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# On ISS with respect to average value of disturbances: a time-delay approach

Efimov D., Fridman E.

**Abstract**—For input-affine nonlinear dynamical systems, an input-to-state stability analysis with respect to (weighted) average values of exogenous perturbations is proposed. The time-delay method is used to represent the system in a suitable form for investigation: a kind of neutral-type differential equation. The introduced approach allows the asymptotic gains with respect to zero-mean periodic signals to be evaluated for nonlinear systems (an analogue of Bode magnitude plot), as well as for integral input-to-state stable systems with periodic inputs. The results are illustrated on the class of homogeneous systems.

## I. INTRODUCTION

Analysis of robustness of stability properties for dynamical systems with respect to external disturbances, measurement noises, variations of parameters, unmodeled dynamics, delays, *etc.*, is one of the central issues in the control and estimation theory [1]. For linear systems all these notions have been well-studied through  $H_\infty$  norms or  $L_2$  gains in 60s of the previous century. For nonlinear systems there are several approaches for robust stability analysis, and one of the most popular frameworks is built around the input-to-state stability (ISS) concept [2]. The advantage of this theory is that it has a list of properties describing different stability characteristics, as well as efficient necessary and sufficient conditions to check these properties in applications (existence of proper Lyapunov functions are usually verified). This framework allows one to make qualitative and quantitative analysis of behavior of the system trajectories with respect to the  $L_\infty$  norm of the input (or an integral of the norm).

In some cases, the analysis should be performed not with respect to the maximal amplitude of the input, but evaluating the bounds on trajectories as a function of the frequency of the disturbance, or its average value on an interval of time (see the respective extensions of the ISS framework [3], [4]). This is often the case in vibration analysis, or when the external inputs are (almost) periodic functions of time. In such a scenario, the time-averaging approach was found to be very efficient [5], and later its different extensions have been proposed [6], [7], [8], also in conjunction with ISS [9], [10] (where the stability property of the system is substantiated from the characteristics of the averaged dynamics). Recent works [11], [12], which are based on the time-delay approach, introduced the idea of a transformation of coordinates, which allowed quantitative estimates to be obtained on the averaging period.

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The objective of this work is to use the latter results to propose a tool for studying ISS property of nonlinear systems with respect to an averaged value of the input, which can also be transformed in a frequency gain analysis (similar to the amplitude Bode plot for linear systems or its counterpart given for nonlinear convergent dynamics in [13]). Applying the transformation from [11], [12], it will be shown that a nonlinear affine in the inputs system can be transformed to the neutral-type delayed dynamics dependent on the average values of the inputs. The obtained theoretical findings are illustrated on a class of homogeneous systems.

For homogeneous dynamical systems, their local and global behaviors coincide, and for stability/instability analysis, Lyapunov or Chetaev function of a homogeneous system can also be chosen homogeneous [14]. In addition, if a system is homogeneous (considering disturbance as an auxiliary variable) and asymptotically stable without disturbances, then it is ISS [15]. These facts make homogeneous systems suitable for analysis and modeling of complex nonlinear compartments, disposing many efficient and helping design procedures. The averaging method has been already used with homogeneous systems in [16], [17], [18], [19], where stability of a system was related with the same property of its averaged dynamics.

The outline of this work is as follows. The preliminary definitions and the homogeneity framework are given in Section II. A method for analysis of robust stability with respect to an average value of perturbations is presented in Section III. Application of the developed theory to homogeneous dynamical systems is considered in Section IV.

## II. PRELIMINARIES

### A. Notation

The real and integer numbers are denoted by  $\mathbb{R}$  and  $\mathbb{Z}$ , respectively,  $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$  and  $\mathbb{Z}_+ = \mathbb{R}_+ \cap \mathbb{Z}$ . Euclidean norm for a vector  $x \in \mathbb{R}^n$  is denoted by  $\|x\|$  (for a matrix  $A \in \mathbb{R}^{n \times m}$ ,  $\|A\|$  corresponds to the induced norm), and the absolute value of  $s \in \mathbb{R}$  by  $|s|$ . We denote by  $C_{[a,b]}^n$ ,  $-\infty < a < b < +\infty$  the Banach space of continuous functions  $\phi : [a, b] \rightarrow \mathbb{R}^n$  with the uniform norm  $\|\phi\|_C = \sup_{a \leq \zeta \leq b} \|\phi(\zeta)\|$  (similarly for  $C_{[a,b]}^n$ ), for  $x : [-T, +\infty) \rightarrow \mathbb{R}^n$  with  $T > 0$  we denote by  $x_t \in C_{[-T,0]}^n$ ,  $t \in \mathbb{R}_+$  its respective restriction  $x_t(s) = x(t+s)$  for  $s \in [-T, 0]$ . For a Lebesgue measurable function of time  $d : [a, b] \rightarrow \mathbb{R}^m$  define the norm  $\|d\|_{[a,b]} = \text{ess sup}_{t \in [a,b]} \|d(t)\|$ , then  $\|d\|_\infty = \|d\|_{[0,+\infty)}$  and the space of  $d$  with  $\|d\|_{[a,b]} < +\infty$  ( $\|d\|_\infty < +\infty$ ) we further denote as  $\mathcal{L}_{[a,b]}^m$  ( $\mathcal{L}_\infty^m$ ).

For a continuously differentiable function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ , its directional derivative along a given vector  $v \in \mathbb{R}^n$  at a point  $x \in \mathbb{R}^n$  is defined as follows:

$$DV(x)v = \frac{\partial V(x)}{\partial x}v.$$

A continuous function  $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to class  $\mathcal{K}$  if it is strictly increasing and  $\sigma(0) = 0$ ; it belongs to class  $\mathcal{K}_\infty$  if it is also unbounded. A continuous function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to class  $\mathcal{KL}$  if  $\beta(\cdot, r) \in \mathcal{K}$  and  $\beta(r, \cdot)$  is a strictly decreasing to zero for any fixed  $r > 0$ .

A sequence of integers,  $1, 2, \dots, n$ , is further denoted by  $\overline{1, n}$ .

### B. Input-to-state stability

Consider a dynamical system:

$$\dot{x}(t) = F(x(t), u(t)), \quad t \geq 0, \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the state and  $u(t) \in \mathbb{R}^m$  is the input,  $u \in \mathcal{L}_\infty^m$ ;  $F : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$  is a continuous function, and it ensures forward existence and uniqueness of solutions of the system at least locally in time (e.g., it is locally Lipschitz continuous out the origin),  $F(0, 0) = 0$ . For any  $x_0 \in \mathbb{R}^n$  and  $u \in \mathcal{L}_\infty^m$  the respective solution is denoted by  $x(t, x_0, u)$  with  $x(0, x_0, u) = x_0$ .

The definitions and results given in this subsection follow [2], [20].

**Definition 1.** The system (1) is called *practically ISS*, if there exist  $\beta \in \mathcal{KL}$ ,  $\gamma \in \mathcal{K}$  and  $q \geq 0$  such that

$$\|x(t, x_0, u)\| \leq \max\{\beta(\|x_0\|, t), \gamma(\|u\|_{[0,t]}), q\}, \quad \forall t \geq 0$$

for all  $x_0 \in \mathbb{R}^n$  and  $u \in \mathcal{L}_\infty^m$ .

If  $q = 0$  then (1) is called *ISS*.

The function  $\gamma$  from Definition 1 quantifies an *asymptotic gain* of (1):

$$\limsup_{t \rightarrow +\infty} \|x(t, x_0, u)\| \leq \max\{\gamma(\|u\|_{[0,t]}), q\}$$

satisfying for all  $x_0 \in \mathbb{R}^n$  and  $u \in \mathcal{L}_\infty^m$ . Another property, followed by (practical) ISS, is (practical) *global stability*: there exist  $\vartheta_x, \vartheta_u \in \mathcal{K}_\infty$  such that

$$\|x(t, x_0, u)\| \leq \max\{\vartheta_x(\|x_0\|), \vartheta_u(\|u\|_{[0,t]}), q\}, \quad \forall t \geq 0$$

for all  $x_0 \in \mathbb{R}^n$  and  $u \in \mathcal{L}_\infty^m$  (obviously, a possible choice is  $\vartheta_x(s) = \beta(s, 0)$  and  $\vartheta_u(s) = \gamma(s)$ ).

