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Abstract. We formulate a security game in a context of mixed armament acquisition, involving a finite set of Nations in strategic relationship, with utility functions which are nonlinear and non-differentiable on the boundary of their sets of definition. Since we want to study the long-term effect of the Nations' investment in nuclear weapons, we focus on the steady-state analysis of the game. This requires us to extend the classical results from Rosen on compact-convex games, to unbounded convex games, relying on the coercivity property of the Nations' utility functions. In addition, we prove the existence and uniqueness, under mild assumptions, of an interior point Nash Equilibrium solution of the game. Simulations are performed in case of a duopoly, highlighting the efficiency loss reduction and stabilizing effect of nuclear armaments by comparison with the conventional-only setting.

Keywords: Security Game · Nash Equilibrium · Uniform Coercivity · Strategic Stability

1 Introduction

The primary goal of Nations is to maximize their perceived security, which requires each Nation to develop an effective deterrence strategy [15]. In classical game theoretic models, deterrence as a military strategy is based upon a standard, iterated two-player prisoner's dilemma model by which cooperation can be assured if a rational player understands the punishment or cost of defection [14, 3, 26]. Models in this area are often inspired by the USA and USSR-allied, or at least Western vs Eastern blocs, nuclear arms race, and formulate the security dilemma in arms procurement as a two-player noncooperative game in normal form [29, 23]. However, many of the actions taken by Nations in pursuit of the maximization of their perceived security – such as arms procurement and the development of new military technologies – might, in turn, lead to a decrease rather than an increase in their perceived security. This dilemma is explained by the fact that the increase in military capability of one Nation might be interpreted as a risk that the arming Nation will use it to perform an attack in the future. In this context, in order to re-establish the balance of power, the other Nations could either increase their own military capabilities or initiate a preemptive strike. Choosing the first option may result in a security spiral, in which several Nations are tied in an arms race, responding to increases in arms procurement and defense expenditure by others by arming themselves more and more heavily, and may lead to war in the long run. Such a situation is known as the security dilemma [11, 12], and is one possible outcome of arms race models.

Standard literature dealing with arms race models [30, 24] focuses on the dynamic process of the arms race over the time or on fitting real data collected over the years using this model. In this setting, game theory is a useful tool to model interactions between Nations with conflicting interests. The classical literature dealing with nuclear strategy using game theory focuses mainly on discrete decision space repeated games [22] and differential games. Within repeated games, iterated prisoners' dilemmas between two Nations [18, 17] have been extensively studied. At each iteration, each Nation has a choice between a high or low level of arms. In the static version, each Nation's dominant strategy is to choose a high level of arms. The Nash Equilibrium outcome of the static prisoner dilemma model for arms race is therefore that both Nations choose high. As a result, the static game outcome is worse for both Nations than if both had chosen a low level of arms. However, in reality, the game is not played once and for all, but is an ongoing series of decisions, i.e., the prisoner dilemma is played repeatedly by the two Nations. This opens the possibility for cooperation to emerge through rewards and punishment strategies, e.g., tit-for-tat [10].

A somewhat equivalent framework stems from Richardson-type models of arms race [16, 27], which provide parametrized systems of differential equations to represent the evolution of the weapons stockpile. These equations can be derived as closed-loop Nash Equilibria of linear-quadratic differential games that implement an interpretable reasoning of players [27, 19, 20, 28]. However, military strategies that hinge on nuclear weapons imply that Nations' evaluation of the effect of each other's actions should be nonlinear, and Richardson-type's models fail to represent accurately the specific nature of nuclear armaments [6, 8]. Indeed, contrary to conventional war, nuclear warfare allows for significant preemptive and retaliatory strikes whose magnitude depend nonlinearly on the initial stockpiles. A two-stage game model may show that a second-strike capability decreases as the exponential of the ratio of the initial stockpiles [5].

If the goal of game-theoretic models is to quantify the outcomes of situations of conflict between armed Nations, non-proliferation policies and containment treaties are often designed with the goal to stabilize the rivalry between Nations and reduce strategic risks. For example, during the Cold War, several agreements between the USA and the USSR, such as the intermediate-range nuclear forces (INF) treaty, helped balance nuclear-able weapons so that each side could survive a preemptive nuclear attack with a sufficiently large stockpile of ballistic missiles to launch a retaliatory strike, in the context of nuclear strategies heavily influenced by the concept of MAD (mutually assured destruction). Nowadays, 9 countries are thought to possess nuclear weapons. Within them, only 5 – the

USA, Russia, China, France and the United Kingdom – have signed the treaty on the non-proliferation of nuclear weapons (NPT), the second most widely ratified treaty after the United Nations charter. Under the NPT, each of the 191 parties 'undertakes to pursue negotiations in good faith on effective measures relating to the cessation of the nuclear arms race at an early date and to nuclear disarmament, and on a treaty on general and complete disarmament under strict and effective international control' [1]. The security architecture is completed by the comprehensive nuclear-test-ban treaty (CTBT) with 177 parties, which prohibits nuclear tests and as such limits the possibility to develop nuclear weapons[1].

