



**HAL**  
open science

## On converse Lyapunov theorem for fixed-time input-to-state stability

Denis Efimov, Andrey Polyakov

► **To cite this version:**

Denis Efimov, Andrey Polyakov. On converse Lyapunov theorem for fixed-time input-to-state stability. SIAM Journal on Control and Optimization, In press, 10.1137/22M1497596 . hal-04274297

**HAL Id: hal-04274297**

**<https://inria.hal.science/hal-04274297v1>**

Submitted on 15 Nov 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

# On converse Lyapunov theorem for fixed-time input-to-state stability

Efimov Denis and Polyakov Andrey

## Abstract

Input-to-state stability is one of the most utilizable robust stability properties for nonlinear dynamical systems, while (nearly) fixed-time convergence is a kind of decay for trajectories of disturbance-free systems that is independent in initial conditions. The presence of both these features for a system can be checked by existence of a proper Lyapunov function. The objective of this work is to provide the conditions for a converse result that (nearly) fixed-time input-to-state stable systems admit a respective Lyapunov function. Similar auxiliary results for uniform finite-time stability and uniform (nearly) fixed-time stability are obtained.

## I. INTRODUCTION

Analysis of stability properties of dynamical systems is an important theoretical problem having applications in many research and development domains [7]. The method of Lyapunov functions constitutes the most popular approach for stability investigation [8], [19], [7], since existence of a proper Lyapunov function is an equivalent (necessary and sufficient) condition of respective stability properties. Establishment of such an equivalence is important: it formulates the canonical representations of Lyapunov functions and conditions to check in the applications (avoiding multiple similar framings). However, the issue of finding a Lyapunov function for a given nonlinear system still has no generic solution.

The problem becomes even more sophisticated if the system of interest is subject to external disturbances or/and the rates of convergence have to be established and quantified. A popular and powerful framework for robust stability analysis is created around the input-to-state stability (ISS) concept [16], [3], which also disposes equivalent characterizations in terms of existence of different Lyapunov functions. For the rates of decay, recently, finite-time and (nearly) fixed-time ones got some popularity [15], [1], [14]: the former corresponds to an exact convergence to the equilibrium with a finite settling time dependent on initial conditions, while the latter is related with existence of the settling-time function independent on initial deviations (globally bounded). The equivalent Lyapunov characterizations of the finite-time stability are well known [1], while for the fixed-time stability they are quite new [11]. The ISS systems having finite-time convergence rates in the disturbance-free setting are introduced in [6] with sufficient conditions in terms of existence of Lyapunov functions, and its necessary counterpart is proposed in [12]. The sufficient characterization of fixed-time ISS is also given in [12] (see also a survey [5]).

This work aims on filling the gap by proposing the necessary Lyapunov conditions for (nearly) fixed-time ISS property. To this end the strategy proposed in [17] will be used: *i*) the existence of Lyapunov functions will be shown for uniformly (nearly) fixed-time stable systems (when the inputs take values in a compact set) in Section III; *ii*) it will be demonstrated that (nearly) fixed-time ISS property (introduced in Section IV) can be related with uniform (nearly) fixed-time stability of an auxiliary system (in Section V); *iii*) under additional mild restrictions these results will be united in Section VI. The difficulty of such a development is twofold: first, systems possessing finite/fixed-time convergence rates are not Lipschitz continuous at the origin; second, a Lyapunov characterization has to guarantee the respective accelerated convergence of disturbance-free trajectories.

## Notation

Denote by  $\mathbb{R}$  the set of real numbers and  $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$ . For  $x \in \mathbb{R}^n$ ,  $\|x\|$  corresponds to its (Euclidean) norm.

For a Lebesgue measurable function of time  $d : \mathbb{R}_+ \rightarrow \mathbb{R}^m$ , define the norms  $\|d\|_{[t_0, t_1]} = \text{ess sup}_{s \in [t_0, t_1]} \|d(s)\|$  for  $[t_0, t_1] \subset \mathbb{R}_+$  and  $\|d\|_\infty = \text{ess sup}_{s \geq 0} \|d(s)\|$ , then the space of such functions with  $\|\cdot\|_\infty < +\infty$  we further denote as  $\mathcal{L}_\infty^m$ .

The class of continuous functions  $\mathbb{X} \rightarrow \mathbb{Y}$  is denoted by  $C(\mathbb{X}, \mathbb{Y})$  for two metric spaces  $\mathbb{X}$  and  $\mathbb{Y}$ .

### Problem statement

So, our goal is to provide the converse Lyapunov function results for the ISS property in (nearly) fixed-time sense for the system:

$$\dot{x}(t) = f(x(t), d(t)), \quad t > 0, \quad x(0) = x_0, \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  and  $d(t) \in \mathbb{R}^m$ ,  $d \in \mathcal{L}_\infty^m$ ;  $f \in C(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$  ensures forward existence and uniqueness of solutions of the system at least locally in time,  $f(0, 0) = 0$ . For any  $x_0 \in \mathbb{R}^n$  and  $d \in \mathcal{L}_\infty^m$  the respective solution is denoted by  $x(t, x_0, d)$  with  $x(0, x_0, d) = x_0$ .

The converse Lyapunov theorem for finite-time ISS has been obtained in [12] under additional restrictions:

### Assumption 1. Let

- (i)  $f \in C(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$  be locally Lipschitz continuous in  $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^m$ ;
- (ii) there exist  $L \in C(\mathbb{R}_+, \mathbb{R}_+)$  such that the inequality

$$\|f(x, d) - f(x, 0)\| \leq L(\|x\|)\|d\|$$

is satisfied for all  $(x, d) \in \mathbb{R}^n \times \mathbb{R}^m$ .

In the sequel we assume that Assumption 1 is verified for (1).

## II. PRELIMINARIES

### A. Comparison functions

A function  $\sigma \in C(\mathbb{R}_+, \mathbb{R}_+)$  belongs to class  $\mathcal{K}$  if it is strictly increasing and  $\sigma(0) = 0$ ; it belongs to class  $\mathcal{K}_\infty$  if it is also unbounded.

A function  $\beta \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$  belongs to class  $\mathcal{KL}$  if  $\beta(\cdot, r) \in \mathcal{K}$  and  $\beta(r, \cdot)$  is decreasing to zero for any fixed  $r > 0$ .

A function  $\chi \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$  belongs to class  $\mathcal{FKL}$  if  $\chi(\cdot, 0) \in \mathcal{K}$ ,  $\chi(r, \cdot)$  is decreasing to zero for any fixed  $r > 0$  and there is  $\mathbb{T} \in \mathcal{K}$  such that  $\chi(r, t) = 0$  for all  $t \geq \mathbb{T}(r)$ . A function  $\chi \in \mathcal{FKL}$  with  $\sup_{r \geq 0} \mathbb{T}(r) < +\infty$  belongs to the class  $\mathcal{FxKL}$ .

Finally,  $\chi \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$  belongs to class  $n\mathcal{FKL}$  if  $\chi(\cdot, 0) \in \mathcal{K}$ ,  $\chi(r, \cdot)$  is decreasing to zero for any fixed  $r > 0$ , and there is  $\mathcal{T} \in \mathcal{K}$  such that  $\chi(r, t) \leq \rho$  for any  $\rho > 0$ , for all  $t \geq \mathcal{T}(\rho^{-1})$  and all  $r \in \mathbb{R}_+$ .

### B. Derivatives and gradient

For  $g \in C(\mathbb{R}, \mathbb{R})$ , the upper right Dini derivative [4] is given by

$$D^+g(t) = \limsup_{h \rightarrow 0^+} \frac{g(t+h) - g(t)}{h}$$

for any  $t \in \mathbb{R}$ . For  $V \in C(\mathbb{R}^n, \mathbb{R})$ , the upper right Dini derivative in the direction of  $v \in \mathbb{R}^n$  at point  $x \in \mathbb{R}^n$  is defined as

$$DV(x)v = \limsup_{h \rightarrow 0^+} \frac{V(x+hv) - V(x)}{h}.$$

For  $V \in C(\mathbb{R}^n, \mathbb{R})$ , the derivative along the solutions of (1) is defined as

$$\dot{V}(t) = D^+V(x(t, x_0, d))$$

for any  $t \geq 0$ ,  $x_0 \in \mathbb{R}^n$  and  $d \in \mathcal{L}_\infty^m$ ; and if  $V$  is locally Lipschitz continuous in  $\mathbb{R}^n \setminus \{0\}$ , then

$$\dot{V}(x, d) := DV(x)f(x, d)$$

for any  $x \in \mathbb{R}^n$  and  $d \in \mathbb{R}^m$  (in such a case  $\dot{V}(t) = \dot{V}(x(t, x_0, d), d(t))$  for almost all  $t \geq 0$ ). If  $V$  is locally Lipschitz continuous, then its generalized gradient is defined as follows [2]:

$$\partial V(x) = \text{co} \left\{ \lim_{i \rightarrow +\infty} \nabla V(x_i) : x_i \rightarrow x, \nabla V(x_i) \text{ exists} \right\},$$

where  $\text{co}\{\cdot\}$  takes the convex hull.