If  $u \equiv 0$ , then from ISS we recover the conventional *global asymptotic stability at the origin*.

**Definition 2.** A continuously differentiable function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  is called *practical ISS-LF* for the system (1) if there exist  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ ,  $\eta, \varrho \in \mathcal{K}$  and  $r \geq 0$  such that for all  $x \in \mathbb{R}^n$  and all  $u \in \mathbb{R}^m$ :

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|),$$

$$V(x) \geq \max\{\varrho(\|u\|), r\} \implies DV(x)F(x, u) \leq -\eta(\|x\|).$$

If  $r = 0$ , then such a function  $V$  is called *ISS-LF*.

**Theorem 1.** *The system (1) is (practically) ISS if and only if it admits a (practical) ISS-LF.*

### C. Homogeneity

For any  $r_i > 0$ ,  $i \in \overline{1, n}$  and  $\lambda > 0$ , define the *vector of weights*  $\mathbf{r} = [r_1, \dots, r_n]$  and the *dilation matrix*  $\Lambda_r(\lambda) = \text{diag}\{\lambda^{r_i}\}_{i=1}^n$ ;  $r_{\min} = \min_{i \in \overline{1, n}} r_i$  and  $r_{\max} = \max_{i \in \overline{1, n}} r_i$ .

**Definition 3.** [21], [14] The function  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  is called *r-homogeneous*, if for any  $x \in \mathbb{R}^n$  the relation

$$h(\Lambda_r(\lambda)x) = \lambda^\nu h(x)$$

holds for some  $\nu \in \mathbb{R}$  and all  $\lambda > 0$ .

The vector field  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is called *r-homogeneous*, if for any  $x \in \mathbb{R}^n$  the relation

$$f(\Lambda_r(\lambda)x) = \lambda^\nu \Lambda_r(\lambda)f(x)$$

holds for some  $\nu \geq -r_{\min}$  and all  $\lambda > 0$ .

In both cases, the constant  $\nu$  is called the *degree* of homogeneity.

A dynamical system

$$\dot{x}(t) = f(x(t)), \quad t \geq 0 \quad (2)$$

is called *r-homogeneous* of degree  $\nu$  if this property is satisfied for  $f$  in the sense of Definition 3.

For any  $x \in \mathbb{R}^n$  and  $\varpi \geq r_{\max}$ , a *homogeneous norm* can be defined as follows

$$\|x\|_r = \left( \sum_{i=1}^n |x_i|^{\varpi/r_i} \right)^{1/\varpi}.$$

For all  $x \in \mathbb{R}^n$ , its Euclidean norm  $\|x\|$  is related with the homogeneous one:

$$\underline{\sigma}_r(\|x\|_r) \leq \|x\| \leq \bar{\sigma}_r(\|x\|_r) \quad (3)$$

for some  $\underline{\sigma}_r, \bar{\sigma}_r \in \mathcal{K}_\infty$  [14]. In the following, due to this ‘‘equivalence’’, stability analysis with respect to the norm  $\|x\|$  can be substituted with analysis for the norm  $\|x\|_r$ . The homogeneous norm has an important property: it is *r-homogeneous* of degree 1, that is  $\|\Lambda_r(\lambda)x\|_r = \lambda\|x\|_r$  for all  $x \in \mathbb{R}^n$  and  $\lambda > 0$ .

Homogeneous systems possess certain robustness with respect to external disturbances.

**Theorem 2.** [15] *Let  $F(\Lambda_r(\lambda)x, \Lambda_{\tilde{r}}(\lambda)u) = \lambda^\nu \Lambda_r(\lambda)F(x, u)$  for all  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$  and all  $\lambda > 0$  with the weights  $\mathbf{r} = [r_1, \dots, r_n] > 0$ ,  $\tilde{\mathbf{r}} = [\tilde{r}_1, \dots, \tilde{r}_m] > 0$  and a degree  $\nu \geq -r_{\min}$ . Assume that the system (1) is asymptotically stable for  $u = 0$ , then (1) is ISS.*

## III. MAIN RESULT

Consider an affine in the inputs nonlinear system:

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad t \geq 0, \quad (4)$$

where  $x(t) \in \mathbb{R}^n$  is the state,  $u(t) \in \mathbb{R}^m$  is the input,  $u \in \mathcal{L}_\infty^m$ ;  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$  are locally Lipschitz continuous functions for all  $x \in \mathbb{R}^n \setminus \{0\}$ , ensuring forward existence and uniqueness of the system solutions at least locally in time,  $f(0) = 0$ .

Our goal is to derive ISS-like estimates for (4) with respect to average or weighted average values of the input  $u$ , and for this purpose we define the signals of interest:

$$\delta_1(t) = \frac{1}{T} \int_{t-T}^t (s-t+T)u(s)ds, \quad \delta_2(t) = \frac{1}{T} \int_{t-T}^t u(s)ds,$$

$$\tilde{\delta}_1(t) = \frac{1}{T} \int_{t-T}^t (s-t+T)\|u(s)\|ds, \quad \tilde{\delta}_2(t) = \frac{1}{T} \int_{t-T}^t \|u(s)\|ds,$$

where  $T > 0$  is the window of averaging (a parameter for our analysis). These quantities are well-defined for  $t \geq T$ , while for  $t \in [0, T)$  in the sequel we will assume that  $u(t-T) = 0$ , then all these averaged variables are defined for all  $t \geq 0$ .

*Remark 1.* Note that  $\delta_1, \delta_2, \tilde{\delta}_1$  and  $\tilde{\delta}_2$  are continuous functions of time  $t \geq 0$ , and the upper bounds of these averaged values are related to the respective norms of  $u$  and the period  $T$ :

$$\|\delta_1(t)\| \leq \|\tilde{\delta}_1(t)\| \leq \frac{T}{2}\|u\|_\infty, \quad \|\delta_2(t)\| \leq \|\tilde{\delta}_2(t)\| \leq \|u\|_\infty.$$

### A. Time-delay approach analysis

Similar to [11], [12], [22], let us define a transformation of state variables:

$$x(t) = \frac{1}{T} \int_{t-T}^t (s-t+T)g(x(s))u(s)ds + z(t), \quad (5)$$

$$x(s) = x(0), \quad u(s) = 0, \quad \forall s < 0,$$

where  $z(t) \in \mathbb{R}^n$  is a new auxiliary residual variable defined for all  $t \geq 0$ . Differentiating (5) we obtain:

$$\dot{z}(t) = f(x(t)) + \frac{1}{T} \int_{t-T}^t g(x(s))u(s)ds \quad (6)$$

$$= f(z(t) + d_1(t)) + d_2(t),$$

where

$$d_1(t) = -\frac{1}{T} \int_{t-T}^t (s-t+T)g(x(s))u(s)ds, \quad (7)$$

$$d_2(t) = \frac{1}{T} \int_{t-T}^t g(x(s))u(s)ds.$$

The right-hand side of (5) depends on the past values of  $x(t)$  (i.e.,  $x(s)$  for  $s \in [t-T, t)$ ). Thus, (5) can be written as  $x(t) - G(x_t, u_t) = z(t)$  for  $G(x_t, u_t) = \frac{1}{T} \int_{t-T}^t (s-t+T)g(x(s))u(s)ds$ , where  $x_t \in C_{[-T, 0]}^n$  with  $x_t(s) = x(t+s)$  and  $u_t(s) = u(t+s)$  for  $s \in [-T, 0]$ . It can be considered as a difference equation with respect to  $x(t)$  with disturbances  $u$  and  $z$ , which is ISS in  $z(t)$  and  $\tilde{\delta}_1(t)$  under introduced restrictions, as it is demonstrated later. To this end note that for any  $x(0) \in \mathbb{R}^n$  and  $u \in \mathcal{L}_\infty^m$  in (5),  $z(t)$  exists on maximal time interval  $\mathcal{T} = [0, t')$  for some  $t' > 0$  and  $z(0) = x(0)$ . By continuity of the solutions  $x(t)$  of (4),  $z(t)$  is a continuous function of time  $t \in \mathcal{T}$  by construction.