1.1 Problem Statement

We model the interactions among Nations possessing both conventional and nuclear weapons, i.e., mixed armament, using noncooperative game theory. Contrary to most papers which focus on the dynamics of arms procurement, we focus on the steady-state analysis of the game, which captures the long-term solution of the dynamical model, and is formulated as a one-shot (static) game. On the quantitative side, a first difficulty appears in modeling the Nations' utility functions such that they reflect the security perceived at a Nation-wide level – thus, depending on the Nation's armament strategy and on that of its rivals. This leads us to define utility functions which are nonlinear and non-differentiable on the boundary of their sets of definition. A second difficulty arises from this modeling choice and requires that we extend classical results from noncooperative game theory, to characterize the game equilibria. On the qualitative side, we aim to assess the effects of long-term investments in nuclear weapons on the efficiency and strategic stability of the Nations' international system.

1.2 Main Contributions

We propose a multipolar security model which addresses the specific logic of nuclear strategies in a context involving mixed armament, i.e., both conventional and nuclear weapons. The model is formulated in a dynamic setting, but since we want to study the stabilizing and security dilemma reduction potential of mixed armament, we focus on the one-shot game steady-state formulation, which captures the long-term effects of the competition among Nations. This paper provides three important contributions to the state of the art. First, considering a new way to model the security perceived at the Nation-wide level, that takes inspiration from [5], we extend the classical results from Rosen [25] on compact-convex games to unbounded convex games relying on the coercivity property of the Nations' utility functions. This allows us, in a second step, to prove the existence of an interior point Nash Equilibrium solution of the one-shot game. The proof of uniqueness of the interior point Nash Equilibrium follows, under mild assumption on the game parameters. In a last step, our model is used to

[1] Currently, while it is already enforced by parties, the CTBT has not yet entered into force because 8 out of 44 Annex 2 Nations have not ratified it.

determine numerically that nuclear weapons have a stabilizing effect and can potentially reduce the efficiency loss by comparison with a conventional-only setting, therefore providing possible guidelines for current treaties refinements.

Paper Organisation

This paper is organized as follows. In Section 1, we provide an overview of the game theoretic approaches for arms procurement competition modeling and how their outcomes can be used to design containment treaties with stabilizing effects. We also introduce the problem statement, and main contributions. In Section 2, the arms building model is formulated as a one-shot noncooperative game. Conditions for the existence and uniqueness of an interior point Nash Equilibrium solution of this game are studied in Section 3. Analytical results and numerical illustrations in case of a duopoly are discussed in Section 4. We conclude in Section 5.

2 The Mixed Armament Competition Model

The international system is first and foremost made up of Nations, that we will indifferently call players. In international relations [2], an organization is said to be unitary if and only if its decisions contain all the information useful for its interaction with the international system. Moreover, an organization is said to be rational if its decision-making process consists in maximizing its (well-ordered) preferences through its actions, given the information available to it and its anticipation of what other organizations might do. Hence, in our work, Nations can be represented as rational unitary players. Note that no hypothesis is made a priori on the order of the preferences.

We aim to study the strategies of Nations to invest in nuclear and conventional weapons in order to maximize their security while minimizing their storage and R&D costs. We start by placing ourselves in a framework of discrete time evolution. Nations seek to ensure their own security within a certain time horizon (finite or infinite), and in anticipation of a crisis. They can acquire conventional and nuclear weapons to maximize their security, while bearing storage and R&D costs. We choose a realistic framework [31], and assume that Nations are primarily defensive and secondarily aggressive. This means that they seek through conventional and nuclear armaments to ensure their own security, and that their marginal gain from arming themselves increases with the levels of adversary armaments.

2.1 Dynamics of Arms Building

Let $\mathcal{N}$ be a finite set of N Nations, and T be the horizon of the game (possibly infinite). We propose a model with continuous decision spaces, i.e., the Nations (as players) do not decide whether to attack or not, but they decide on the power drawn from their investment in nuclear and conventional weapons. Models with continuous decision spaces prove themselves more realistic [4, 15, 27, 28]. We might assume that the power drawn for each category of armament depends linearly on the Nation's arms production and that the coefficients relating the arms production to the power drawn out of it differ between the categories of weapons. In absence of numerical values for these coefficients, we will not consider this level of details in the present work. In what follows, $\tau = \{\text{nuc}\}$ or $\{\text{conv}\}$ and denotes nuclear or conventional.

