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Security Analysis of SDiTH

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- 1 Security Analysis in NIST specification
- 2 Correction of the analysis
 - First problem
 - Second problem
 - Third problem
- 3 Improved attack with projective space

Syndrom Decoding problem

Let us recall the Syndrome Decoding problem

Problem (Syndrome Decoding $SD(\mathbf{H}, \mathbf{s}, t)$)

Given a matrix $\mathbf{H} \in \mathbb{F}_q^{(n-k) \times n}$, a syndrome $\mathbf{s} \in \mathbb{F}_q^{n-k}$ and a distance $t \in \llbracket 0, n \rrbracket$, one wants to find a vector $\mathbf{e} \in \mathbb{F}_q^n$ such that $\mathbf{H}\mathbf{e} = \mathbf{s}$ and $|\mathbf{e}| = t$.

SDiTH security

Based on d-split syndrom decoding problem (a key recovery attack on SDiTH is as hard as solving the d-split Syndrome Decoding problem)

Problem (d-split Syndrome Decoding $SD(d, \mathbf{H}, \mathbf{s}, t)$)

Given a matrix $\mathbf{H} \in \mathbb{F}_q^{(n-k) \times n}$, a syndrome $\mathbf{s} \in \mathbb{F}_q^{n-k}$ and a distance $t \in \llbracket 0, n \rrbracket$, one wants to find a vector $\mathbf{e} = (\mathbf{e}_1, \dots, \mathbf{e}_d) \in (\mathbb{F}_q^{n/d})^d$ such that $\mathbf{H}\mathbf{e} = \mathbf{s}$ and $\forall i \in \llbracket 1, d \rrbracket, |\mathbf{e}_i| = t/d$.

SDiTH framework

Solve the $SD(d, \mathbf{H}, \mathbf{s}, t)$ problem with :

- a random \mathbf{H}
- \mathbf{s} from an injected solution ($\mathbf{s} = \mathbf{H}\mathbf{e}$ such that $|\mathbf{e}_i| = t/d$)

Reduce $SD(d)$ to $SD(1)$?

In the specification, they use [Feneuil,Joux,Rivain,2022] results to obtain the following bound

Reduction from $SD(d)$ to $SD(1)$

$$T_{SD(d)} \geq \frac{\binom{n/d}{t/d}^d}{\binom{n}{t}} T_{SD(1)}$$

Idea :

- Consider an algorithm \mathcal{A} which solves $SD(d)$
- Let (\mathbf{H}, s, t) be an instance of $SD(1)$
- Run \mathcal{A} on $(\mathbf{H}_\sigma, s_\sigma, t)$ with a random permutation at each step
- There is a probability $\frac{\binom{n/d}{t/d}^d}{\binom{n}{t}}$ of obtaining an instance of $SD(d)$ at each step.

Problem

This bound is tight only when $t \rightarrow 0$ (we will see why)

How they analyze security

What algorithm used to solve SD(1) ?

- 1 ISDs (Information Set Decoding) as usual
- 2 In the case of SDiTH the field size is big $q = 256$ or $q = 251$
- 3 They claim that it is enough to stop at Stern's algorithm [Stern,1989]
- 4 Asymptotically with n and q [May,Meurer,Thomae,2011] and [Becker,Joux,May,Meurer,2012] are equivalent to [Prange,1962] ([Canto Torres,2016])
- 5 But non asymptotically ?

Security assessment in the specification

- Based on Stern algorithm
- They perform non-asymptotic analysis to gain security bits
- They use Peters' tricks [Peters,2010] to optimize the non-asymptotic complexity of Stern

Stern algorithm

Input : $\mathbf{H} \in \mathbb{F}_q^{(n-k) \times n}$, $\mathbf{s} \in \mathbb{F}_q^{n-k}$ and $t \in \llbracket 0, n \rrbracket$

Output : $\mathbf{e} \in \mathbb{F}_q^n$ such that $\mathbf{H}\mathbf{e} = \mathbf{s}$ and $|\mathbf{e}| = t$

Parameters : $p \in \llbracket 0, \frac{\min(t,k)}{2} \rrbracket$ and $\ell \in \llbracket 0, n - k - t + 2p \rrbracket$

Repeat as many time as necessary :