The starting point for our analysis is ISS of (6) with respect to the auxiliary inputs  $d_1$  and  $d_2$ :

**Assumption 1.** *There exists a continuously differentiable function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  such that*

$$\alpha_1(\|z\|) \leq V(z) \leq \alpha_2(\|z\|),$$

$$V(z) \geq \max\{\sigma_1(\|d_1\|), \sigma_2(\|d_2\|)\} \Rightarrow$$

$$DV(z)(f(z+d_1) + d_2) \leq -\alpha(\|z\|),$$

for all  $z, d_1, d_2 \in \mathbb{R}^n$ , with some  $\alpha_1, \alpha_2, \alpha \in \mathcal{K}_\infty$  and  $\sigma_1, \sigma_2 \in \mathcal{K}$ .

These conditions imply that  $V$  is an ISS-LF for the system (6) with disturbances  $d_1, d_2 \in \mathcal{L}_\infty^n$  and, hence, due to Theorem 1 this system is ISS with respect to these perturbations.

Since in (7) there is additionally a nonlinear gain  $g$  multiplying the input  $u$ , supplementary constraints have to be imposed. Note that by continuity of  $g$ , its norm can be bounded by a growing function of  $\|x\|$ , hence, there exist  $\rho \in \mathcal{K}$  and  $\rho_0 \geq 0$  such that

$$\|g(x)\| \leq \rho(\|x\|) + \rho_0, \quad \forall x \in \mathbb{R}^n. \quad (8)$$

**Assumption 2.** *There exist  $\theta_\delta, \theta_z \in \mathcal{K}_\infty$  such that*

$$\theta_z(s) + (\rho(s) + \rho_0)\theta_\delta(s) < s, \quad \forall s > 0.$$

**Assumption 3.** *There exist  $\gamma_V \in \mathcal{K}$ ,  $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$  such that*

$$\xi \geq \gamma_V(\zeta) \Rightarrow \sigma_i(\gamma_i(\xi)(\rho(\zeta) + \rho_0)) \leq \xi, \quad i = 1, 2, \quad \forall \xi, \zeta \in \mathbb{R}_+,$$

where  $\sigma_1, \sigma_2 \in \mathcal{K}$  are given in Assumption 1.

In these assumptions, some functions from classes  $\mathcal{K}$  and  $\mathcal{K}_\infty$  are introduced, whose existence guarantees ISS of (4) with respect to averaged signals  $\tilde{\delta}_1$  and  $\tilde{\delta}_2$ , as it will be shown below (assumptions 1 and 3 imply ISS of (4) with respect to the input  $u$  if  $\gamma_V(s) \geq \alpha_1(s)$  for all  $s \in \mathbb{R}_+$ ).

**Lemma 1.** *Let Assumption 2 hold and given  $x(0) \in \mathbb{R}^n$ ,  $z \in \mathcal{L}_\infty^n \cap C_{[0, +\infty)}^n$ ,  $u \in \mathcal{L}_\infty^m$  the relation (5) be satisfied for all  $t \in \mathcal{T}$ . Then  $\mathcal{T} = \mathbb{R}_+$  and there exists  $\beta_1 \in \mathcal{K}\mathcal{L}$  such that*

$$\|x(t)\| \leq \max\{\beta_1(\|x(0)\|, t), \theta_z^{-1}(\|z\|_{[0, t)}), \theta_\delta^{-1}(\|\tilde{\delta}_1\|_{[0, t)})\}, \quad (9)$$

for all  $t \geq 0$ , all  $x(0) \in \mathbb{R}^n$ ,  $z \in \mathcal{L}_\infty^n \cap C_{[0, +\infty)}^n$  and  $u \in \mathcal{L}_\infty^m$ .

*Proof.* Using (8) an estimate follows from (5) for all  $t \in \mathcal{T}$ :

$$\|x(t)\| \leq \frac{1}{T} \int_{t-T}^t (s-t+T)\|g(x(s))u(s)\|ds + \|z(t)\|$$

$$\leq (\rho(|x_t|_C) + \rho_0)\tilde{\delta}_1(t) + \|z(t)\|.$$

Under Assumption 2, this estimate leads to the following relation:

$$\|x(t)\| \geq \max\{\theta_z^{-1}(\|z(t)\|), \theta_\delta^{-1}(\tilde{\delta}_1(t))\} \Rightarrow$$

$$\|x(t)\| \leq \theta_z(|x_t|_C) + (\rho(|x_t|_C) + \rho_0)\theta_\delta(|x_t|_C) = \mu(|x_t|_C)$$

with  $\mu \in \mathcal{K}_\infty$ . The contraction property,  $\mu(|x_t|_C) < |x_t|_C$  for all  $|x_t|_C \neq 0$ , is introduced in Assumption 2, and it implies convergence of  $x(t)$  while the latter implication holds. Indeed, assume that  $\mathcal{T} = \mathcal{T}_z \cup \mathcal{T}_x$ , where

$$t \in \mathcal{T}_z : \|x(t)\| \leq \max\{\theta_z^{-1}(\|z\|_\infty), \theta_\delta^{-1}(\|\tilde{\delta}_1\|_\infty)\},$$

$$t \in \mathcal{T}_x : \|x(t)\| \geq \max\{\theta_z^{-1}(\|z\|_\infty), \theta_\delta^{-1}(\|\tilde{\delta}_1\|_\infty)\},$$

then due to continuity of  $x(t)$  there exists a sequence of nonempty intervals such that  $\mathcal{T}_x = \bigcup_{j \in \mathbb{Z}_+} [t_s^j, t_e^j]$  (all isolated points can be placed in  $\mathcal{T}_z$ ). Denote  $\tau_i = \min\{\arg \max_{s \in [-T, 0]} \|x(t+s)\|\}$  (we use  $\min\{\cdot\}$  to avoid

non-uniqueness of  $\arg \max$  operation), *i.e.*,  $|x_t|_C = \|x(\tau_t)\|$ , and for some  $j \in \mathbb{Z}_+$  consider  $t \in [t_s^j, t_e^j]$ , then

$$\begin{aligned} \|x(t)\| &\leq \mu(\|x(\tau_t)\|) \leq \mu \circ \mu(|x_{\tau_t}|_C) \leq \mu \circ \mu(\|x(\tau_{\tau_t})\|) \leq \\ &\quad \text{ceil}\left(\frac{t-t_s^j}{T}\right) \\ &\dots \leq \mu \overbrace{\circ \dots \circ}^{|x_{t_s^j}|_C}, \end{aligned}$$

where  $\text{ceil}(s)$  is the function of truncation to the smallest integer bigger than  $s \in \mathbb{R}$ . Hence, there is  $\beta_1 \in \mathcal{KL}$  such that

$$\|x(t)\| \leq \beta_1(|x_{t_s^j}|_C, t - t_s^j)$$

for all  $t \in [t_s^j, t_e^j]$  and any  $j \in \mathbb{Z}_+$ . Note that  $\beta_1(s, 0) = s$  by the construction, then for the value of  $|x_{t_s^j}|_C$  two options are possible: either  $t_s^j = 0$  and  $|x_{t_s^j}|_C = |x_0|_C = \|x(0)\|$ , or  $t_s^j > 0$  and  $|x_{t_s^j}|_C = \max\{\theta_z^{-1}(\|z\|_\infty), \theta_\delta^{-1}(\|\tilde{\delta}_1\|_\infty)\}$  by continuity. Therefore, the following estimate is satisfied for all  $t \in \mathcal{T}$ :

$$\|x(t)\| \leq \max\{\beta_1(\|x(0)\|, t), \theta_z^{-1}(\|z\|_\infty), \theta_\delta^{-1}(\|\tilde{\delta}_1\|_\infty)\},$$

implying that  $\mathcal{T} = \mathbb{R}_+$ , which finishes the proof.  $\square$

