$.
- $I \models R_i(x_{j_1}, \ldots, x_{j_{\alpha_i}})$ iff $(I(x_{j_1}), \ldots, I(x_{j_{\alpha_i}})) \in R_i$.
- $I \models \varphi_1 \wedge \varphi_2$, $I \models \varphi_1 \vee \varphi_2$ and $I \models \neg\varphi$ iff (respectively) $(I \models \varphi_1) \wedge (I \models \varphi_2)$, $(I \models \varphi_1) \vee (I \models \varphi_2)$, and $I \not\models \varphi$.
- $I \models \exists x_i\, \varphi(x_1, \ldots, x_k, X_1, \ldots, X_\ell)$ iff there exists $n \in \mathbb{N}$ such that $I[x_i = n] \models \varphi(x_1, \ldots, x_k, X_1, \ldots, X_\ell)$.
- $I \models \forall x_i\, \varphi(x_1, \ldots, x_k, X_1, \ldots, X_\ell)$ iff for every $n \in \mathbb{N}$, one has $I[x_i = n] \models \varphi(x_1, \ldots, x_k, X_1, \ldots, X_\ell)$.

In the two last rules, the notation $I[x_i = n]$, where $n \in \mathbb{N}$, stands for the extension of the interpretation I to one additional first-order variable x_i that takes the value n, i.e., the interpretation such that $I[x_i = n](x_j) = I(x_j)$ for all $j \in [1, k]$ such that $j \neq i$, $I[x_i = n](x_i) = n$, and $I[x_i = n](X_j) = I(X_j)$ for all $j \in [1, \ell]$.

A formula is said to be *satisfiable* if it admits a model.

2.2 Automata

A finite-word or infinite-word automaton is a tuple $\mathcal{A} = (\Sigma, Q, \Delta, Q_0, F)$ where Σ is a finite *alphabet*, Q is a finite set of *states*, $\Delta \subseteq Q \times (\Sigma \cup \{\varepsilon\}) \times Q$ is a *transition relation*, $Q_0 \subseteq Q$ is a set of *initial states*, and $F \subseteq Q$ is a set of *accepting states*.

A *path* of $\mathcal{A}$ from q_0 to q_m, with $q_0, q_m \in Q$ and $m \geq 0$, is a finite sequence $\pi = (q_0, a_0, q_1); (q_1, a_1, q_2); \ldots; (q_{m-1}, a_{m-1}, q_m)$ of transitions from Δ. The finite word $w \in \Sigma^*$ *read* by π is $w = a_0 a_1 \ldots a_{m-1}$; the existence of such a path

is denoted by $q_0 \xrightarrow{w} q_m$. A *cycle* is a non-empty path from a state to itself. If $\mathcal{A}$ is a finite-word automaton, then a path π from q_0 to q_m is *accepting* if $q_m \in F$. A word $w \in \Sigma^*$ is *accepted* from the state q_0 if there exists an accepting path originating from q_0 that reads w.

For infinite-word automata, we use a Büchi acceptance condition for the sake of simplicity, but the results of this paper straightforwardly generalize to other types of infinite-word automata. If $\mathcal{A}$ is an infinite-word automaton, then a *run* of $\mathcal{A}$ from a state $q_0 \in Q$ is an infinite sequence $\sigma = (q_0, a_0, q_1); (q_1, a_1, q_2); \ldots$ of transitions from Δ. This run reads the infinite word $w = a_0 a_1 \ldots \in \Sigma^\omega$. The run σ is *accepting* if the set $inf(\sigma)$ formed by the states q_i that occur infinitely many times in σ is such that $inf(\sigma) \cap F \neq \emptyset$, i.e., there exists a state in F that is visited infinitely often by σ. A word $w \in \Sigma^\omega$ is *accepted* from the state $q_0 \in Q$ if there exists an accepting run from q_0 that reads w.