Let $u_n^\tau(t)$ be the power drawn from the quantity of weapons of class τ produced by Nation $n \in \mathcal{N}$ and $x_n^\tau(t)$ be the power drawn from its stockpile of weapons of class τ at time period t. We define $u_n(t) \stackrel{\text{def}}{=} \big(u_n^\tau(t)\big)_\tau$, the column vector which contains the power drawn from the production of Nation n for each class of weapons, $u(t) \stackrel{\text{def}}{=} \text{col}\big((u_n(t))_n\big)$ be the stack of the power drawn from the weapons production by the N Nations. Similarly, $x_n(t) \stackrel{\text{def}}{=} \big(x_n^\tau(t)\big)_\tau$ is the column vector which contains the power drawn from stockpile of Nation n for each class of weapons, and $x(t) \stackrel{\text{def}}{=} \text{col}\big((x_n(t))_n\big)$ be the stack of the power drawn from the stockpiles of the N Nations. Furthermore, we define the following sequences: $x \stackrel{\text{def}}{=} (x(t))_t$ and $u_n \stackrel{\text{def}}{=} (u_n(t))_t$. The dynamics of Nation n's power drawn from stockpile of class τ weapons is defined as follows:

$$\forall n \in \mathcal{N}, \quad x_n^\tau(t+1) = x_n^\tau(t) + u_n^\tau(t) - \nu x_n^\tau(t), \tag{1}$$

with an obsolescence rate $\nu \geq 0$. The players' utility at time period t writes:

$$J_n\big(u_n(t), x(t)\big) = f_n\big(x(t)\big) - c_n^S\big(x_n(t)\big) - c_n^D\big(u_n(t)\big), \tag{2}$$

where $f_n(\cdot)$ is the security function of player n, $c_n^S(\cdot)$ its storage (operational) cost and $c_n^D(\cdot)$ its development (R&D) cost. The players may have local constraints, e.g., finite budget, maximum capacity of production, etc., which may evolve dynamically. We denote $\mathcal{U}_n(t) \subset \mathbb{R}_+^2$ the set of $u_n(t)$ that preserve these constraints, i.e., the feasibility set of player n. Each player $n \in \mathcal{N}$ maximizes the sum $J_n(\cdot)$ over time of its future utility, with discounting rate $\rho_n \in]0; 1[$, $n \in \mathcal{N}$:

$$\begin{aligned} V_n(x) &= \max_{u_n(t) \in \mathcal{U}_n(t), \forall t} \bar{J}_n\big(u_n, x\big), \\ \text{where} \quad \bar{J}_n(u_n, x) &\stackrel{\text{def}}{=} \sum_{t=0}^{T} \rho_n^t J_n\big(u_n(t), x(t)\big). \end{aligned} \tag{3}$$

We aim to study this model in the long run, with $\nu = 0$. In what follows, we denote $x_n \stackrel{\text{def}}{=} \lim_{t \to +\infty} x_n(t)$ player n's steady state, with $\lim_{t \to +\infty} u(t) = 0$. Using a slight abuse of notation, the static model is given by:

$$V_n(x) = \max_{x_n \in \mathbb{R}_+^2} \bar{J}_n(x) \quad \text{where} \qquad \bar{J}_n(x) \stackrel{\text{def}}{=} J_n\big(0, x\big). \tag{4}$$

with the decision variable now being Nation n's power drawn from its stockpiles, $x_n \in \mathcal{U}_n$. Let $x_{-n} \stackrel{\text{def}}{=} \text{col}\big((x_m)_{m \neq n}\big), \forall n$ be the vector that contains the stack of the power drawn from the stockpiles of weapons of all Nations in $\mathcal{N}$ except n.

2.2 Strategic Relationship

For each Nation, the international system is divided into opponents, neutrals and allies (commercial or military). A commercial alliance is a binding agreement allowing transfer of military capabilities between parties. A military alliance is a binding agreement establishing a coalition of military interests and capabilities between parties. To simplify the setting, we will not consider alliances in the following. In what follows, we make the assumption that the decisions of Nations are not affected by those of neutrals.

We now want to structure the set of Nations, by taking into account how their decisions impact those of the others. For each Nation $n \in \mathcal{N}$, the set of opponents $\sigma(n) \subset \mathcal{N}$ is defined by a binary relation of hostility, represented by the hostility function $\sigma(\cdot)$. It is reasonable to assume that a Nation is never its own opponent and that any opponent of a Nation must treat the considered Nation as an opponent in return. Therefore, we assume that the relation of hostility is irreflexive and symmetric, i.e.:

$$\begin{aligned} &\forall n \in \mathcal{N}, && n \notin \sigma(n), \\ &\forall n, m \in \mathcal{N}, && m \in \sigma(n) \Longleftrightarrow n \in \sigma(m). \end{aligned}$$

We introduce the strategic relation as the reflexive-transitive closure of the hostility relation:

Definition 1 (Strategic relation). *We say that two Nations n and m are in strategic relationship and write $n \sim m$ if a (possibly degenerate) sequence of hostility relations connects n to m, i.e., if there exists $k \in \mathbb{N}$ such that $m \in \sigma^k(n)$.*

Lemma 1. *The strategic relation is an equivalence relation.*

Proof. By definition, the strategic relation is symmetric. It is transitive because $\forall n, m, l \in \mathcal{N}$ such that $n \neq m, m \neq l, n \neq l$, $n \sim m$ and $m \sim l$ mean that there exist $k, k' \in \mathbb{N}$ such that $m = \sigma^k(n)$ and $l = \sigma^{k'}(m)$. Then, by composition of the strategic relation, we get that $l = \sigma^{k+k'}(n)$. Therefore, the strategic relation is transitive. Now, $\forall n \in \mathcal{N}$, $\sigma^0(n) = \{n\}$, thus the strategic relation is an equivalence relation. □

Lemma 1 implies that the decisions of Nations belonging to the same equivalence class are independent of the decisions of the Nations out of it. Hence, we can split the set of players into a finite number of equivalence classes. This allows to decompose the main noncooperative game into the same number of noncooperative games, that can be solved independently. In what follows, we assume this procedure has already been done, and focus on a game in which the N Nations share strategic relations:

Assumption 1 *All Nations in $\mathcal{N}$ are in strategic relationship.*

In particular, if the game is nontrivial and consists of at least two players, then every Nation has an opponent.

2.3 Target Value

For Nation $n \in \mathcal{N}$, the target value of adverse armament of class τ is denoted by $x^{\tau}_{\sigma(n)}$, which is a function of x^{τ}_{-n}. Later, we will call it Nation n target value. This value represents the level of threat caused by the set of opponents. Several modeling choices are possible. For instance, Nations preparing for a one-off crisis will evaluate the target value as the maximum level of armaments of class τ of the set of opponents. In any case, it must satisfy the following assumption:

Assumption 2 *There exist κ^-, $\kappa^+ > 0$ input coefficients that do not depend on the power drawn from the weapon stockpiles, such that:*

$$\kappa^- \max_{m \in \sigma(n)} x^{\tau}_m \leq x^{\tau}_{\sigma(n)} \leq \kappa^+ \max_{m \neq n} x^{\tau}_m.$$

Remark 1. Target values defined as the maximum, the sum or the mean over $\sigma(n)$ of the power drawn from the weapon stockpiles satisfy Assumption 2:

$$\max_{m \in \sigma(n)} x^{\tau}_m \leq \max_{m \in \sigma(n)} x^{\tau}_m \leq \max_{m \neq n} x^{\tau}_m,$$

$$\max_{m \in \sigma(n)} x^{\tau}_m \leq \sum_{m \in \sigma(n)} x^{\tau}_m \leq N \max_{m \neq n} x^{\tau}_m,$$

$$\frac{1}{N} \max_{m \in \sigma(n)} x^{\tau}_m \leq \frac{\sum_{m \in \sigma(n)} x^{\tau}_m}{|\sigma(n)|} \leq \max_{m \neq n} x^{\tau}_m.$$

Let us discuss briefly the interpretation of Assumption 2. The right hand side of the inequality ensures that Nations use armaments levels of others to compute the level of threat they face, while the left hand side ensures that Nations do not ignore the greatest opponent in the computation of the threat they face.

2.4 Security Functions

The perceived security of a Nation $n \in \mathcal{N}$ is a function of the power drawn from its own weapon stockpile x_n and of its target value $x_{\sigma(n)}$, where $x_{\sigma(n)}$ is the column vector which contains the target value of Nation n in each weapon class. As Nations seek to maximize their perceived security, it increases with their effective weapon stockpile and decreases with the opponents' effective weapon stockpiles. Using deterrence terminology, the effective stockpile value can be defined as the maximum available quantity of armaments, a Nation is both credible, and capable to use in a potential conflict. We shall assume that Nations have a credible threat in using the military capabilities we consider. However, the effectiveness of the weapon stockpiles may depend on the quality and type of armaments as well as on the military doctrine of each Nation. We distinguish between two classes of armaments: nuclear (strategic) weapons allowing for preemptive strikes, and conventional (tactical) weapons which can be used on the battlefield. Therefore, Nation n's security function can be decomposed as follows:

$$f_n(x_n, x_{-n}) = \sum_{\tau} \alpha^{\tau} f^{\tau}_n(x_n, x_{-n}), \quad \text{where } \sum_{\tau} \alpha^{\tau} = 1,\ \alpha^{\tau} \geq 0,\ \forall \tau. \qquad (5)$$