In the framework of the trajectory based approach [23], a similar ISS result was obtained for  $\mu(s) = \mu_0 s$  with  $\mu_0 \in (0, 1)$  and the exponential convergence, then Lemma 1 can be seen as an extension of that method to the case of asymptotic convergence. Now let us investigate the behavior of  $z(t)$  in (6). Under Assumption 1, for  $d_1$  and  $d_2$  defined by (7), considering  $V(z)$  as an ISS-LF candidate for (6), the following relation is valid:

$$V(z(t)) \geq \max\{\sigma_1(\|d_1(t)\|), \sigma_2(\|d_2(t)\|)\} \Rightarrow$$

$$\dot{V}(t) = DV(z(t))(f(z(t) + d_1(t)) + d_2(t)) \leq -\alpha(\|z(t)\|),$$

Recalling (8),

$$\|d_1(t)\| \leq (\rho(|x_t|_C) + \rho_0) \tilde{\delta}_1(t),$$

$$\|d_2(t)\| \leq (\rho(|x_t|_C) + \rho_0) \tilde{\delta}_2(t).$$

In addition, under Assumption 3 the following implications are satisfied:

$$V(z(t)) \geq \max\{\gamma_V(|x_t|_C), \gamma_1^{-1}(\tilde{\delta}_1(t)),$$

$$\gamma_2^{-1}(\tilde{\delta}_2(t))\} \Rightarrow$$

$$V(z(t)) \geq \max\{\sigma_1\left((\rho(|x_t|_C) + \rho_0) \tilde{\delta}_1(t)\right),$$

$$\sigma_2\left((\rho(|x_t|_C) + \rho_0) \tilde{\delta}_2(t)\right)\} \Rightarrow$$

$$V(z(t)) \geq \max\{\sigma_1(\|d_1(t)\|), \sigma_2(\|d_2(t)\|)\},$$

which implies due to Definition 2 that  $V$  is an ISS-LF for (6) with respect to the inputs  $|x_t|_C$ ,  $\tilde{\delta}_1(t)$  and  $\tilde{\delta}_2(t)$ :

$$V(z) \geq \max\{\gamma_V(|x_t|_C), \gamma_1^{-1}(\tilde{\delta}_1), \gamma_2^{-1}(\tilde{\delta}_2)\} \Rightarrow \dot{V} \leq -\alpha(\|z\|)$$

for all  $z \in \mathbb{R}^n$ ,  $\tilde{\delta}_1, \tilde{\delta}_2 \in \mathbb{R}_+$  and  $x_t \in C_{[-T, 0]}^n$ . Therefore, according to Theorem 1, the following estimate is valid:

$$\|z(t)\| \leq \alpha_1^{-1} \circ \max\{\beta_2(\|z(0)\|, t), \gamma_V(\|x\|_{[0, t]})\}, \quad (10)$$

$$\gamma_1^{-1}(\|\tilde{\delta}_1\|_{[0, t]}), \gamma_2^{-1}(\|\tilde{\delta}_2\|_{[0, t]})\}$$

for all  $t \geq 0$  and some  $\beta_2 \in \mathcal{KL}$  (note that again  $\beta_2(s, 0) = s$ ).

We are in position to formulate the main result.

**Theorem 3.** *Let Assumptions 1–3 be satisfied for (4), then there exists  $\beta_3 \in \mathcal{KL}$  such that*

$$\begin{aligned} \|x(t)\| &\leq \max\{\beta_3(\|x(0)\|, t), \theta_\delta^{-1}(\|\tilde{\delta}_1\|_{[0, t]})\}, \\ \theta_z^{-1} \circ \alpha_1^{-1} \circ \gamma_1^{-1}(\|\tilde{\delta}_1\|_{[0, t]}) &\circ \theta_z^{-1} \circ \alpha_1^{-1} \circ \gamma_2^{-1}(\|\tilde{\delta}_2\|_{[0, t]}) \end{aligned}$$

for all  $t \geq 0$ , all  $x(0) \in \mathbb{R}^n$  and  $u \in \mathcal{L}_\infty^m$  provided that

$$\theta_z^{-1} \circ \alpha_1^{-1} \circ \gamma_V(s) < s, \quad \forall s > 0,$$

meaning that (4) is ISS with respect to  $\tilde{\delta}_1, \tilde{\delta}_2 \in \mathcal{L}_\infty^1$  that are averages of  $|u|$ .

*Proof.* Since all conditions of Lemma 1 are verified, the estimate (9) is valid for  $t \in \mathcal{T} \subseteq \mathbb{R}_+$ . Above, it has been shown that the estimate (10) is also true for all  $t \in \mathcal{T}$ . Using the standard small-gain arguments [24], [25], first, the global stability of the feedback interconnection (5), (6) can be proven, *i.e.*, there exist functions  $\psi_i \in \mathcal{K}_\infty$  for  $i = 0, 1, 2$  such that

$$\max\{\|x(t)\|, \|z(t)\|\} \leq \max\{\psi_0 \circ \max\{\|x(0)\|, \|z(0)\|\},$$

$$\psi_1(\|\tilde{\delta}_1\|_\infty), \psi_2(\|\tilde{\delta}_2\|_\infty)\}, \quad \forall t \geq 0,$$

for all  $x(0), z(0) \in \mathbb{R}^n$  and  $u \in \mathcal{L}_\infty^m$ , implying also that  $\mathcal{T} = \mathbb{R}_+$ . Under the observation that  $\|z(0)\| \leq (\rho(\|x(0)\|) + \rho_0) \|\tilde{\delta}_1\|_\infty + \|x(0)\|$ , the latter estimate can be rewritten as

$$\max\{\|x(t)\|, \|z(t)\|\} \leq \max\{\tilde{\psi}_0(\|x(0)\|),$$

$$\tilde{\psi}_1(\|\tilde{\delta}_1\|_\infty), \tilde{\psi}_2(\|\tilde{\delta}_2\|_\infty)\}$$

for all  $t \geq 0$ ,  $x(0) \in \mathbb{R}^n$  and  $u \in \mathcal{L}_\infty^m$ , for suitably modified  $\tilde{\psi}_0, \tilde{\psi}_1 \in \mathcal{K}_\infty$ . Second, the asymptotic gain function can be derived (we are further interested only in the behavior of  $x(t)$  due to its physical meaning in (4)):

$$\limsup_{t \rightarrow +\infty} \|x(t)\| \leq \max\{\theta_\delta^{-1}(\|\tilde{\delta}_1\|_\infty),$$

$$\theta_z^{-1} \circ \alpha_1^{-1} \circ \max\{\gamma_1^{-1}(\|\tilde{\delta}_1\|_\infty), \gamma_2^{-1}(\|\tilde{\delta}_2\|_\infty)\}\}.$$

If the system (4) admits both properties, global stability and asymptotic gain, then it is ISS [26, Lemma 2.7].  $\square$

*Remark 2.* All assumptions, 1–3, as well as the small-gain condition in Theorem 3, are formulated globally (*i.e.*, for all  $x \in \mathbb{R}^n$  and  $s > 0$ ). Restricting validity of the introduced inequalities it is possible to get the respective local stability results.

In the conditions of Theorem 3, the system (4) is not obligatory ISS in  $u$ , while the power ISS or the moving average ISS properties from [3] are equivalent to ISS under Assumption 1. In [4], ISS of (4) with respect to  $\tilde{\delta}_2$  was established starting from ISS property in  $u$ .

Note that rewriting (6) in the coordinates  $x(t)$ ,

$$\begin{aligned} \frac{d}{dt} \left( x(t) - \frac{1}{T} \int_{t-T}^t (s-t+T)g(x(s))u(s)ds \right) \\ = f(x(t)) + \frac{1}{T} \int_{t-T}^t g(x(s))u(s)ds, \end{aligned}$$

we get a special neutral-type time-delay system. Then the main result of this work can be interpreted as an extension of

Lyapunov-Razumikhin ISS result of [27] to this neutral-type system.

The results of Theorem 3 can be easily applied if the function  $g(x)$  is just bounded, then  $\rho \in \mathcal{K}$  from (8) admits a restriction:

$$\rho(s) \leq \rho_1 < +\infty, \forall s > 0,$$

for some  $\rho_1 > 0$ , and we can take  $\theta_\delta(s) = \frac{\theta_z(s)}{\rho_1 + \rho_0}$  with  $\theta_z(s) < \frac{s}{2}$ ,  $\gamma_i(s) = \frac{\sigma_i^{-1}(s)}{\rho_1 + \rho_0}$  with  $i = 1, 2$  and  $\gamma_V(s) < \alpha_1 \circ \theta_z(s)$ . In addition, if  $g$  is a constant function, then a more constructive result can be derived given below.