For both finite-word and infinite-word automata, a word w is *accepted* by $\mathcal{A}$ if it is accepted from an initial state $q_0 \in Q_0$. The set of all words accepted from a state $q \in Q$ (resp. by $\mathcal{A}$) forms the *language* $L(\mathcal{A}, q)$ accepted from q (resp. $L(\mathcal{A})$ accepted by $\mathcal{A}$). An automaton accepting $L(\mathcal{A}, q)$ can be derived from $\mathcal{A}$ by setting Q_0 equal to $\{q\}$. The language of finite-words w read by paths from q_1 to q_2, with $q_1, q_2 \in Q$, is denoted by $L(\mathcal{A}, q_1, q_2)$; a finite-word automaton accepting this language can be obtained from $\mathcal{A}$ by setting Q_0 equal to $\{q_1\}$ and F equal to $\{q_2\}$. A language is said to be *regular* (resp. *ω-regular*) if it can be accepted by a finite-word (resp. an infinite-word) automaton.

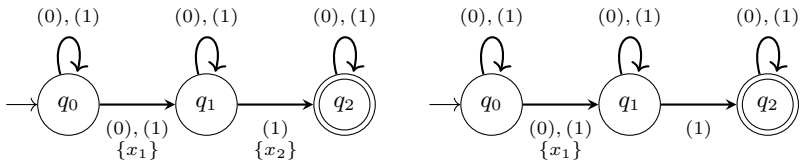
3 Deciding Satisfiability

3.1 Encoding Interpretations

In order to decide whether a formula $\varphi(x_1, \ldots, x_k, X_1, \ldots, X_\ell)$ is satisfiable, Büchi introduced the idea of building an automaton that accepts the set of all models of φ, encoded in a suitable way, and then checking whether it accepts a non-empty language [5, 6].

A simple encoding scheme consists in representing the value of first-order variables x_i in *unary notation*: A number $n \in \mathbb{N}$ is encoded by the infinite word $0^n 1 0^\omega$ over the alphabet $\{0, 1\}$, i.e., by a word in which the symbol 1 occurs only once, at the position given by n. This leads to a compatible encoding scheme for the values of second-order variables X_j: a predicate $P \subseteq \mathbb{N}$ is encoded by the infinite word $a_0 a_1 a_2 \ldots$ such that for every $n \in \mathbb{N}$, $a_n \in \{0, 1\}$ satisfies $a_n = 1$ iff $n \in P$, i.e., if $P(n)$ holds.

Encodings for the values of first-order variables $x_1, \ldots, x_k$ and second-order variables $X_1, \ldots, X_\ell$ can be combined into a single word over the alphabet $\Sigma = \{0, 1\}^{k+\ell}$: A word $w \in \Sigma^\omega$ encodes an interpretation I for those variables iff $w = (a_{0,1}, \ldots, a_{0,k+\ell})(a_{1,1}, \ldots, a_{1,k+\ell}) \ldots$, where for each $i \in [1, k]$, $a_{0,i} a_{1,i} \ldots$ encodes $I(x_i)$, and for each $j \in [1, \ell]$, $a_{0,k+j} a_{1,k+j} \ldots$ encodes $I(X_j)$. Note that not all infinite words over Σ form valid encodings: For each first-order variable x_i, an encoding must contain exactly one occurrence of the symbol 1 for the i-th



(a) $x_1 < x_2 \wedge X_1(x_2)$ (b) $\exists x_2(x_1 < x_2 \wedge X_1(x_2))$

Fig. 1: Automata recognizing sets of models.

component of its tuple symbols. Assuming that the set of variables is clear from the context, we write $e(I)$ for the encoding of I with respect to those variables.

3.2 Automata Recognizing Sets of Models

Let S be a set of interpretations for k first-order and ℓ second-order variables. The set of encodings of the elements of S forms a language L over the alphabet $\{0,1\}^{k+\ell}$. If this language is ω-regular, then we say that an automaton $\mathcal{A}$ that accepts L *recognizes*, or *represents*, the set S. Such an automaton can be viewed as having $k+\ell$ input tapes reading symbols from $\{0,1\}$, each of these tapes being associated to a variable. Equivalently, we can write the label of a transition $(q_1, (a_1, \ldots, a_{k+\ell}), q_2) \in \Delta$ as $\overset{(a_{k+1},\ldots,a_{k+\ell})}{V}$ where V is the set of the variables x_i, with $i \in [1,k]$, for which $a_i = 1$. In other words, each transition label distinct from ε specifies the set of first-order variables whose value corresponds to this transition, and provides one symbol for each second-order variable. For each $x_i \in V$, we then say that x_i is *associated* to the transition. Note that every transition for which $V \neq \emptyset$ can only be followed at most once in an accepting run. Any automaton recognizing a set of valid encodings can therefore easily be transformed into one in which such transitions do not appear in cycles, and that accepts the same language.

