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Abstract: This report presents two parametric implicit strategies for discretizing the mixed formulation of linear wave equations, based for the first on the Störmer-Verlet scheme and for the second on the Crank-Nicolson scheme. If the initial conditions are well chosen, these two strategies are equivalent, and equivalence formulas allow the variables of one scheme to be reconstructed from the other. These two schemes are also equivalent to the θ-scheme discretization of the wave equation formulated at order 2. However, the first scheme has a lower complexity than the second one. Numerical results illustrate these equivalences and these efficiency comparisons.

Key-words: numerical analysis, implicit discretization, mixed formulation

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Deux stratégies de discrétisation implicites paramétriques de la formulation mixte des équations d'ondes linéaires

Résumé : Ce rapport présente deux stratégies implicites paramétriques de discrétisation de la formulation mixte des équations d'onde linéaire, basées pour la première sur le schéma de Störmer-Verlet et pour la seconde sur le schéma de Crank-Nicolson. Si les conditions initiales sont bien choisies, ces deux stratégies sont équivalentes et des formules d'équivalence permettent de reconstruire les variables d'un schéma à partir de l'autre. Ces deux schémas sont également équivalents à la discrétisation par θ-schéma de l'équation des ondes formulée à l'ordre 2. Cependant, la complexité du premier est inférieure à celle du second. Des résultats numériques illustrent ces équivalences et ces comparaisons d'efficacité.

Mots-clés : analyse numérique, discrétisation implicite, formulation mixte

Contents

1 General model, space discretization and usual time integration

The mixed formulation of linear wave equations is one possible modeling of wave propagation phenomena. It is chosen over its second order formulation counterpart in situations where the unknowns are more relevant to the physical context [Makridakis and Monk, 1995, Lanteri and Scheid, 2013], or where these unknowns are more natural for modeling purposes (as for instance coupling with other parts [Banjai et al., 2015]), or even where it is not possible to formulate the equations as a second order equation (as for instance in presence of intricate dissipative [Bilbao and Harrison, 2016] or nonlinear phenomena [He and Sun, 2020]).

Numerical methods to solve this system are numerous and can rely on several analysis tools. In this work, we want to introduce and compare two parametrized time discretisation strategies, and we therefore suppose that the spatial discretisation is done with usual methods such as Finite Differences [Zuazua, 2005], Finite Elements [Brezzi and Fortin, 1991], Finite Volumes [Eymard et al., 2000], or any other method that provides the following semi discrete system, along with some necessary bounds on the space discretisation error (typically those supposed in [Chabassier, 2023]). More precisely, we assume that, after introducing a small space discretization parameter h, and following the steps that lead to the semi discrete system, the semi discrete solution (p_h, v_h) is sought in finite dimensional spaces $U_h \subset U \subset P$ and $D_h \subset D$, where P and D are two Hilbert spaces naturally prompted by the equation as the variational spaces for both variables:

$$\begin{cases} p_h(0) = p_{h,0} & (1.1a) \\ v_h(0) = v_{h,0} & (1.1b) \\ \dot{p}_h + B_h^* v_h = f_h & (1.1c) \\ \dot{v}_h - B_h p_h = \dot{g}_h & (1.1d) \end{cases}$$

where $B_h : U_h \to D_h$ is a discrete operator and $B_h^* : D_h \to U_h$ is its adjoint, and $p_{h,0}$, $v_{h,0}$, f_h and g_h are discrete representations of the initial conditions and sources in U_h and D_h respectively. We also denote I_h the identity operator of U_h.

For the time discretization, we introduce the time step $\Delta t > 0$. Two usual discretization strategies are particularly spread and used in the community. The usual Störmer-Verlet (leap-frog) scheme [Hairer et al., 2003] is an explicit and conditionally stable strategy, and Crank-Nicolson scheme [Hairer et al., 2006] is an implicit and unconditionally stable strategy. They are recalled below.

1.1 Störmer-Verlet (SV)

Time is discretized on staggered time grids with the time step Δt. The unknown p_h is sought for on a regular time grid $t^n = n\Delta t$ for $n \in [0, N]$ while the unknown v_h is sought for on the interleaved regular time grid $t^{n+\frac{1}{2}} = (n+\frac{1}{2})\Delta t$ for $n \in [0, N]$. Let us introduce the discrete operators δ and μ defined as

$$\delta v_h^n = \frac{v_h^{n+\frac{1}{2}} - v_h^{n-\frac{1}{2}}}{\Delta t}, \quad \mu v_h^n = \frac{v_h^{n+\frac{1}{2}} + v_h^{n-\frac{1}{2}}}{2}, \quad \delta p_h^{n+\frac{1}{2}} = \frac{p_h^{n+1} - p_h^n}{\Delta t}, \quad \mu p_h^{n+\frac{1}{2}} = \frac{p_h^{n+1} + p_h^n}{2} \tag{1.2}$$

The usual Störmer-Verlet scheme is defined as

$$\begin{cases} p_h^0 = p_h(0) & (1.3a) \\ v_h^{\frac{1}{2}} = v_h(0) + \dfrac{\Delta t}{2}\left[\delta g_h^{\frac{1}{2}} + B_h p_h(0)\right] & (1.3b) \\ \delta p_h^{n+\frac{1}{2}} + B_h^* v_h^{n+\frac{1}{2}} = f_h^{n+\frac{1}{2}}, \qquad 0 \le n \le N & (1.3c) \\ \delta v_h^n - B_h p_h^n = \mu\delta g_h^n, \qquad 1 \le n \le N & (1.3d) \end{cases}$$

Note that the initial conditions are chosen in order to guarantee second order accuracy in time even in the presence of initial data and nonzero source terms at initial time.

1.2 Crank-Nicolson (CN)

We seek both unknowns q_h and w_h a regular time grid $t^n = n\Delta t$ for $n \in [0, N]$. Let us introduce the discrete operators δ and μ defined as

$$\delta q_h^{n+\frac{1}{2}} = \frac{q_h^{n+1} - q_h^n}{\Delta t}, \quad \mu q_h^{n+\frac{1}{2}} = \frac{q_h^{n+1} + q_h^n}{2}, \quad \delta w_h^{n+\frac{1}{2}} = \frac{w_h^{n+1} - w_h^n}{\Delta t}, \quad \mu w_h^{n+\frac{1}{2}} = \frac{w_h^{n+1} + w_h^n}{2} \tag{1.4}$$

The usual Crank-Nicolson scheme reads

$$\begin{cases} q_h^0 = p_h(0) & (1.5a) \\ w_h^0 = v_h(0) & (1.5b) \\ \delta q_h^{n+\frac{1}{2}} + B_h^* \mu w_h^{n+\frac{1}{2}} = f_h^{n+\frac{1}{2}}, \qquad 0 \le n \le N & (1.5c) \\ \delta w_h^{n+\frac{1}{2}} - B_h \mu q_h^{n+\frac{1}{2}} = \delta g_h^{n+\frac{1}{2}}, \qquad 0 \le n \le N & (1.5d) \end{cases}$$

2 Implicit parametrisation of the SV and CN schemes

Let $\theta \ge 0$ be a real number. We now consider two implicit parametrized perturbations of the Störmer-Verlet (leap-frog) and Crank-Nicolson schemes. The treatment of the source terms is tailored so that the comparison of the resulting discrete series is as straighforward as possible.