### III. UNIFORM COMPLETE NEARLY FIXED-TIME STABILITY

Denote  $\mathcal{D} = \{d \in \mathcal{L}_\infty^m : \|d\|_\infty \leq 1\}$  and  $\mathbb{D} = \{d \in \mathbb{R}^m : \|d\| \leq 1\}$  in order to analyze the behavior of (1) with a bounded input  $d$ . The following uniform stability properties and their necessary Lyapunov characterizations derived below constitute an intermediate step for the main result of this work:

**Definition 1.** 1) The system (1) is called uniformly globally asymptotically stable (uGAS) at the origin [10], if there exists  $\beta \in \mathcal{KL}$  such that

$$\|x(t, x_0, d)\| \leq \beta(\|x_0\|, t) \quad \forall t \geq 0$$

for all  $x_0 \in \mathbb{R}^n$  and  $d \in \mathcal{D}$ .

2) A uGAS system (1) is called uniformly finite-time stable (uFTS) at the origin if for all  $x_0 \in \mathbb{R}^n$  and  $d \in \mathcal{D}$  there is  $T_{x_0, d} \in [0, T(x_0)]$  such that

$$x(t, x_0, d) = 0 \quad \forall t \geq T_{x_0, d}, \quad x(t, x_0, d) \neq 0 \quad \forall t \in [0, T_{x_0, d}),$$

where  $T : \mathbb{R}^n \rightarrow \mathbb{R}_+$  is a locally bounded settling-time function  $T(x_0) = \sup_{d \in \mathcal{D}} T_{x_0, d}$ .

3) A uGAS system (1) is called uniformly nearly fixed-time stable (unFXTS) at the origin if for any  $\rho > 0$  there is  $T_\rho \in \mathbb{R}_+$  such that

$$\|x(t, x_0, d)\| \leq \rho \quad \forall t \geq T_\rho$$

for all  $x_0 \in \mathbb{R}^n$  and  $d \in \mathcal{D}$ .

4) A unFXTS system (1) is called uniformly fixed-time stable (uFXTS) at the origin if additionally  $\sup_{\rho > 0} T_\rho < +\infty$ .

Equivalently, uFTS, unFXTS or uFXTS of (1) can be formulated as the uGAS property with  $\beta \in \mathcal{FKL}$ ,  $\beta \in n\mathcal{F}x\mathcal{KL}$  or  $\beta \in \mathcal{F}x\mathcal{KL}$ , respectively. Another definition of uFTS (for time-varying systems) can be found in [13].

To formulate the converse Lyapunov function theorem for u(n)FXTS we will consider a special subclass of these systems:

**Definition 2.** A u(n)FXTS system is called complete if for any  $x_0 \in \mathbb{R}^n \setminus \{0\}$  and any  $d \in \mathcal{D}$  there exists a unique solution of the system  $\dot{x}^*(t) = -f(x^*(t), d(t))$  with  $x^*(0) = x_0$ , denoted as  $x^*(t, x_0, d)$  and defined for  $t \in [0, \tau_{x_0, d})$  such that

$$\limsup_{t \rightarrow \tau_{x_0, d}} \|x^*(t, x_0, d)\| = +\infty,$$

with  $\tau : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}_+$  being a locally bounded escape-time function satisfying  $\tau(x_0) = \sup_{d \in \mathcal{D}} \tau_{x_0, d}$ .

Necessary and sufficient Lyapunov characterizations of uGAS property (for the case of locally Lipschitz continuous  $f$ ) can be found in [10].

**Definition 3.**  $V \in C(\mathbb{R}^n, \mathbb{R}_+)$  is called unFXTS-Lyapunov function (LF) for the system (1) if there exist  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ ,  $\alpha > 1$  and  $c > 0$  such that

$$\begin{aligned} \alpha_1(\|x\|) &\leq V(x) \leq \alpha_2(\|x\|), \quad \forall x \in \mathbb{R}^n, \\ \dot{V}(t) &\leq -cV^\alpha(x(t, \xi, d)) \end{aligned}$$

for almost all  $t \geq 0$ , all  $\xi \in \mathbb{R}^n$  and all  $d \in \mathcal{D}$ . If  $\alpha \in [0, 1)$ , then such a  $V$  is called uFTS-LF.

For  $\alpha = 1$ ,  $V$  has at least asymptotic (exponential) convergence, which for  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  allows only uGAS property of (1) to be concluded. If  $V$  is locally Lipschitz continuous in  $\mathbb{R}^n \setminus \{0\}$ , then the last condition of Definition 3 can be replaced with

$$\dot{V}(x, d) \leq -cV^\alpha(x)$$

for almost all  $x \in \mathbb{R}^n$  and all  $d \in \mathbb{D}$ .

It is straightforward that existence of a uFTS/unFxTS-LF implies the respective stability property, this claim can be proven using standard arguments following the properties of LFs given in Definition 3. Note that uGAS property in (1) is realizable provided that  $f(0, d) = 0$  for any  $d \in \mathbb{D}$ , then Assumption 1 implies a uniform in the second argument local Lipschitz continuity of  $f$  on  $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{D}$ .

**Theorem 1.** *If the system (1) is complete unFxTS with a continuous escape-time function  $\tau$ , then there exists a unFxTS-LF.*

*Proof.* The argumentation is split on the following steps. 1) The result is recalled that for a locally Lipschitz  $f$  there is a Lyapunov function. 2) Since  $f$  is not Lipschitz continuous at the origin, it is shown how to construct an auxiliary system possessing this property globally. 3) It is demonstrated that a global Lyapunov function  $W$  for the auxiliary Lipschitz system is also one for the original dynamics (1). 4) Due to assumed completeness of unFxTS property, the system (1) in backward time is studied, and based on the escape-time function, a Lyapunov function  $V_1$  is designed for big deviations of the state with desired upper estimate on the derivative. 5) The upper estimate on the derivative of  $W$  is refined, and a local Lyapunov function  $V_2$  is introduced based on  $W$ . 6) The Lyapunov functions  $V_1$  and  $V_2$  are united.

**I** If  $f$  is uniformly in the second argument locally Lipschitz continuous on  $\mathbb{R}^n \times \mathbb{D}$  (i.e., there exists  $\hat{L} \in C(\mathbb{R}_+, \mathbb{R}_+)$  such that the inequality

$$\|f(x_1, d) - f(x_2, d)\| \leq \hat{L}(\|x_1\| + \|x_2\|)\|x_1 - x_2\|$$

is satisfied for all  $x_1, x_2 \in \mathbb{R}^n$  and  $d \in \mathbb{D}$ ), since any unFxTS system is uGAS, then there is a smooth LF  $W \in C(\mathbb{R}^n, \mathbb{R}_+)$  satisfying for all  $x \in \mathbb{R}^n$  and  $d \in \mathbb{D}$ :

$$\alpha'_1(\|x\|) \leq W(x) \leq \alpha'_2(\|x\|), \quad \dot{W}(x, d) \leq -\eta(W(x)) \quad (2)$$

for some  $\alpha'_1, \alpha'_2 \in \mathcal{K}_\infty$  and  $\eta \in \mathcal{K}$  (see, for example, Theorem 2.9 in [10]). Define two positive constants,  $r$  and  $R$ , such that  $0 < 2r < R$ , and denote  $\Omega_r = \{x \in \mathbb{R}^n : W(x) \geq r\}$ .

**II** If  $f$  is not Lipschitz at the origin (under Assumption 1), define functions (see also Section 8 in [9])

$$g_\delta(v) = \frac{1}{v} \int_v^{2v} \sup_{s \leq \|x\| \leq \delta} \max_{i=1, \dots, n} \sup_{d \in \mathbb{D}} \sup_{\ell_i \in \partial f_i(x, d)} \|\ell_i\| ds,$$

$$\mu_\delta(v) := \min \left\{ 1, \frac{1 + \frac{1}{v} \int_v^{2v} \sup_{\|x\| \leq s, d \in \mathbb{D}} \|f(x, d)\| ds \, b(v)v}{1 + \frac{1}{\delta} \int_\delta^{2\delta} \sup_{\|x\| \leq s, d \in \mathbb{D}} \|f(x, d)\| ds \, \delta} \right\},$$

$$b(v) := \begin{cases} 1 & g_\delta(0) < +\infty \\ \frac{g_\delta(\delta)}{g_\delta(v)} & \text{otherwise} \end{cases}$$

for any  $v \in \mathbb{R}_+$  and some  $\delta > R$ . Note that by design,  $D^+ \mu_\delta(0) < +\infty$ ,  $\mu_\delta$  differentiable on  $(0, \delta)$  and globally Lipschitz continuous, increasing, bounded by 1, equals to 1 when  $v \geq \delta$ , strictly positive outside of the origin and  $\mu_\delta(0) = 0$ . Define a vector field

$$f_\delta(x, d) := \begin{cases} f(x, d), & \text{if } \|x\| \geq \delta \\ \mu_\delta(\|x\|)f(x, d), & \text{if } \|x\| < \delta \end{cases},$$

which we claim is uniformly locally Lipschitz continuous by constitution on  $\mathbb{R}^n \times \mathbb{D}$ . Indeed, the function  $f$  possesses this

property outside of the origin by the imposed hypothesis, and for almost all  $\|x\| < \delta$  we have that