An example of an automaton recognizing the set of models of the formula $\varphi(x_1, x_2, X_1) = x_1 < x_2 \wedge X_1(x_2)$ is given in Figure 1a. For the sake of clarity, labels of transitions sharing the same origin and destination are grouped together, and empty sets of variables are omitted.

3.3 Decision Procedure

For the automata-based approach to be applicable, it must be possible to construct elementary automata recognizing the models of atomic formulas. This is clearly the case for atoms of the form $x_i = x_j$, $X_i = X_j$ and $X_i(x_j)$, and this property must also hold for each relation R_i that belongs to the structure of interest; in other words, the atomic formula $R_i(x_1, x_2, \ldots, x_{\alpha_i})$ must admit a set of models whose encoding is ω-regular. With the positional encoding of natural numbers, this is the case in particular for the order relation $x_i < x_j$ and

the successor relation $x_j = x_i + 1$. Note that one can easily add supplementary variables to an automaton, by inserting a new component in the tuples of its alphabet, and making this component read a symbol 1 at any single position of a run for first-order variables, and any symbol at any position for second-order ones. Reordering the variables is a similarly immediate operation.

After automata recognizing the models of atomic formulas have been obtained, the next step consists in combining them recursively by following the syntactic structure of the formula to be decided. Let us denote by L_φ the language of encodings of all the models of a formula φ, i.e., $L_\varphi = \{e(I) \mid I \models \varphi\}$.

For the Boolean operator $\wedge$, we have $L_{\varphi_1 \wedge \varphi_2} = L_{\varphi_1} \cap L_{\varphi_2}$, where φ_1 and φ_2 are formulas over the same free variables. Similarly, we have $L_{\varphi_1 \vee \varphi_2} = L_{\varphi_1} \cup L_{\varphi_2}$. The case of the complement operator $\neg$ is slightly more complicated, since the complement of a language of encodings systematically contains words that do not validly encode an interpretation. The set of models of a formula $\neg\varphi$ is encoded by the language $\overline{L_\varphi} \cap L_{valid}$, where L_{valid} is the language of all valid encodings consistent with the free variables of φ. It is easily seen that this language is ω-regular.

It remains to compute the effect of quantifiers. The language $L_{\exists x_i \varphi}$ can be derived from L_φ by removing the i-th component from each tuple symbol, i.e., by applying a mapping $\Pi_{\neq i} : \Sigma^{k+\ell} \to \Sigma^{k+\ell-1} : (a_1, \ldots, a_{k+\ell}) \mapsto (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{k+\ell})$ to each symbol of the alphabet. Indeed, the models of $\exists x_i \, \varphi$ correspond exactly to the models of φ in which the variable x_i is removed. In the rest of this paper, we will use the notation $\Pi_{\neq i}(w)$, where w is a finite or infinite word, to express the result of applying $\Pi_{\neq i}$ to each symbol in w. If L is a language, then we write $\Pi_{\neq i}(L)$ for the language $\{\Pi_{\neq i}(w) \mid w \in L\}$.

Finally, universal quantification can be reduced to existential quantification: For computing $L_{\forall x_i \varphi}$, we use the equivalence $\forall x_i \, \varphi \equiv \neg \exists x_i \, \neg\varphi$ which yields $L_{\forall x_i \varphi} = \overline{L_{\exists x_i \neg \varphi}} \cap L_{valid}$.