2.1 Parametrized Störmer-Verlet (θ-SV)

The unknown p_h is sought for on a regular time grid $t^n = n\Delta t$ for $n \in [0, N]$ while the unknown v_h is sought for on the interleaved regular time grid $t^{n+\frac{1}{2}} = (n + \frac{1}{2})\Delta t$ for $n \in [0, N]$. The proposed scheme reads

$$\begin{cases} v_h^{\frac{1}{2}} = v_h(0) + \dfrac{\Delta t}{2} \left[\delta g_h^{\frac{1}{2}} + B_h p_h(0) \right] & (2.1a) \\ p_h^0 = p_h(0) & (2.1b) \\ (I_h + \theta \Delta t^2 B_h^* B_h) \delta p_h^{n+\frac{1}{2}} + B_h^* v_h^{n+\frac{1}{2}} = f_h^{n+\frac{1}{2}}, \qquad 0 \le n \le N & (2.1c) \\ \delta v_h^n - B_h p_h^n = \mu \delta g_h^n, \qquad 1 \le n \le N & (2.1d) \end{cases}$$

We see that the usual SV scheme (1.3) is recovered from the choice $\theta = 0$. This is the only choice that makes the scheme explicit. This scheme satisfies an energy identity which prompts the stability condition for some values of θ.

Proposition 2.1. *Any solution to* (2.1) *satisfies, for* $n \in [0, N]$*,*

$$\delta \mathcal{E}_h^{n+\frac{1}{2}} = \left(f_h^{n+\frac{1}{2}} + B_h^* \mu g_h^{n+\frac{1}{2}} \right) \cdot \mu p_h^{n+\frac{1}{2}} \quad \textit{where} \quad \mathcal{E}_h^n = \frac{1}{2} \|p_h^n\|^2_{\widetilde{I}_h} + \frac{1}{2} \|\mu v_h^n - \mu^2 g_h^n\|^2 \tag{2.2}$$

where

$$\widetilde{I}_h = I_h + (\theta - \frac{1}{4}) \Delta t^2 B_h^* B_h \tag{2.3}$$

Proof. Take the scalar product of (2.1c) with $\mu p_h^{n+\frac{1}{2}}$ and of μ (2.1d) with $\mu v_h^n - \mu^2 g_h^n$. □

Proposition 2.2 (Stability condition). *If* $\theta \ge \frac{1}{4}$*, the discrete energy* $\mathcal{E}_h^n$ *is always positive. If* $\theta < \frac{1}{4}$*, the discrete energy is positive if*

$$\eta \le 1, \qquad \textit{where} \quad \eta = \frac{\Delta t}{2} \sqrt{1 - 4\theta} \sqrt{\rho(B_h^* B_h)} \tag{2.4}$$

Proof. This directly follows from the definition of $\widetilde{I}_h$. □

2.2 Parametrized Crank-Nicolson (θ-CN)

We seek both unknowns q_h and w_h a regular time grid $t^n = n\Delta t$ for $n \in [0, N]$. The proposed scheme reads

$$\begin{cases} w_h^0 = v_h(0) & (2.5a) \\ q_h^0 = p_h(0) & (2.5b) \\ \left(I_h + (\theta - \frac{1}{4})\Delta t^2 B_h^* B_h\right) \delta q_h^{n+\frac{1}{2}} + B_h^* \mu w_h^{n+\frac{1}{2}} = f_h^{n+\frac{1}{2}}, \qquad 0 \le n \le N & (2.5c) \\ \delta w_h^{n+\frac{1}{2}} - B_h \mu q_h^{n+\frac{1}{2}} = \delta g_h^{n+\frac{1}{2}}, \qquad 0 \le n \le N & (2.5d) \end{cases}$$

We see that the usual CV scheme (1.5) is recovered from the choice $\theta = \frac{1}{4}$. No choice of θ makes the scheme explicit. This scheme satisfies an energy identity which prompts the stability condition for some values of θ.

PROPOSITION 2.3. *Any solution to* (2.5) *satisfies, for* $n \in [0, N]$,

$$\delta \mathcal{E}_h^{n+\frac{1}{2}} = f_h^{n+\frac{1}{2}} \cdot \mu p_h^{n+\frac{1}{2}} + \delta g_h^{n+\frac{1}{2}} \cdot \mu w_h^{n+\frac{1}{2}} \quad \textit{where} \quad \mathcal{E}_h^n = \frac{1}{2}\|q_h^n\|^2_{\widetilde{I}_h} + \frac{1}{2}\|w_h^n\|^2 \tag{2.6}$$

where

$$\widetilde{I}_h = I_h + (\theta - \frac{1}{4})\Delta t^2 B_h^* B_h \tag{2.7}$$

Proof. Take the scalar product of (2.5c) with $\mu q_h^{n+\frac{1}{2}}$ and of (2.5d) with $\mu w_h^{n+\frac{1}{2}}$. □

PROPOSITION 2.4 (Stability condition). *If* $\theta \ge \frac{1}{4}$, *the discrete energy* $\mathcal{E}_h^n$ *is always positive. If* $\theta < \frac{1}{4}$, *the discrete energy is positive if*

$$\eta \le 1, \qquad \textit{where} \quad \eta = \frac{\Delta t}{2}\sqrt{1 - 4\theta}\sqrt{\rho(B_h^* B_h)} \tag{2.8}$$

Proof. This directly follows from the definition of $\widetilde{I}_h$. □

2.3 Relation between θ-SV and θ-CN

The following proposition allows to reconstruct the fields of the θ-SV scheme (2.1) from the fields obtained by the θ-CN scheme (2.5).

