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# Input-to-state stability analysis of heat equation with boundary finite-time control <sup>★</sup>

Xin-Xin Han <sup>a</sup>, Denis Efimov <sup>b</sup>, Andrey Polyakov <sup>b</sup>, Kai-Ning Wu <sup>c</sup>

<sup>a</sup>*Department of Mathematics, Nanjing Forestry University, Nanjing, Jiangsu 210037, China*

<sup>b</sup>*Inria, Univ. Lille, CNRS, UMR 9189 - CRISTAL, F-59000 Lille, France*

<sup>c</sup>*Department of Mathematics, Harbin Institute of Technology at Weihai, Weihai 264209, China*

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## Abstract

This paper addresses the problem of input-to-state stabilization for heat equation with external input via boundary control strategy. Following [29], based on the backstepping approach, a switching boundary control law depending on the system state is designed. By estimating the upper bound of kernel functions, switching levels are determined and commutation law is further constructed. For the chosen switched control, the well-posedness property is verified. It is proved that the resulting system is input-to-state stable, and the solutions of the system will not exceed the highest admissible level dependent on the disturbance amplitude. Meanwhile, a stronger result is also obtained, that is the finite-time stability for the disturbance-free system. The numerical example is provided to support the derived result.

*Key words:* heat equation; input-to-state stability; switching; boundary control; backstepping.

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## 1 Introduction

Analysis and design of stable dynamical processes have always been in the focus of control theory research. In the uncertainty-free case the rates of convergence to the desired mode constitute one of the main characteristics of a system such as asymptotic stability [9], finite-time and fixed-time stability [30], etc. However, in real world, the system under consideration may be disturbed by various perturbations or noises. In this situation, it is of vital importance to proceed to the robustness analysis. The concept of input-to-state stability (ISS) introduced by E. Sontag in [35] plays a fundamental role in the robust stability analysis, which provides a framework to tackle the stability with respect to disturbance inputs. Conceptually, ISS is characterized by the dynamical be-

havior of the system being bounded with respect to arbitrary bounded input perturbations, and also asymptotically stable in the absence of external inputs. Nowadays, fruitful results in ISS for finite-dimensional systems have been carried out, see for instance [7, 36].

On the other hand, the problem of ISS for infinite-dimensional systems is as important as that for finite-dimensional systems. To name a few examples, reaction process [5], robotic arms [11], aircraft wings [33], etc., all require the robust control methods in practical applications. The investigation of ISS for infinite-dimensional systems has also received a considerable attention (see e.g., [20, 25]). In [16] the authors developed the ISS theory to the infinite-dimensional systems for the first time. Afterwards, some qualitative results of ISS were established [12, 26]. Aiming at the infinite-dimensional systems governed by PDE, plenty of works on the ISS analysis and control have focused on the distributed disturbances [39, 22]. Meanwhile for the situation involving boundary disturbances, various methodologies have been proposed more recently [20, 46]. For instance, spectral decomposition and finite difference scheme [18], monotonicity-based approach [24], Lyapunov methods [44], De Giorgi iteration [45], etc. The aforementioned ISS results indicate that the trajectories of the system in the disturbance-free case are asymptotically stable.

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*Email addresses:* [hitwhhxx@163.com](mailto:hitwhhxx@163.com) (Xin-Xin Han), [denis.efimov@inria.fr](mailto:denis.efimov@inria.fr) (Denis Efimov), [andrey.polyakov@inria.fr](mailto:andrey.polyakov@inria.fr) (Andrey Polyakov), [wkn@hit.edu.cn](mailto:wkn@hit.edu.cn) (Kai-Ning Wu).

In particular, many real-world application settings also require the disturbance-free system to guarantee finite-time stability [29, 6]. Compared with the asymptotic stability, it is recognized that finite-time stability plays a more important role following some specific control engineering demands, such as finite-time deployment for large networks of multi-agent systems [23], finite-time operation duration of solid propellant rockets [34], etc. Nevertheless, the issue that whether ISS and finite-time stability can be satisfied (or be guaranteed by some control approaches) for the disturbed and disturbance-free PDE systems respectively has not been sufficiently explored, which is the motivation of this study.

Currently, control placements of PDE systems from the perspective of its spatial distribution are either distributed over the entire spatial domain (which we usually call the distributed control [40]), or only placed at the boundary of the spatial domain (that is the boundary control [31]), or applied at some specific points in the spatial domain (which is commonly known as pointwise control [1, 2, 32] or interior point control [4]). Among them boundary control, as a novel approach to effectively save the implementation cost, has gradually come into researchers' mind [10, 14, 13, 41]. One can refer to the foundational work from Miroslav Krstic by backstepping method [21, 3]. Some existing results on the ISS boundary stabilization can be found in [42, 43, 45] and references therein. Nevertheless, regarding the dual requirements of ISS boundary stabilization for the resulting PDE system and finite-time boundary stabilization for the corresponding disturbance-free system, serious technical challenges arise in the boundary feedback controller design and such stability analysis comparing to the asymptotic ISS case. In recent years, some efforts have been made to stabilize the (disturbance-free) parabolic PDE systems in finite time via different boundary control design methods [10, 6, 29]. However, when disturbances enter the system, ISS analysis under the boundary finite-time control methods is a problem that are worthy of probing. For example, the results of [10] guarantee finite-time stability through boundary control and time-varying kernel functions, whose utilization in the backstepping transformations with the presence of disturbances leads to unbounded gains for perturbations in the target dynamics. The results of [6, 29] are based on switching strategy with piecewise constant kernel functions for different boundary actuations, and growing to infinity control gains and rate of commutation are serious obstacles for analysis of these systems in the presence of exogenous inputs (the definition and existence of solutions is an issue). For these reasons, ISS analysis via boundary finite-time control strategy becomes complicated and furthermore, the achievement of the aforementioned dual requirements is still open. We are going to fill this gap in the present work.

This paper extends the results in [29] to the system with disturbance input, steering the solution of the closed-

loop system to achieve ISS-like stabilization. Based on [29] a piecewise linear boundary controller with a state-dependent switching law is presented. By estimating the bound of kernel functions, switching levels are determined and switching law is constructed. The main result states that the solution of the system cannot exceed the highest level dependent on the disturbance amplitude based on the switching framework we designed, which makes it possible to achieve ISS property. It is worth mentioning that a stronger ISS result is obtained in this paper. Specifically, in addition to establishing an ISS-like estimate in the presence of disturbance, finite-time stability is also guaranteed for the disturbance-free case. The strategy of hybrid boundary control and analysis methods of ISS may be applicable for some other types of PDE systems.

The paper is organized as follows. Section 2 provides the preliminaries. The model description is proposed in Section 3.1. With the switching law described in Section 3.2, the well-posedness analysis is further discussed in Section 3.3. In Section 4, the main result of ISS-like stabilization is derived via rigorous computations. The numerical simulation is presented in Section 5 to illustrate the theoretical result. Section 6 concludes this study.