#### B. A static input gain

A constant input gain,  $g(x) = B \in \mathbb{R}^{n \times m}$ , represents an interesting practical case, where rather technical Assumptions 2, 3 and the small-gain condition can be skipped (they characterize the properties of  $g$  as a function of  $x$ ):

**Theorem 4.** *Let Assumption 1 be satisfied for (4) with  $g(x) \equiv B$ , then there exists  $\beta_4 \in \mathcal{K}\mathcal{L}$  such that*

$$\|x(t)\| \leq \max\{\beta_4(\|x(0)\|, t), \alpha_1^{-1} \circ \sigma_1(\|B\| \|\delta_1\|_{[0,t]}), \alpha_1^{-1} \circ \sigma_2(\|B\| \|\delta_2\|_{[0,t]})\} + \|B\| \|\delta_1\|_{[0,t]}$$

for all  $t \geq 0$ , all  $x(0) \in \mathbb{R}^n$  and  $u \in \mathcal{L}_\infty^m$ , meaning that (4) is ISS with respect to  $\delta_1, \delta_2 \in \mathcal{L}_\infty^m \cap C_{[0,+\infty)}^m$  that are averages of  $u$ .

*Proof.* In such a case the equations (5) and (6) take the form:

$$x(t) = B\delta_1(t) + z(t)$$

and

$$\dot{z}(t) = f(z(t) - B\delta_1(t)) + B\delta_2(t),$$

respectively. Hence, from the first equation, which is static here, we immediately get:

$$\|x(t)\| \leq \|B\| \|\delta_1(t)\| + \|z(t)\|$$

for all  $t \geq 0$ , while applying Assumption 1 to the dynamics of  $z$ , its ISS property can be substantiated, *i.e.*, there exists  $\tilde{\beta}_4 \in \mathbb{R}^n$  such that

$$\|z(t)\| \leq \alpha_1^{-1} \circ \max\{\tilde{\beta}_4(\|z(0)\|, t), \sigma_1(\|B\| \|\delta_1\|_{[0,t]}), \sigma_2(\|B\| \|\delta_2\|_{[0,t]})\}, \forall t \geq 0,$$

for all  $z(0) \in \mathbb{R}^n$  and  $\delta_1, \delta_2 \in \mathcal{L}_\infty^m$ . Combining both inequalities, due to  $\|z(0)\| = \|x(0)\|$  we obtain:

$$\|x(t)\| \leq \|B\| \|\delta_1\|_{[0,t]} + \alpha_1^{-1} \circ \max\{\tilde{\beta}_4(\|x(0)\|, t), \sigma_1(\|B\| \|\delta_1\|_{[0,t]}), \sigma_2(\|B\| \|\delta_2\|_{[0,t]})\}, \forall t \geq 0,$$

for all  $x(0) \in \mathbb{R}^n$  and  $\delta_1, \delta_2 \in \mathcal{L}_\infty^m$ , leading to the required estimate.  $\square$

*Remark 3.* Note that we may have  $\delta_2 \in \mathcal{L}_\infty^m$  while  $u \notin \mathcal{L}_\infty^m$ .

For a  $T$ -periodic time signal  $u$  with zero mean value (in such a case  $\delta_2(t) = 0$  for all  $t \geq 0$ ), the following corollary is a direct consequence of Theorem 4 and Remark 1:

**Corollary 1.** *Let Assumption 1 be satisfied for  $d_2 = 0$  and  $g(x) \equiv B$  in (4). Then for all  $x(0) \in \mathbb{R}^n$  and all  $T$ -periodic signals  $u \in \mathcal{L}_\infty^m$  with zero mean value the following holds:*

$$\limsup_{t \rightarrow +\infty} \|x(t)\| \leq \alpha_1^{-1} \circ \sigma_1(\|B\| \frac{T}{2} \|u\|_\infty) + \|B\| \frac{T}{2} \|u\|_\infty.$$

*Remark 4.* If the signal  $u$  is  $T$ -periodic with non-zero average, then practical asymptotic gain can be obtained. Local versions of the above results follow the respective restrictions in Assumption 1 (*e.g.*, if  $V$  is a local ISS-LF).

Defining the frequency  $\omega = \frac{2\pi}{T}$ , this result allows us to introduce an analogue of Bode magnitude plot for the nonlinear system (4) with  $g(x) \equiv B$ :

$$H(\omega) = 20 \log \left( \alpha_1^{-1} \circ \sigma_1(\|B\| \frac{\pi}{\omega} \|u\|_\infty) + \|B\| \frac{\pi}{\omega} \|u\|_\infty \right),$$

which indicates that for a constant input amplitude, increasing the frequency improves the filtering abilities for the class of nonlinear systems satisfying the conditions of Corollary 1. Comparing with [13] (see also [28]), where a frequency response of convergent systems is studied, and to find the response it is suggested either to solve a PDE or to make simulations, in the present work an upper estimate  $H(\omega)$  of the response is obtained dependent on the functions  $\alpha_1$  and  $\sigma_1$  defining an ISS-LF of (6), which are easier to derive.

**Example 1.** In [13], the frequency response of

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + x_2^2(t), \\ \dot{x}_2(t) &= -x_2(t) + u(t), \end{aligned}$$

was analytically derived,  $x(t) = [x_1(t) \ x_2(t)]^\top \in \mathbb{R}^2$ . To apply Corollary 1, consider in Assumption 1 a Lyapunov function  $V(x) = \frac{1}{2}x_1^2 + \frac{\varkappa}{2}x_2^2 + \frac{\ell}{4}x_2^4$  for some  $\varkappa > 0$  and  $\ell > 0$ , then an estimate  $DV(x)f(x + d_1) \leq -(1 - 1.5\epsilon)x_1^2 - \frac{\varkappa}{2}x_2^2 - (\frac{\ell}{4} - \epsilon^{-1})x_2^4 + 0.5\epsilon^{-1}(2d_{12}^2 + |d_{11}|)^2 + \frac{\varkappa}{2}d_{12}^2 + \frac{\ell}{4}d_{12}^4$  for  $d_1 = [d_{11} \ d_{12}]^\top \in \mathbb{R}^2$  and some  $\epsilon \in (0, \frac{2}{3})$  can be obtained, which for  $\ell > 4\epsilon^{-1}$  gives

$$\begin{aligned} \alpha_1(s) &= \frac{\min\{1, \varkappa\}}{2} s^2, \\ \sigma_1(s) &= 2 \frac{(\frac{\ell}{4} + \frac{4}{\epsilon}) s^4 + \max\{\epsilon^{-1}, \frac{\varkappa}{2}\} s^2}{\min\{1, 2 - 3\epsilon, 1 - \frac{4}{\ell\epsilon}\}}. \end{aligned}$$

An interesting feature of the obtained result is that in (4), in order to have a well-defined asymptotic gain with respect to an average value of the input  $u$ , according to Assumption 1, the system should be ISS with respect to a "measurement noise" signal  $d_1$ . However, the noise sensitivity of a system usually has other origins and mechanisms than the robustness in the exogenous state disturbances leading to insightful results, which is demonstrated in the next example.

**Example 2.** Consider a system

$$\dot{x}(t) = -k|x(t)|^\kappa \text{sign}(x(t)) + \sin(\omega t),$$

where  $x(t) \in \mathbb{R}$  and  $k, \kappa > 0$  are parameters. Assumption 1 is satisfied for  $V(x) = x^2$  with

$$\begin{aligned} \alpha_1(s) = \alpha_2(s) &= s^2, \quad \sigma_1(s) = \epsilon_1^{-2} s^2, \quad \sigma_2(s) = \epsilon_2^{-\frac{2}{\kappa}} s^{\frac{2}{\kappa}}, \\ \alpha(s) &= 2\epsilon_3 s^{1+\kappa}, \end{aligned}$$