- 1 draw $I \subseteq \llbracket 1, n \rrbracket$ of size k uniformly at random
- 2 $J \leftarrow \llbracket 1, n \rrbracket \setminus I$
- 3 $\mathbf{P} \leftarrow \mathbf{H}_J^{-1} \mathbf{H}_I$
- 4 $\mathbf{y} \leftarrow \mathbf{H}_J^{-1} \mathbf{s}$
- 5 $\mathbf{P}' \leftarrow$ the ℓ first rows of \mathbf{P} and $\mathbf{y}' \leftarrow$ the ℓ first positions of \mathbf{s}
- 6 **Bet that :**
 - $\mathbf{e}_I = (\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{F}_q^{k/2}$ such that $|\mathbf{x}_1| = |\mathbf{x}_2| = p$
 - $\mathbf{e}_J = (\mathbf{0}_\ell, \mathbf{x}_3) \in \mathbb{F}_q^\ell \times \mathbb{F}_q^{n-k-\ell}$ such that $|\mathbf{x}_3| = t - 2p$

How to find e with betting?

With the bet, we have :

$$\begin{pmatrix} \mathbf{P}'\mathbf{x}_2 \\ \mathbf{x}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{y}' - \mathbf{P}'\mathbf{x}_1 \\ \mathbf{y}'' - \mathbf{P}''(\mathbf{x}_1 + \mathbf{x}_2) \end{pmatrix}$$

- 7 $\mathcal{L}_1 \leftarrow \left\{ \mathbf{y}' - \mathbf{P}'\mathbf{x}_1 : \mathbf{x}_1 \in \mathbb{F}_q^{\lfloor k/2 \rfloor} \times 0^{k - \lfloor k/2 \rfloor} \text{ and } |\mathbf{x}_1| = p \right\}$
- $\mathcal{L}_2 \leftarrow \left\{ \mathbf{P}'\mathbf{x}_2 : \mathbf{x}_2 \in 0^{\lfloor k/2 \rfloor} \times \mathbb{F}_q^{k - \lfloor k/2 \rfloor} \text{ and } |\mathbf{x}_2| = p \right\}$
- 8 For all $(\mathbf{y}' - \mathbf{P}'\mathbf{x}_1, \mathbf{P}'\mathbf{x}_2) \in \mathcal{L}_1 \times \mathcal{L}_2$ such that $\mathbf{y}' - \mathbf{P}'\mathbf{x}_1 = \mathbf{P}'\mathbf{x}_2$
 - If $|\mathbf{P}(\mathbf{x}_1 + \mathbf{x}_2) - \mathbf{y}| = t - 2p$, return e , such that
 - $\mathbf{e}_I \stackrel{\text{def}}{=} \mathbf{x}_1 + \mathbf{x}_2$
 - $\mathbf{e}_J \stackrel{\text{def}}{=} \mathbf{y} - \mathbf{P}(\mathbf{x}_1 + \mathbf{x}_2)$

Stern complexity

$$T_{\text{SD}(1)} = \left(\frac{T_{\text{Gauss}} + T_{\text{lists}} + T_{\text{checks}}}{(1 - (1 - p_e)^{N_{\text{sol}}})} \right) \log_2(q)$$

- 1 T_{Gauss} : complexity of Gaussian elimination
- 2 T_{lists} : complexity of constructing lists
- 3 T_{checks} : complexity of collision checking
- 4 p_e : probability of finding a particular solution
- 5 N_{sol} : expected number of solutions to the problem

Peters trick to construct \mathcal{L}_1 and \mathcal{L}_2

- 1 $\mathcal{L}_1 = \left\{ \mathbf{P}'\mathbf{x}_1 - \mathbf{y}' : \mathbf{x}_1 \in \mathbb{F}_q^{\lfloor k/2 \rfloor} \text{ and } |\mathbf{x}_1| = p \right\}$ and
 $\mathcal{L}_2 = \left\{ \mathbf{P}'\mathbf{x}_2 : \mathbf{x}_2 \in \mathbb{F}_q^{k/2 - \lfloor k/2 \rfloor} \times \times \text{ and } |\mathbf{x}_2| = p \right\}$
- 2 Choose an enumeration order σ on $\left\{ \mathbf{x} \in \mathbb{F}_q^{\lfloor k/2 \rfloor} : |\mathbf{x}| = p \right\}$ such that $\mathbf{x}_{\sigma(i)}$ and $\mathbf{x}_{\sigma(i+1)}$ differ by 1 on at most two **successive** coordinates depending on the case :
 - 1 **On the same support** : $\mathbf{x}_{\sigma(i)} = (\dots, d, \dots, 0) \Rightarrow \mathbf{x}_{\sigma(i+1)} = (\dots, d+1, \dots, 0) \Rightarrow$ **One coordinate**
 - 2 **Change of support** : $\mathbf{x}_{\sigma(i)} = (\dots, q-1, 0, \dots) \Rightarrow \mathbf{x}_{\sigma(i+1)} = (\dots, 0, 1, \dots, 0) \Rightarrow$ **Two successive coordinates**
- 3 Precompute the sums of two successive columns of \mathbf{P}'
- 4 Construct an element of the list \Rightarrow one addition of one column of size ℓ (except the first which requires $p-1$ or p addition of columns depending of the list)