**Notation:**  $\mathbb{R}$  is the set of real numbers and  $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ ,  $|x|$  corresponds to absolute value of  $x \in \mathbb{R}$ .  $\mathbb{Z}$ ,  $\mathbb{Z}^+$  and  $\mathbb{Z}^-$  represent the sets of integers, nonnegative integers and negative integers, respectively. Denote  $f'(x) = \frac{df(x)}{dx}$  for a differentiable function  $f : \mathbb{R} \rightarrow \mathbb{R}$ . For any interval  $I \subseteq \mathbb{R}^+$ , let  $C^k(I, X)$  with  $k \in \mathbb{Z}^+$  be the space consisting of all  $k$ -order continuously differentiable functions defined on  $I$ , which take values in the space  $X$ . Denote by  $L^2(I, \mathbb{R})$  the Lebesgue space of square integrable functions  $f : I \rightarrow \mathbb{R}$  equipped with the  $L^2$ -norm  $\|f\| = \sqrt{\int_I f^2(x)dx}$ . Let  $Q_{(a,b)} = (a, b) \times (0, 1)$  for  $(a, b) \subset \mathbb{R}^+$  (or  $Q_t$  for  $(a, b) = (0, t)$  with  $t \in \mathbb{R}^+ \cup \{\infty\}$ ), we use  $\mathcal{L}^\infty(Q_{(a,b)}) := \mathcal{L}^\infty(Q_{(a,b)}, \mathbb{R})$  to denote the set of essentially bounded functions  $d : Q_{(a,b)} \rightarrow \mathbb{R}$  with corresponding norm  $\|d\|_{\mathcal{L}^\infty(Q_{(a,b)})} = \text{ess sup}_{(s,x) \in (a,b) \times (0,1)} |d(s,x)| < \infty$ .  $J_m(p)$  and  $I_m(p)$  with  $m \in \mathbb{Z}$  denote the Bessel and modified Bessel functions of the first kind, respectively, of the order  $m$  for complex argument  $p$ .

## 2 Preliminaries

We recall that a continuous function  $\theta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  belongs to class- $\mathcal{K}$  if it satisfies  $\theta(0) = 0$  and it is strictly increasing;  $\theta$  is a class- $\mathcal{K}_\infty$  function if it belongs to class- $\mathcal{K}$  and  $\theta(s) \rightarrow \infty$  as  $s \rightarrow \infty$  also. A continuous function  $\psi : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a class- $\mathcal{KL}$  function if  $\psi(\cdot, t)$  belongs to class- $\mathcal{K}$  for each fixed  $t \geq 0$ , and it decreases to zero as  $t \rightarrow \infty$  for each fixed  $s \geq 0$ .

**Lemma 2.1** ([8, Ch. 1, pp. 47-48] Wirtinger's inequality) For any bounded interval  $[0, l] \subset \mathbb{R}^+$  and  $z \in C^1([0, l], \mathbb{R})$  with  $z(0) = z(l) = 0$ :

$$\int_0^l z^2(\xi) d\xi \leq \frac{l^2}{\pi^2} \int_0^l z_\xi^2(\xi) d\xi.$$

### 3 Model description and well-posedness analysis

Consider the following scalar system described by heat equation:

$$\begin{cases} u_t(t, x) = u_{xx}(t, x) + d(t, x), & t > 0, x \in (0, 1), \\ u(t, 0) = 0, u(t, 1) = U(t), & t > 0, \\ u(0, x) = u_0(x), & x \in (0, 1), \end{cases} \quad (1)$$

where  $u(t, x) \in \mathbb{R}$  is the system state at time  $t \in \mathbb{R}^+$  and spatial position  $x \in (0, 1)$ ,  $d(t, x) \in \mathbb{R}$  is the external input which belongs to  $\mathcal{L}^\infty(Q_\infty)$ . The initial condition  $u_0 \in L^2((0, 1), \mathbb{R})$  and satisfies zero order compatibility condition  $u_0(0) = 0$  and  $u_0(1) = U(0)$ , where  $U(t) \in \mathbb{R}$  is the boundary control input.

Next we will formulate the definition of ISS estimate in  $L^2$  norm.

**Definition 3.1** It is said that system (1) admits an ISS estimate in  $L^2$  norm with respect to the initial value  $u_0 \in L^2((0, 1), \mathbb{R})$  and input  $d \in \mathcal{L}^\infty(Q_\infty)$ , if there exist functions  $\nu \in \mathcal{K}$  and  $\beta \in \mathcal{KL}$  satisfying

$$\|u(t, \cdot)\| \leq \beta(\|u_0\|, t) + \nu(\|d\|_{\mathcal{L}^\infty(Q_t)})$$

for all  $t > 0$ .

In this work, the main goal is to design a novel hybrid boundary feedback control law, via backstepping method, that provides the system (1) with ISS. The hybrid boundary controller is given as

$$U(t) = \int_0^1 K(1, y, 2^{\sigma(t)}) u(t, y) dy \quad (2)$$

and  $U(t) = 0$  if  $\|u(t, \cdot)\| = 0$ , where  $K : D \rightarrow \mathbb{R}$  defined in  $D := \{(x, y, i) \in [0, 1]^2 \times \mathbb{Z}^+ : y \leq x\}$  is the kernel of the backstepping transformation,  $\sigma : \mathbb{R}^+ \rightarrow \mathbb{Z}$  is a function being constant for  $t \in [t_k, t_{k+1})$ , and  $t_k$  for  $k \in \mathbb{Z}^+$  with  $t_0 = 0$  is the sequence of time instances for control commutation. The shape of the kernel  $K$  and the values of  $\sigma(t)$  and  $t_k$  will be specified later.

#### 3.1 Backstepping transformation and inverse transformation

Following the standard framework of backstepping approach [21], we search for the invertible Volterra integral

transformation

$$w(t, x) = u(t, x) - \int_0^x K(x, y, 2^{\sigma(t)}) u(t, y) dy \quad (3)$$

that maps system (1) into the following target system

$$\begin{cases} w_t(t, x) = w_{xx}(t, x) - 2^{\sigma(t)} w(t, x) + d(t, x) \\ \quad - \int_0^x K(x, y, 2^{\sigma(t)}) d(t, y) dy, \\ w(t, 0) = 0, w(t, 1) = 0, \end{cases} \quad (4)$$

with  $w(t_k, x) = u(t_k, x) - \int_0^x K(x, y, 2^{\sigma(t_k)}) u(t_k, y) dy$  and  $t \in [t_k, t_{k+1})$  ( $\sigma(t)$  is a constant on this interval of time).

After utilizing integration by parts and taking into account boundary conditions, substitution of transformation (3) into target system (4) and original system (1) leads to the following PDE that the kernel  $K(x, y, 2^{\sigma(t)})$  satisfies

$$\begin{cases} K_{xx}(x, y, 2^{\sigma(t)}) - K_{yy}(x, y, 2^{\sigma(t)}) - 2^{\sigma(t)} K(x, y, 2^{\sigma(t)}) = 0, \\ K(x, 0, 2^{\sigma(t)}) = 0, \\ \frac{d}{dx} K(x, x, 2^{\sigma(t)}) = -2^{\sigma(t)-1}. \end{cases} \quad (5)$$

Here and after, we use the notation  $\frac{d}{dx} K(x, x, 2^{\sigma(t)}) := K_x(x, y, 2^{\sigma(t)})|_{y=x} + K_y(x, y, 2^{\sigma(t)})|_{y=x}$ .

Note that after a straightforward calculation [10, 34], the solution of PDE (5), with the help of modified Bessel function  $I_1(\cdot)$ , is given by

$$K(x, y, 2^{\sigma(t)}) = -y 2^{\sigma(t)} \frac{I_1\left(\sqrt{2^{\sigma(t)}(x^2 - y^2)}\right)}{\sqrt{2^{\sigma(t)}(x^2 - y^2)}}. \quad (6)$$

**Remark 3.1** Based on some properties of modified Bessel function  $I_1(\cdot)$ , the well-posedness of (5) is ensured by explicitly giving an analytical solution (6). Note that it holds only for constant  $\sigma(t)$  instead of a function  $\sigma(t)$  of  $t$ . In other words,  $K(x, y, 2^{\sigma(t)})$  is a solution of PDE (5) when  $\sigma(t)$  is a constant, i.e.,  $t \in [t_k, t_{k+1})$ . Therefore, target system (4) holds for  $t \in [t_k, t_{k+1})$ , and the state function of the first equation of target system (4) will jump if  $\sigma(t)$  switches to next mode. It is worth remarking that for the original system (1), it remains a continuous solution in  $L^2$  norm even though there exist some jumps in the right boundary of the spatial domain. This will be proven in the following subsection and also illustrated in the numerical simulation.