PROPOSITION 2.5. *Let* (w_h, q_h) *be the solution to* (2.5). *Then,* (v_h, p_h) *solution to* (2.1) *satisfy*

$$\begin{cases} p_h^n = q_h^n, \quad n \ge 0 & (2.9a) \\ v_h^{n+\frac{1}{2}} = \mu w_h^{n+\frac{1}{2}} - \dfrac{\Delta t^2}{4} B_h \delta q_h^{n+\frac{1}{2}}, \quad n \ge 0 & (2.9b) \end{cases}$$

Proof. Define the field $x_h^{n+\frac{1}{2}}$ for all $n \ge 0$ as

$$x_h^{n+\frac{1}{2}} = \mu w_h^{n+\frac{1}{2}} - \frac{\Delta t^2}{4} B_h \delta q_h^{n+\frac{1}{2}} \tag{2.10}$$

From the definition of x_h we can derive the value of δx_h^n:

$$\begin{aligned} \delta x_h^n &= \delta \mu w_h^n - \frac{\Delta t^2}{4} B_h \delta^2 q_h^n && (2.11) \\ &= B_h \left(\mu^2 - \frac{\Delta t^2}{4}\delta^2\right) q_h^n + \delta \mu g_h^n \quad \text{using (2.5d)} && (2.12) \\ &= B_h q_h^n + \delta \mu g_h^n && (2.13) \end{aligned}$$

If we use the definition of x_h in (2.5c), we get the system

$$\begin{cases} q_h^0 = p_h(0) & (2.14a) \\ x_h^{\frac{1}{2}} = \mu w_h^{\frac{1}{2}} - \dfrac{\Delta t^2}{4} B_h \delta q_h^{\frac{1}{2}} & (2.14b) \\ \left(I_h + \theta \Delta t^2 B_h^* B_h\right) \delta q_h^{n+\frac{1}{2}} + B_h^* x_h^{n+\frac{1}{2}} = f_h^{n+\frac{1}{2}}, \qquad 0 \le n \le N & (2.14c) \\ \delta x_h^{n+\frac{1}{2}} - B_h q_h^{n+\frac{1}{2}} = \delta \mu g_h^{n+\frac{1}{2}}, \qquad 0 \le n \le N & (2.14d) \end{cases}$$

Finally, from (2.5d) taken for $n = 0$, we replace $\delta w_h^{\frac{1}{2}}$ and $\mu q_h^{\frac{1}{2}}$ with $\mu w_h^{\frac{1}{2}}$ and $\delta q_h^{\frac{1}{2}}$ and remaining terms to get

$$\frac{2}{\Delta t} \mu w_h^{\frac{1}{2}} - \frac{2}{\Delta t} w_h^0 - B_h \left[\frac{\Delta t}{2} \delta q_h^{\frac{1}{2}} + q_h^0\right] = \delta g_h^{\frac{1}{2}} \tag{2.15}$$

We recognize the terms forming the expression of $x_h^{\frac{1}{2}}$ to get that

$$x_h^{\frac{1}{2}} = w_h^0 + \frac{\Delta t}{2}\left[\delta g_h^{\frac{1}{2}} + B_h q_h^0\right] \tag{2.16}$$

In the end, the couple (x_h, q_h) is solution to

$$\begin{cases} q_h^0 = p_h(0) & (2.17a) \\ x_h^{\frac{1}{2}} = w_h^0 + \dfrac{\Delta t}{2}\left[\delta g_h^{\frac{1}{2}} + B_h q_h^0\right] & (2.17b) \\ \left(I_h + \theta \Delta t^2 B_h^* B_h\right) \delta q_h^{n+\frac{1}{2}} + B_h^* x_h^{n+\frac{1}{2}} = f_h^{n+\frac{1}{2}}, \qquad 0 \le n \le N & (2.17c) \\ \delta x_h^{n+\frac{1}{2}} - B_h q_h^{n+\frac{1}{2}} = \delta \mu g_h^{n+\frac{1}{2}}, \qquad 0 \le n \le N & (2.17d) \end{cases}$$

which is strictly identical to (2.1). □

The following proposition gives the formula to reconstruct the fields of the θ-CN scheme (2.5) from the fields of the θ-SV scheme (2.1).

PROPOSITION 2.6. *Let (v_h, p_h) be the solution to (2.1). Then, (w_h, q_h) solution to (2.5) satisfy*

$$\begin{cases} q_h^n = p_h^n, \quad \forall n \ge 0 & (2.18a) \\ w_h^n = \mu v_h^n - \dfrac{\Delta t^2}{4} \delta^2 g_h^n, \quad \forall n \ge 1 & (2.18b) \end{cases}$$

Proof. It is also possible to prove the proposition without using the field u. Indeed, let (v_h, p_h) be the solution to the θ-SV scheme (2.1), and define for all $n \ge 1$,

$$y_h^n = \mu v_h^n - \frac{\Delta t^2}{4} \delta^2 g_h^n \tag{2.19}$$

Let us compute

$$\begin{aligned} \delta y_h^{n+\frac{1}{2}} - B_h \mu p_h^{n+\frac{1}{2}} &= \delta\left(\mu v_h - \frac{\Delta t^2}{4}\delta^2 g_h\right)^{n+\frac{1}{2}} - B_h \mu p_h^{n+\frac{1}{2}} && (2.20) \\ &= \mu\left(\delta v_h\right)^{n+\frac{1}{2}} - \frac{\Delta t^2}{4}\delta^3 g_h^n - B_h \mu p_h^{n+\frac{1}{2}} && (2.21) \\ &=_{(2.1d)} \mu\left(\mu \delta g_h + B_h p_h\right)^{n+\frac{1}{2}} - \frac{\Delta t^2}{4}\delta^3 g_h^n - B_h \mu p_h^{n+\frac{1}{2}} && (2.22) \\ &= \delta\left(\mu^2 - \frac{\Delta t^2}{4}\delta^2\right) g_h^{n+\frac{1}{2}} && (2.23) \\ &= \delta g_h^{n+\frac{1}{2}} && (2.24) \end{aligned}$$

On the other hand, let $\widetilde{I}_h = \big(I_h + (\theta - \frac{1}{4})\Delta t^2 B_h^* B_h\big)$. We have

$$\begin{aligned}
\widetilde{I}_h \delta p_h^{n+\frac{1}{2}} + B_h^* \mu y_h^{n+\frac{1}{2}} &= \widetilde{I}_h \delta p_h^{n+\frac{1}{2}} + B_h^* \mu \left[\mu v_h - \frac{\Delta t^2}{4} \delta^2 g_h \right]^{n+\frac{1}{2}} && (2.25)\\
&= \widetilde{I}_h \delta p_h^{n} + B_h^* \mu^2 v_h^{n+\frac{1}{2}} - B_h^* \frac{\Delta t^2}{4} \mu \delta^2 g_h^{n+\frac{1}{2}} && (2.26)\\
&=_{(2.31)} \widetilde{I}_h \delta p_h^{n} + B_h^* \left[v_h^{n+\frac{1}{2}} + \frac{\Delta t^2}{4} \delta^2 v_h^{n+\frac{1}{2}} \right] - B_h^* \frac{\Delta t^2}{4} \mu \delta^2 g_h^{n+\frac{1}{2}} && (2.27)\\
&=_{\delta(2.1d)} \widetilde{I}_h \delta p_h^{n} + B_h^* v_h^{n+\frac{1}{2}} + \frac{\Delta t^2}{4} B_h^* \left[\mu \delta^2 g_h^{n+\frac{1}{2}} + B_h \delta p_h^{n+\frac{1}{2}} \right] - B_h^* \frac{\Delta t^2}{4} \mu \delta^2 g_h^{n+\frac{1}{2}} && (2.28)\\
&= \big(I_h + \theta \Delta t^2 B_h^* B_h\big)\, \delta p_h^n + B_h^* v_h^{n+\frac{1}{2}} && (2.29)\\
&=_{(2.1c)} f_h^{n+\frac{1}{2}} && (2.30)
\end{aligned}$$

where we used the identity

$$X^{n+\frac{1}{2}} = \mu^2 X^{n+\frac{1}{2}} - \frac{\Delta t^2}{4} \delta^2 X^{n+\frac{1}{2}} \tag{2.31}$$