$$T_{\text{lists}} = \ell k + \ell(2p-1) + \ell(|\mathcal{L}_1| + |\mathcal{L}_2|) \quad (1.1)$$

Second and third tricks

The second trick comes into collision checking

$$|\mathbf{P}(\mathbf{x}_1 - \mathbf{x}_2) - \mathbf{y}| = t - 2p \quad (1.2)$$

- Verify that the Hamming weight of a certain vector of length $n - k - \ell$ equals $t - 2p$
- Remarks that the vector is actually a random vector when it does not come from a wanted solution
- On the wrong candidates, check the Hamming weight over $\frac{q}{q-1}(t - 2p + 1)$ positions on average
- Checking on each coordinate for each position : $2p$ additions, $2p \frac{q-2}{q-1}$ multiplications

$$T_{\text{checks}} = \frac{q}{q-1}(t - 2p + 1)2p \left(1 + \frac{q-2}{q-1}\right) \frac{|\mathcal{L}_1| |\mathcal{L}_2|}{q^\ell}$$

Third trick

The third trick concerns Gaussian elimination but they neglect it because they claim that its cost is negligible

Their complexity result

Let's summarize their results :

Complexité

- $L_1 := |\mathcal{L}_1| = \binom{\lfloor \frac{k}{2} \rfloor}{p} (q-1)^p$
- $L_2 := |\mathcal{L}_2| = \binom{k - \lfloor \frac{k}{2} \rfloor}{p} (q-1)^p$
- $T_{\text{Gauss}} = \frac{1}{2} (n-k)^2 (n+k)$
- $T_{\text{lists}} = \ell k + \ell (2p-1) + \ell (|\mathcal{L}_1| + |\mathcal{L}_2|)$
- $T_{\text{checks}} = \frac{q}{q-1} (t-2p+1) 2p \left(1 + \frac{q-2}{q-1}\right) \frac{|\mathcal{L}_1| |\mathcal{L}_2|}{q^\ell}$
- $p_e = \frac{L_1 L_2 \binom{n-k-\ell}{t-2p}}{\binom{n}{t} (q-1)^{2p}}$
- $N_{\text{sol}} = 1$

Practical results

Table – SDiTH security in the specification

Parameter Sets	Category	Security Specification				SD Parameters				
		p	l	$T_{SD(1)} \frac{\binom{n/d}{t/d}^d}{\binom{n}{t}}$	target	q	n	k	t	d
SDitH-L1-gf256	I	1	2	143.46	143	256	230	126	79	1
SDitH-L1-gf251	I	1	2	143.45	143	251	230	126	79	1
SDitH-L3-gf256	III	2	5	207.67	207	256	352	193	120	2
SDitH-L3-gf251	III	2	5	207.61	207	251	352	193	120	2
SDitH-L5-gf256	V	2	5	272.35	272	256	480	278	150	2
SDitH-L5-gf251	V	2	5	272.29	272	251	480	278	150	2

The choice of a non-asymptotic analysis

Advantage

Bit gain on security

Inconvenience

- Very accurate calculations \Rightarrow easy to make mistakes
- We are exposed to more attacks which bring non-asymptotic gains

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They do not take into account the N_{sol} factor

Definition

- Let $N_{\text{sol}}(d)$ denote the expected number of solutions of $\text{SD}(d)$
- $d_{\text{GV}}(n, k) \stackrel{\text{def}}{=} \inf \left(\left\{ t \in \left[\left[0, n - \frac{n}{q} \right] \right] : \binom{n}{t} (q-1)^t \geq q^{n-k} \right\} \right)$

- 1 With their parameters, they take $t = d_{\text{GV}}(n, k)$, they assume that the factor $N_{\text{sol}}(1)$ is one (we only have the injected solution)
- 2 It is only true asymptotically with n
- 3 With their range of parameters $N_{\text{sol}}(1) \in [412, 1056] \Rightarrow$ between 8.7 and 10 bits lost