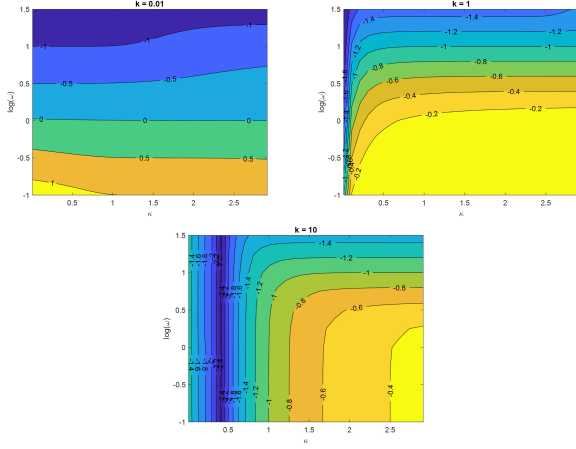


Figure 1. Plot of  $a_k$  for different value of  $k$

where  $\epsilon_1 \in (0, 1)$ ,  $\epsilon_2, \epsilon_3 > 0$  and  $k(1 - \epsilon_1)^\kappa \geq \epsilon_2 + \epsilon_3$ . Therefore,

$$H(\omega) = 20 \log \left( (1 + \epsilon_1^{-1}) \frac{\pi}{\omega} \right)$$

and optimizing the value of  $\epsilon_1$  we can get a lower bound  $\underline{H}(\omega) = 20 \log \left( 2 \frac{\pi}{\omega} \right)$  independent in  $k$  and  $\kappa$ , which may indicate that the obtained bound is qualitative representing the worst-case estimate. To evaluate its conservativeness, we simulate this system on the time interval  $[0, 500]$  using the explicit Euler method with the time step  $h = 0.005$ , for different values of  $\omega$ ,  $\kappa$ ,  $k$  and calculating the map

$$a_k(\omega, \kappa) = \log \left[ \max_{400 \leq t \leq 500} |x(t)| \right]$$

we obtain the results given in Fig. 1 for  $k = 0.01, 1, 100$ . These results surprisingly demonstrate indeed a weak dependence on  $\kappa$ , and also a limited sensitivity for the decay in  $k$ , while increasing  $\omega$  leads to lower values of  $a_k$  confirming the theoretical findings.

### C. Integral ISS

Another interesting consequence of Corollary 1 is that if the system (4) is integral ISS (iISS) [2], [20], then it has an asymptotic gain with respect to  $\eta \left( \int_0^t \gamma(\|u(s)\|) ds \right)$  for some  $\eta, \gamma \in \mathcal{K}_\infty$  (and not with respect to  $\gamma(\|u\|_\infty)$  as in the ISS case), and it is easy to observe that  $\int_0^t \gamma(\|u(s)\|) ds$  for any  $\gamma \in \mathcal{K}_\infty$  and a bounded non-zero periodic  $u$  is an unbounded signal in  $t$ . Therefore, iISS property, formally, does not provide a reliable conclusion about the boundedness of solutions of (4) in such a setting. However, if the conditions of Corollary 1 are verified, it could be the case that the system is still ISS with respect to  $d_1$  (Assumption 1 is satisfied for  $d_2 = 0$ ), then the theory proposed in this work allows to guarantee the boundedness with respect to periodic signals for iISS systems.

**Example 3.** For an illustration consider a scalar system

$$\dot{x}(t) = -k\varphi(x(t)) + \sin(\omega t),$$

where  $x(t) \in \mathbb{R}$ ,  $k > 0$  is a parameter, and  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is a bounded continuous function such that  $\varphi(s)s > 0$  for all

$s \in \mathbb{R} \setminus \{0\}$ . Due to boundedness of  $\varphi$  this system is only iISS with respect to an additive input. However, Assumption 1 is satisfied for  $d_2 = 0$  with  $V(x) = x^2$  and

$$\begin{aligned} \alpha_1(s) &= \alpha_2(s) = s^2, \quad \sigma_1(s) = \epsilon_1^{-2} s^2, \\ \alpha(s) &= 2ks\varphi_{\epsilon_1}(s), \end{aligned}$$

where  $\epsilon_1 \in (0, 1)$  is a parameter, and

$$\varphi_{\epsilon_1}(x) = \begin{cases} \min_{s \in [(1-\epsilon_1)x, (1+\epsilon_1)x]} f(s) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ \max_{s \in [(1+\epsilon_1)x, (1-\epsilon_1)x]} f(s) & \text{if } x < 0 \end{cases}$$

is a continuous function verifying the passivity condition  $\varphi_{\epsilon_1}(s)s > 0$  for all  $s \in \mathbb{R} \setminus \{0\}$ . Therefore, again

$$H(\omega) = 20 \log \left( (1 + \epsilon_1^{-1}) \frac{\pi}{\omega} \right)$$

and optimizing the value of  $\epsilon_1$  we can get a lower bound  $\underline{H}(\omega) = 20 \log \left( 2 \frac{\pi}{\omega} \right)$ .

## IV. APPLICATION TO HOMOGENEOUS SYSTEMS

Let us consider a subclass of (4) with homogeneous functions  $f$  and  $g$ , which allows us to replace Assumptions 1–3 by the following one (denote  $g(x) = [g_1(x) \dots g_m(x)]$ , where  $g_i(x)$  is a column of  $g$  for  $i \in \overline{1, m}$ , then  $g_{i,j}(x)$  is the element of  $g_i$  for  $j \in \overline{1, n}$ ):

**Assumption 4.** *The vector field  $f$  and functions  $g_{i,j}$  for  $i \in \overline{1, m}$ ,  $j \in \overline{1, n}$  are  $\mathbf{r}$ -homogeneous of degrees  $\nu \geq -r_{\min}$  and  $\varsigma_i > 0$ , respectively, and the system (4) is asymptotically stable for  $u \equiv 0$ .*

Following [21], [29], under Assumption 4, for the system (4) there is a twice continuously differentiable and  $\mathbf{r}$ -homogeneous of degree  $\mu > r_{\max}$  Lyapunov function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ , which implies fulfillment of the following inequalities:

$$\begin{aligned} c_1 \|x\|_r^\mu &\leq V(x) \leq c_2 \|x\|_r^\mu, \quad DV(x)f(x) \leq -c_3 \|x\|_r^{\mu+\nu}, \\ \sup_{\|x\|_r=1} \left\| \frac{\partial V(x)}{\partial x} \right\| &\leq c_4 \end{aligned}$$

for all  $x \in \mathbb{R}^n$  and some  $c_i > 0$ ,  $i = 1, \dots, 4$ . Moreover, following [15] and using (3) there exist  $\omega_1 > 0$  and  $\omega_2 > 0$  (dependent only on the properties of  $f$  and  $c_3, c_4$ ) such that

$$\begin{aligned} \|x\|_r &\geq \max\{\omega_1 \|d_1\|_r, \omega_2 \|d_2\|_{\bar{r}}\} \Rightarrow \\ DV(x)(f(x + d_1) + d_2) &\leq -\frac{c_3}{2} \|x\|_r^{\mu+\nu} \end{aligned}$$

for all  $x, d_1, d_2 \in \mathbb{R}^n$  (see Theorem 2),  $\bar{\mathbf{r}} = \mathbf{r} + \nu$ . Hence, Assumption 1 is verified for this  $V$  with  $\alpha(s) = \frac{c_3}{2} s^{\mu+\nu}$ ,  $\sigma_i(s) = c_2 \omega_i^\mu s^\mu$  for  $i = 1, 2$ , where the homogeneous norm is used instead of the standard one (recall that these norms are equivalent due to (3)), then  $|x_t|_C = \sup_{-T \leq \varsigma \leq 0} \|x(t + \varsigma)\|_r$  in this section. As a consequence, to verify Assumptions 2–3 we will use also the homogeneous norm, then to simplify the computations let  $m = 1$  and  $\varsigma = \varsigma_1$ . From Assumption 4 we get that  $\|g(x)\| \leq c_5 \|x\|_r^\varsigma$  for all  $x \in \mathbb{R}^n$  and some  $c_5 > 0$ , therefore,  $\rho(s) = c_5 \underline{\sigma}_r^{-\varsigma}(s)$  and  $\rho_0 = 0$  in (8). Next,



let us repeat the steps made in the previous section in order to replace the conditions in Assumptions 2 and 3.