Extrinsic and intrinsic perceived security. For each class τ of armaments, we decompose f_n^τ into two parts: the extrinsic part, which represents the loss of perceived security due to the environment, and the intrinsic part, which represents the gain of perceived security due to the possession of armaments of class τ. The extrinsic perceived security of Nation n brought by class τ armaments coincides with its opponents' effective stockpile of class τ. Now, as Nations are primarily defensive and secondarily aggressive, the intrinsic perceived security of Nation n brought by class τ armaments is modeled as a linear function of Nation n's effective stockpile value of armaments of class τ. Its slope is a concave function $\phi(\cdot)$ of the weapons they are supposed to deter. Nuclear strategies are generally dual-purpose, contrary to conventional strategies. Thus, the slope is $\phi\left(\frac{x^{\text{conv}}_{\sigma(n)}+x^{\text{nuc}}_{\sigma(n)}}{\xi_n}\right)$ for nuclear, and $\phi\left(\frac{x^{\text{conv}}_{\sigma(n)}}{\xi_n}\right)$ for conventional, where $\xi_n > 0$ is a threat threshold which captures Nation n's peacefulness. To go further we need to evaluate the effective stockpile value for nuclear and conventional armaments.

Effective stockpiles. Consider first conventional weapons. It is reasonable to assume that Nations are able to use at some point in a potential conflict all of the conventional weapons they possess. Therefore, Nation n's power drawn from effective stockpile of conventional weapons is equal to the power drawn from its actual stockpile of conventional weapons x_n^{conv}.

Now, consider nuclear weapons. Depending on multiple factors, Nations can be cautious and prepare for an adversarial first strike by seeking a second-strike capability, or they can hinge on their preemptive first-strike capability to deter their counterparts. Therefore, stochastic multi-stage games can be used to model the succession of strikes [5]. We use this setting to derive the closed form expression of the expected number of remaining weapons in case of a conflict between two armed Nations, after a single preemptive strike.

Following [5], we consider two Nations called n and $m \in \mathcal{N}, n \neq m$. Nation n holds $W_n \in \mathbb{N}$ weapons. It is attacking Nation m, which holds $W_m \in \mathbb{N}^*$ weapons. Nation n aims to destroy with a preemptive strike as much as possible of Nation m's weapons. Assume that Nation n's weapons have an accuracy $\lambda_n > 0$, i.e., each weapon has a probability $p_n = 1 - e^{-\lambda_n}$ of destroying its target. Assume that both weapons and targets are indistinguishable. Let W_m^r be the random variable that gives the number of remaining weapons for Nation m after a single preemptive strike of Nation n. Let $K \overset{\text{def}}{=} \left\lfloor \frac{W_n}{W_m} \right\rfloor$ and $w \overset{\text{def}}{=} W_n - KW_m$.

Proposition 1. *After an optimal single preemptive strike by Nation n, the expected number of remaining weapons for Nation m is:*

$$\begin{aligned}\mathbb{E}[W_m^r] &= (W_m - w)\exp(-K\lambda_n) + w\exp\left(-(K+1)\lambda_n\right),\\ &= W_m \exp\left(-\lambda_n \frac{W_n}{W_m}\right), \qquad \textit{if } K \in \mathbb{N}.\end{aligned}$$

Proof. Let $\mathcal{W}_n = [\![1, W_n]\!]$ and $\mathcal{W}_m = [\![1, W_m]\!]$. For all $k \in \mathcal{W}_m$, let $g(k) \in \mathcal{W}_n$ be the number of weapons assigned by Nation n to target k. We order the targets such that $g(\cdot)$ is decreasing. Notice that W_m^r is a function of $g(\cdot)$, but to simplify the notation we omit this dependence. Nation n has to solve the following optimization problem:

$$\min_{g(\cdot)} \ \mathbb{E}\left[W_m^r\right], \qquad \text{s.t.} \sum_{k\in\mathcal{W}_m} g(k) = W_n, \tag{6}$$

which consists in finding a minimizer of a function over a non-empty finite set, whence the minimizer exists. We argue that the (unique) minimizer is given by

$$g(k) = \begin{cases} K+1 & \text{if } 1 \leq k \leq W_n - KW_m, \\ K & \text{else.} \end{cases} \tag{7}$$

First, consider a strike on a single target $k \in \mathcal{W}_m$. Up to a reordering, let $(X_l)_{l\in[\![1,g(k)]\!]}$ be a collection of Bernoulli independent and identically distributed random variables with parameter p_n. $X_l = 1$ means that weapon l destroys its target k, with probability $p_n = 1 - e^{-\lambda_n}$. Let $Y_k = \sum_{l=1}^{g(k)} X_l$. Target k remains intact if and only if $Y_k = 0$, with probability $\mathbb{P}\left(Y_k = 0\right) = e^{-g(k)\lambda_n}$. Nation m has $W_m^r = \sum_{k\in\mathcal{W}_m} \mathbf{1}_{\{Y_k=0\}}$ remaining weapons. Therefore:

$$\mathbb{E}\left[W_m^r\right] = \textstyle\sum_{k\in\mathcal{W}_m} \mathbb{E}\left[\mathbf{1}_{\{Y_k=0\}}\right] = \sum_{k\in\mathcal{W}_m} \mathbb{P}\left(Y_k = 0\right) = \sum_{k\in\mathcal{W}_m} e^{-g(k)\lambda_n}.$$