On the other hand, the form of inverse backstepping transformation is determined as

$$u(t, x) = w(t, x) + \int_0^x L(x, y, 2^{\sigma(t)}) w(t, y) dy \quad (7)$$

for  $t \in [t_k, t_{k+1})$ , where the kernel  $L(x, y, 2^{\sigma(t)})$  of the inverse transformation satisfies the following PDE:

$$\begin{cases} L_{xx}(x, y, 2^{\sigma(t)}) - L_{yy}(x, y, 2^{\sigma(t)}) + 2^{\sigma(t)} L(x, y, 2^{\sigma(t)}) = 0, \\ L(x, 0, 2^{\sigma(t)}) = 0, \\ \frac{d}{dx} L(x, x, 2^{\sigma(t)}) = -2^{\sigma(t)-1}. \end{cases} \quad (8)$$

Similarly, it can be verified [10, 34] that PDE (8) has a well-posed solution of the form

$$L(x, y, 2^{\sigma(t)}) = -y 2^{\sigma(t)} \frac{J_1\left(\sqrt{2^{\sigma(t)}(x^2 - y^2)}\right)}{\sqrt{2^{\sigma(t)}(x^2 - y^2)}},$$

where  $J_1(\cdot)$  is the Bessel function of the first kind of order 1.

For further analysis, we provide the following estimates of the backstepping transformation and inverse transformation.

**Proposition 3.1** *For the transformations (3) and (7), the following estimates are satisfied*

$$\begin{aligned} \|w(t, \cdot)\| &\leq \Omega_1(2^{\sigma(t)}) \|u(t, \cdot)\|; \\ \|u(t, \cdot)\| &\leq \Omega_2(2^{\sigma(t)}) \|w(t, \cdot)\| \end{aligned} \quad (9)$$

for all  $t \in [t_k, t_{k+1})$ , where

$$\begin{aligned} \Omega_1(s) &= 1 + \frac{s \sqrt{\int_0^1 \int_0^x y^2 \left( I_0(\sqrt{s(x^2 - y^2)}) - I_2(\sqrt{s(x^2 - y^2)}) \right)^2 dy dx}}{2}, \\ \Omega_2(s) &= 1 + \frac{s \sqrt{\int_0^1 \int_0^x y^2 \left( J_0(\sqrt{s(x^2 - y^2)}) + J_2(\sqrt{s(x^2 - y^2)}) \right)^2 dy dx}}{2}. \end{aligned}$$

**Proof.** Using the Cauchy-Schwarz inequality and  $I_k(p) = \frac{p}{2k} (I_{k-1}(p) - I_{k+1}(p))$  [37], we have

$$\begin{aligned} \|w(t, \cdot)\| &\leq \|u(t, \cdot)\| + \left\| \int_0^x K(x, y, 2^{\sigma(t)}) u(t, y) dy \right\| \\ &\leq \left( 1 + \sqrt{\int_0^1 \int_0^x K^2(x, y, 2^{\sigma(t)}) dy dx} \right) \\ &\quad \times \|u(t, \cdot)\| \\ &= \left( 1 + \frac{2^{\sigma(t)} \sqrt{\int_0^1 \int_0^x y^2 (I_0(p) - I_2(p))^2 dy dx}}{2} \right) \\ &\quad \times \|u(t, \cdot)\| \\ &= \Omega_1(2^{\sigma(t)}) \|u(t, \cdot)\|, \end{aligned}$$

where  $p = \sqrt{2^{\sigma(t)}(x^2 - y^2)}$ .

Similarly, the estimate of  $\|u(t, \cdot)\|$  in (9) can be obtained further by using  $J_k(p) = \frac{p}{2k} (J_{k-1}(p) + J_{k+1}(p))$  [37].

### 3.2 Switching law design

Define switching level  $\eta_i$  for  $i \in \mathbb{Z}$  recursively as  $\Omega(2^i)\eta_i = \eta_{i-1}$  with  $\eta_0 = 1$ , where  $\Omega(s) = \Omega_1(s)\Omega_2(s)$ . In order to ensure system (1) to be ISS under the hybrid boundary feedback control law (2), we design the switching signal  $\sigma(t)$  following the rule:

$$\sigma(t) = \begin{cases} i + 1, & \text{if } \sigma(t) = i \text{ and } \|u(t, \cdot)\| \leq \eta_{i+1} \\ i, & \text{if } \sigma(t) = i \text{ and } \eta_{i+1} < \|u(t, \cdot)\| < \eta_{i-1} \\ i - 1, & \text{if } \sigma(t) = i \text{ and } \|u(t, \cdot)\| \geq \eta_{i-1} \end{cases} \quad (10)$$

for  $t < t^*$  and  $\sigma(t) = 0$  for all  $t \geq t^*$ , where  $t^* = \inf\{t \geq 0 \mid \|u(t, \cdot)\| = 0\}$ . Here, switching level  $\eta_{\sigma(t)}$  determines the switching point  $t_k, k \in \mathbb{Z}^+$  satisfying  $t_{k+1} = \inf\{t \in (t_k, t^*) \mid \|u(t, \cdot)\| = \eta_{\sigma(t)+1} \text{ or } \|u(t, \cdot)\| = \eta_{\sigma(t)-1}\}$ ,  $t_0 = 0$ .

**Remark 3.2** *It is obvious from (10) that  $\sigma(t)$  is a function of  $\|u(t, \cdot)\|$ . In the sequel, we present the relation between these two functions (see Proposition 3.2), which will be helpful for the following robustness analysis. Furthermore, for the function  $\Omega(s)$  that determines the switching level, we can see that  $\Omega(s)$  ( $s > 0$ ) is an increasing function and  $\Omega(s) \rightarrow +\infty$  as  $s \rightarrow +\infty$ . Indeed, a straightforward calculation yields  $\Omega'_1(s) = (1 + s\sqrt{h_1(s)})' = \sqrt{h_1(s)} + \frac{1}{2\sqrt{h_1(s)}} \int_0^1 \int_0^x y^2 \frac{I_1(\sqrt{s(x^2 - y^2)})I_2(\sqrt{s(x^2 - y^2)})}{\sqrt{s(x^2 - y^2)}} dy dx$  utilizing recurrence relation  $I'_k(p) = I_{k+1}(p) + \frac{k}{p} I_k(p)$  [37],*