Therefore we recover that (y_h, p_h) is solution to a system similar to (2.5). It remains to prove that the initial conditions match. It is the case for p_h and it suffices to pose $y_h^0 = v_h(0)$ to ensure the full equivalence, hence $y_h^n = w_h^n$. □

In other words, the two formulations are equivalent, and we have formulas to recover one from the other.

3 Equivalence to second order θ-scheme

In this section, the solutions to (2.1) and to (2.5) are written as post processed quantities based on two different fields u_1 and u_2 which are in fact solutions to θ-schemes with appropriate initial data. Then, these two fields are shown to be equal, which provides an alternate proof to Prop. 2.6.

3.1 Field associated to the parametrized Störmer-Verlet (θ-SV)

Proposition 3.1. *The unknowns $(p_h^n, v_h^{n+\frac{1}{2}})$ solution to (2.1) can be written as:*

$$p_h^n = \delta u_1^n, \qquad v_h^{n+\frac{1}{2}} = B_h u_1^{n+\frac{1}{2}} + \mu g_h^{n+\frac{1}{2}} \tag{3.1}$$

where u_1^n is defined on the interleaved grid $t^{n+\frac{1}{2}}$ such that for all $0 \le n \le N$,

$$\begin{cases}
u_1^{-\frac{1}{2}} = u_1^{\frac{1}{2}} - \Delta t p_h^0 & (3.2a)\\
u_1^{\frac{1}{2}} = (B_h^* B_h)^{-1} B_h^* \left[v_h^{\frac{1}{2}} - \mu g_h^{\frac{1}{2}} \right] & (3.2b)\\
\delta^2 u_1^{n+\frac{1}{2}} + B_h^* B_h \{u_1\}_\theta^{n+\frac{1}{2}} = f_h^{n+\frac{1}{2}} - B_h^* \mu g_h^{n+\frac{1}{2}}, \quad 0 \le n \le N & (3.2c)
\end{cases}$$

Proof. Let us introduce a field u_1 defined on the interleaved grid $t^{n+\frac{1}{2}}$ such that for all $0 \le n \le N$,

$$\delta u_1^n = p_h^n, \quad u_1^{\frac{1}{2}} = (B_h^* B_h)^{-1} B_h^* \left[v_h^{\frac{1}{2}} - \mu g_h^{\frac{1}{2}} \right] \tag{3.3}$$

Then

$$\begin{cases}
(I_h + \theta \Delta t^2 B_h^* B_h) \delta^2 u_1^{n+\frac{1}{2}} + B_h^* v_h^{n+\frac{1}{2}} = f_h^{n+\frac{1}{2}}, \qquad 1 \le n \le N & (3.4a)\\
\delta v_h^n - B_h \delta u_1^n = \mu \delta g_h^n, \qquad 1 \le n \le N & (3.4b)
\end{cases}$$

We get from summation of B_h^* applied to the second equation from 1 to n that

$$B_h^* v_h^{n+\frac{1}{2}} = B_h^* B_h u_1^{n+\frac{1}{2}} + B_h^* \mu g_h^{n+\frac{1}{2}} + \underbrace{B_h^* v_h^{\frac{1}{2}} - B_h^* B_h u_1^{\frac{1}{2}} - B_h^* \mu g_h^{\frac{1}{2}}}_{=0} \tag{3.5}$$

Hence

$$\left(I_h + \theta\Delta t^2 B_h^* B_h\right)\delta^2 u_1^{n+\frac{1}{2}} + B_h^* B_h u_1^{n+\frac{1}{2}} = f_h^{n+\frac{1}{2}} - B_h^*\mu g_h^{n+\frac{1}{2}} \tag{3.6}$$

Which also can be written as the usual θ-scheme on u_1

$$\begin{cases} u_1^{-\frac{1}{2}} = u_1^{\frac{1}{2}} - \Delta t p_h^0 & (3.7a)\\ u_1^{\frac{1}{2}} = \left(B_h^* B_h\right)^{-1} B_h^*\left[v_h^{\frac{1}{2}} - \mu g_h^{\frac{1}{2}}\right] & (3.7b)\\ \delta^2 u_1^{n+\frac{1}{2}} + B_h^* B_h\{u_1\}_\theta^{n+\frac{1}{2}} = f_h^{n+\frac{1}{2}} - B_h^*\mu g_h^{n+\frac{1}{2}}, \quad 0 \le n \le N & (3.7c) \end{cases}$$

Moreover,

$$v_h^{n+\frac{1}{2}} = B_h u_1^{n+\frac{1}{2}} + \mu g_h^{n+\frac{1}{2}} + e_v^{\frac{1}{2}}, \quad \text{where } e_v^{\frac{1}{2}} = v_h^{\frac{1}{2}} - B_h u_1^{\frac{1}{2}} - \mu g_h^{\frac{1}{2}} \tag{3.8}$$

But

$$B_h^* B_h u_1^{\frac{1}{2}} = B_h^*\left[v_h^{\frac{1}{2}} - \mu g_h^{\frac{1}{2}}\right] = B_h^* e_v^{\frac{1}{2}} + B_h^* B_h u_1^{\frac{1}{2}} \Rightarrow B_h^* e_v^{\frac{1}{2}} = 0 \Rightarrow B_h B_h^* e_v^{\frac{1}{2}} = 0 \Rightarrow e_v^{\frac{1}{2}} = 0 \tag{3.9}$$

□

3.2 Field associated to the parametrized Crank-Nicolson (θ-CN)

PROPOSITION 3.2. *The unkowns w_h^n and q_h^n solution to (2.5) can be written as:*

$$q_h^n = \delta u_2^n, \qquad w_h^n = B_h\mu u_2^n + g_h^n \tag{3.10}$$

where the field u_2 is defined on the interleaved grid $t^{n+\frac{1}{2}}$ such that for all $1 \le n \le N$,

$$\begin{cases} u_2^{-\frac{1}{2}} = u_2^{\frac{1}{2}} - \Delta t q_h^0 & (3.11a)\\ u_2^{\frac{1}{2}} = \left(B_h^* B_h\right)^{-1}\left[B_h^* w_h^0 - B_h^* g_h^0\right] + \dfrac{\Delta t}{2} q_h^0 & (3.11b)\\ \delta^2 u_2^{n+\frac{1}{2}} + B_h^* B_h\{u_2\}_\theta^{n+\frac{1}{2}} = f_h^{n+\frac{1}{2}} - B_h^*\mu g_h^{n+\frac{1}{2}}, \quad 0 \le n \le N & (3.11c) \end{cases}$$