They consider that Gaussian elimination is negligible

In the case of **SDitH-L1-gf256** with $q, n, k, t, d = 256, 230, 126, 79, 1$

Their optimal cases :

- $T_{\text{lists}} = 15.974459404437539$
- $T_{\text{checks}} = 20.231483911937776$
- $T_{\text{Gauss}} : 20.883072906964095$
- $T = 143.4589084369141$

The optimal case if we neglect Gaussian elimination :

- $T_{\text{lists}} = 16.559421905158693$
- $T_{\text{checks}} = 12.231483911937778$
- $T_{\text{Gauss}} : 20.883072906964095$
- $T = 140.4941924784691$

It's the opposite

- The cost of Gaussian elimination is even exactly their dominant term
- We have to consider **[Bernstein,Lange,Peters,2008]** and **[Peters,2010]** to improve T_{Gauss}

The reduction bound is not tight

They do not take into account N_{sol}

- **Without** N_{sol} : $p_e(1) \geq \frac{\binom{n/d}{t/d}^d}{\binom{n}{t}} p_e(d)$
- **With** N_{sol} : $p_e(1)N_{\text{sol}}(1) \geq p_e(d) \frac{\binom{n/d}{t/d}^d}{\binom{n}{t}} \left(1 + (N_{\text{sol}}(d) - 1) \frac{\binom{n}{t}}{\binom{n/d}{t/d}^d} \right)$

The precise reduction from $\text{SD}(d)$ to $\text{SD}(1)$ give

- 1 $N_{\text{sol}}(d) = 1 \Rightarrow p_e(1)N_{\text{sol}}(1) \geq \frac{\binom{n/d}{t/d}^d}{\binom{n}{t}} p_e(d)N_{\text{sol}}(d)$
- 2 $N_{\text{sol}}(d) \gg 1 \Rightarrow p_e(1)N_{\text{sol}}(1) \geq p_e(d)N_{\text{sol}}(d)$

$N_{\text{sol}}(d) \in \llbracket 30, 748 \rrbracket$ so we are in the second case giving

$$T_{\text{SD}(d)} \geq T_{\text{SD}(1)}$$

Consequences

Their bound is therefore optimistic, with the real bound they gain security bits

Stern algorithm adapted to $SD(d)$

it's interesting to directly solve the problem $SD(d)$ here. We therefore adapt Stern to $SD(d)$

Principle of adaptation

- 1 Split the information set I as $I_1 \cup \dots \cup I_d$ where I_i is of size $\frac{k}{d}$ and is randomly chosen in I .
- 2 Make a bet on each of them.

The fact that

$$T_I(d) \leq T_I(1)$$

makes us gain in complexity compared to the theoretical bound.

Result after correction

Table – Correction of the analysis

Parameter Sets	Security Specification			Correction			SD Parameters				
	p	l	$T_{SD(1)} \frac{\binom{n/d}{t/d}^d}{\binom{n}{t}}$	p	l	$T_{SD(d)}$	q	n	k	t	d
SDitH-L1-gf256	1	2	143.46	1	2	134.61	256	230	126	79	1
SDitH-L1-gf251	1	2	143.45	1	2	133.61	251	230	126	79	1
SDitH-L3-gf256	2	5	207.67	2	5	206.16	256	352	193	120	2
SDitH-L3-gf251	2	5	207.61	2	5	205.02	251	352	193	120	2
SDitH-L5-gf256	2	5	272.35	2	5	271.30	256	480	278	150	2
SDitH-L5-gf251	2	5	272.29	2	5	269.81	251	480	278	150	2

Remark

We have not yet corrected the complexity of Gaussian elimination, we have to consider **[Bernstein,Lange,Peters,2008]** and **[Peters,2010]**

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Observation

Suppose we are trying to solve the problem SD(1) in the case of a $\mathbf{0}$ syndrome.

We get a good solution if and only if we find \mathbf{x}_1 and \mathbf{x}_2 such that

$$\mathbf{P}'\mathbf{x}_1 = \mathbf{P}'\mathbf{x}_2 \text{ and } |\mathbf{P}(\mathbf{x}_1 - \mathbf{x}_2)| = t - 2p \quad (3.1)$$

We calculate $(q - 1)$ times too much solution

$(\mathbf{x}_1, \mathbf{x}_2)$ checks (3.1) \Leftrightarrow for all α non zero $(\alpha\mathbf{x}_1, \alpha\mathbf{x}_2)$ checks (3.1)

How to exploit that ?