where  $h_1(s) = \int_0^1 \int_0^x y^2 \left( \frac{I_1(\sqrt{s(x^2 - y^2)})}{\sqrt{s(x^2 - y^2)}} \right)^2 dy dx$ . Since  $I_2(p) \geq 0$  and  $\frac{I_1(p)}{p} = \frac{I_0(p) - I_2(p)}{2} \geq 0$  for  $p \geq 0$  (it can be easily obtained by the representation  $I_k(p) = \frac{1}{\pi} \int_0^\pi e^{p \cos \theta} \cos(k\theta) d\theta$  [38]), we derive that  $\Omega'_1(s) \geq 0$  and thus  $\Omega_1(s)$  increases. In addition, it can be deduced by  $J'_k(p) = -J_{k+1}(p) + \frac{k}{p} J_k(p)$  [37] that  $\Omega'_2(s) = \sqrt{h_2(s)} - \frac{1}{2\sqrt{h_2(s)}} \int_0^1 \int_0^x y^2 \frac{J_1(\sqrt{s(x^2 - y^2)})J_2(\sqrt{s(x^2 - y^2)})}{\sqrt{s(x^2 - y^2)}} dy dx$ , where  $h_2(s) = \int_0^1 \int_0^x y^2 \left( \frac{J_1(\sqrt{s(x^2 - y^2)})}{\sqrt{s(x^2 - y^2)}} \right)^2 dy dx$ . Then we infer from  $J_1(p) = -J'_0(p)$  [37] that  $\Omega'_2(s) \geq 0$  is equivalent to  $h_3(s) := \int_0^1 \int_0^x y^2 \frac{J_0(\sqrt{s(x^2 - y^2)})J_1(\sqrt{s(x^2 - y^2)})}{\sqrt{s(x^2 - y^2)}} dy dx = \frac{1}{2s} \left( \frac{1}{2} - \int_0^1 \int_0^x J_0^2(\sqrt{s(x^2 - y^2)}) dy dx \right) \geq 0$ . Notice that  $|J_k(p)| \leq 1$  by the representation  $J_k(p) = \frac{1}{\pi} \int_0^\pi \cos(k\theta - p \sin(\theta)) d\theta$  [37], thus  $h_3(s) \geq 0$  holds and  $\Omega_2(s)$  is also increasing. To sum up, the monotonicity of  $\Omega(s) = \Omega_1(s)\Omega_2(s)$  follows immediately from the fact that  $\Omega_1(s) > 0$  and  $\Omega_2(s) > 0$ . On the other hand, using the representation  $I_k(p) = \frac{1}{\pi} \int_0^\pi e^{p \cos \theta} \cos(k\theta) d\theta$  [38], we know that  $I_0(p) - I_2(p) = \frac{1}{\pi} \int_0^\pi e^{p \cos \theta} (1 - \cos(2\theta)) d\theta \geq \frac{1}{\pi} \int_0^{\frac{\pi}{2}} (1 - \cos(2\theta)) d\theta = \frac{1}{2}$  for  $p \geq 0$ . Since

$\Omega_2(s) \geq 1$ , then  $\Omega(s) \geq \Omega_1(s) \geq 1 + \frac{s}{8\sqrt{3}} \rightarrow +\infty$  for  $s \rightarrow +\infty$ . Hence,  $\Omega(s) \rightarrow +\infty$  as  $s \rightarrow +\infty$ .

**Proposition 3.2** *There exist  $\underline{\eta}, \bar{\eta} \in \mathcal{K}_\infty$  such that  $\underline{\eta}(2^{-\sigma(t)}) \leq \eta_{\sigma(t)} \leq \bar{\eta}(2^{-\sigma(t)})$  for all  $t \in [0, t^*]$  provided that  $\sigma(t)$  is defined by (10) and thus we have*

$$\frac{1}{2\underline{\eta}^{-1}(\|u(t, \cdot)\|)} \leq 2^{\sigma(t)} \leq \frac{2}{\bar{\eta}^{-1}(\|u(t, \cdot)\|)} \quad (11)$$

holds for all  $t \in [0, t^*]$ .

**Proof.** According to the switching law (10), we have for all  $t \in [0, t^*]$  that  $\eta_{\sigma(t)+1} \leq \|u(t, \cdot)\| \leq \eta_{\sigma(t)-1}$  holds. On the other hand, based on the relation  $\Omega(2^{\sigma(t)})\eta_{\sigma(t)} = \eta_{\sigma(t)-1}$ , we can get  $\eta_{\sigma(t)} = \prod_{k=1}^{\sigma(t)} \frac{1}{\Omega(2^k)}$  when  $\sigma(t) \in \mathbb{Z}^+$  and  $\eta_{\sigma(t)} = \prod_{k=\sigma(t)+1}^0 \Omega(2^k)$  when  $\sigma(t) \in \mathbb{Z}^-$ , and  $\eta_{\sigma(t)} \rightarrow 0$  as  $\sigma(t) \rightarrow +\infty$  or  $\eta_{\sigma(t)} \rightarrow +\infty$  as  $\sigma(t) \rightarrow -\infty$  (it is a decreasing step function). Therefore, there exists a non-decreasing function  $\tilde{\eta} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that  $\eta_{\sigma(t)} = \tilde{\eta}(2^{-\sigma(t)})$ . Indeed, in order to find two class  $\mathcal{K}_\infty$  functions  $\underline{\eta}, \bar{\eta}$ , which form lower and upper bounds for  $\tilde{\eta}$ , we observe that  $\tilde{\eta}(2^{-\sigma(t)}) = \eta_{\sigma(t)} \rightarrow 0$  as  $2^{-\sigma(t)} \rightarrow 0$  ( $\sigma(t) \rightarrow +\infty$ ) and  $\tilde{\eta}(2^{-\sigma(t)}) = \eta_{\sigma(t)} \rightarrow +\infty$  as  $2^{-\sigma(t)} \rightarrow +\infty$  ( $\sigma(t) \rightarrow -\infty$ ). In this case,  $\tilde{\eta}(2^{-\sigma(t)})$  is an increasing step function with respect to argument  $2^{-\sigma(t)}$ . Thus the relation  $\underline{\eta}(2^{-\sigma(t)}) \leq \eta_{\sigma(t)} \leq \bar{\eta}(2^{-\sigma(t)})$  holds. Furthermore, it implies that  $\underline{\eta}(2^{-\sigma(t)-1}) \leq \|u(t, \cdot)\| \leq \bar{\eta}(2^{-\sigma(t)+1})$  for all  $t \in [0, t^*]$ . Consequently,  $\frac{1}{2\underline{\eta}^{-1}(\|u(t, \cdot)\|)} \leq 2^{\sigma(t)} \leq \frac{2}{\bar{\eta}^{-1}(\|u(t, \cdot)\|)}$  is valid for all  $t \in [0, t^*]$ .

### 3.3 Well-posedness analysis

Before stating the main result, let us discuss the well-posedness of the closed-loop system (1) with (2) and (10). First we give the following definition about the solution of system (1) with (2) and (10).

**Definition 3.2** *The solution of system (1) with (2) and (10) is a tuple  $(u, \sigma) : u \in C^0(\mathbb{R}^+, L^2((0, 1), \mathbb{R}))$  and  $\sigma : \mathbb{R}^+ \rightarrow \mathbb{Z}$  satisfying the switching rule (10). Here,  $u$  is a solution to the system (1) understood in the weak sense, that is the following equality*

$$\begin{aligned} & - \int_0^1 \xi(0, x)u_0(x)dx - \int_0^T \int_0^1 \xi_t(t, x)u(t, x)dxdt \\ & + \int_0^T \xi_x(t, 1) \int_0^1 K(1, y, 2^{\sigma(t)})u(t, y)dydt \\ & - \int_0^T \int_0^1 \xi_{xx}(t, x)u(t, x)dxdt - \int_0^T \int_0^1 d(t, x)\xi(t, x)dxdt = 0 \end{aligned}$$

holds for a.e.  $T \in \mathbb{R}^+$  and for each  $\xi \in C^2([0, T] \times [0, 1])$  with compact support in  $[0, T] \times [0, 1]$  and  $\xi$  vanished on  $[0, T] \times \{0, 1\}$ .