$$\begin{aligned} \left\| \frac{\partial f_\delta(x, d)}{\partial x} \right\| &= \left\| \frac{\partial \mu_\delta(\|x\|) f(x, d)}{\partial x} \right\| \\ &\leq \left\| \frac{\partial \mu_\delta(\|x\|)}{\partial x} \right\| \|f(x, d)\| + \mu_\delta(\|x\|) \left\| \frac{\partial f(x, d)}{\partial x} \right\|, \end{aligned}$$

where

$$\frac{\partial f(x, d)}{\partial x} \in \begin{bmatrix} \ell_1^\top \\ \vdots \\ \ell_n^\top \end{bmatrix}, \quad \ell_i \in \partial f_i(x, d), \quad i = 1, \dots, n$$

and  $\left\| \frac{\partial f(x, d)}{\partial x} \right\| \leq g_\delta(\|x\|)$  by construction, then the direct computations show that

$$\left\| \frac{\partial f_\delta(x, d)}{\partial x} \right\| \leq C < +\infty$$

for almost all  $\|x\| < \delta$  and all  $d \in \mathbb{D}$ . Since  $f_\delta$  is Lipschitz continuous on  $\mathbb{R}^n \setminus \{0\}$  then for any  $x_1, x_2$  satisfying  $\|x_i\| < \delta$ ,  $i = 1, 2$  and  $\exists \lambda \in [0, 1] : \lambda x_1 + (1 - \lambda)x_2 = 0$  we have

$$\|f_\delta(x_1, d) - f_\delta(x_2, d)\| = \left\| \int_0^1 (x_1 - x_2)^\top \frac{\partial f_\delta(x_1 + tx_2, d)}{\partial x} dt \right\| \leq C \|x_1 - x_2\|.$$

On the other hand, for  $\|x\| < \delta$  we get:

$$\begin{aligned} \|f_\delta(x, d)\| &\leq \|\mu_\delta(\|x\|) f(x, d)\| \\ &= \frac{\|f(x, d)\|}{1 + \frac{1}{\delta} \int_\delta^{2\delta} \sup_{\|z\| \leq s, d \in \mathbb{D}} \|f(z, d)\| ds} \cdot \frac{1 + \frac{1}{\|x\|} \int_{\|x\|}^{2\|x\|} \sup_{\|z\| \leq s, d \in \mathbb{D}} \|f(z, d)\| ds}{\delta} b(\|x\|) \|x\| \\ &\leq \frac{1 + \frac{1}{\|x\|} \int_{\|x\|}^{2\|x\|} \sup_{\|z\| \leq s, d \in \mathbb{D}} \|f(z, d)\| ds}{\delta} \|x\| \leq \frac{1 + \sup_{\|z\| \leq 2\|x\|, d \in \mathbb{D}} \|f(z, d)\|}{\delta} \|x\| \\ &\leq C' \|x\|, \quad C' = \frac{1 + \sup_{\|z\| \leq 2\delta, d \in \mathbb{D}} \|f(z, d)\|}{\delta}. \end{aligned}$$

If  $\exists \lambda \in [0, 1] : \lambda x_1 + (1 - \lambda)x_2 = 0$ , then  $\lambda \|x_1 - x_2\| = \|x_2\|$  and  $(1 - \lambda)\|x_1 - x_2\| = \|x_1\|$ , hence,

$$\|f_\delta(x_1, d) - f_\delta(x_2, d)\| \leq C' \|x_1\| + C' \|x_2\| = C'(1 - \lambda)\|x_1 - x_2\| + C'\lambda \|x_1 - x_2\| = C' \|x_1 - x_2\|.$$

So  $f_\delta$  is locally Lipschitz for all  $x \in \mathbb{R}^n$  and  $d \in \mathbb{D}$ .

**III** Now let us consider the system

$$\dot{x} = f_\delta(x, d),$$

where  $f_\delta$  is, as above, a locally Lipschitz continuous function uniformly in the second argument, and it is unFxTS since (1) has this property (multiplication by a continuous bounded strictly positive scalar function  $\mu_\delta$  does not influence the stability, it acts as a time re-scaling). Following the converse results on existence of uGAS LFs, for this auxiliary system there exists a smooth  $W \in C(\mathbb{R}^n, \mathbb{R}_+)$  such that (2) is verified for all  $x \in \mathbb{R}^n$  and  $d \in \mathbb{D}$ , for some  $\alpha'_1, \alpha'_2 \in \mathcal{K}_\infty$  and  $\eta \in \mathcal{K}$ . Note that in such a case

$$0 \geq -\eta(W(x)) \geq \dot{W}(x, d) = DW(x) f_\delta(x, d) = \mu_\delta(\|x\|) DW(x) f(x, d) \geq DW(x) f(x, d)$$

for all  $\|x\| < \delta$  and  $d \in \mathbb{D}$ . Therefore, an unFxTS system (1) admits an LF  $W$  satisfying (2) for all  $x \in \mathbb{R}^n$  and  $d \in \mathbb{D}$  under Assumption 1.

**IV** Consider the system (1) in the “backward” time:

$$\dot{x}^*(t) = -f(x^*(t), d^*(t)), \quad t \geq 0,$$

then due to the complete unFxTS property (see Definition 2), for any  $x_0 \in \Omega_r$  and  $d^* \in \mathcal{D}$  there is a unique solution  $x^*(t, x_0, d^*)$  that is defined on some bounded interval of time  $t \in [0, \tau_{x_0, d^*}] \subseteq [0, \tau(x_0))$  (i.e.,  $\tau(x_0) \in [0, T_r]$  for all  $x_0 \in \Omega_r$ , where  $T_r > 0$  is a finite constant), and  $\limsup_{t \rightarrow \tau_{x_0, d^*}} \|x^*(t, x_0, d^*)\| = +\infty$ . For a continuous function  $\tau$ , by its definition,

$$\tau(x^*(t, x_0, d^*)) \leq \max\{0, \tau(x_0) - t\},$$

then

$$D^+\tau(x^*(t, x_0, d^*)) \leq -1$$

for almost all  $t \in [0, \tau_{x_0, d^*})$ , all  $x_0 \in \Omega_r$  and  $d^* \in \mathcal{D}$ . For any  $x_0 \in \Omega_r$ ,  $d^* \in \mathcal{D}$  and  $T \in [0, \tau_{x_0, d^*})$ , there exists  $d \in \mathcal{D}$  such that

$$x^*(t, x_0, d^*) = x(T - t, x^*(T, x_0, d^*), d)$$

for all  $t \in [0, T]$ , or in other words, solution  $x$  of (1) in negative time argument coincides with  $x^*$  on any finite interval of time for a properly selected inputs and  $x_0 \in \Omega_r$ . Hence,

$$\frac{\partial \tau(x(t, x_0, d))}{\partial t} = D^+\tau(x(t, x_0, d)) \geq 1$$

for almost all  $t \geq 0$ , any  $x_0 \in \Omega_r$  and  $d \in \mathcal{D}$ . For  $x \in \Omega_r$  and  $\alpha > 1$ , introduce a candidate LF function

$$V_1(x) = \rho \tau(x)^{\frac{1}{1-\alpha}}$$

with some  $\rho > 0$  specified later. The function  $V_1$  is radially unbounded due to the properties of  $\tau$  and since  $\alpha > 1$ , its time derivative computed along trajectories of (1) can be rewritten as follows:

$$\begin{aligned} \dot{V}_1(t) &= -\frac{\rho}{\alpha - 1} \tau(x(t, x_0, d))^{\frac{\alpha}{1-\alpha}} \frac{\partial \tau(x(t, x_0, d))}{\partial t} \\ &\leq -\frac{\rho}{\alpha - 1} \tau(x(t, x_0, d))^{\frac{\alpha}{1-\alpha}} = -\frac{\rho^{1-\alpha}}{\alpha - 1} V_1^\alpha(x(t, x_0, d)) \end{aligned}$$

for almost all  $t \geq 0$  such that  $x(t, x_0, d) \in \Omega_r$  and  $d \in \mathcal{D}$ .

**V** Moreover, consider the related to the dynamics of  $W$  a scalar comparison system in the backward time:

$$\dot{y}(t) = \eta(y(t)), \quad t \in [0, \tau_\eta(y(0))]$$

with initial conditions  $y(0) \in (0, R]$ ; the escape-time function  $\tau_\eta : (0, R] \rightarrow \mathbb{R}_+$  to the level  $y = R$  is defined as follows

$$\tau_\eta(y(0)) = t^* \geq 0 : y(t^*) = R.$$

Then, by construction,

$$\tau_\eta(y) = \int_y^R \frac{ds}{\eta(s)}, \quad \frac{d\tau_\eta(y)}{dy} = \frac{-1}{\eta(y)} < 0$$

for all  $y \in (0, R]$ . The latter meant  $\tau_\eta$  is a monotone function and

$$\lim_{y \rightarrow 0} \frac{1}{\tau_\eta(y)} = \lim_{y \rightarrow 0} \frac{1}{\int_y^R \frac{ds}{\eta(s)}} = \mathfrak{b} \in [0, +\infty).$$

Applying argumentation similar as for the function  $\tau$  we derive

$$\frac{\partial \tau_\eta(y(t))}{\partial t} = \frac{d\tau_\eta(y)}{dy} \Big|_{y=y(t)} \dot{y}(t) = -1$$

for all  $t \in [0, \tau_\eta(y_0)]$ . Returning to the dynamics of  $W$ , if  $W(x(t, x_0, d)) = W(t) = y(t)$ , while  $\dot{W}(t) \leq -\dot{y}(t)$  by design, we get

$$\frac{d\tau_\eta(y)}{dy} \Big|_{y=W(t)} \dot{W}(t) \geq 1$$

for all instants of time such that  $W(t) \in (0, R]$ . Define a local Lyapunov function candidate

$$V_2(x) = \left( \frac{1}{\tau_\eta(W(x))} - \mathbf{b} \right)^{\frac{1}{\alpha-1}}$$

for  $W(x) \leq R$ , then

$$\dot{V}_2(x, d) \leq -\frac{1}{\alpha-1} \left( \frac{1}{\tau_\eta(W(x))} - \mathbf{b} \right)^{\frac{\alpha}{\alpha-1}} = -\frac{1}{\alpha-1} V_2^\alpha(x)$$

for  $0 < W(x) \leq R$  and  $d \in \mathbb{D}$ .