Proof. Let us introduce a field u_2 defined on the interleaved grid $t^{n+\frac{1}{2}}$ such that for all $1 \le n \le N$,

$$\delta u_2^n = q_h^n, \quad u_2^{\frac{1}{2}} = \left(B_h^* B_h\right)^{-1}\left[B_h^* w_h^0 - B_h^* g_h^0\right] + \frac{\Delta t}{2} q_h^0 \tag{3.12}$$

$$\begin{cases} \left(I_h + (\theta - \dfrac{1}{4})\Delta t^2 B_h^* B_h\right)\delta^2 u_2^{n+\frac{1}{2}} + B_h^*\mu w_h^{n+\frac{1}{2}} = f_h^{n+\frac{1}{2}} & (3.13a)\\ \delta w_h^{n+\frac{1}{2}} - B_h\mu\delta u_2^{n+\frac{1}{2}} = \delta g_h^{n+\frac{1}{2}} & (3.13b) \end{cases}$$

We get from summation of B_h^* applied to the second equation from 0 to n that

$$B_h^* w_h^n = B_h^* B_h\mu u_2^n + B_h^* g_h^n + \underbrace{B_h^* w_h^0 - B_h^* B_h\mu u_2^0 - B_h^* g_h^0}_{=0} \tag{3.14}$$

Hence,

$$\left(I_h + (\theta - \frac{1}{4})\Delta t^2 B_h^* B_h\right)\delta^2 u_2^{n+\frac{1}{2}} + B_h^* B_h\mu^2 u_2^{n+\frac{1}{2}} = f_h^{n+\frac{1}{2}} - B_h^*\mu g_h^{n+\frac{1}{2}} \tag{3.15}$$

Which also can be written as the usual θ-scheme on u_2

$$\begin{cases} u_2^{-\frac{1}{2}} = u_2^{\frac{1}{2}} - \Delta t q_h^0 & (3.16a)\\ u_2^{\frac{1}{2}} = \left(B_h^* B_h\right)^{-1}\left[B_h^* w_h^0 - B_h^* g_h^0\right] + \dfrac{\Delta t}{2} q_h^0 & (3.16b)\\ \delta^2 u_2^{n+\frac{1}{2}} + B_h^* B_h\{u_2\}_\theta^{n+\frac{1}{2}} = f_h^{n+\frac{1}{2}} - B_h^*\mu g_h^{n+\frac{1}{2}}, \quad 0 \le n \le N & (3.16c) \end{cases}$$

Moreover,

$$w_h^n = B_h \mu u_2^n + g_h^n + e_w^0, \quad \text{where } e_w^0 = w_h^0 - B_h \mu u_2^0 - g_h^0 \tag{3.17}$$

But

$$B_h^* w_h^0 - B_h^* B_h \mu u_2^0 - B_h^* g_h^0 = 0 \Rightarrow B_h \left[B_h^* w_h^0 - B_h^* B_h \mu u_2^0 - B_h^* g_h^0 \right] = 0 \tag{3.18}$$

$$\Rightarrow w_h^0 - B_h \mu u_2^0 - g_h^0 = 0 = e_w^0 \quad \text{by inverting } B_h B_h^* \tag{3.19}$$

which proves the relation between u_2 and (q_h^n, w_h^n). □

3.3 Equivalence of the fields

The similarity between (3.2) and (3.11) is striking. Only the initial conditions seem to differ, but the following proposition shows that they are in fact the same.
PROPOSITION 3.3. *The fields u_1 defined by* (3.2) *and u_2 defined by* (3.11) *are equal.*

Proof. The difference $\Delta_u = u_1 - u_2$ is solution to the following system of equations:

$$\begin{cases} \Delta_u^{-\frac{1}{2}} = u_1^{-\frac{1}{2}} - u_2^{-\frac{1}{2}} & \text{(3.20a)} \\ \Delta_u^{\frac{1}{2}} = u_1^{\frac{1}{2}} - u_2^{\frac{1}{2}} & \text{(3.20b)} \\ \delta^2 \Delta_u^{n+\frac{1}{2}} + B_h^* B_h \{\Delta_u\}_\theta^{n+\frac{1}{2}} = 0, \quad 0 \le n \le N & \text{(3.20c)} \end{cases}$$

where

$$u_1^{\frac{1}{2}} = (B_h^* B_h)^{-1} B_h^* \left[v_h^{\frac{1}{2}} - \mu g_h^{\frac{1}{2}} \right], \quad u_1^{-\frac{1}{2}} = u_1^{\frac{1}{2}} - \Delta t p_h^0 \tag{3.21}$$

and

$$u_2^{\frac{1}{2}} = (B_h^* B_h)^{-1} B_h^* \left[w_h^0 - g_h^0 \right] + \frac{\Delta t}{2} q_h^0, \quad u_2^{-\frac{1}{2}} = u_2^{\frac{1}{2}} - \Delta t q_h^0 \tag{3.22}$$

From discrete energy considerations [Chabassier and Imperiale, 2017], and given that the source term is null in (3.20c), we get that

$$\|\Delta_u^{n+\frac{1}{2}}\| \le \|\Delta_u^{-\frac{1}{2}}\| + t^{n+\frac{1}{2}} \sqrt{2\mathcal{E}_\Delta^0} \tag{3.23}$$

where

$$\mathcal{E}_\Delta^0 = \frac{1}{2} \left\| \delta \Delta_u^0 \right\|^2 + \frac{1}{2} \left\| \mu \Delta_u^0 \right\|_{\widetilde{I}_{h,\theta}}^2, \quad \text{with } \widetilde{I}_{h,\theta} = I_h + \Delta t^2 (\theta - \frac{1}{4}) B_h^* B_h \tag{3.24}$$