An estimate follows from (5) for all  $t \in \mathcal{T}$ :

$$\begin{aligned} \|x(t)\| &\leq \frac{1}{T} \int_{t-T}^t (s-t+T) \|g(x(s))u(s)\| ds + \|z(t)\| \\ &\leq \frac{1}{T} \int_{t-T}^t (s-t+T) \|u(s)\| \|g(x(s))\| ds + \|z(t)\| \\ &\leq c_5 |x_t|_{\mathcal{C}} \tilde{\delta}_1(t) + \|z(t)\| \end{aligned}$$

for  $x(0) \in \mathbb{R}^n$ ,  $z \in \mathcal{L}_{\infty}^n$  and  $u \in \mathcal{L}_{\infty}^m$ . This estimate leads to the following relation:

$$\begin{aligned} \|x(t)\|_r &\geq \max\{\theta_z^{-1}(\|z(t)\|), \theta_{\delta}^{-1}(\tilde{\delta}_1(t))\} \Rightarrow \\ \|x(t)\| &\leq \theta_z(|x_t|_{\mathcal{C}}) + c_5 |x_t|_{\mathcal{C}} \theta_{\delta}(|x_t|_{\mathcal{C}}) \Rightarrow \\ \|x(t)\|_r &\leq \underline{\sigma}_r^{-1}(\theta_z(|x_t|_{\mathcal{C}}) + c_5 |x_t|_{\mathcal{C}} \theta_{\delta}(|x_t|_{\mathcal{C}})) \\ &< |x_t|_{\mathcal{C}}, \end{aligned}$$

which is valid provided that

$$\underline{\sigma}_r^{-1}(\theta_z(s) + c_5 s^{\varsigma} \theta_{\delta}(s)) < s, \quad \forall s > 0. \quad (11)$$

Repeating the same computations as before, the following inequality is satisfied for all  $t \geq 0$ , all  $x(0) \in \mathbb{R}^n$ ,  $z \in \mathcal{L}_{\infty}^n$  and  $u \in \mathcal{L}_{\infty}^m$ :

$$\|x(t)\|_r \leq \max\{\beta_1(\|x(0)\|_r, t), \theta_z^{-1}(\|z\|_{\infty}), \theta_{\delta}^{-1}(\|\tilde{\delta}_1\|_{\infty})\}$$

for some  $\beta_1 \in \mathcal{KL}$ . Recalling the definition of  $d_1$  and  $d_2$  in (7), we obtain

$$\begin{aligned} \underline{\sigma}_r(\|d_1(t)\|_r) &\leq c_5 |x_t|_{\mathcal{C}} \tilde{\delta}_1(t), \\ \underline{\sigma}_{\bar{r}}(\|d_2(t)\|_{\bar{r}}) &\leq c_5 |x_t|_{\mathcal{C}} \tilde{\delta}_2(t). \end{aligned}$$

Next, the following implications should be satisfied:

$$\begin{aligned} V(z(t)) &\geq \max\{\gamma_V(|x_t|_{\mathcal{C}}), \gamma_1^{-1}(\tilde{\delta}_1(t)), \gamma_2^{-1}(\tilde{\delta}_2(t))\} \Rightarrow \\ V(z(t)) &\geq \max\{c_2 \omega_1^{\mu} \underline{\sigma}_r^{-\mu} (c_5 |x_t|_{\mathcal{C}} \tilde{\delta}_1(t)), \\ &\quad c_2 \omega_2^{\mu} \underline{\sigma}_{\bar{r}}^{-\mu} (c_5 |x_t|_{\mathcal{C}} \tilde{\delta}_2(t))\} \Rightarrow \\ V(z(t)) &\geq \max\{c_2 \omega_1^{\mu} \|d_1(t)\|_r^{\mu}, c_2 \omega_2^{\mu} \|d_2(t)\|_{\bar{r}}^{\mu}\}, \end{aligned}$$

leading to the property:

$$\begin{aligned} \xi \geq \gamma_V(\zeta) &\Rightarrow c_2 \omega_1^{\mu} \underline{\sigma}_r^{-\mu} (c_5 \zeta^{\varsigma} \gamma_1(\xi)) \leq \xi, \\ c_2 \omega_2^{\mu} \underline{\sigma}_{\bar{r}}^{-\mu} (c_5 \zeta^{\varsigma} \gamma_2(\xi)) &\leq \xi, \quad \forall \xi, \zeta \in \mathbb{R}_+. \end{aligned} \quad (12)$$

Therefore, according to Theorem 1, the following estimate is valid for all  $t \geq 0$ :

$$\begin{aligned} \|z(t)\|_r &\leq c_1^{-1/\mu} \max\{\beta_2(\|z(0)\|_r, t), \gamma_V(\sup_{s \in [0, t]} \|x(s)\|_r), \\ &\quad \gamma_1^{-1}(\|\tilde{\delta}_1\|_{[0, t]}), \gamma_2^{-1}(\|\tilde{\delta}_2\|_{[0, t]})\}^{1/\mu}, \end{aligned}$$

for all  $z(0) \in \mathbb{R}^n$ ,  $x \in \mathcal{L}_{\infty}^n$  and  $u \in \mathcal{L}_{\infty}^m$ , and a suitably defined  $\beta_2 \in \mathcal{KL}$ . Finally, the small-gain condition of Theorem 3 takes the form:

$$\theta_z^{-1} \circ \bar{\sigma}_r (c_1^{-1/\mu} \gamma_V^{1/\mu}(s)) < s, \quad \forall s > 0. \quad (13)$$

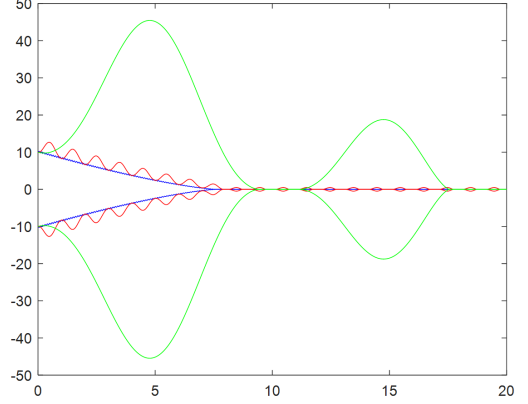


Figure 2. An illustration for Corollary 2

Recall from [30] that for  $r_{\max} \leq 1$  and  $\varpi = 1$  (this property can be always provided by scaling the weights in  $\mathbf{r}$ ) the following choice is possible

$$\underline{\sigma}_r^{-1}(s) = n \begin{cases} s^{r_{\max}} & s \leq 1 \\ s^{1/r_{\min}} & s > 1 \end{cases}, \quad \bar{\sigma}_r(s) = \sqrt{n} \begin{cases} s^{r_{\min}} & s \leq 1 \\ s^{r_{\max}} & s > 1 \end{cases}.$$

Let us perform verification of the conditions (11)–(13) (Assumptions 2–3 and the small-gain condition) for  $s > 1$ . After straightforward computations (applying Jensen's inequality) we get that (11) is satisfied for  $\theta_z(s) = c_6 s^{r_{\min}}$  and  $\theta_{\delta}(s) = c_6 s^{r_{\min} - \varsigma}$  with some sufficiently small  $c_6 > 0$  under constraint  $\varsigma < r_{\min}$ ; the condition (12) is valid for  $\gamma_V(s) = c_7 s^{r_{\min} + \nu}$  and  $\gamma_1(s) = c_8 s^{r_{\min}/2\mu}$ ,  $\gamma_2(s) = c_8 s^{r_{\min} + 2\nu}/2\mu$  for some  $c_7 > 0$  and  $c_8 > 0$  if  $\nu \geq 0$ , and  $\gamma_V(s) = c_7 s^{r_{\min} + \nu}/2\mu$ ,  $\gamma_1(s) = c_8 s^{r_{\min}/2\mu}$ ,  $\gamma_2(s) = c_8 s^{r_{\min} + \nu}/2\mu$  otherwise; finally the small-gain relation (13) can be checked for  $s > 1$  if  $\varsigma \leq \frac{r_{\min}}{2} \min\{r_{\min}, r_{\min} + \nu\}$ . Since  $r_{\max} = 1$  is a possible/maximal choice, the restriction on the degree of  $g$  is  $\varsigma \leq \frac{r_{\min}}{2} \min\{r_{\min}, r_{\min} + \nu\}$ , and the following result can be deduced from Theorem 3:

**Corollary 2.** *For the case  $m = 1$ , the system (4) under Assumption 4 with  $\varsigma \leq \frac{r_{\min}}{2} \min\{r_{\min}, r_{\min} + \nu\}$  is practically ISS with respect to the inputs  $\tilde{\delta}_1, \tilde{\delta}_2 \in \mathcal{L}_{\infty}^1$ .*

**Example 4.** Consider a system

$$\dot{x}(t) = -k|x(t)|^{\kappa} \text{sign}(x(t)) + |x(t)|^{\varsigma} \text{sign}(x(t))u(t),$$

where  $x(t) \in \mathbb{R}$  and  $k, \kappa, \varsigma > 0$  are parameters. Assumption 4 is satisfied for  $\mathbf{r} = 1$  and  $\nu = \kappa - 1$ , and according to the established result,  $\varsigma \leq \frac{\kappa}{2}$ . Obviously, if  $\kappa > \varsigma$  then the system is ISS with respect to  $u \in \mathcal{L}_{\infty}^1$  and for  $u(t) = A \sin(\frac{2\pi}{T}t)$ , for some  $A > 0$  and  $T > 0$ , this system is practically ISS with respect to  $\tilde{\delta}_1(t) \leq \frac{AT}{2}$  and  $\tilde{\delta}_2(t) = \frac{2A}{\pi}$ . The results of simulation for  $k = 1$ ,  $\kappa = 0.25$ ,  $\varsigma = 1/3$  and  $A = 5$  are presented in Fig. 2, where blue, red and green curves correspond to  $T = 0.1, 1, 10$ , respectively. A kind of finite-time convergence to the origin is observed since the term  $g(x(t))u(t)$  acts as a stabilizer for the instants of time when  $u(t)$  is negative.