3.4 Operations over Automata

We now discuss how the operations over languages mentioned in Section 3.3 can be computed over infinite-word automata. Given automata $\mathcal{A}_1$ and $\mathcal{A}_2$, automata $\mathcal{A}_1 \cap \mathcal{A}_2$ and $\mathcal{A}_1 \cup \mathcal{A}_2$ accepting respectively $L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$ and $L(\mathcal{A}_1) \cup L(\mathcal{A}_2)$ can be obtained by the so-called product construction. The idea consists in building an automaton $\mathcal{A}$ that simulates the combined behavior of $\mathcal{A}_1$ and $\mathcal{A}_2$ on identical input words. The states of $\mathcal{A}$ need to store additional information about the accepting states that are visited in $\mathcal{A}_1$ and $\mathcal{A}_2$. For $\mathcal{A}_1 \cap \mathcal{A}_2$, one ensures that each accepting run of $\mathcal{A}$ correspond to an accepting run in both $\mathcal{A}_1$ and $\mathcal{A}_2$. For $\mathcal{A}_1 \cup \mathcal{A}_2$, the condition is that the run should be accepting in $\mathcal{A}_1$ or $\mathcal{A}_2$, or both. A complete description of the product construction for Büchi automata is given in [16].

Modifying the alphabet of an automaton in order to implement the effect of an existential quantification is a simple operation. As an example, Figure 1b shows an automaton recognizing the set of models of $\exists x_2(x_1 < x_2 \wedge X_1(x_2))$, obtained by removing all occurrences of the variable x_2 from transition labels.

Testing whether the language accepted by an automaton is not empty amounts to checking the existence of a reachable cycle that visits at least one accepting state, which is simple as well.

The only problematic operation is complementation, which consists in computing from an automaton $\mathcal{A}$ an automaton that accepts the language $\overline{L(\mathcal{A})}$. Although it preserves ω-regularity, this operation is difficult to perform on Büchi automata [13, 17]. In the context of our decision procedure, it is only useful for applying universal quantifiers. Indeed, other instances of the negation operator in formulas can be pushed inwards until they are applied to atomic formulas, and it is easy to construct the complement of the elementary automata recognizing the models of those atomic formulas, provided that for each relation R_i in the structure of interest, an automaton recognizing $\{(x_1, \ldots, x_{\alpha_i}) \in \mathbb{N}^{\alpha_i} \mid \neg R_i(x_1, \ldots, x_{\alpha_i})\}$ is available. In order to eliminate the need for complementation, we develop in the next section a direct algorithm for computing the effect of universal quantifiers on automata recognizing sets of models.

4 Universal Quantification

4.1 Principles

Let $\varphi(x_1, \ldots, x_k, X_1, \ldots, X_\ell)$, with $k > 0$ and $\ell \geq 0$, be a formula. Our goal is to compute an automaton $\mathcal{A}'$ accepting $L_{\forall x_i \varphi}$, given an automaton $\mathcal{A}$ accepting L_φ and $i \in [1, k]$.

By definition of universal quantification, we have $I \models \forall x_i \, \varphi$ iff $I[x_i = n] \models \varphi$ holds for every $n \in \mathbb{N}$. In other words, $L_{\forall x_i \varphi}$ contains $e(I)$ iff L_φ contains $e(I[x_i = n])$ for every $n \in \mathbb{N}$. Conceptually, we can then obtain $L_{\forall x_i \varphi}$ by defining for each $n \in \mathbb{N}$ the language $S_n = \{e(I) \mid e(I[x_i = n]) \in L_\varphi\}$, which yields $L_{\forall x_i \varphi} = \bigcap_{n \in \mathbb{N}} S_n$.

An automaton $\mathcal{A}'$ accepting $L_{\forall x_i \varphi}$ can be obtained as follows. Each language S_n, with $n \in \mathbb{N}$, is accepted by an automaton $\mathcal{A}_n$ derived from $\mathcal{A}$ by restricting the transitions associated to x_i to be followed only after having read exactly n symbols. In other words, the accepting runs of $\mathcal{A}_n$ correspond to the accepting runs of $\mathcal{A}$ that satisfy this condition. After imposing this restriction, the variable x_i is removed from the set of variables managed by the automaton, i.e., the operator $\Pi_{\neq i}$ is applied to the language that this automaton accepts, so as to get $S_n = L(\mathcal{A}_n)$. The automaton $\mathcal{A}'$ then corresponds to the infinite intersection product of the automata $\mathcal{A}_n$ for all $n \in \mathbb{N}$, i.e., an automaton that accepts the infinite intersection $\bigcap_{n \in \mathbb{N}} S_n$. We show in the next section how to build $\mathcal{A}'$ by means of a finite computation.