**Theorem 3.1** *Assume that  $u_0 \in L^2((0, 1), \mathbb{R})$  and  $d \in \mathcal{L}^\infty(Q_\infty)$ . Then closed-loop system (1) with (2) and (10) has a unique solution  $(u, \sigma) : u \in C^0(\mathbb{R}^+, L^2((0, 1), \mathbb{R}))$  and  $\sigma : \mathbb{R}^+ \rightarrow \mathbb{Z}$ .*

**Proof.** For simplicity of notation, we denote  $U(t) = \mathfrak{K}_{\sigma(t)}u(t, \cdot)$ , where  $\mathfrak{K}_{\sigma(t)}$  is a linear operator of the form  $\mathfrak{K}_{\sigma(t)}v = \int_0^1 K(1, y, 2^{\sigma(t)})v(y)dy$  for  $v \in L^2((0, 1), \mathbb{R})$ . Since  $|I_k(p)| \leq e^p \leq e^{\sqrt{2^{\sigma(t)}}}$  where  $p = \sqrt{2^{\sigma(t)}(x^2 - y^2)}$  ( $0 \leq y \leq x \leq 1$ ), we can get  $|K(x, y, 2^{\sigma(t)})| = |y2^{\sigma(t)} \frac{I_0(y) - I_2(y)}{2}| \leq 2^{\sigma(t)} e^{\sqrt{2^{\sigma(t)}}}$  and thus  $|\mathfrak{K}_{\sigma(t)}u(t, \cdot)| \leq 2^{\sigma(t)} e^{\sqrt{2^{\sigma(t)}}} \|u(t, \cdot)\| := M_{\sigma(t)} \|u(t, \cdot)\|$  holds. In fact,  $\sigma(t)$  is a function of  $u(t, \cdot)$  and depends on the number of switching. Defining  $q$  as the number of commutations in the system (1), in order to construct the resulting switching signal  $\sigma(t)$ , we introduce the notation  $\sigma^q(t)$  to represent the corresponding switching signal (step) function when the number of commutation is  $q$ .

In what follows, we use the fixed point argument [17]. Set  $\mathcal{U}_0^q(t, x) = 0$  and let  $\mathcal{U}_n^q(t, x)$  ( $n \geq 1$ ) be the unique solution of

$$\begin{cases} \mathcal{U}_{n,t}^q(t, x) = \mathcal{U}_{n,xx}^q(t, x) + d(t, x), \\ \mathcal{U}_n^q(t, 0) = 0, \mathcal{U}_n^q(t, 1) = \mathfrak{K}_{\sigma^q(t)} \mathcal{U}_{n-1}^q(t, \cdot), \\ \mathcal{U}_n^q(0, x) = u_0(x). \end{cases}$$

Define  $\mathcal{E}_n^q(t, x) = \mathcal{U}_{n+1}^q(t, x) - \mathcal{U}_n^q(t, x)$ , then we obtain

$$\begin{cases} \mathcal{E}_{n,t}^q(t, x) = \mathcal{E}_{n,xx}^q(t, x), \\ \mathcal{E}_n^q(t, 0) = 0, |\mathcal{E}_n^q(t, 1)| \leq M_{\sigma^q(t)} \|\mathcal{E}_{n-1}^q(t, \cdot)\|, \\ \mathcal{E}_n^q(0, x) = 0. \end{cases}$$

Step 0 ( $q = 0$ ): For the given initial value  $u_0(x)$ , two scenarios are possible: either  $\|u_0\| = 0$ , then by rule (10)  $\sigma(t) = 0$  for all  $t \geq t_0$ . Since  $d \in \mathcal{L}^\infty(Q_\infty)$ , the considered system has a unique solution  $u \in C^0(\mathbb{R}^+, L^2((0, 1), \mathbb{R}))$  and thus the pair  $(u, \sigma)$  exists [27, Def. 2.3, Ch. 4, pp. 106]; or  $\|u_0\| \neq 0$ , then  $\sigma^0(t)$  is determined by  $\|u_0\|$  and now  $\sigma^0(t)$  (also for  $M_{\sigma^0(t)}$ ) is a constant for  $t \geq 0$ . Using the proof of Lemma 6 in [6], we conclude that sequence  $\{\mathcal{U}_n^0(t, \cdot)\}$  will converge to its limit  $\mathcal{U}_\infty^0(t, \cdot)$  and system (1) has a unique solution  $\mathcal{U}_\infty^0 \in C^0(\mathbb{R}^+, L^2((0, 1), \mathbb{R}))$ . In this case, we determine a pair  $(\mathcal{U}_\infty^0(t, \cdot), \sigma^0(t))$  for  $t \geq 0$ .

Step 1 ( $q = 1$ ): Note that from rule (10) the first switching will happen either when  $\mathcal{U}_\infty^0(t, \cdot)$  reaches the switching level  $\eta_{\sigma^0(t)+1}$  or  $\eta_{\sigma^0(t)-1}$  or there exists the point

$t^* = \inf\{t \geq t_0 \mid \|\mathcal{U}_\infty^0(t, \cdot)\| = 0\}$ . Define switching point  $t_1 = \inf\{t > t_0 \mid \|\mathcal{U}_\infty^0(t, \cdot)\| = \eta_{\sigma^0(t)+1}$  or  $\|\mathcal{U}_\infty^0(t, \cdot)\| = \eta_{\sigma^0(t)-1}\}$ . If  $t^* < t_1$ , the first switching happens at  $t^*$  and by (10)  $\sigma(t) = 0$  for  $t \geq t^*$  (for  $t \in [t_0, t^*)$ ,  $\sigma(t) = \sigma^0(t)$ ), then the solution  $\mathcal{U}_\infty^0(t, \cdot)$  will exit or stay at the origin after  $t^*$ . To conclude, system (1) has a unique solution  $u \in C^0(\mathbb{R}^+, L^2((0, 1), \mathbb{R}))$  and the pair  $(u, \sigma)$  is determined. If  $t^* \geq t_1$ , the first switching happens at  $t_1$ . For  $t \in [t_0, t_1)$ , we have  $\sigma^1(t) = \sigma^0(t)$ ; and for  $t \geq t_1$ ,  $\sigma^1(t)$  is determined by  $\|\mathcal{U}_\infty^0(t_1, \cdot)\|$ . To sum up,  $\sigma^1(t), t \geq 0$  is defined by  $\mathcal{U}_\infty^0(t, \cdot)$ . It is obvious that  $\sigma^1(t), t \geq 0$  is a piecewise constant and then we have  $M_{\sigma^1(t)} \leq C$  for some constant  $C > 0$ . Following the similar line as that in [6], one can also verify the existence of the unique solution  $\mathcal{U}_\infty^1 \in C^0(\mathbb{R}^+, L^2((0, 1), \mathbb{R}))$  and a pair  $(\mathcal{U}_\infty^1(t, \cdot), \sigma^1(t))$  for  $t \geq 0$  is then confirmed.

Iterating the aforementioned procedure, we can obtain some pairs  $(\mathcal{U}_\infty^q(t, \cdot), \sigma^q(t))$  for each  $q$  and the unique solution  $(u(t, \cdot), \sigma(t))$  of system (1) with (2) and (10) can be found.

#### 4 Main result

We are in a position to state the criterion of ISS via hybrid boundary control in term of the designed state-dependent switching law. We will omit the arguments  $(t, x)$  wherever no confusion arises.

**Theorem 4.1** *For any initial switching mode  $\sigma(0) \in \mathbb{Z}$  with  $\|u_0\| \in (\eta_{\sigma(0)+1}, \eta_{\sigma(0)})$  or  $\|u_0\| = 0$ , system (1) with the hybrid boundary control law (2) under the switching rule (10) admits an ISS estimate in  $L^2$  norm with respect to the initial value  $u_0 \in L^2((0, 1), \mathbb{R})$  and input  $d \in \mathcal{L}^\infty(Q_\infty)$ , which is given by (20). Moreover, the system in the case of  $d(t, x) = 0$  is finite-time stable, and the settling-time function  $\hat{T}$  satisfies  $\hat{T} < 3 \sum_{j=\sigma(0)}^{+\infty} \frac{q_j}{2^j} - \frac{q_{\sigma(0)}}{2^{\sigma(0)-1}} < +\infty$ , where  $q_j = \ln(\Omega(2^j))$ .*

**Proof.** Let  $w_n$  be the classical solutions of system (4) with disturbance functions  $d_n \in C^\infty(\mathbb{R}^+ \times (0, 1), \mathbb{R})$  satisfying  $\|d_n(t, \cdot) - d(t, \cdot)\| \rightarrow 0$  as  $n \rightarrow +\infty$  uniformly on any compact interval of time, then  $\|w_n(t, \cdot) - w(t, \cdot)\| \rightarrow 0$  as  $n \rightarrow +\infty$  uniformly on any compact interval of time [27, Th. 2.7, Ch. 4, pp. 108], where  $w$  is the unique mild solution of system (4), which, due to the transformation (7), corresponds to the unique weak solution of system (1). Take the Lyapunov functional  $V(w_n(t, \cdot)) = \frac{1}{2} \int_0^1 w_n^2(t, x) dx$ . In what follows, our proof is carried out on two intervals  $[0, t^*)$  and  $[t^*, +\infty)$ , where  $t^* = \inf\{t \geq 0 \mid \|u(t, \cdot)\| = 0\}$ .