Let $g(\cdot)$ be a minimizer of problem (6). Let us show that $g(0) \leq g(W_m)+1$ using on a proof by contradiction. Assume the converse: $g(0) \geq g(W_m) + 2$. As $g(\cdot)$ is decreasing, there exists l, l' such that $g(0) = g(l) > g(l+1) \geq g(l') = g(W_m)$ and l' is minimal. Define $g^\sharp(\cdot)$ by

$$g^\sharp(l) = g(l) - 1, \quad g^\sharp(l') = g(l+1), \quad g^\sharp(k) = g(k),\ \forall k \in \mathcal{W}_m \setminus \{l, l'\},$$

$g^\sharp$ is decreasing and sums to W_n. Note that $g(l')+1 = g(W_m)+1 < g(0) = g(l)$. Whence, one can compute:

$$\mathbb{E}\left[W_m^r\right]\big|_{g^\sharp} - \mathbb{E}\left[W_m^r\right]\big|_{g} = \left(e^{\lambda_n} - 1\right)\left(e^{-\lambda_n g(l)} - e^{-\lambda_n (g(l')+1)}\right) < 0,$$

which contradicts that g be minimal. Whence, the minimizer g is decreasing and satisfies $g(0) \leq g(W_m) + 1$, i.e., it is given by (7). In turn, the minimum is:

$$\mathbb{E}\left[W_m^r\right] = (W_m - w)\exp\left(-K\lambda_n\right) + w\exp\left(-(K+1)\lambda_n\right).$$

Now, if $W_n = KW_m$, i.e., $w = 0$, then $\mathbb{E}\left[W_m^r\right] = W_m \exp\left(-K\lambda_n\right)$. $\square$

Nation n's opponents have $x_{\sigma(n)}^\tau$ weapons of class τ. Thus Proposition 1 shows that the effective power drawn from stockpile value of class τ weapons, after a single preemptive strike by Nation n with nuclear weapons, is given by $x_{\sigma(n)}^\tau \exp\left(-\lambda_n \frac{x_n^{\text{conv}}}{x_{\sigma(n)}^\tau}\right)$.

Generic security function. Combining all of the above, we can define a generic security function for both conventional and nuclear armaments. Let

$a, b \in \{0,1\}$. Define Nation n's security function for conventional armaments as:

$$f_n^{\text{conv}}(x_n, x_{-n}) = x_n^{\text{conv}} \exp\left(-\lambda_{\sigma(n)} \frac{x_{\sigma(n)}^{\text{nuc}}}{x_n^{\text{conv}}}\right)^a \phi\left(\frac{x_{\sigma(n)}^{\text{conv}}}{\xi_n}\right) - x_{\sigma(n)}^{\text{conv}} \exp\left(-\lambda_n \frac{x_n^{\text{nuc}}}{x_{\sigma(n)}^{\text{conv}}}\right)^b, \tag{8}$$

and Nation n's security function for nuclear armaments as

$$\begin{aligned} f_n^{\text{nuc}}(x_n, x_{-n}) = \quad & x_n^{\text{nuc}} \exp\left(-\lambda_{\sigma(n)} \frac{x_{\sigma(n)}^{\text{nuc}}}{x_n^{\text{nuc}}}\right)^a \phi\left(\frac{x_{\sigma(n)}^{\text{conv}} + x_{\sigma(n)}^{\text{nuc}}}{\xi_n}\right) \\ & - x_{\sigma(n)}^{\text{nuc}} \exp\left(-\lambda_n \frac{x_n^{\text{nuc}}}{x_{\sigma(n)}^{\text{nuc}}}\right)^b, \end{aligned} \tag{9}$$

where $\lambda_{\sigma(n)}$ is a measure of the accuracy of the nuclear weapons of Nation n's opponents. In general, Nations might choose among four doctrine choices, shaping their security functions. Indeed, each Nation will choose whether it relies on its vulnerable second-strike capability ($a = 1$) or not ($a = 0$); and whether it relies on its preemptive first strike capability ($b = 1$) or not ($b = 0$). We focus on the case $(a, b) = (0, 1)$, more representative of modern nuclear military capabilities. Therefore, applying (5), the security function of Nation $n \in \mathcal{N}$ will be given by:

$$\begin{aligned} f_n(x_n, x_{-n}) = \alpha^{\text{nuc}} & \Big[x_n^{\text{nuc}} \phi\Big((x_{\sigma(n)}^{\text{conv}} + x_{\sigma(n)}^{\text{nuc}})/\xi_n\Big) - x_{\sigma(n)}^{\text{nuc}} \exp\Big(-\lambda_n x_n^{\text{nuc}} / x_{\sigma(n)}^{\text{nuc}}\Big)\Big] \\ & + \alpha^{\text{conv}} \Big[x_n^{\text{conv}} \phi\Big(x_{\sigma(n)}^{\text{conv}}/\xi_n\Big) - x_{\sigma(n)}^{\text{conv}} \exp\Big(-\lambda_n x_n^{\text{nuc}} / x_{\sigma(n)}^{\text{conv}}\Big)\Big], \end{aligned} \tag{10}$$

with $\alpha^{\text{nuc}} + \alpha^{\text{conv}} = 1$, α^{nuc}, $\alpha^{\text{conv}} \geq 0$, λ_n is Nation n's nuclear armaments accuracy, and $\xi_n > 0$ is a Nation-specific constant which models Nation n's peacefulness. Remind that $\phi(\cdot)$ is a concave, strictly increasing function. We also assume $\phi(0) = 0$, $\phi'(0) = 1$, $\phi^{(3)}(0) > 0$ and that $\phi(x) = o(x)$ at infinity.

Remark 2. In what follows, we will choose $\phi : s \mapsto \ln(1+s)$ in numerical computations. Furthermore, for technical purposes, let $k > 0$, $\phi^k : s \mapsto \min\{\phi(s), \phi(k)\}$ and denote $\bar{J}_n^k(\cdot)$ the utility function defined as $\bar{J}_n(\cdot)$ by replacing $\phi(\cdot)$ by $\phi^k(\cdot)$. Let $\nabla_n \cdot \bar{J}_n(x) \stackrel{\text{def}}{=} \frac{\langle \nabla_{x_n} \bar{J}_n(x), x_n \rangle}{\|x_n\|}$ be the directional derivative of $\bar{J}_n(\cdot)$ along x_n. Also, observe that under Assumption 2, the following inequality holds:

$$\nabla_n \cdot \bar{J}_n(x_n, x_{-n}) \leq \widehat{\lambda} + \phi\left(\sqrt{2}\frac{\kappa_+}{\widehat{\xi}}\|\widehat{x}\|\right) - \check{C}\|x_n\|, \quad \forall n \in \mathcal{N} \tag{11}$$

where $\widehat{x} = (\widehat{x}_\tau)_\tau$ with $\widehat{x}^\tau = \max_{m \in \mathcal{N}} x_m^\tau$, $\check{C} = \min_{m \in \mathcal{N},\, \tau} \check{C}_m^\tau$, $\widehat{\lambda} = \max_{m \in \mathcal{N}} \lambda_m$, and $\widehat{\xi} = \max_{m \in \mathcal{N}} \xi_m$. An analogous inequality holds for $\big(\bar{J}_n^k(\cdot), \phi^k(\cdot)\big)$.

We observe that $f_n(., x_{-n})$ is a $\mathcal{C}^\infty$ function on $\big(\mathbb{R}_+^*\big)^2$. It is also well-defined on $\partial(\mathbb{R}_+^*)^2 \setminus \{0\}$. Moreover, around this set, it is continuous and bounded thus one can extend it by continuity at 0 by the value 0. However, it is not differentiable on $\partial(\mathbb{R}_+^*)^2$. Therefore, we can not apply directly standard results from game theory which require that the players' utility functions are defined on a closed-convex domain and differentiable. The analysis of the one-shot game will therefore require to deal, first, with the degenerate case of the boundary and, second, to develop theoretical tools to study objective functions defined on open unbounded domains.

3 Analysis of the One-Shot Game

Let $\mathcal{U} \overset{\text{def}}{=} \prod_n \mathcal{U}_n$. We define $\mathcal{G} \overset{\text{def}}{=} \left(\mathcal{N}, \mathcal{U}, (\bar{J}_n)_n\right)$ as the one-shot (static) game involving a set $\mathcal{N}$ of N armed Nations. For all $n \in \mathcal{N}$, Nation n's utility function is derived from (4), with target value satisfying Assumption 2. The security functions are defined in (5), and we assume quadratic cost functions, $c_n^S(\cdot)$, $\forall n$:

$$c_n^S(x_n) = \frac{1}{2} C_n^{\text{conv}} \left(x_n^{\text{conv}}\right)^2 + \frac{1}{2} C_n^{\text{nuc}} \left(x_n^{\text{nuc}}\right)^2 ,$$

where $C_n^{\text{conv}}, C_n^{\text{nuc}} > 0$ are storage marginal costs for conventional and nuclear weapons respectively, assuming $c_n^D(0) = 0$. Nation n's utility function takes the form: $\bar{J}_n(x_n, x_{-n}) = f_n(x) - c_n^S(x_n)$. We will analyze the outcome of the noncooperative game $\mathcal{G}$ relying on the classical Nash Equilibrium, as solution concept.