**VI** Finally, let us unite the LF candidates  $V_1$  and  $V_2$  to obtain a strict LF  $V \in C(\mathbb{R}^n, \mathbb{R}_+)$  as

$$V(x) = \begin{cases} V_2(x) & \text{if } W(x) \leq r, \\ V_1(x) & \text{if } W(x) \geq 2r, \\ \lambda(x)V_1(x) + (1 - \lambda(x))V_2(x) & \text{otherwise,} \end{cases}$$

where  $\lambda(x) = r^{-1}(W(x) - r)$  with  $\dot{\lambda} \leq -r^{-1}\eta(W(x)) < 0$ , and  $\rho$  is chosen in a way to guarantee that  $V_1(x) > V_2(x)$  for all  $x \in \Omega_{2r} \setminus \Omega_r$ . Such a design of  $V$  ensures all the requirements imposed on a unFxTS-LF due to the proven properties of  $V_1, V_2$ , and since

$$\dot{V} \leq -\frac{1}{\alpha-1} \begin{cases} V_2^\alpha(x) & \text{if } W(x) \leq r, \\ \rho^{1-\alpha} V_1^\alpha(x) & \text{if } W(x) \geq 2r, \\ \rho^{1-\alpha} \lambda(x) V_1^\alpha(x) + (1 - \lambda(x)) V_2^\alpha(x) & \text{otherwise.} \end{cases}$$

Hence, there exists  $c > 0$  such that

$$\dot{V}(t) \leq -cV^\alpha(x(t, \xi, d))$$

for almost all  $t \geq 0$ , all  $\xi \in \mathbb{R}^n \setminus \{0\}$  and  $d \in \mathcal{D}$  as required.  $\square$

For the sufficient part, if we assume that there is a unFxTS-LF, then the system (1) is unFxTS. Consider the system behavior in the backward time, then we obtain that

$$-\frac{\partial V(x)}{\partial x} f(x, d) \geq cV^\alpha(x),$$

from which we conclude that the escape-time function should be upper bounded by a continuous function, and hence it is also continuous at infinity (and it is difficult to substantiate continuity of  $\tau$  used in Theorem 1).

**Theorem 2.** *If the system (1) is uFTS with a continuous settling-time function  $T$ , then there exists a uFTS-LF.*

*Proof.* For a continuous settling-time function  $T$ , by its definition we have:

$$T(x(t, x_0, d)) \leq \max\{0, T(x_0) - t\}$$

for all  $t \geq 0$ , then

$$\frac{\partial T(x(t, x_0, d))}{\partial t} = D^+ T(x(t, x_0, d)) \leq -1$$

for all  $t \in [0, T_{x_0, d})$  and all  $x_0 \in \mathbb{R}^n$  and  $d \in \mathcal{D}$ . Let  $V(x) = T^{\frac{1}{1-\alpha}}(x)$  for some  $\alpha \in [0, 1)$ , then

$$\dot{V}(t) \leq -\frac{1}{1-\alpha} T^{\frac{\alpha}{1-\alpha}}(x(t, x_0, d)) = -cV^\alpha(x(t, x_0, d)), \quad c = \frac{1}{1-\alpha},$$

for almost all  $t \geq 0$ , all  $x_0 \in \mathbb{R}^n$  and  $d \in \mathcal{D}$ . If such a LF  $V$  is radially unbounded, then it is uFTS-LF as needed; otherwise, as in the proof of Theorem 1 it can be combined with a LF that exists since a uFTS system is also UGAS.  $\square$

Obviously, if there is a uFTS-LF  $V$  as in Definition 3, then the system is uFTS, and from the estimate for  $V$  we get that  $T(x) \leq \frac{V^{1-\alpha}(x)}{c(1-\alpha)}$  for all  $x \in \mathbb{R}^n$ , which implies continuity of  $T$  at the origin.



*Remark 1.* If in the conditions of theorems 1 and 2 we additionally assume that  $\tau$  and  $T$  are locally Lipschitz continuous in  $\mathbb{R}^n \setminus \{0\}$ , respectively, then the obtained LFs  $V$  also inherit the same property.

The proofs of theorems 1 and 2 demonstrate the main technical difficulties of the studied problem: the lack of Lipschitz continuity at the origin in (1) and the special forms of upper estimates on  $\dot{V}$  that should be established.

#### IV. ROBUST STABILITY

The stability concepts of interest and their Lyapunov characterizations are formulated in the following definitions completing [16], [6], [12]:

**Definition 4.** The system (1) is called ISS, if there exist  $\beta \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}$  such that

$$\|x(t, x_0, d)\| \leq \beta(\|x_0\|, t) + \gamma(\|d\|_{[0,t]})$$

for all  $x_0 \in \mathbb{R}^n$ ,  $d \in \mathcal{L}_\infty^m$  and  $t \geq 0$ . It is called FTS-ISS or (n)FxT-ISS if  $\beta \in \mathcal{FKL}$  or  $\beta \in \mathcal{F}x\mathcal{KL}$  ( $\beta \in n\mathcal{F}x\mathcal{KL}$ ), respectively.

If  $d = 0$ , then we recover the conventional FTS/(n)FxTS properties [5] (since  $\mathbb{D} = \{0\}$  is singleton, the uniformity required in Definition 1 can be omitted).

**Definition 5.** [16] The system (1) is said to possess the asymptotic gain (AG) property, if there exists  $\gamma \in \mathcal{K}$  such that

$$\lim_{t \rightarrow +\infty} \|x(t, x_0, d)\| \leq \gamma(\|d\|_\infty)$$

for all  $x_0 \in \mathbb{R}^n$  and  $d \in \mathcal{L}_\infty^m$ .

**Definition 6.** [16] The system (1) is said to have the global stability (GS) property, if there exist  $\sigma_1, \sigma_2 \in \mathcal{K}$  such that

$$\|x(t, x_0, d)\| \leq \max\{\sigma_1(\|x_0\|), \sigma_2(\|d\|_\infty)\}$$

for all  $x_0 \in \mathbb{R}^n$ ,  $d \in \mathcal{L}_\infty^m$  and  $t \geq 0$ .

A system (1), with a locally Lipschitz continuous  $f$ , has AG and GS properties if and only if it is ISS [3].

**Definition 7.** A  $V \in C(\mathbb{R}^n, \mathbb{R}_+)$ , locally Lipschitz continuous on  $\mathbb{R}^n \setminus \{0\}$ , is called ISS-LF for the system (1) if there exist  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  and  $\eta, \varrho \in \mathcal{K}$  such that for almost all  $x \in \mathbb{R}^n$  and all  $d \in \mathbb{R}^m$ :

$$\begin{aligned} \alpha_1(\|x\|) &\leq V(x) \leq \alpha_2(\|x\|), \\ V(x) \geq \varrho(\|d\|) &\implies \dot{V}(x, d) \leq -\eta(V(x)). \end{aligned}$$

It is called FTS-ISS-LF, nFxT-ISS-LF or FxT-ISS-LF if  $\eta(s) = cs^a$ ,  $\eta(s) = cs^b$  or  $\eta(s) = c_1s^a + c_2s^b$ , respectively, for  $c, c_1, c_2 > 0$ ,  $a \in [0, 1)$  and  $b > 1$ .

The relations between FTS-ISS-LF and FTS-ISS property have been investigated in [6] and [12], sufficient and necessary parts, respectively.

**Theorem 3.** *If there exists a (n)FxT-ISS-LF for the system (1), then it is (n)FxT-ISS with the AG  $\gamma = \alpha_1^{-1} \circ \varrho$ . If additionally (1) is complete (n)FxTS for  $d \equiv 0$ , then the escape-time function is continuous at infinity; while in FxTS case, the settling-time function is continuous at the origin.*

*Proof.* The proof of this theorem is rather standard and can be partly found in the existing literature [16], [6], [12]. For completeness of the paper, consider the FxT-ISS case. If there exists a FxT-ISS-LF for (1), then

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \quad \forall x \in \mathbb{R}^n$$

for some  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ , and  $\|x\| \geq \varrho(\|d\|)$  implies that

$$DV(x)f(x, d) \leq -c_1V^a(x) - c_2V^b(x), \quad (3)$$

where  $c_1, c_2 > 0$ ,  $a \in [0, 1)$  and  $b > 1$ , hence, the system is ISS. Further, the following steps are performed below: 1) an estimate is derived for the values of  $V$  sufficiently higher than the amplitude of  $d$ ; 2) an estimate is obtained for small values of  $V$ ; 3) these estimates are combined; 4) the continuity of the settling-time and the escape-time functions is discussed.