For  $t \in [0, t^*)$  (in this case,  $\|u_0\| \neq 0$  by default, otherwise we have  $t^* = 0$ ), calculating the derivative of sys-

tem for  $w_n$  on each subinterval  $[t_k, t_{k+1})$  gives

$$\begin{aligned} & \dot{V}(w_n(t, \cdot)) \\ &= \int_0^1 \left( w_n w_{n,xx} - 2^{\sigma(t)} w_n^2 + w_n d_n \right. \\ & \quad \left. - w_n \int_0^x K(x, y, 2^{\sigma(t)}) d_n(t, y) dy \right) dx. \end{aligned} \quad (12)$$

Applying integration by parts and utilizing the boundary conditions to  $w_n$  with  $d_n$ , we obtain by virtue of Lemma 2.1 that

$$\begin{aligned} \int_0^1 w_n w_{n,xx} dx &= w_n w_{n,x} \Big|_0^1 - \int_0^1 w_{n,x}^2 dx \\ &\leq -\pi^2 \int_0^1 w_n^2 dx. \end{aligned} \quad (13)$$

Substituting (13) into (12) yields

$$\begin{aligned} & \dot{V}(w_n(t, \cdot)) \\ &\leq -\pi^2 \int_0^1 w_n^2 dx - 2^{\sigma(t)} \int_0^1 w_n^2 dx + \int_0^1 w_n d_n dx \\ & \quad - \int_0^1 w_n \int_0^x K(x, y, 2^{\sigma(t)}) d_n(t, y) dy dx \\ &\leq -(2^{\sigma(t)} + \pi^2) \|w_n(t, \cdot)\|^2 + \|w_n(t, \cdot)\| \cdot \|d_n(t, \cdot)\| \\ & \quad + \|w_n(t, \cdot)\| \cdot \left\| \int_0^x K(x, y, 2^{\sigma(t)}) d_n(t, y) dy \right\| \\ &\leq -(2^{\sigma(t)} + \pi^2) \|w_n(t, \cdot)\|^2 + \|w_n(t, \cdot)\| \cdot \|d_n(t, \cdot)\| \\ & \quad \times \left( 1 + \frac{2^{\sigma(t)} \sqrt{\int_0^1 \int_0^x y^2 (I_0(p) - I_2(p))^2 dy dx}}{2} \right) \\ &\leq -2\pi^2 V(w_n(t, \cdot)) + \Omega_1(2^{\sigma(t)}) \|d_n(t, \cdot)\| \\ & \quad \times \sqrt{2V(w_n(t, \cdot))} \end{aligned} \quad (14)$$

for  $t \in [t_k, t_{k+1})$ , where  $p = \sqrt{2^{\sigma(t)}(x^2 - y^2)}$ .

Notice that the solution of ODE  $\dot{\zeta}(t) = -a\zeta(t) + b(t)\sqrt{\zeta(t)}$ ,  $t > t_k$  with the initial value  $\zeta(t_k) > 0$  is  $\zeta(t) = (e^{-\frac{a}{2}(t-t_k)} [\sqrt{\zeta(t_k)} + \frac{1}{2} \int_{t_k}^t e^{\frac{a}{2}(s-t_k)} b(s) ds])^2$ . Setting  $a = 2\pi^2$  and  $b(t) = \sqrt{2}\Omega_1(2^{\sigma(t)})\|d_n(t, \cdot)\|$  and applying the comparison principle to (14), we obtain

$$\begin{aligned} & \sqrt{V(w_n(t, \cdot))} \\ &\leq e^{-\pi^2(t-t_k)} \left[ \sqrt{V(w_n(t_k, \cdot))} + \frac{\Omega_1(2^{\sigma(t)})}{\sqrt{2}} \int_{t_k}^t (e^{\pi^2(s-t_k)} \right. \\ & \quad \left. \times \|d_n(s, \cdot)\|) ds \right], \quad t \in [t_k, t_{k+1}). \end{aligned} \quad (15)$$

Letting  $n \rightarrow \infty$  in (15), we derive

$$\begin{aligned} & \sqrt{V(w(t, \cdot))} \\ & \leq e^{-\pi^2(t-t_k)} \sqrt{V(w(t_k, \cdot))} + \frac{\Omega_1(2^{\sigma(t)})}{\sqrt{2\pi^2}} \|d\|_{\mathcal{L}^\infty(Q_t)} \\ & \quad \times (1 - e^{-\pi^2(t-t_k)}), \quad t \in [t_k, t_{k+1}). \end{aligned}$$

Then it follows from (9) that

$$\begin{aligned} & \frac{1}{\Omega_2(2^{\sigma(t)})} \|u(t, \cdot)\| \leq \|w(t, \cdot)\| \\ & \leq e^{-\pi^2(t-t_k)} \Omega_1(2^{\sigma(t)}) \|u(t_k, \cdot)\| + \frac{\Omega_1(2^{\sigma(t)})}{\pi^2} \|d\|_{\mathcal{L}^\infty(Q_t)} \\ & \quad \times (1 - e^{-\pi^2(t-t_k)}), \quad t \in [t_k, t_{k+1}), \end{aligned}$$

that is

$$\begin{aligned} & \|u(t, \cdot)\| \\ & \leq \Omega(2^{\sigma(t)}) \left[ e^{-\pi^2(t-t_k)} \|u(t_k, \cdot)\| + \frac{1}{\pi^2} (1 - e^{-\pi^2(t-t_k)}) \right. \\ & \quad \left. \times \|d\|_{\mathcal{L}^\infty(Q_t)} \right], \quad t \in [t_k, t_{k+1}). \end{aligned} \quad (16)$$

Recalling the relation (11) and using the properties of  $\Omega(s)$ , it holds that

$$\begin{aligned} \alpha(\|u(t, \cdot)\|) & = \frac{\|u(t, \cdot)\|}{\Omega\left(\frac{2}{\bar{\eta}^{-1}(\|u(t, \cdot)\|)}\right)} \leq \frac{\|u(t, \cdot)\|}{\Omega(2^{\sigma(t)})} \\ & \leq e^{-\pi^2(t-t_k)} \|u(t_k, \cdot)\| + \frac{1}{\pi^2} (1 - e^{-\pi^2(t-t_k)}) \|d\|_{\mathcal{L}^\infty(Q_t)} \\ & \leq e^{-\pi^2(t-t_k)} \|u(t_k, \cdot)\| + \frac{1}{\pi^2} \|d\|_{\mathcal{L}^\infty(Q_t)}, \quad t \in [t_k, t_{k+1}) \end{aligned} \quad (17)$$

where  $\alpha(s) = \frac{s}{\Omega\left(\frac{2}{\bar{\eta}^{-1}(s)}\right)}$  and  $\alpha \in \mathcal{K}_\infty$ .

Therefore, an ISS-like estimate has been obtained for each interval of time  $[t_k, t_{k+1})$ , and now our goal is to derive a similar estimate for all  $t \in [0, t^*)$ . To this end, the following analysis will be proceeded.