4.2 Construction

The idea of the construction is to make $\mathcal{A}'$ simulate the join behavior of the automata $\mathcal{A}_n$, for all $n \in \mathbb{N}$, on the same input words. This can be done by making each state of $\mathcal{A}'$ correspond to one state q_n in each $\mathcal{A}_n$, i.e., to an infinite

tuple $(q_0, q_1, \ldots)$. By definition of $\mathcal{A}_n$, there exists a mapping $\mu : Q_n \to Q$, where Q_n and Q are respectively the sets of states of $\mathcal{A}_n$ and $\mathcal{A}$, such that whenever a run of $\mathcal{A}_n$ visits q_n, the corresponding run of $\mathcal{A}$ on the same input word visits $\mu(q_n)$.

If two automata $\mathcal{A}_{n_1}$ and $\mathcal{A}_{n_2}$, with $n_1, n_2 \in \mathbb{N}$, are (respectively) in states q_{n_1} and q_{n_2} such that $\mu(q_{n_1}) = \mu(q_{n_2})$, then they share the same future behaviors, except for the requirement to follow a transition associated to x_i after having read (respectively) n_1 and n_2 symbols. It follows that the states of $\mathcal{A}'$ can be characterized by sets of states of $\mathcal{A}$: The infinite tuple $(q_0, q_1, \ldots)$ is described by the set $\{\mu(q_i) \mid i \in \mathbb{N}\}$. Each element of this set represents the current state of one or several automata among the $\mathcal{A}_n$. This means that the number of these automata that are in this current state is not counted. We will establish that this abstraction is precise and leads to a correct construction.

During a run of $\mathcal{A}'$, each transition with a label other than ε must correspond to a transition reading the same symbol in every automaton $\mathcal{A}_n$, which in turn can be mapped to a transition of $\mathcal{A}$. In the automaton $\mathcal{A}_n$ for which n is equal to the number of symbols already read during the run, this transition of $\mathcal{A}$ is necessarily associated to x_i, by definition of $\mathcal{A}_n$. It follows that every transition of $\mathcal{A}'$ with a non-empty label is characterized by a set of transitions of $\mathcal{A}$, among which one of them is associated to x_i.

We are now ready to describe formally the construction of $\mathcal{A}'$, leaving for the next section the problem of determining which of its runs should be accepting or not: From the automaton $\mathcal{A} = (\Sigma, Q, \Delta, Q_0, F)$, we construct $\mathcal{A}' = (\Sigma', Q', \Delta', Q'_0, F')$ such that

- $\Sigma' = \Pi_{\neq i}(\Sigma)$.
- $Q' = 2^Q \setminus \{\emptyset\}$.
- Δ' contains
  - the transitions $(q'_1, (a'_1, \ldots, a'_{k+\ell-1}), q'_2)$ for which there exists a set $T \subseteq \Delta$ that satisfies the following conditions:
    - $q'_1 = \{q_1 \mid (q_1, (a_1, \ldots, a_{k+\ell}), q_2) \in T\}$.
    - $q'_2 = \{q_2 \mid (q_1, (a_1, \ldots, a_{k+\ell}), q_2) \in T\}$.
    - For all $(q_1, (a_1, \ldots, a_{k+\ell}), q_2) \in T$, one has $a'_j = a_j$ for all $j \in [1, i-1]$, and $a'_j = a_{j+1}$ for all $j \in [i, k+\ell-1]$.
    - There exists exactly one $(q_1, (a_1, \ldots, a_{k+\ell}), q_2) \in T$ such that $a_i = 1$.
  - the transitions $(q'_1, \varepsilon, q'_2)$ for which there exists a transition $(q_1, \varepsilon, q_2) \in \Delta$ such that
    - $q_1 \in q'_1$.
    - $q'_2 = q'_1 \cup \{q_2\}$ or $q'_2 = (q'_1 \setminus \{q_1\}) \cup \{q_2\}$.
- $Q'_0 = 2^{Q_0} \setminus \{\emptyset\}$.
- $F' = Q'$ for now. The problem of characterizing more finely the accepting runs will be addressed in the next section.