**I** Let us define the set  $\mathcal{V}_d = \{x : V(x) \geq \alpha_2 \circ \varrho(\|d\|_\infty)\}$ . We have  $\alpha_2(\|x\|) \geq V(x) \geq \alpha_2 \circ \varrho(\|d\|_\infty)$  for any  $x \in \mathcal{V}_d$ , which implies that  $\|x\| \geq \varrho(\|d\|)$  and by (3),  $\mathbb{R}^n \setminus \mathcal{V}_d$  is an invariant and attractive set. Using the comparison lemma and direct integration, it is straightforward to obtain a class- $\mathcal{F}x\mathcal{K}\mathcal{L}$  function  $\beta(r, t)$  such that

$$\|x(t, x_0, d)\| \leq \beta(\|x_0\|, t) \quad \text{while } x(t, x_0, d) \in \mathcal{V}_d, \quad (4)$$

where  $\beta(r, t) = 0$  for all  $t \geq \tilde{T}_\beta$  and all  $r \in \mathbb{R}_+$ , for some  $\tilde{T}_\beta > 0$ .

**II** If  $x \notin \mathcal{V}_d$  then  $V(x) < \alpha_2 \circ \varrho(\|d\|_\infty)$  and therefore  $\|x\| \leq \vartheta(\|d\|_\infty)$ , where  $\vartheta = \alpha_1^{-1} \circ \alpha_2 \circ \varrho$ . In addition,  $\mathbb{R}^n \setminus \mathcal{V}_d$  is invariant so that

$$\|x(t, x_0, d)\| \leq \vartheta(\|d\|_\infty) \quad \text{while } x(t, x_0, d) \notin \mathcal{V}_d, \quad (5)$$

**III** Combining (4) and (5) gives

$$\|x(t, x_0, d)\| \leq \beta(\|x_0\|, t) + \vartheta(\|d\|_\infty) \quad \forall t \geq 0,$$

for any  $x_0 \in \mathbb{R}^n$  and  $d \in \mathcal{L}_\infty^m$ , and FxT-ISS for (1) is obtained.

**IV** The continuity at the origin of the settling-time function follows the continuity of its upper estimate derived for LF: i.e.,  $DV(x)f(x, 0) \leq -c_1 V^a(x)$  for almost all  $x \in \mathbb{R}^n$  implies  $T(x_0) \leq \frac{V^{1-a}(x_0)}{(1-a)c_1}$  for any initial condition  $x_0 \in \mathbb{R}^n$ , then  $T$  is continuous at the origin (recall  $T(0) = 0$ ) since  $V$  is continuous,  $V(0) = 0$  and  $a \in [0, 1)$ . In the complete (n)FxT-ISS case, for any  $x_0 \in \mathbb{R}^n \setminus \{0\}$  there is a solution of the system  $\dot{x} = -f(x, 0)$ , and continuity of the escape-time function at infinity follows the estimates for LF:  $-DV(x)f(x, 0) \geq c_2 V^b(x)$  for almost all  $x \in \mathbb{R}^n$  implies  $\tau(x_0) \leq \frac{V^{1-b}(x_0)}{(b-1)c_2}$  for any  $x_0 \in \mathbb{R}^n \setminus \{0\}$ , then  $\tau(x_0) \rightarrow 0$  for  $|x_0| \rightarrow +\infty$  due to positive definiteness and radial unboundedness of  $V$  and since  $b > 1$ .  $\square$

*Remark 2.* The stability properties with accelerated convergence were introduced iteratively over decades and, in some cases, with perplexing names. To clarify their roles and relations, note that if for  $d \equiv 0$  the system (1) is globally asymptotically stable at the origin having unique solutions in forward time, then for any point  $x_0 \in \mathbb{R}^n$  there exists the trajectory passing by it, and we can define the time of convergence from this initial condition  $x_0$  to the origin by  $T_0(x_0)$ , and the time of arrival of the respective trajectory to  $x_0$  from infinity by  $T_\infty(x_0)$ . The following table describes the relations between different properties used in this work:

Stability for $d = 0$	$T_0(x_0)$	$T_\infty(x_0)$	Stability for $d \in \mathcal{L}_\infty^m$	Class of $\mathcal{K}\mathcal{L}$ functions	Examples for $x, d \in \mathbb{R}$
uGAS	infinite	infinite	ISS	$\mathcal{K}\mathcal{L}$	$f(x, d) = -x + d$
uFTS	finite	infinite	FTS-ISS	$\mathcal{F}\mathcal{K}\mathcal{L}$	$f(x, d) = -\sqrt[3]{x} + d$
unFxTS	infinite	finite	nFxTS-ISS	$n\mathcal{F}x\mathcal{K}\mathcal{L}$	$f(x, d) = -x^3 + d$
uFxTS	finite	finite	FxTS-ISS	$\mathcal{F}x\mathcal{K}\mathcal{L}$	$f(x, d) = -\sqrt[3]{x} - x^3 + d$

## V. A UNIFORMLY GAS SYSTEM

The result given in this section will be used to make a bridge between u(n)FxTS and (n)FxT-ISS properties.

Further consider an auxiliary system:

$$\dot{z}(t) = f(z(t), \gamma^{-1} \circ \phi(\|z(t)\|)\delta(t)), \quad t \geq 0, \quad (6)$$

where  $z(t) \in \mathbb{R}^n$  is the state as before,  $z_0 = z(0) \in \mathbb{R}^n$ ,  $\gamma \in \mathcal{K}_\infty$  is the AG function given for (1) in Definition 5 (which exists assuming that (1) is ISS, and it can be always chosen from the class  $\mathcal{K}_\infty$ , then the inverse is well-defined),  $\phi(s) = 0.5 \min(s, \beta^{-1}(0.25s, 0))$  is a function from the class  $\mathcal{K}_\infty$  where  $\beta$  also comes from the ISS property definition, and  $\delta \in \mathcal{D}$  is a uniformly bounded input. Obviously, the origin is an equilibrium of (6) since  $f(0, \cdot) = 0$ . We can always modify the AG  $\gamma$  of (1) such that  $\gamma^{-1} \circ \phi$  will be locally Lipschitz continuous away from zero. In the sequel this property is assumed to hold. In addition, for any  $z_0 \in \mathbb{R}^n$  and  $\delta \in \mathcal{D}$  the corresponding solution  $z(t, z_0, \delta)$  of (6) is defined on some interval of

time  $[0, T_{z_0}^{\max})$  and it coincides with the solution  $x(t, z_0, d)$  of (1) for  $d(t) = \gamma^{-1} \circ \phi(\|z(t, z_0, \delta)\|)\delta(t)$  on this interval of time. Let us demonstrate a stronger relation between solutions of the systems (1) and (6) (as in [16] for the pure ISS case):

**Proposition 1.** *The following relations are true:*

- i) If the system (1) is (n)FxT-ISS with the decay  $\beta \in \mathcal{F}x\mathcal{K}\mathcal{L}$  ( $\beta \in n\mathcal{F}x\mathcal{K}\mathcal{L}$ ) and AG  $\gamma$ , then the system (6) is u(n)FxTS.*
- ii) If the system (6) is u(n)FxTS with the decay  $\beta \in \mathcal{F}x\mathcal{K}\mathcal{L}$  ( $\beta \in n\mathcal{F}x\mathcal{K}\mathcal{L}$ ), then the system (1) is (n)FxT-ISS with the same decay  $\beta$  and the AG  $\beta(\phi^{-1} \circ \gamma(\cdot), 0)$ .*

Similar relations hold true for uFTS of (6) and FTS-ISS of (1) (the proof is the same as below).