Adaptation of Stern's algorithm in projective space

Impact on complexity

We divide the size of the lists by a $(q - 1)$ factor \Rightarrow we hope to gain a factor $(q - 1)$ on the complexity

What if the syndrome is non-zero ?

Reduction of the Syndrome Decoding problem to the low weight code words search

The reduction :

We look for \mathbf{e} such that $\mathbf{H}\mathbf{e} = \mathbf{s}$ and $|\mathbf{e}| = t$.

Let \mathcal{C} be the code associated with the parity check matrix \mathbf{H} .

- 1 Find \mathbf{z} such that $\mathbf{H}\mathbf{z} = \mathbf{s}$ ($\mathbf{z} = (\mathbf{0}, \mathbf{s})$)
- 2 Let \mathbf{H}_z a parity check matrix of the code $\langle \mathcal{C}, \mathbf{z} \rangle$.
- 3 Find \mathbf{e} such that $\mathbf{H}_z \mathbf{e} = \mathbf{0}$

Analysis

The algorithm finds \mathbf{e} such that $\mathbf{H}\mathbf{e} = \alpha \mathbf{s}$ (for a $\alpha \in \mathbb{F}_q$).

(So just divide by α and we solved the problem)

Proof on board

What about the complexity

- The reduction cost almost nothing (a factor $\frac{q}{q-1}$)
- We reduced the problem to dimension $k + 1 \Rightarrow$ possible loss

Notations

Projective space

For a space \mathcal{E} over \mathbb{F}_q

- $\forall \mathbf{x}, \mathbf{y} \in \mathcal{E}, \quad \mathbf{x} \sim \mathbf{y} \iff \exists \alpha \in \mathbb{F}_q^*, \mathbf{x} = \alpha \mathbf{y}$
- $[\mathbf{x}] \stackrel{\text{def}}{=} \{\mathbf{y} \in \mathcal{E} : \mathbf{y} \sim \mathbf{x}\}$
- $\mathcal{E}/\sim \stackrel{\text{def}}{=} \{[\mathbf{x}] : \mathbf{x} \in \mathcal{E}\}$

Choose the right representatives

To exploit the idea, we want to take the lists in projective space to divide their **size** as well as the **number of collisions** by $(q - 1)$:

$$\mathcal{L}'_1 = \left\{ [\mathbf{P}'\mathbf{x}_1] : [\mathbf{x}_1] \in \mathbb{F}_q^{\lfloor \frac{k+1}{2} \rfloor} / \sim \text{ and } |\mathbf{x}_1| = p \right\}$$

$$\mathcal{L}'_2 = \left\{ [\mathbf{P}'\mathbf{x}_2] : [\mathbf{x}_2] \in \mathbb{F}_q^{k+1 - \lfloor \frac{k+1}{2} \rfloor} / \sim \text{ and } |\mathbf{x}_2| = p \right\}$$

We must take precautions

- A collision of equivalence classes does not necessarily induce a collision
 $[\mathbf{P}'\mathbf{x}_1] = [\mathbf{P}'\mathbf{x}_2] \not\Rightarrow \mathbf{P}'\mathbf{x}_1 = \mathbf{P}'\mathbf{x}_2$
- Only a proportion of $\frac{1}{q-1}$ of the pairs $(\mathbf{x}'_1, \mathbf{x}'_2) \in ([\mathbf{x}_1], [\mathbf{x}_2])$ induce a collision
- We cannot do a naive search for such a couple otherwise we lose the gain of the factor $(q - 1)$

The solution

We build good representatives who induce a collision

Particular representative

For all $[x] \in \mathbb{F}_q^{\frac{k+1}{2}} / \sim$ we define \bar{x} as the unique representative of $[x]$ such that :

- If $P'x = 0$: the first non-zero symbol of \bar{x} is 1
- If $P'x \neq 0$: the first non-zero symbol of $P'\bar{x}$ is 1

We now have the desired property

Property

$$[P'x_1] = [P'x_2] \Leftrightarrow P'\bar{x}_1 = P'\bar{x}_2$$

The algorithm

The only things that change with the classic Stern algorithm are the following :