Due to (10),  $\|u(t_k, \cdot)\| = \eta_{\sigma(t)}$  for  $k > 0$  and  $\|u(t_0, \cdot)\| = \|u_0\|$ . Hence, for  $k > 0$  from (16) we have the condition of boundedness for a trajectory:

$$e^{-\pi^2(t-t_k)} \eta_{\sigma(t)} + \frac{1}{\pi^2} (1 - e^{-\pi^2(t-t_k)}) \|d\|_{\mathcal{L}^\infty(Q_t)} \leq \eta_{\sigma(t)}$$

or equivalently,  $\frac{1}{\pi^2} \|d\|_{\mathcal{L}^\infty(Q_t)} \leq \eta_{\sigma(t)}$ . Therefore, for  $\|d\|_{\mathcal{L}^\infty(Q_t)} \neq 0$ , there exists  $i^* \in \mathbb{Z}$  such that

$$\eta_{i^*+1} \leq \frac{1}{\pi^2} \|d\|_{\mathcal{L}^\infty(Q_t)} \leq \eta_{i^*}$$

for any input  $d$  with a finite norm  $\|d\|_{\mathcal{L}^\infty(Q_t)}$ . Moreover,

$$\mu(2^{-i^*}) \leq \pi^2 \underline{\eta}(2^{-i^*-1}) \leq \pi^2 \eta_{i^*+1} \leq \|d\|_{\mathcal{L}^\infty(Q_t)}$$

for some  $\mu \in \mathcal{K}_\infty$ , then

$$\eta_{i^*} \leq \varrho(\|d\|_{\mathcal{L}^\infty(Q_t)})$$

for  $\varrho(s) = \bar{\eta} \circ \mu^{-1}(s)$ , where the functions  $\eta$  and  $\bar{\eta}$  from the class  $\mathcal{K}_\infty$  were defined in the proof of Proposition 3.2. Note that due to (16) the trajectory  $u(t, \cdot)$  cannot exceed the level  $\eta_{i^*-1}$  for the given input  $d$  provided that  $\|u_0\| < \eta_{i^*-1}$ .

Based on the above analysis, we have the estimate  $\|u(t_k, \cdot)\| \leq \max\{\|u_0\|, \eta_{i^*}\}$ . Then it follows from (17) that

$$\begin{aligned} & \alpha(\|u(t, \cdot)\|) \\ & \leq e^{-\pi^2(t-t_k)} \max\{\|u_0\|, \eta_{i^*}\} + \frac{1}{\pi^2} \|d\|_{\mathcal{L}^\infty(Q_t)} \\ & \leq e^{-\pi^2(t-t_k)} \|u_0\| + \kappa(\|d\|_{\mathcal{L}^\infty(Q_t)}), \quad t \in [t_k, t_{k+1}) \end{aligned}$$

where  $\kappa(s) = \varrho(s) + \frac{s}{\pi^2}$  and  $\kappa \in \mathcal{K}_\infty$ .

Finally, we will demonstrate that there exists a finite upper bound  $T$  of  $t_k$ . To this end let us first exploit a known conclusion for the convergence time of system (1) without external input. It is shown in [29] that system (1) under the hybrid boundary control law (2) with the switching rule (10) in the case of  $d(t, x) = 0$  is finite-time stable, and the convergence time to the origin is  $\hat{T} \leq 3 \sum_{j=\sigma(0)}^{+\infty} \frac{q_j}{2^j} - \frac{q_{\sigma(0)}}{2^{\sigma(0)-1}} < +\infty$ , where  $q_j = \ln(\Omega(2^j))$ . Hence, there exists an upper estimate  $\bar{T} \in \mathcal{K}$  of the settling time, i.e.,  $\|u(t, \cdot)\| = 0$  for  $t \geq \bar{T}(\|u_0\|)$ , and  $\bar{T} = \bar{T}(\|u_0\|)$  depends on the initial value  $\|u_0\|$ . It is clear that  $T < \hat{T} \leq \bar{T}$ .

Therefore, we derive

$$\alpha(\|u(t, \cdot)\|) \leq e^{-\pi^2(t-\bar{T}(\|u_0\|))} \|u_0\| + \kappa(\|d\|_{\mathcal{L}^\infty(Q_t)})$$

for  $t \in [0, t^*)$ , which implies that

$$\begin{aligned} & \|u(t, \cdot)\| \\ & \leq \alpha^{-1}(2e^{-\pi^2(t-\bar{T}(\|u_0\|))} \|u_0\|) + \alpha^{-1}(2\kappa(\|d\|_{\mathcal{L}^\infty(Q_t)})) \\ & = \beta(\|u_0\|, t) + \theta_1(\|d\|_{\mathcal{L}^\infty(Q_t)}) \end{aligned} \quad (18)$$

for  $t \in [0, t^*)$ , where  $\beta(s, t) = \alpha^{-1}(2se^{-\pi^2(t-\bar{T}(s))})$  and  $\theta_1(x) = \alpha^{-1}(2\kappa(x))$ . Moreover,  $\theta_1 \in \mathcal{K}_\infty$  and  $\beta \in \mathcal{KL}$  since  $\alpha^{-1} \in \mathcal{K}_\infty$ .

Hence, the ISS estimate (18) of the solutions to system (1) with the hybrid boundary control law (2) under the switching rule (10) is derived for  $t \in [0, t^*)$ .



Next we establish an ISS estimate for  $t \in [t^*, +\infty)$ . According to the switching rule (10), we have  $\sigma(t) = 0$  for all  $t \geq t^*$ . Proceeding in the similar way as (12)-(16) and bearing  $\|u(t^*, \cdot)\| = 0$  in mind, we get

$$\begin{aligned} \|u(t, \cdot)\| &\leq \frac{\Omega(1)}{1 + \pi^2} (1 - e^{-(1+\pi^2)(t-t^*)}) \|d\|_{\mathcal{L}^\infty(Q_t)} \\ &\leq \frac{\Omega(1)\|d\|_{\mathcal{L}^\infty(Q_t)}}{1 + \pi^2} = \theta_2(\|d\|_{\mathcal{L}^\infty(Q_t)}), \quad t \geq t^* \end{aligned} \quad (19)$$

where  $\theta_2(x) = \frac{\Omega(1)}{1+\pi^2}x$  and  $\theta_2 \in \mathcal{K}_\infty$ .

Combining (18) and (19) yields the following estimate

$$\begin{aligned} \|u(t, \cdot)\| &\leq \beta(\|u_0\|, t) + \max\{\theta_1(\|d\|_{\mathcal{L}^\infty(Q_t)}), \theta_2(\|d\|_{\mathcal{L}^\infty(Q_t)})\} \\ &= \beta(\|u_0\|, t) + \theta(\|d\|_{\mathcal{L}^\infty(Q_t)}), \quad t \geq 0 \end{aligned} \quad (20)$$

where  $\theta(x) = \max\{\theta_1(x), \theta_2(x)\}$  and  $\theta \in \mathcal{K}_\infty$ . This completes the proof.