The rule for the transitions $(q'_1, (a'_1, \ldots, a'_{k+\ell-1}), q'_2)$ ensures that for each $q_1 \in q'_1$, each automaton $\mathcal{A}_n$ that is simulated by $\mathcal{A}'$ has the choice of following any possible transition originating from q_1 that has a label consistent with

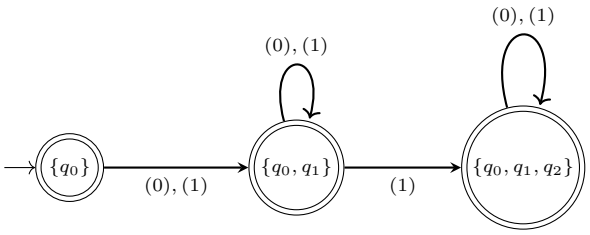


Fig. 2: First step of construction for $\forall x_1 \exists x_2 (x_1 < x_2 \wedge X_1(x_2))$.

$(a'_1, \ldots, a'_{k+\ell-1})$. One such automaton must nevertheless follow a transition associated to the quantified variable x_i. The rule for the transitions $(q'_1, \varepsilon, q'_2)$ expresses that one automaton $\mathcal{A}_n$, or any number of identical copies of this automaton, must follow a transition labeled by ε, while the other automata stay in their current state.

As an example, applying this construction to the automaton in Figure 1b, as a first step of the computation of a representation of $\forall x_1 \exists x_2 (x_1 < x_2 \wedge X_1(x_2))$, yields the automaton given in Figure 2. For the sake of clarity, unreachable states and states from which the accepted language is empty are not depicted.

4.3 A Criterion for Accepting Runs

The automaton $\mathcal{A}'$ defined in the previous section simulates an infinite combination of automata $\mathcal{A}_n$, for all $n \in \mathbb{N}$. By construction, every accepting run of this infinite combination corresponds to a run of $\mathcal{A}'$.

The reciprocal property is not true, in the sense that there may exist a run of $\mathcal{A}'$ that does not match an accepting run of the infinite combination of automata $\mathcal{A}_n$. Consider for instance a run of the automaton in Figure 2 that ends up cycling in the state $\{q_0, q_1, q_2\}$, reading 0^ω from that state. Recall that for this example, the automaton $\mathcal{A}$ that undergoes the universal quantification operation is the one given in Figure 1b. The run that we have considered can be followed in $\mathcal{A}'$, but cannot be accepting in every $\mathcal{A}_n$. Indeed, in this example, the transition of $\mathcal{A}_n$ reading the $(n+1)$-th symbol of the run corresponds, by definition of this automaton, to the transition of $\mathcal{A}$ that is associated to the quantified variable x_1. By the structure of $\mathcal{A}$, this transition is necessarily followed later in any accepting run by one that reads the symbol 1, which implies that no word of the form $u \cdot 0^\omega$, with $u \in \{0,1\}^*$, can be accepted by a run of $\mathcal{A}_n$ such that $n \geq |u|$. This represents the fact that the words accepted by all $\mathcal{A}_n$ correspond to the encodings of predicates that are true infinitely often.

One thus needs a criterion for characterizing the runs of $\mathcal{A}'$ that correspond to combinations of accepting runs in all automata $\mathcal{A}_n$.