*Proof.* *i)* If the system (1) is (n)FxT-ISS, then according to Definition 4 (by time invariance of the system we can replace in that definition 0 by  $t_0$  and  $t$  by  $t - t_0$ )

$$\|x(t)\| = \|x(t - t_0, x(t_0), d)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma(\|d\|_{[t_0, t]}),$$

for all  $t \geq t_0$ , for some  $\beta \in \mathcal{F}x\mathcal{K}\mathcal{L}$  ( $\beta \in n\mathcal{F}x\mathcal{K}\mathcal{L}$ ),  $\gamma \in \mathcal{K}_\infty$  and all  $x(t_0) \in \mathbb{R}^n$ ,  $d \in \mathcal{L}_\infty^m$ ,  $t_0 \geq 0$ . Since substituting  $d(t) = \gamma^{-1} \circ \phi(\|x(t)\|)\delta(t)$  in the system (1) we get (6), for the latter with  $z(t) = z(t - t_0, z(t_0), \delta)$  and any  $z(t_0) \in \mathbb{R}^n$ ,  $\delta \in \mathcal{D}$  we obtain:

$$\begin{aligned} \|z(t)\| &\leq \beta(\|z(t_0)\|, t - t_0) + \gamma\left(\sup_{s \in [t_0, t]} \|\gamma^{-1} \circ \phi(\|z(s)\|)\delta(s)\| \right) \\ &\leq \beta(\|z(t_0)\|, t - t_0) + \gamma\left(\sup_{s \in [t_0, t]} \gamma^{-1} \circ \phi(\|z(s)\|) \right) \\ &= \beta(\|z(t_0)\|, t - t_0) + \phi\left(\sup_{s \in [t_0, t]} \|z(s)\| \right) \end{aligned}$$

for all  $t \in [t_0, T_{z(t_0)}^{\max})$ . Take  $t_0 = 0$ , then from the last inequality and the definition of  $\phi$ ,

$$\begin{aligned} \sup_{s \in [0, t]} \|z(s)\| &\leq \sup_{s \in [0, t]} \{\beta(\|z_0\|, s) + 0.5 \sup_{\sigma \in [0, s]} \|z(\sigma)\|\} \\ &\leq \beta(\|z_0\|, 0) + 0.5 \sup_{s \in [0, t]} \|z(s)\|, \end{aligned}$$

leading to

$$0.5 \sup_{s \in [0, t]} \|z(s, z_0, \delta)\| \leq \beta(\|z_0\|, 0)$$

for any  $t \geq 0$ , all  $z_0 \in \mathbb{R}^n$  and all  $\delta \in \mathcal{D}$ , which implies global uniform boundedness of the solutions of the system (6). Therefore,  $T_{z_0}^{\max} = +\infty$  for all  $z_0 \in \mathbb{R}^n$  and all  $\delta \in \mathcal{D}$ . Choosing  $t_0 = \frac{t}{2}$  and repeating the same manipulations we get:

$$\begin{aligned} \sup_{s \in [\frac{t}{2}, t]} \|z(s)\| &\leq \sup_{s \in [\frac{t}{2}, t]} \{\beta(\|z(0.5s)\|, 0.5s) + 0.5 \sup_{\sigma \in [\frac{t}{2}, s]} \|z(\sigma)\|\} \\ &\leq \sup_{s \in [\frac{t}{2}, t]} \beta(2\beta(\|z_0\|, 0), 0.5s) + 0.5 \sup_{s \in [\frac{t}{4}, t]} \|z(s)\|, \end{aligned}$$

which is true for any  $t \geq 0$ , all  $z_0 \in \mathbb{R}^n$  and all  $\delta \in \mathcal{D}$ . Then, for  $t \rightarrow +\infty$  (i.e., considering the asymptotic behavior of (6)) it results in

$$\lim_{t \rightarrow +\infty} \sup_{s \in [\frac{t}{2}, t]} \|z(s)\| \leq 0.5 \lim_{t \rightarrow +\infty} \sup_{s \in [\frac{t}{4}, t]} \|z(s)\|$$

that can be satisfied for  $\lim_{t \rightarrow +\infty} \|z(t)\| = 0$  only, and (6) is uGAS.

For  $\beta \in \mathcal{F}x\mathcal{K}\mathcal{L}$ , denote by  $\mathbb{T} \in \mathcal{K}$  the settling time function satisfying  $\beta(s, \mathbb{T}(s)) = 0$  for any  $s \in \mathbb{R}_+$ . Note that locally, for sufficiently small  $s \in [0, 1]$ , the function  $\beta(s, t)$  can be upper bounded by  $\beta(\mathbb{T}^{-1} \circ \max\{0, 2\mathbb{T}(s) - t\}, 0)$  (with a bigger settling time), then we can always assume, with a slight ambiguity in the notation, that  $\beta$  is (locally) presented in a canonical form  $\beta(s, t) = \chi \circ \max\{0, \mathbb{T}(s) - t\}$  for some  $\chi \in \mathcal{K}_\infty$  (then  $\chi(s) \geq \beta(\mathbb{T}^{-1}(s), 0)$  and  $\phi(s) = 0.5 \min(s, \mathbb{T}^{-1} \circ \chi^{-1}(0.25s))$ ). Denote by  $\mathfrak{T} \in \mathcal{K}$  another time decay function satisfying  $2\beta(s, \mathfrak{T}(s)) \geq \beta(s, 0)$  (i.e.,  $\mathfrak{T}(s) \leq \mathbb{T}(s) - \chi^{-1} \circ \frac{1}{2}\chi \circ \mathbb{T}(s)$ ), then

locally, for sufficiently small  $s \in [0, 1]$ , there exists  $\kappa \in (0, 1)$  such that  $\mathfrak{T}(s) \geq \kappa \mathbb{T}(s)$ . Due to proven uGAS property, select arbitrary  $t_0 \in \mathbb{R}_+$  and  $\delta \in \mathcal{D}$ , and a sufficiently small  $z(t_0) \in \mathbb{R}^n$  such that  $\beta$ ,  $\mathbb{T}$  and  $\mathfrak{T}$  verify the above restrictions, then

$$\|z(t)\| \leq \sup_{s \in [t_0, t]} \|z(s)\| \leq 2\beta(\|z(t_0)\|, 0) \leq 4\beta(\|z(t_0)\|, \mathfrak{T}(\|z(t_0)\|))$$

for all  $t \in [t_0, t_0 + \mathbb{T}(\|z(t_0)\|)]$ , and for  $t = t_0 + \mathbb{T}(\|z(t_0)\|)$ , by continuity:

$$\begin{aligned} \|z(t_0 + \mathbb{T}(\|z(t_0)\|))\| &\leq \phi \left( \sup_{s \in [t_0, t_0 + \mathbb{T}(\|z(t_0)\|)]} \|z(s)\| \right) \\ &\leq \mathbb{T}^{-1} \circ \max\{0, \mathbb{T}(\|z(t_0)\|) - \mathfrak{T}(\|z(t_0)\|)\}. \end{aligned}$$

Define iteratively the series of time instants  $t_i = t_{i-1} + \mathbb{T}(\|z(t_{i-1})\|)$  for all  $i \geq 1$ , then we get the estimate:

$$\begin{aligned} \|z(t_0 + \sum_{i \geq 0} \mathbb{T}(\|z(t_i)\|))\| &\leq \mathbb{T}^{-1} \circ \max\{0, \mathbb{T}(\|z(t_0)\|) - \sum_{i \geq 0} \mathfrak{T}(\|z(t_i)\|)\} \\ &\leq \mathbb{T}^{-1} \circ \max\{0, \mathbb{T}(\|z(t_0)\|) - \kappa \sum_{i \geq 0} \mathbb{T}(\|z(t_i)\|)\}, \end{aligned}$$

and either  $\kappa \sum_{i \geq 0} \mathbb{T}(\|z(t_i)\|)$  is equal or bigger than  $\mathbb{T}(\|z(t_0)\|)$  implying that  $\|z(t_0 + \mathbb{T}(\|z(t_0)\|))\| = 0$  or  $\sum_{i \geq 0} \mathbb{T}(\|z(t_i)\|) < \kappa^{-1} \mathbb{T}(\|z(t_0)\|)$  that due to the properties of  $\mathbb{T}$  results in  $\|z(t_0 + \sum_{i \geq 0} \mathbb{T}(\|z(t_i)\|))\| = 0$ , which was necessary to show.

In order to substantiate the nearly fixed-time convergence, recall existence of the convergence time  $\mathcal{T} \in \mathcal{K}$  such that  $\beta(s, \mathcal{T}(\rho^{-1})) \leq \rho$  for arbitrary  $\rho > 0$  for any  $s \in \mathbb{R}_+$ . With this notation, any function  $\beta(s, t)$  from class  $n\mathcal{F}x\mathcal{K}\mathcal{L}$  can be upper bounded by  $\frac{2\beta(s, 0)}{1 + \mathcal{T}^{-1}(t)\beta(s, 0)}$  for  $s > 0$  sufficiently big (with a bigger convergence time), then we can always assume, with a slight ambiguity in the notation, that  $\beta$  is written in a canonical form  $\beta(s, t) = \frac{\chi(s)}{1 + \mathcal{T}^{-1}(t)\chi(s)}$  for some  $\chi \in \mathcal{K}_\infty$  and the argument of  $s$  sufficiently big (then  $\chi(s) \geq \beta(s, 0) \geq s$  and  $\phi(s) = 0.5 \min(s, \chi^{-1}(0.25s))$ ). Denote by  $\mathfrak{T} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  an auxiliary time decay function satisfying  $2\beta(s, \mathfrak{T}(s)) \geq \beta(s, 0)$  (i.e.,  $\mathcal{T}^{-1} \circ \mathfrak{T}(s) \leq \frac{1}{\chi(s)}$ ). In such a case, a time of convergence  $\mathcal{S} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  can be defined from the interval  $\left( \frac{1}{2} \chi^{-1} \left( \frac{1}{\frac{1}{\chi(s)} + \mathcal{T}^{-1} \circ \mathfrak{T}(s)} \right) \right)^{-1} - \frac{1}{\chi(s)} \leq \mathcal{T}^{-1} \circ \mathcal{S}(s) \leq \frac{2}{\chi^{-1}(0.5\chi(s))} - \frac{1}{\chi(s)}$  and providing the property  $\frac{\chi(s)}{1 + \mathcal{T}^{-1} \circ \mathcal{S}(s)\chi(s)} \leq \frac{1}{2} \chi^{-1} \left( \frac{\chi(s)}{1 + \mathcal{T}^{-1} \circ \mathfrak{T}(s)\chi(s)} \right)$  ( $\mathcal{S}$  is a decreasing function). Select arbitrary  $t_0 \in \mathbb{R}_+$  and  $\delta \in \mathcal{D}$ , and  $z(t_0) \in \mathbb{R}^n$  with  $\|z(t_0)\| \geq \rho$  for a sufficiently big  $\rho > 0$  such that  $\beta$ ,  $\mathcal{S}$  and  $\mathfrak{T}$  verify the above restrictions, then