**Remark 4.1** *It is worth stressing that continuous boundary feedback control  $U(t) = \int_0^1 K(1, y, \lambda)u(t, y)dy$  with constant decay rate  $\lambda > 0$  can only provide the exponential ISS property, which indicates the exponential stability for the disturbance-free system. In order to achieve finite-time stability for the disturbance-free system, the kernel gain in the boundary feedback control law is required to vary in time, that is in the form of  $K(1, y, g(t))$  with a function  $g(\cdot)$ . This idea is verified by effectively treating the problem of finite-time stabilization for various reaction-diffusion systems, see [29, 10, 6] for more details. Highly inspired by [29], the present paper is the attempt to develop a framework of ISS control issue for PDE systems. The design of switching linear boundary feedback (2) that depends on system state leads to the boundedness of the system trajectory as proven in Theorem 4.1 (i.e., the state norm will not exceed the highest level), and finally the resulting system turns out to be ISS, as well as finite-time stable in the disturbance-free case.*

**Remark 4.2** *In the process of proving Theorem 4.1, it has been shown that the solution of the close-loop system (1) will converge to zero if there is no disturbance input, which corresponds to Theorem 1 in [29]. Consequently, this paper extends available results in [29] to the heat equation with external input or disturbance in the distributed form. It is well known that, some disturbances or uncertainties are expressed as boundary disturbances [28], such as the presence of control actuator errors that can be modeled by a PDE system with Dirichlet boundary conditions  $u(t, 0) = 0, u(t, 1) = U(t) + v(t)$  ( $v(t)$  is the unknown boundary disturbance). At present, some methods such as the eigenfunction expansion [18, 19], finite-difference scheme [19], monotonicity method [24], De Giorgi iteration [45], non-coercive Lyapunov function*

*method [15], etc., are presented, which provides ideas to deal with the ISS analysis in the case of Dirichlet boundary inputs.*

## 5 Numerical example

Consider the scalar system (1) with disturbance input  $d(t, x) = 2 \cos(5x) \sin(5t)$ . The initial value is chosen to be  $u_0(x) = 9x(1-x)$ ,  $x \in [0, 1]$  which satisfies compatibility conditions. We simulate numerically the state of system (1) by using the implicit Euler scheme with the step size  $h = 0.02$  of space variable and the step size  $\tau = h^2$  of time variable. First-order forward difference scheme and second-order centred difference scheme are applied in the time variable and space variable, respectively. Note that the solution of open-loop system (i.e.,  $U(t) = 0$ ) with  $d(t, x) = 0$  decays exponentially to zero as  $t \rightarrow \infty$ . To illustrate this fact, the  $L^2$  norm  $\|u(t, \cdot)\|$  for the open-loop system (1) with  $d(t, x) = 0$  is shown in Fig. 3(b) (blue line) in a logarithmic scale. It can be seen that open-loop system (1) without disturbance input is exponentially stable. On the other side, in the presence of disturbance input  $d(t, x)$ , the simulation result for the  $L^2$  norm is plotted in Fig. 1, and thus the open-loop system (1) is asymptotically ISS.

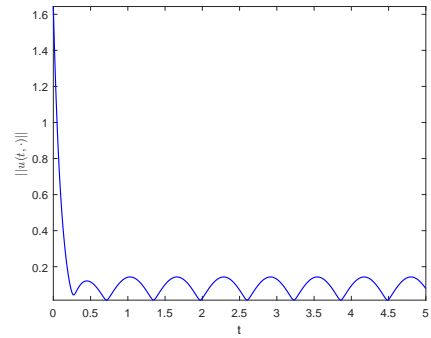


Fig. 1. Numerical solution of the open-loop system (1) with disturbance input  $d(t, x) = 2 \cos(5x) \sin(5t)$ .

Now the hybrid boundary feedback control (2) with the corresponding switching law (10) are utilized to stabilize the system (1) by using the kernel function (6). The  $L^2$  norm  $\|u_0\|$  of the initial value is numerically calculated as 1.6432 (here, we adopt the trapezoidal rule for the numerical integration) and satisfies  $\|u_0\| \in (\eta_{-3}, \eta_{-4}) = (1.6163, 1.6752]$ . Set the initial mode  $\sigma(0) = -4$ . Thus, the closed-loop system (1) is ISS according to Theorem 4.1. Fig. 2 depicts the dynamical behavior of closed-loop system (1), in which Fig. 2(a) shows the state  $u(t, x)$  of the system and Fig. 2(b) shows the  $L^2$  norm of the state. It is observed that after 6 switches, the  $L^2$  norm of the system gradually decreases at first and finally converges to the bounded neighborhood of the origin. Compared with Fig. 1, we also notice from Fig. 2(b) that the  $L^2$  norm of closed-loop system (1) remain confined in a smaller vicinity of the origin, which is less influenced by

disturbance input  $d(t, x)$ . Moreover, Fig. 3 shows the state response of the system (1) with switching boundary control  $U(t)$  in the absence of disturbance input. Similarly, the spatio-temporal evolution of the state  $u(t, x)$  and the time evolution of  $L^2$  norm  $\|u(t, \cdot)\|$  with a logarithmic scale (red line) respectively, are plotted in Fig. 3(a) and Fig. 3(b). It is clearly seen that the closed-loop system (1) with  $d(t, x) = 0$  is finite-time stable. Obviously, the comparison result shows that under switching boundary control  $U(t)$ , the convergence rate for the system (1) without external input is faster than exponential convergence and the disturbance attenuation is improved.

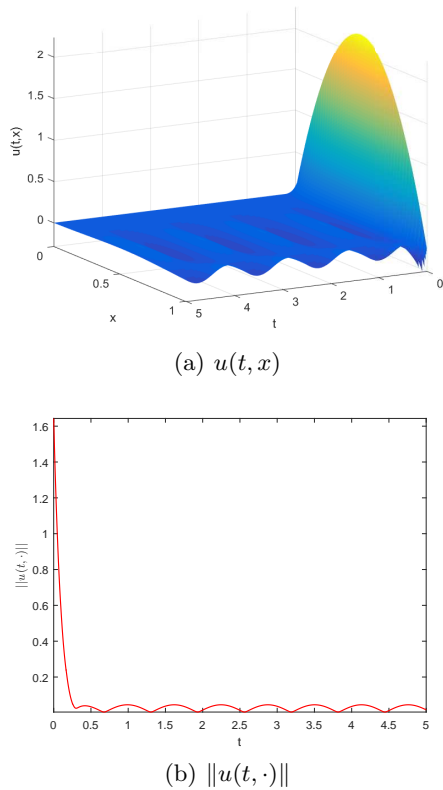


Fig. 2. State response of the closed-loop system (1) with disturbance input  $d(t, x) = 2 \cos(5x) \sin(5t)$ .

## 6 Conclusion

In this paper, we dealt with the switching boundary feedback control towards ISS for heat equation. The hybrid linear boundary control law was developed based on the standard backstepping method. Under a constructive switching rule, we have shown that closed-loop system can successfully achieve ISS. Meanwhile, a stronger result has also been obtained, that is the finite-time stability can be guaranteed for the disturbance-free system. Besides linear switching boundary feedback control in this paper, some other novel and challenging control

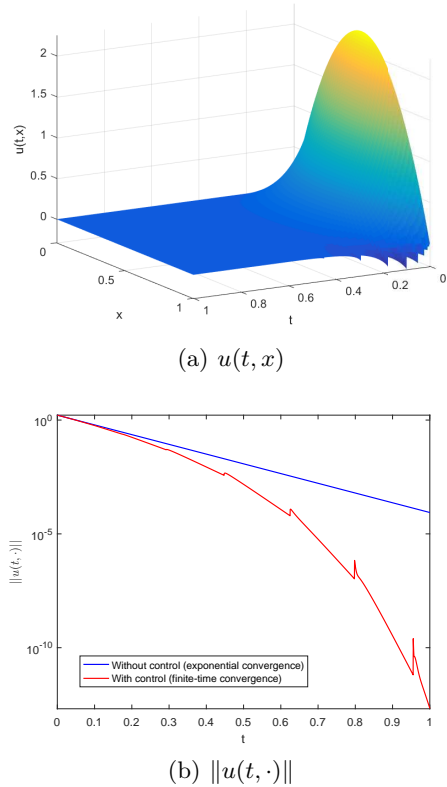


Fig. 3. State response of the closed-loop system (1) with disturbance input  $d(t, x) = 0$ .

schemes can deal with the problem of ISS (such as design of a suitable characterization that  $g(t)$  satisfies for the kernel gain with the form  $K(1, y, g(t))$  as aforementioned), which will be of interest for future researches.

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