It is known [11] that two ω-regular languages over the alphabet Σ are equal iff they share the same set of *ultimately periodic* words, i.e., words of the form $u \cdot v^\omega$ with $u \in \Sigma^*$ and $v \in \Sigma^+$. It follows that it is sufficient to characterize the accepting runs of $\mathcal{A}'$ that read ultimately periodic words. The automaton

$\mathcal{A}'$ accepts a word $u \cdot v^\omega$ iff every $\mathcal{A}_n$, with $n \in \mathbb{N}$, admits an accepting run that reads this word. Note that such a run also matches a run of $\mathcal{A}$, and that this run of $\mathcal{A}$ always ends up following a cycle from an accepting state to itself.

Our solution takes the following form. For each state q of $\mathcal{A}$, we define a language $U_q \subseteq \Sigma^+$ of non-empty words u such that $\mathcal{A}$ accepts u^ω from q, after dismissing the input tape associated to the quantified variable x_i. The alphabet Σ is thus equal to $\{0,1\}^{k+\ell-1}$. Remember that each state q' of $\mathcal{A}'$ is defined as a subset of states of $\mathcal{A}$, corresponding to the current states in the combination of copies of $\mathcal{A}$ that are jointly simulated by $\mathcal{A}'$. In order for the word u^ω to be accepted by $\mathcal{A}'$ from q', it should therefore be accepted by $\mathcal{A}$ from each state $q \in q'$, i.e., u must belong to all the languages U_q such that $q \in q'$.

It must also be possible to read u^ω from the state q' of $\mathcal{A}'$. We impose a stronger condition, by requiring that there exists a cycle from q' to itself labeled by u. This condition leads to a correct acceptance criterion.

In summary, the language $U'_{q'} = L(\mathcal{A}', q', q') \cap \bigcap_{q \in q'} U_q$ characterizes the words u such that u^ω must be accepted from the state q' of $\mathcal{A}'$. Note that for this property to hold, it is not necessary for the language U_q to contain all words u such that $u^\omega \in \Pi_{\neq i}(L(\mathcal{A}, q))$, but only some number of copies u^p, where $p > 0$ is bounded, of each such u. In other words, the finite words u whose infinite repetition is accepted from q do not have to be the shortest possible ones.

Once the language $U'_{q'}$ has been obtained, we build a *widget*, in the form of an infinite-word automaton accepting $(U'_{q'})^\omega$, along the state q' of $\mathcal{A}'$, and add a transition labeled by ε from q' to the initial state of this widget. This ensures that every path that ends up in q' can be suitably extended into an accepting run. Such a widget does not have to be constructed for every state q' of $\mathcal{A}'$: Since the goal is to accept from q' words of the form u^ω, we can require that at least one state $q \in q'$ is accepting in $\mathcal{A}$. We then only build widgets for the states q' that satisfy this requirement.

4.4 Computation Steps

The procedure for modifying $\mathcal{A}'$ in order to make it accept the runs that match those of the infinite combination of automata $\mathcal{A}_n$, outlined in the previous section, can be carried out by representing the regular languages U_q and $U'_{q'}$ by finite-state automata. The construction proceeds as follows:

1. For each state $q \in Q$ of $\mathcal{A}$, build a finite-word automaton $\mathcal{A}_q$ that accepts all the non-empty words u for which there exists a path $q \xrightarrow{v} q$ of $\mathcal{A}$ that visits at least one accepting state $q_F \in F$, such that $u = \Pi_{\neq i}(v)$. This automaton can be constructed in a similar way as one accepting $\Pi_{\neq i}(L(\mathcal{A}, q, q))$ (cf. Sections 2.2 and 3.4), keeping one additional bit of information in its states for determining whether an accepting state has already been visited or not.
2. For each pair of states $q_1, q_2 \in Q$ of $\mathcal{A}$, build a finite-word automaton $\mathcal{A}_{q_1,q_2}$ accepting the language $\Pi_{\neq i}(L(\mathcal{A}, q_1, q_2))$ (cf. Sections 2.2 and 3.4).
3. For each state $q \in Q$ of $\mathcal{A}$, build an automaton $\mathcal{A}_{U_q} = \bigcup_{r \in Q} (\mathcal{A}_{q,r} \cap \mathcal{A}_r)$ accepting the finite-word language U_q.