$$\|z(t)\| \leq \sup_{s \in [t_0, t]} \|z(s)\| \leq 2\beta(\|z(t_0)\|, 0) \leq 4\beta(\|z(t_0)\|, \mathfrak{T}(\|z(t_0)\|))$$

for all  $t \in [t_0, t_0 + \mathcal{S}(\|z(t_0)\|)]$ , and for  $t = t_0 + \mathcal{S}(\|z(t_0)\|)$ , by continuity:

$$\begin{aligned} \|z(t_0 + \mathcal{S}(\|z(t_0)\|))\| &\leq \beta(\|z(t_0)\|, \mathcal{S}(\|z(t_0)\|)) + \phi \left( \sup_{s \in [t_0, t_0 + \mathcal{S}(\|z(t_0)\|)]} \|z(s)\| \right) \\ &\leq \frac{1}{2} \chi^{-1} \left( \frac{\chi(\|z(t_0)\|)}{1 + \mathcal{T}^{-1} \circ \mathfrak{T}(\|z(t_0)\|)\chi(\|z(t_0)\|)} \right) + \frac{1}{2} \chi^{-1} \left( \frac{\chi(\|z(t_0)\|)}{1 + \mathcal{T}^{-1} \circ \mathfrak{T}(\|z(t_0)\|)\chi(\|z(t_0)\|)} \right) \\ &= \chi^{-1} \left( \frac{1}{\frac{1}{\chi(\|z(t_0)\|)} + \mathcal{T}^{-1} \circ \mathfrak{T}(\|z(t_0)\|)} \right). \end{aligned}$$

Define iteratively the series of time instants  $t_i = t_{i-1} + \mathcal{S}(\|z(t_{i-1})\|)$  for all  $i \geq 1$ , then we get the estimate:

$$\|z(t_0 + \sum_{i \geq 0} \mathcal{S}(\|z(t_i)\|))\| \leq \chi^{-1} \left( \frac{1}{\frac{1}{\chi(\|z(t_0)\|)} + \sum_{i \geq 0} \mathcal{T}^{-1} \circ \mathfrak{T}(\|z(t_i)\|)} \right),$$

which implies uniform convergence since  $\sum_{i \geq 0} \mathcal{S}(\|z(t_i)\|)$  and  $\sum_{i \geq 0} \mathcal{T}^{-1} \circ \mathfrak{T}(\|z(t_i)\|)$  stay bounded while  $\|z(t_i)\| \geq \rho$ , which due to proven uGAS property implies unFXTS of (6).

ii) Conversely, let (6) be u(n)FXTS, then according to Definition 1:

$$\|z(t - t_0, z(t_0), \delta)\| \leq \beta(\|z(t_0)\|, t - t_0), \quad \forall t \geq t_0,$$

for some  $\beta \in \mathcal{FKL}$  ( $\beta \in n\mathcal{FKL}$ ) and all  $z(t_0) \in \mathbb{R}^n$ ,  $\delta \in \mathcal{D}$ ,  $t_0 \geq 0$ . Consider the system (1), take any  $x_0 \in \mathbb{R}^n$  and any  $d \in \mathcal{L}_\infty^m$ , assume that  $\|d\|_\infty \leq \gamma^{-1} \circ \phi(\|x(t)\|)$  for  $t \in [0, t_1)$  with some (possibly infinite)  $t_1 \geq 0$ , then there is  $\delta \in \mathcal{D}$  such that  $d(t) = \gamma^{-1} \circ \phi(\|x(t)\|)\delta(t)$  for all  $t \in [0, t_1)$  and, hence, using the properties of (6) we obtain:

$$\|x(t, x_0, d)\| \leq \beta(\|x_0\|, t), \quad \forall t \in [0, t_1).$$

Next, assume that  $\|d\|_\infty > \gamma^{-1} \circ \phi(\|x(t)\|)$  for  $t \in [t_1, t_2)$  with some  $t_2 > t_1$  (obviously, always  $t_1 + t_2 > 0$  for any  $x_0 \in \mathbb{R}^n$  and  $d \in \mathcal{L}_\infty^m$ ), hence,

$$\|x(t, x_0, d)\| \leq \phi^{-1} \circ \gamma(\|d\|_\infty), \quad \forall t \in [t_1, t_2).$$

Finally, let again  $\|d\|_\infty \leq \gamma^{-1} \circ \phi(\|x(t)\|)$  for  $t \in [t_2, t_3)$  with some  $t_3 > t_2$ , similarly

$$\|x(t, x_0, d)\| \leq \beta(\|x(t_2)\|, t - t_2) \leq \beta(\phi^{-1} \circ \gamma(\|d\|_\infty), 0), \quad \forall t \in [t_2, t_3)$$

since in this case  $\|x(t_2)\| = \phi^{-1} \circ \gamma(\|d\|_\infty)$  by construction. Next, all these steps can be repeated iteratively if necessary (they cover all possible scenarios), and it is clear that for all  $t \geq 0$

$$\|x(t, x_0, d)\| \leq \beta(\|x_0\|, t) + \beta(\phi^{-1} \circ \gamma(\|d\|_\infty), 0),$$

since by definition  $\beta(s, 0) \geq s$  for all  $s \geq 0$ , which implies (n)FxT-ISS property of (1).  $\square$

## VI. MAIN RESULTS

For a (n)FxT-ISS dynamical system (1), the result of Proposition 1 guarantees that (6) is u(n)FxTS, then the material of the previous sections allows the complete robust accelerated stability notions to be introduced:

**Definition 8.** A (n)FxT-ISS system (1) is called complete if (6) is complete u(n)FxTS.

The main outcomes of our work are formulated in the theorems below.

**Theorem 4.** *If the system (1) is complete nFxT-ISS with a locally Lipschitz continuous on  $\mathbb{R}^n \setminus \{0\}$  escape-time function  $\tau$ , then there exists a nFxT-ISS-LF.*

*Proof.* Our objective is to construct a nFxT-ISS-LF for a complete nFxT-ISS system (1). In such a case, by Definition 8, the auxiliary system (6) is complete unFxTS with a locally Lipschitz continuous on  $\mathbb{R}^n \setminus \{0\}$  escape-time function  $\tau$ , then by Theorem 1 there is a unFxTS-LF  $V \in C(\mathbb{R}^n, \mathbb{R}_+)$  for the system (6), locally Lipschitz continuous on  $\mathbb{R}^n \setminus \{0\}$ : there exist  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ ,  $\alpha > 1$  and  $c > 0$  such that for all  $z \in \mathbb{R}^n$  and  $\delta \in \mathbb{D}$ :

$$\begin{aligned} \alpha_1(\|z\|) &\leq V(z) \leq \alpha_2(\|z\|), \\ \dot{V}(z, \gamma^{-1} \circ \phi(\|z\|)\delta) &\leq -cV^\alpha(z). \end{aligned}$$

Returning to the system (1), we obtain:

$$\|x\| \geq \phi^{-1} \circ \gamma(\|d\|) \implies \dot{V}(x, d) \leq -cV^\alpha(x),$$

hence,  $V$  is the required nFxT-ISS-LF.  $\square$

**Theorem 5.** *If the system (1) is complete FxT-ISS with a locally Lipschitz continuous on  $\mathbb{R}^n \setminus \{0\}$  escape-time function  $\tau$  and settling-time function  $T$ , then there exists a FxT-ISS-LF.*

*Proof.* The proof has four main steps. First, using converse arguments, a LF  $U(x)$  is constructed that shows local FTS (which is locally equivalent to FxTS) of the unperturbed system (1). Second, it is shown that this LF  $U(x)$  is actually an FxT-ISS-LF Lyapunov function if  $\|x\| < \rho$ , for any  $\rho > 0$  (with the asymptotic gain dependent on  $\rho$ ). Third, applying the substantiated converse results for nFxT-ISS (which follows from FxT-ISS), another LF  $W(x)$  is designed for  $\|x\| \geq \delta$  for any  $\delta \in (0, \rho)$ . Finally, a desired global FxT-ISS-LF is constructed by uniting  $U$  and  $W$ .