- ⑦ $\mathcal{L}'_1 \leftarrow \left\{ \mathbf{P}'\overline{\mathbf{x}}_1 : [\mathbf{x}_1] \in \left(\mathbb{F}_q^{\lfloor \frac{k+1}{2} \rfloor} \times 0^{k+1-\lfloor \frac{k+1}{2} \rfloor} \right) / \sim \text{ and } |\mathbf{x}_1| = p \right\}$
- $\mathcal{L}'_2 \leftarrow \left\{ \mathbf{P}'\overline{\mathbf{x}}_2 : [\mathbf{x}_2] \in \left(0^{\lfloor \frac{k+1}{2} \rfloor} \times \mathbb{F}_q^{k+1-\lfloor \frac{k+1}{2} \rfloor} \right) / \sim \text{ and } |\mathbf{x}_2| = p \right\}$
- ⑧ The collision checking is done with $(\overline{\mathbf{x}}_1, \overline{\mathbf{x}}_2)$

The possible losses on the theoretical gain of the factor $(q - 1)$:

problem one

We must adapt Peters' tips to our new algorithm \Rightarrow potential losses

problem two

We go from dimension k to dimension $(k + 1)$ \Rightarrow potential losses

Adaptation of tricks

The tricks on list construction :

- Use as before an enumeration order on $\left\{ \hat{\mathbf{x}} \in \mathbb{F}_q^{(k+1)/2} : \hat{\mathbf{x}} = (0, \dots, 0, 1, \dots) \text{ and } |\hat{\mathbf{x}}| = p \right\}$ to compute $\mathbf{P}'\hat{\mathbf{x}}$ in one addition of one columns by elements.
- Normalize $\mathbf{P}'\hat{\mathbf{x}}$ by its first non-zero coordinate to get $\mathbf{P}'\bar{\mathbf{x}}$ (We lose a factor of 2 here on the trick)

$$T'_{\text{lists}} \approx \frac{2}{q} T_{\text{lists}}$$

The second trick on collision checking :

We just need to normalize $\hat{\mathbf{x}}$ on collisions to get $\bar{\mathbf{x}}$ which takes $2p$ operations per collisions (negligible). We can then apply the trick as before.

$$T'_{\text{checks}} \approx \frac{1}{q} T_{\text{checks}}$$

The real gain

For the other complexity values, we have :

- $T'_{\text{Gauss}} \approx T_{\text{Gauss}}$
- $N'_{\text{sol}} \approx N_{\text{sol}}$
- $p'_e \approx p_e/4 \quad \left(\frac{\binom{n-k-1-\ell}{t-2p}}{\binom{n-k-\ell}{t-2p}} \approx \frac{1}{4} \text{ for the range of parameters} \right)$

Expected gain

- We lose 2 bits due to the transition from dimension k to dimension $k + 1$.
- We lose 1 bits if T'_{lists} control T'_{checks}
- We should have an additional gain between 5 and 6 bits with the projections

Remark

We can adapt the Stern projectif algorithm to the problem $\text{SD}(d)$ as previously

Result

Table – SDiTH security with our results

	Security Specification			Correction			d-split-Stern-proj			SD Parameters				
	p	l	$T_{SD(1)} \frac{\binom{n/d}{t/d}^d}{\binom{n}{t}}$	p	l	$T_{SD(d)}$	p	l	$T_{SD(d)}$	q	n	k	t	d
L1-gf256	1	2	143.46	1	2	134.61	2	4	130.07	256	230	126	79	1
L1-gf251	1	2	143.45	1	2	133.91	2	4	129.36	251	230	126	79	1
L3-gf256	2	5	207.67	2	5	206.16	2	4	200.73	256	352	193	120	2
L3-gf251	2	5	207.61	2	5	205.02	2	4	199.65	251	352	193	120	2
L5-gf256	2	5	272.35	2	5	271.30	2	4	266.77	256	480	278	150	2
L5-gf251	2	5	272.29	2	5	269.81	2	4	265.35	251	480	278	150	2

Table – Bit gain on security

Parameter Sets	Gains		
	Gain-Correction	Gain-proj	Total Gain
SDitH-L1-gf256	8.84	4.54	13.39
SDitH-L1-gf251	9.54	4.54	14.09
SDitH-L3-gf256	1.51	5.43	6.94
SDitH-L3-gf251	2.59	5.36	7.96
SDitH-L5-gf256	1.04	4.53	5.57
SDitH-L5-gf251	2.48	4.46	6.94

The loss of 0.5 bits on the expected results is linked to the fact that the Gaussian elimination is not negligible.

Work in coming

- 1 Improvement of T_{Gauss} with **[Bernstein,Lange,Peters,2008]** and **[Peters,2010]**
- 2 Apply the adaptation of projective spaces to **[Dumer,1991]** to see if we obtain a gain
- 3 Try to generalize the idea to the Grassmannian (generalization of projective spaces with several dimensions)