Since $p_h^0 \equiv q_h^0$ (see eqs. (2.1b) and (2.5b)), we get that

$$\Delta_u^{\frac{1}{2}} = u_1^{\frac{1}{2}} - u_2^{\frac{1}{2}} = u_1^{-\frac{1}{2}} - u_2^{-\frac{1}{2}} = \Delta_u^{-\frac{1}{2}} \tag{3.25}$$

and

$$\Delta_u^{\frac{1}{2}} = (B_h^* B_h)^{-1} B_h^* \left[v_h^{\frac{1}{2}} - \mu g_h^{\frac{1}{2}} - w_h^0 + g_h^0 \right] - \frac{\Delta t}{2} q_h^0 \tag{3.26}$$

$$= (B_h^* B_h)^{-1} B_h^* \left[v_h(0) + \frac{\Delta t}{2} \delta g_h^{\frac{1}{2}} + \frac{\Delta t}{2} B_h p_h(0) - \mu g_h^{\frac{1}{2}} - v_h(0) + g_h^0 - \frac{\Delta t}{2} B_h p_h(0) \right] \tag{3.27}$$

$$= (B_h^* B_h)^{-1} B_h^* \left[\frac{g_h^1 - g_h^0}{2} - \frac{g_h^1 + g_h^0}{2} + g_h^0 \right] \tag{3.28}$$

$$= 0 \tag{3.29}$$

Hence, for all $n \ge 0$,

$$\|\Delta_u^{n-\frac{1}{2}}\| \le 0 \Rightarrow \Delta_u^{n-\frac{1}{2}} \equiv 0 \tag{3.30}$$

□

From this proposition, we can now introduce a unique field $u \equiv u_1 \equiv u_2$. This field allows to derive a reconstruction of the fields of the θ-CN scheme (2.5) from the fields obtained by the θ-SV scheme (2.1). This prompts a second proof to Prop. 2.6:

Proof 2 of Prop 2.6, using the fields. The fields p_h, q_h, v_h, w_h can be reconstructed from u as:

$$p_h^n = \delta u^n, \quad q_h^n = \delta u^n, \quad \forall n \geq 0 \tag{3.31}$$

$$v_h^{n+\frac{1}{2}} = B_h u^{n+\frac{1}{2}} + \mu g_h^{n+\frac{1}{2}}, \quad w_h^n = B_h \mu u^n + g_h^n, \quad \forall n \geq 0 \tag{3.32}$$

Hence, $p_h^n = q_h^n$ and

$$\mu v_h^n = B_h \mu u^n + \mu^2 g_h^n = w_h^n - g_h^n + \mu^2 g_h^n = w_h^n + \frac{\Delta t^2}{4}\delta^2 g_h^n \tag{3.33}$$

□

4 Numerical results

4.1 Test case

This section presents very simple numerical results that illustrate the reconstruction formulas given in this work. An impulse source is put at the entry of a 1D pipe in which waves propagate. More precisely, the pressure $p \equiv p(x,t)$ and the velocity $v = v(x,t)$ are solution to

$$\begin{cases} p(0) = 0 & \text{(4.1a)} \\ v(0) = 0 & \text{(4.1b)} \\ m_p(x)\dot{p} + \partial_x v = 0 & \text{(4.1c)} \\ m_v(x)\dot{v} + \partial_x p = 0 & \text{(4.1d)} \\ v(0) = \lambda(t) & \text{(4.1e)} \\ v(L) = 0 & \text{(4.1f)} \end{cases}$$

Following the usual mixed finite elements theory [Brezzi and Fortin, 1991], the variables are sought to in variational spaces of unequal regularity.

$$\mathcal{H} = \{(p,v) \in H^1([0,L]) \times L^2([0,L])\} \tag{4.2}$$

The considered variational formulation is to seek $(p,v) \in \mathcal{H}$ such that for all $(\tilde{p},\tilde{v}) \in \mathcal{H}$,

$$\begin{cases} \displaystyle\int_0^L m_p(x)\dot{p}\tilde{p} - \int_0^L v\partial_x\tilde{p} - \lambda(t)\tilde{p}(0) = 0 & \text{(4.3a)} \\ \displaystyle\int_0^L m_v(x)\dot{v}\tilde{v} + \int_0^L \partial_x p\,\tilde{v} = 0 & \text{(4.3b)} \end{cases}$$

and the spatial discretization is done as described in [Tournemenne and Chabassier, 2019] by choosing conformal spectral finite elements of the same order on each element of the mesh, allowing jumps between elements for the variable v. This specific choice is not classical but is shown to converge in [Thibault, 2023, Chap 2]. This prompts the matricial formulation

$$\begin{cases} p_h(0) = 0 & \text{(4.4a)} \\ v_h(0) = 0 & \text{(4.4b)} \\ \dot{p}_h + B_h^* v_h = -\lambda_0 E_{h,0} & \text{(4.4c)} \\ \dot{v}_h - B_h p_h = 0 & \text{(4.4d)} \end{cases}$$

This is a special case of (1.1) with a source term which has a contribution from the edge. In the following numerical example, we take $L = 1$, we use a $\mathcal{C}^\infty$ source

$$\lambda(t) = \exp\left(-\frac{1}{1 - \left|\dfrac{2t - t_0}{t_0}\right|^2}\right), \quad 0 < t < t_0, \quad t_0 = 30 \tag{4.5}$$

and the parameters are chosen as

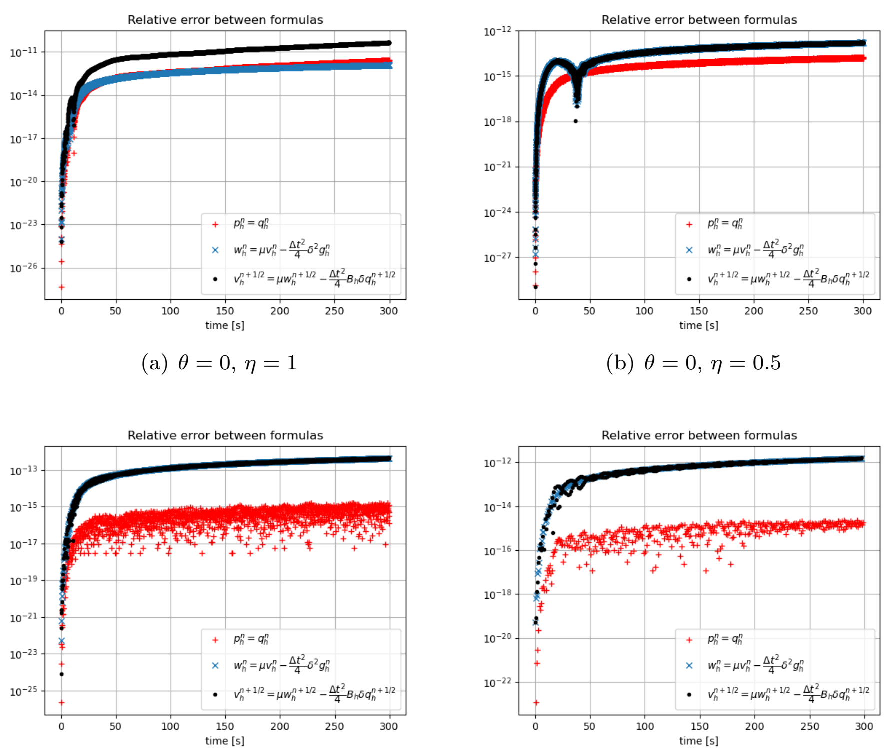
$$m_p(x) = \pi R(x)^2, \quad m_v(x) = \frac{1}{\pi R(x)^2}, \quad R(x) = R_0 + x\frac{R_1 - R_0}{L}, \quad R_0 = 3 \times 10^{-3}, \quad R_1 = 8 \times 10^{-3} \tag{4.6}$$