**I** Since (1) is FxT-ISS, when  $d = 0$  there exists some  $T(x)$  such that  $\|x(t, x, 0)\| = 0$  for all  $t \geq T(x)$ , and  $T(x) \leq T_{\max} < +\infty$ , for any  $x \in \mathbb{R}^n$ . If  $T(x)$  is a locally Lipschitz function, then for any  $\rho > 0$  it is possible to define a function  $U(x) := T(x)^{\frac{1}{1-a}}$ , with  $a \in [0, 1)$ , satisfying  $\alpha'_1(\|x\|) \leq U(x) \leq \alpha'_2(\|x\|)$  for some  $\alpha'_1, \alpha'_2 \in \mathcal{K}_\infty$  and all  $x \in \Omega_\rho = \{x \in \mathbb{R}^n : \|x\| < \rho\}$ , being such that

$$DU(x)f(x, 0) \leq -c_1 U^a(x) \quad (7)$$

for some  $c_1 > 0$  and for almost all  $x \in \Omega_\rho$ .

**II** Since  $T(x)$  is locally Lipschitz continuous away from the origin,  $U(x)$  has the same property and  $\|DU(x)\| \leq \kappa + \psi(\|x\|)$  for almost all  $x \in \Omega_\rho$ , for some  $\kappa \in \mathbb{R}_+$  and  $\psi \in \mathcal{K}$ . By Assumption 1,  $\|f(x, d) - f(x, 0)\| \leq L(\|x\|)\|d\|$  for the given  $L : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ . Thus,

$$\|DU(x)(f(x, d) - f(x, 0))\| \leq (\kappa + \psi(\|x\|))L(\|x\|)\|d\|$$

for almost all  $x \in \Omega_\rho$  and all  $d \in \mathbb{R}^m$ . Let  $\rho > \|x\| \geq \varphi_\rho^{-1}(\|d\|)$ , where

$$\varphi_\rho(\|x\|) := \frac{c_1 \alpha'_1(\|x\|)^a}{2(\kappa + \psi(\rho)) \sup_{s \in [0, \rho]} L(s)} \quad (8)$$

is a function from class  $\mathcal{K}_\infty$ , then

$$\|DU(x)(f(x, d) - f(x, 0))\| \leq \frac{c_1}{2} U^a(x)$$

and using the inequality (7) we have that

$$DU(x)f(x, d) \leq -\frac{c_1}{2} U^a(x)$$

provided that  $\|x\| \geq \varphi_\rho^{-1}(\|d\|)$  for almost all  $x \in \Omega_\rho$  and all  $d \in \mathbb{R}^m$ .

**III** Since complete FxT-ISS property implies complete nFxT-ISS one, and the local Lipschitz continuity on  $\mathbb{R}^n \setminus \{0\}$  of the respective escape-time function  $\tau$  is assumed, by Theorem 4, there is a nFxT-ISS-LF  $W \in C(\mathbb{R}^n, \mathbb{R}_+)$ , locally Lipschitz continuous on  $\mathbb{R}^n \setminus \{0\}$ . Hence, for  $x \notin \Omega_\delta$  with any  $\delta \in (0, \rho)$  and  $d \in \mathbb{R}^m$ :

$$\begin{aligned} \alpha_1(\|x\|) &\leq W(x) \leq \alpha_2(\|x\|), \\ \|x\| \geq \phi^{-1} \circ \gamma(\|d\|) &\implies \dot{W}(x, d) \leq -c' W^b(x) \end{aligned}$$

with  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ ,  $b > 1$  and  $c' > 0$ .

**IV** Let us define the function

$$\tilde{V}(x) := s(U(x))W(x) + (1 - s(U(x)))U(x),$$

where  $s \in C(\mathbb{R}_+, \mathbb{R}_+)$  satisfies

$$s(r) = \begin{cases} 1 & \text{if } r \geq \alpha'_2(\rho) \\ 0 & \text{if } r \leq \alpha'_1(\delta) \end{cases}$$

and  $\dot{s}(r) = \frac{\partial s(r)}{\partial r} > 0$  for all  $r \in (\alpha'_1(\delta), \alpha'_2(\rho))$ . Assume that  $U(x) \leq W(x)$  for all  $x \in \{x \in \mathbb{R}^n : \alpha'_1(\delta) \leq U(x) \leq \alpha'_2(\rho)\}$  (both functions  $U(x)$  and  $W(x)$  are continuous, positive definite and radially unbounded, then we can adopt such a hypothesis without being restrictive, since multiplying  $W(x)$  by a constant as in the proof of Theorem 1 we can always assure its fulfillment), then we have that

$$\begin{aligned} \dot{\tilde{V}}(x, d) &= s\dot{W}(x, d) + (1 - s)DU(x)f(x, d) \\ &\quad + \dot{s}(U(x))DU(x)f(x, d)(W(x) - U(x)), \end{aligned}$$

and gathering all the previous estimates, we arrive to

$$\|x\| \geq \chi(\|d\|) \implies \dot{\tilde{V}}(x, d) \leq -\alpha_4(\|x\|), \quad (9)$$

where  $\chi(r) := \max\{\phi^{-1} \circ \gamma(r), \varphi_\rho^{-1}(r)\}$  and  $\alpha_4 \in \mathcal{K}_\infty$  such that

$$\alpha_4(\|x\|) \geq \begin{cases} c'[\tilde{V}(x)]^b, & U(x) \geq \alpha'_2(\rho) \\ \frac{c}{2}[\tilde{V}(x)]^a, & U(x) \leq \alpha'_1(\delta) \end{cases}.$$

Consequently,  $\tilde{V}$  is a FxT-ISS-LF for (1). □

## VII. CONCLUSION

Several converse Lyapunov theorems are formulated for uFTS, u(n)FxTS and (n)FxTS-ISS properties. The open research directions include relaxation of restrictions given in Assumption 1, avoiding the Lipschitz continuity requirement for  $\tau$  and  $T$  introduced in theorems 4 and 5, investigations of integral ISS property with accelerated convergence rates, or extension of these results to the systems with outputs (to the input-to-output stability concept; note that the necessary and sufficient conditions for output finite-time stability can be found in [18]).

## REFERENCES

- [1] S.P. Bhat and D.S. Bernstein. Finite time stability of continuous autonomous systems. *SIAM J. Control Optim.*, 38(3):751–766, 2000.
- [2] F. H. Clarke. Generalized gradients and applications. *Trans. Am. Math. Soc.*, 205:247–262, 1975.
- [3] S.N. Dashkovskiy, D.V. Efimov, and E.D. Sontag. Input to state stability and allied system properties. *Automation and Remote Control*, 72(8):1579–1614, 2011.
- [4] B.K. Driver. *Analysis Tools with Applications*. Springer, 2003.
- [5] Denis Efimov and Andrey Polyakov. Finite-time stability tools for control and estimation. *Foundations and Trends® in Systems and Control*, 9(2-3):171–364, 2021.
- [6] Y. Hong, Z. Jiang, and G. Feng. Finite-time input-to-state stability and applications to finite-time control design. *SIAM Journal on Control and Optimization*, 48(7):4395–4418, 2010.
- [7] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, third edition, 2002.
- [8] N. N. Krasovskii. *Stability of motion. Applications of Lyapunov's second method to differential systems and equations with delay*. Translated by J. L. Brenner. Stanford University Press, Stanford, Calif., 1963.
- [9] Daniel Liberzon, Eduardo D. Sontag, and Yuan Wang. Universal construction of feedback laws achieving ISS and integral-ISS disturbance attenuation. *Systems & Control Letters*, 46(2):111–127, 2002.
- [10] Yuandan Lin, Eduardo D. Sontag, and Yuan Wang. A smooth converse Lyapunov theorem for robust stability. *SIAM J. Control Optimization*, 34(1):124–160, 1996.
- [11] Francisco Lopez-Ramirez, Denis Efimov, Andrey Polyakov, and Wilfrid Perruquetti. Conditions for fixed-time stability and stabilization of continuous autonomous systems. *Systems & Control Letters*, 129:26–35, 2019.
- [12] Francisco Lopez-Ramirez, Denis Efimov, Andrey Polyakov, and Wilfrid Perruquetti. Finite-time and fixed-time input-to-state stability: Explicit and implicit approaches. *Systems & Control Letters*, 130, 2020.
- [13] Y. Orlov. Finite time stability and robust control synthesis of uncertain switched systems. *SIAM Journal of Control and Optimization*, 43(4):1253–1271, 2005.
- [14] A. Polyakov. Nonlinear feedback design for fixed-time stabilization of linear control systems. *IEEE Transactions on Automatic Control*, 57(8):2106–2110, 2012.
- [15] E. Roxin. On finite stability in control systems. *Rendiconti del Circolo Matematico di Palermo*, 15:273–283, 1966.
- [16] E.D. Sontag. Input to state stability: Basic concepts and results. In P. P. Nistri and G. Stefani, editors, *Nonlinear and Optimal Control Theory*, pages 163–220. Springer-Verlag, Berlin, 2007.
- [17] Eduardo D. Sontag and Yuan Wang. On characterizations of the input-to-state stability property. *Systems & Control Letters*, 24(5):351–359, 1995.
- [18] Konstantin Zimenko, Denis Efimov, Andrey Polyakov, and Artem Kremlev. On necessary and sufficient conditions for output finite-time stability. *Automatica*, 125:109427, 2021.
- [19] V. I. Zubov. *Methods of A.M. Lyapunov and Their Applications*. Noordhoff, Leiden, 1964.