## V. CONCLUSIONS

Using the time-delay approach, the conditions of ISS are investigated for nonlinear affine in the input systems with respect to an average value of the disturbances. For periodic signals and static input gains, a kind of Bode amplitude plot is derived for this class of nonlinear dynamics. For integral ISS systems, the gains are obtained for periodic inputs implying boundedness of solutions. For the case of homogeneous systems, it is shown that stability can be preserved with respect to an average value of the input even in the case when degree of the perturbation gain is higher than the degree of the nominal dynamics. The results are illustrated by simple examples. Relaxing conservatism of the imposed assumptions can be considered as a direction for further research.

## REFERENCES

- [1] H. K. Khalil, *Nonlinear Systems*. Prentice Hall, third ed., 2002.
- [2] E. Sontag, "Input to state stability: Basic concepts and results," in *Nonlinear and Optimal Control Theory* (P. P. Nistri and G. Stefani, eds.), pp. 163–220, Springer-Verlag, Berlin, 2007.
- [3] D. Angeli and D. Nesić, "Power characterizations of input-to-state stability and integral input-to-state stability," *IEEE Transactions on Automatic Control*, vol. 46, no. 8, pp. 1298–1303, 2001.
- [4] E. I. Grøtli, E. Panteley, A. Chaillet, and J. T. Gravdahl, "Robustness of ISS systems to inputs with limited moving average: Application to spacecraft formations," *International Journal of Robust and Nonlinear Control*, vol. 26, no. 4, pp. 816–833, 2016.
- [5] N. N. Bogoliubov and Y. A. Mitropolsky, *Asymptotic Methods in the Theory of Non-Linear Oscillations*. Monogr. Adv. Math. Phys., Delhi/New York: Hindustan Publishing Corp./Gordon and Breach Science Publishers, 1961.
- [6] P. Kokotović and H. Khalil, *Singular perturbations in systems and control*. IEEE press, 1986.
- [7] J. Peuteman and D. Aeyels, "Exponential stability of nonlinear time-varying differential equations and partial averaging," *Math. Control Signals Systems*, vol. 15, pp. 42–70, 2002.
- [8] F. Mazenc, M. Malisoff, and M. de Queiroz, "Further results on strict Lyapunov functions for rapidly time-varying nonlinear systems," *Automatica*, vol. 42, no. 10, pp. 1663–1671, 2006.
- [9] D. Nesić and P. Dower, "A note on input-to-state stability and averaging of systems with inputs," *IEEE Transactions on Automatic Control*, vol. 46, no. 11, pp. 1760–1765, 2001.
- [10] A. Teel, L. Moreau, and D. Nesić, "A unified framework for input-to-state stability in systems with two time scales," *IEEE Transactions on Automatic Control*, vol. 48, no. 9, pp. 1526–1544, 2003.
- [11] E. Fridman and J. Zhang, "Averaging of linear systems with almost periodic coefficients: A time-delay approach," *Automatica*, vol. 122, p. 109287, 2020.
- [12] J. Zhang and E. Fridman, "L2-gain analysis via time-delay approach to periodic averaging with stochastic extension," *Automatica*, vol. 137, p. 110126, 2022.
- [13] A. Pavlov, N. van de Wouw, and H. Nijmeijer, "Frequency response functions for nonlinear convergent systems," *IEEE Transactions on Automatic Control*, vol. 52, no. 6, pp. 1159–1165, 2007.
- [14] D. Efimov and A. Polyakov, "Finite-time stability tools for control and estimation," *Foundations and Trends in Systems and Control*, vol. 9, no. 2-3, pp. 171–364, 2021.
- [15] E. Bernuau, A. Polyakov, D. Efimov, and W. Perruquetti, "Verification of ISS, iISS and IOSS properties applying weighted homogeneity," *System & Control Letters*, vol. 62, no. 12, pp. 1159–1167, 2013.
- [16] J. Peuteman and D. Aeyels, "Averaging results for homogeneous differential equations," *IFAC Proceedings Volumes*, vol. 31, no. 18, pp. 65–69, 1998. 5th IFAC Conference on System Structure and Control 1998 (SSC'98), Nantes, France, 8-10 July.
- [17] J. Peuteman and D. Aeyels, "Averaging results and the study of uniform asymptotic stability of homogeneous differential equations that are not fast time-varying," *SIAM Journal on Control and Optimization*, vol. 37, no. 4, pp. 997–1010, 1999.
- [18] A. Y. Aleksandrov, E. B. Aleksandrova, and A. P. Zhabko, "Stability analysis for a class of nonlinear nonstationary systems via averaging," *Nonlinear Dynamics and Systems Theory*, vol. 13, no. 4, pp. 332–343, 2013.
- [19] A. Y. Aleksandrov and A. A. Tikhonov, "Rigid body stabilization under time-varying perturbations with zero mean values," *Cybernetics and Physics*, vol. 7, no. 1, pp. 5–10, 2018.
- [20] S. Dashkovskiy, D. Efimov, and E. Sontag, "Input to state stability and allied system properties," *Automation and Remote Control*, vol. 72, no. 8, pp. 1579–1614, 2011.
- [21] V. Zubov, "On systems of ordinary differential equations with generalized homogeneous right-hand sides," *Izvestia vuzov. Matematika (in Russian)*, vol. 1, pp. 80–88, 1958.
- [22] X. Yang and E. Fridman, "A robust time-delay approach to continuous-time extremum seeking for multi-variable static map," in *Proc. 62nd IEEE Conference on Decision and Control (CDC)*, (Singapore), 2023.
- [23] F. Mazenc and M. Malisoff, "Trajectory based approach for the stability analysis of nonlinear systems with time delays," *IEEE Transactions on Automatic Control*, vol. 60, no. 6, pp. 1716–1721, 2015.
- [24] Z. Jiang, A. Teel, and L. Praly, "Small-gain theorem for ISS systems and applications," *Math Control Signals Syst*, vol. 7, no. 2, pp. 95–120, 1994.
- [25] S. Dashkovskiy, B. Rüffer, and F. Wirth, "An ISS small gain theorem for general networks," *Math. Control Signals Syst.*, vol. 19, pp. 93–122, 2007.
- [26] E. D. Sontag and Y. Wang, "On characterizations of the input-to-state stability property," *Systems & Control Letters*, vol. 24, no. 5, pp. 351–359, 1995.
- [27] A. R. Teel, "Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem," *IEEE Trans. Automat. Control*, vol. 43, no. 7, pp. 960–964, 1998.
- [28] M. Margaliot, E. D. Sontag, and T. Tuller, "Contraction after small transients," *Automatica*, vol. 67, pp. 178–184, 2016.
- [29] L. Rosier, "Homogeneous Lyapunov function for homogeneous continuous vector field," *Systems & Control Letters*, vol. 19, pp. 467–473, 1992.
- [30] D. Efimov, R. Ushirobira, J. A. Moreno, and W. Perruquetti, "Homogeneous Lyapunov functions: From converse design to numerical implementation," *SIAM Journal on Control and Optimization*, vol. 56, no. 5, pp. 3454–3477, 2018.