We use 4th order finite elements of length 0.5 so that 2 elements compose the mesh. We set a time step Δt chosen according to a chosen value of $\eta \leq 1$ defined in (2.8) and (2.4) (which are the same definition) when $\theta < \frac{1}{4}$. When $\theta \geq \frac{1}{4}$, Δt is chosen as 10 times the maximal allowed value of the explicit case. In other words,

$$\begin{cases} \Delta t = \dfrac{\eta}{\sqrt{1-4\theta}} \dfrac{2}{\sqrt{\rho(B_h^* B_h)}} & \text{if } \theta < \dfrac{1}{4} \quad (4.7a) \\ \Delta t = 10 \dfrac{2}{\sqrt{\rho(B_h^* B_h)}} & \text{if } \theta \geq \dfrac{1}{4} \quad (4.7b) \end{cases}$$

4.2 Verification of the reconstruction formulas

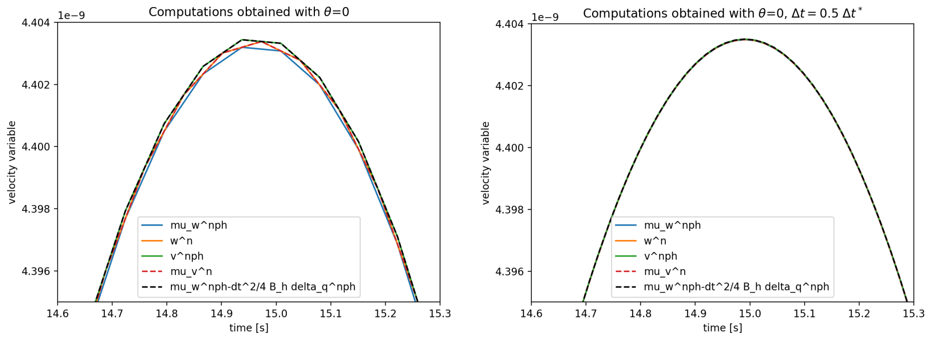
We first run the θ-SV scheme. Then, as a post processing step, we compute the quantities defined in (2.18) and compare them numerically to the ones obtained by running directly θ-CN. The same process is done by running first the θ-CN scheme, then computing the quantities defined in (2.9) and comparing them numerically to the ones obtained by running directly θ-SV.
The relative error between both sides of equations (2.18) and (2.9) are computed and displayed in Fig. 1 for $\theta \in \{0, 0.25\}$, $\eta \in \{0.5, 1\}$. We observe that the relations are indeed respected up to machine precision.



(a) $\theta = 0$, $\eta = 1$ (b) $\theta = 0$, $\eta = 0.5$

(c) $\theta = 0.25$, $\eta = 0.5$ (d) $\theta = 0.25$, $\eta = 1$

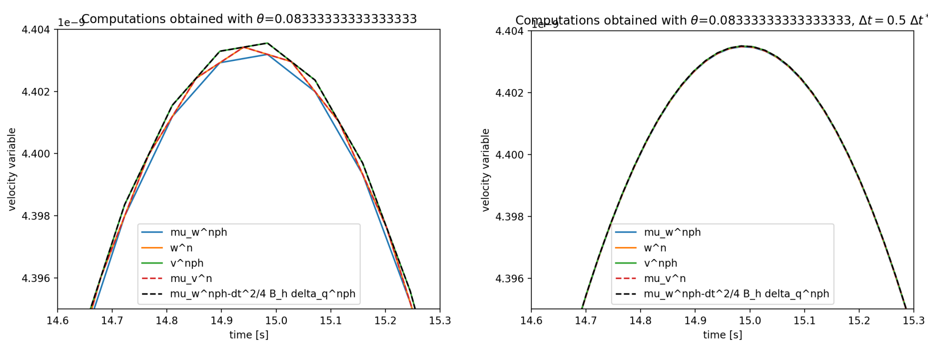
Figure 1: Evolution of different ways of computing the velocity along time, $\theta \in \{0, 0.25\}$, $\eta \in \{0.5, 1\}$.



(a) $\eta = 1$ (b) $\eta = 0.5$

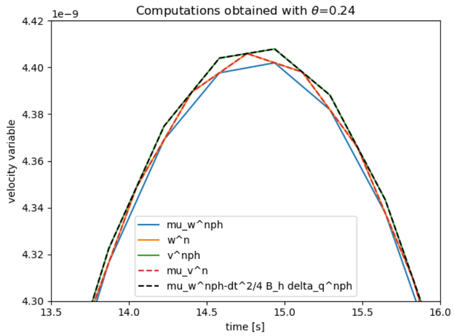
Figure 2: Evolution of different ways of computing the velocity along time, $\theta = 0$, $\eta \in \{0.5, 1\}$.

The results obtained for $\eta \in \{0.5, 1\}$ and $\theta = 0$, $1/12$, 0.24, 0.25 and 0.5 are shown respectively in Fig. 2, 3, 4, 5 and 6. In those figures, the values are zoomed in for better visibility. We see that both relations (2.18) and (2.9) are respected.

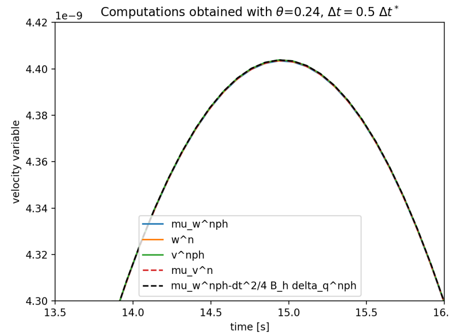


(a) $\eta = 1$ (b) $\eta = 0.5$

Figure 3: Evolution of different ways of computing the velocity along time, $\theta = 1/12$, $\eta \in \{0.5, 1\}$.