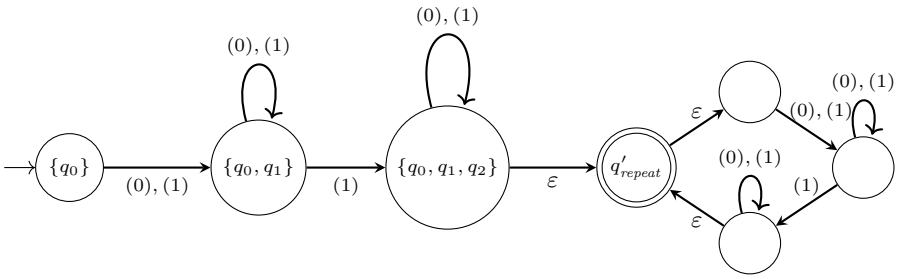


Fig. 3: Automaton recognizing the set of models of $\forall x_1 \exists x_2 (x_1 < x_2 \wedge X_1(x_2))$.

4. For each state q' of $\mathcal{A}'$ such that $q' \cap F \neq \emptyset$, where F is the set of accepting states of $\mathcal{A}$, build a finite-word automaton $\mathcal{A}'_{U'_{q'}} = \mathcal{A}'_{q'} \cap \bigcap_{q \in q'} \mathcal{A}_{U_q}$ accepting $U'_{q'}$, where $\mathcal{A}'_{q'}$ is an automaton accepting $L(\mathcal{A}', q', q')$ (cf. Section 2.2).
5. Then, turn each automaton $\mathcal{A}'_{U'_{q'}}$ into an infinite-word automaton $\mathcal{A}'_{(U'_{q'})^\omega}$ accepting $(U'_{q'})^\omega$:
   (a) Create a new state q'_{repeat}.
   (b) Add a transition $(q'_{repeat}, \varepsilon, q_0)$ for each initial state q_0, and a transition $(q_F, \varepsilon, q'_{repeat})$ for each accepting state q_F, of $\mathcal{A}'_{U'_{q'}}$.
   (c) Make q'_{repeat} the only initial and accepting state of $\mathcal{A}'_{(U'_{q'})^\omega}$.
6. For each state q' of $\mathcal{A}'$ considered at Step 4, add the widget $\mathcal{A}'_{U'_{q'}}$ alongside q', by incorporating its sets of states and transitions into those of $\mathcal{A}'$, and adding a transition $(q', \varepsilon, q'_{repeat})$. In the resulting automaton, mark as the only accepting states the states q'_{repeat} of all widgets.

This procedure constructs an automaton that accepts the language $L_{\forall x_i \varphi}$. Applied to the automaton $\mathcal{A}'$ in Figure 2, it produces the result shown in Figure 3. For the sake of clarity, the states from which the accepted language is empty have been removed. A detailed description of the computation steps for this example and the proof of correctness of the construction are given in [1].

5 Conclusions

This paper introduces a method for directly computing the effect of a first-order universal quantifier on an infinite-word automaton recognizing the set of models of a formula. It is applicable when the first-order variables range over the natural numbers and their values are encoded in unary notation. Among its potential applications, it provides a solution for deciding the first-order theory $\langle \mathbb{N}, < \rangle$ extended with uninterpreted unary predicates.

The operation on regular languages that corresponds to the effect of a universal first-order quantifier has already been studied at the theoretical level [12]. Our contribution is to provide a practical algorithm for computing it, that does not

require to complement infinite-word automata. This algorithm has an exponential worst-case time complexity, which is unavoidable since there exist automata for which universal quantification incurs an exponential blowup in their number of states (see [1] for details). The main advantage over the complementation-based approach is however that this exponential cost is not systematic, since only a fraction of the possible subsets of states typically need to be constructed.

Our solution is open to many possible improvements, one of them being to extend the algorithm so as to quantify over several first-order variables in a single operation. For future work, we plan to generalize this algorithm to automata over more expressive structures, such as the automata over linear orders defined in [4]. This would make it possible to obtain an implementable decision procedure for, e.g., the first-order theory $\langle \mathbb{R}, < \rangle$ with uninterpreted unary predicates [10].

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