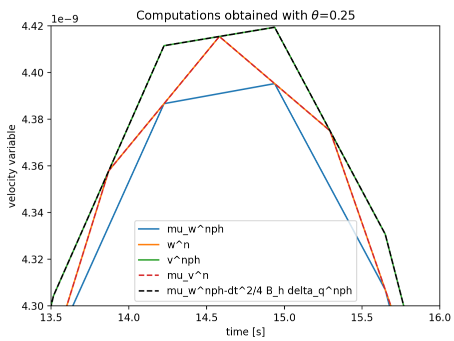


(a) $\eta = 1$

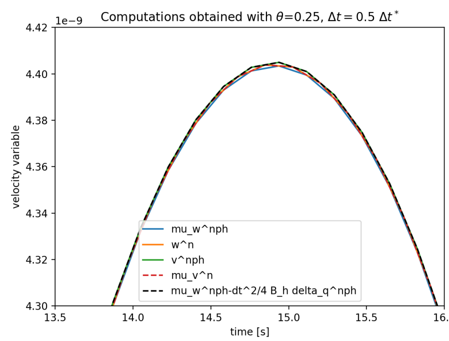


(b) $\eta = 0.5$

Figure 4: Evolution of different ways of computing the velocity along time, $\theta = 0.24$, $\eta \in \{0.5, 1\}$.

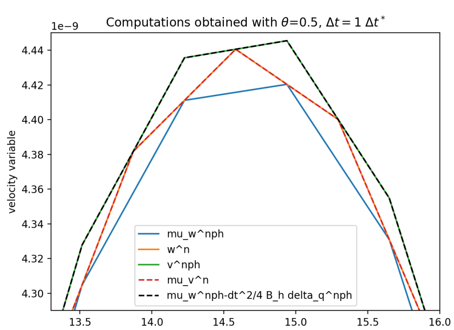


(a) $\eta = 1$

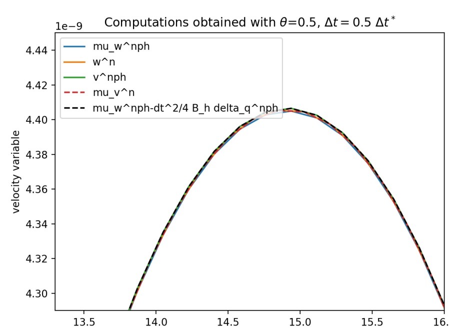


(b) $\eta = 0.5$

Figure 5: Evolution of different ways of computing the velocity along time, $\theta = 0.25$, $\eta \in \{0.5, 1\}$.



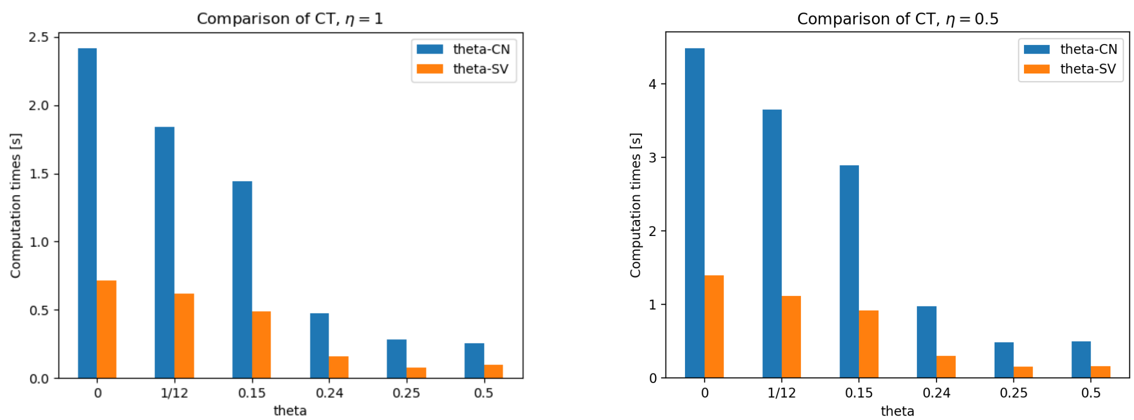
(a) $\eta = 1$



(b) $\eta = 0.5$

Figure 6: Evolution of different ways of computing the velocity along time, $\theta = 0.5$, $\eta \in \{0.5, 1\}$.

4.3 Computational efficiency



(a) $\eta = 1$ (b) $\eta = 0.5$

Figure 7: Comparison of computation times

Finally, we compare the computation times of θ-SV and θ-CN for the same value of θ. Since we have seen that both schemes are in fact equivalent (up to a post-processing step), the less computationally intensive should be preferred.

Solving (2.1) requires at each time step one matrix-vector product with the inverse of the matrix

$$I_h + \theta \Delta t^2 B_h^* B_h \tag{4.8}$$

which is a sparse matrix of size $n_p \times n_p$. Moreover, at each time step, the scheme also requires one matrix-vector product by B_h^* and one matrix-vector product by B_h.

Solving (2.5) requires at each time step the inversion of the matrix

$$\begin{pmatrix} I_h + \left(\theta - \frac{1}{4}\right) \Delta t^2 B_h^* B_h & \frac{\Delta t}{2} B_h^* \\ -\frac{\Delta t}{2} B_h & I_h^w \end{pmatrix} \tag{4.9}$$

which is a sparse matrix of size $(n_p + n_w) \times (n_p + n_w)$. Moreover, at each time step, the scheme also requires one matrix-vector product by B_h^* and one matrix-vector product by B_h.
In both cases, the matrix to invert can be factorized once before the time iterations. The θ-SV scheme should always require less computation.
In Fig. 7, the computation time for the two strategies are compared for the same choice of θ and η. As θ increases, the time step of the computation increases, except for $\theta = \frac{1}{4}$ and $\theta = \frac{1}{2}$ where Δt is the same, as described in (4.7). Every simulation is run 5 times in order to average the uncertainties. We observe that the computation time of the θ-SV is always smaller (about 1/3 in this specific 1D case) than the one of θ-CN, which is coherent with the complexity description.

Remark 4.1. In some sense, the θ-SV scheme can be seen as an efficient algorithm to solve the θ-CN scheme. In particular, the $\frac{1}{4}$-SV scheme is a cheap way of solving the CN scheme.

5 Conclusion and prospects

The two parametrized implicit discretization strategies presented in this work have been shown to be equivalent, if the initial conditions are well chosen. This equivalence has been illustrated by numerical results, as well as the induced computational cost. A a post processing step allows to reconstruct the same field from both schemes, that is also solution to a classical θ-scheme. This work focused on numerical and computational aspects of these schemes, but this equivalence prompts the question of their analysis of stability and convergence. Indeed, one might now expect similar results as those obtained for the θ-scheme applied to the second order equation (see [Chabassier and Imperiale, 2021]). However, the derivation of optimal proofs of stability and space/time convergence for the Störmer-Verlet scheme applied to the mixed formulation is not a straightforward question. Existing stability proofs for the usual Störmer-Verlet scheme (1.3), which are reviewed in [Chabassier, 2023], do not apply when the time step reaches its greatest admissible value for stability. In [Chabassier, 2023] the author proposed alternative proofs of stability and convergence that also apply to this case and that use regularity assumptions similar to those of [Chabassier and Imperiale, 2021]. It is an open question to generalize those results to the implicit parametrized schemes proposed here. The equivalence of the two formulations shown in this work could help harmonize the required hypotheses.

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