Definition 2 (Nash Equilibrium). *A Nash Equilibrium $x^* = (x_n^*)_n$ of $\mathcal{G}$ is a vector of power drawn from the stockpiles of weapons, such that $\bar{J}_n(x^*) \geq \bar{J}_n(x_n, x_{-n}^*), \forall x_n \in \mathcal{U}_n, \forall n \in \mathcal{N}$.*

We recall below the definition of a concave game.

Definition 3 (Concave N-player game). *Let $E \overset{def}{=} \prod_{n=1}^N E_n$ be a product of Euclidean spaces $(E_n)_n$. Let $\mathcal{U} \subset E$ be a convex subset of E and for all $n \in \mathcal{N}$ assume the utility functions $\bar{J}_n : \mathcal{U} \to \mathbb{R}$ are continuous. The game $\mathcal{G}$, where each player $n \in \mathcal{N}$ solves the parametrized optimization problem:*

$$\max_{x_n \in E_n} \bar{J}_n(x_n, x_{-n}) \quad s.t. \; x \in \mathcal{U}, \quad \forall n \in \mathcal{N},$$

is called a concave game if for all $n \in \mathcal{N}$, $x_n \mapsto \bar{J}_n(x_n, x_{-n})$ is a concave function for each fixed value of x_{-n} such that $x \in \mathcal{U}$. If $\mathcal{U}$ is compact, then it is called a compact-concave game.

A classical result [25, Theorem 1] ensures the existence of Nash Equilibria for concave N-player games whose joint strategy set is a compact convex. In the follow up, we extend this result to the case where: (i) the joint strategy set is a closed convex; and, (ii) the players' utility functions satisfy a coercivity property. Finally, we prove the existence of an interior point Nash Equilibrium for the N-player static game $\mathcal{G}$.

Remark 3. Note that contrary to the finite-horizon game involving a finite number of time steps in the dynamic game, decision variables cannot be normalized in the one-shot game. Indeed, as we study the equilibria in the long-run, i.e., steady states, without requiring *a priori* budget constraints, there is no reason that Nash Equilibria take bounded values when time goes to infinity, e.g., in the case of a security spiral.

The following proposition deals with the *degenerate cases* where, at a Nash Equilibrium, a Nation would disarm and give up a weapons class. We show that this kind of solution requires all Nations to disarm for the same weapons class.

Proposition 2 (Disarmament). *Let x be a Nash Equilibrium. Let $\tau \in \{nuc, conv\}$ and $n \in \mathcal{N}$. The following statements hold true:*

1. *If $x^\tau_{\sigma(n)} = 0$, then for all $m \in \sigma(n)$, $x^\tau_m = 0$;*
2. *If $x^\tau_n = 0$, then for all $m \in \mathcal{N}$, $x^\tau_m = 0$.*

Proof. To prove the first statement, let $x^\tau_{\sigma(n)} = 0$. Then, for all $m \in \sigma(n)$,

$$0 \le x^\tau_m \le \max_{\ell \in \sigma(n)} x^\tau_\ell \le \frac{x^\tau_{\sigma(n)}}{\kappa^-} = 0, \text{ by Assumption 2.}$$

Therefore, $\forall m \in \sigma(n)$, $x^\tau_m = 0$. Now, we want to prove the second statement. First, let us show that if $x^\tau_n = 0$, then $\forall m \in \sigma(n)$, $x^\tau_m = 0$ by contraposition. Assume that there exists $m \in \sigma(n)$ such that $x^\tau_m > 0$, which implies $x^\tau_{\sigma(n)} > 0$ using the first statement. Now, as $x^\tau_{\sigma(n)} > 0$, a direct computation yields a lower bound on the gradient of the security function at $(x^{\text{nuc}}_n, x^{\text{conv}}_n) \in (\mathbb{R}_+)^2$: $\frac{\partial f_n}{\partial x^\tau_n}(x_n, x_{-n}) \ge \alpha^\tau \phi\left(\frac{x^\tau_{\sigma(n)}}{\xi_n}\right) > 0$, because $\phi(\cdot)$ is strictly increasing and $\phi(0) = 0$. Note that, at $x^\tau_n = 0$, $\frac{\partial c^S_n}{\partial x^\tau_n}(x_n) = C^\tau_n x^\tau_n = 0$. Therefore,

$$\frac{\partial \bar{J}_n}{\partial x^\tau_n}(x_n, x_{-n}) = \frac{\partial f_n}{\partial x^\tau_n}(x_n, x_{-n}) - \frac{\partial c^S_n}{\partial x^\tau_n}(x_n) = \frac{\partial f_n}{\partial x^\tau_n}(x_n, x_{-n}) > 0.$$

Hence, all strategies with $x^\tau_n = 0$ are dominated, i.e., if x is a Nash Equilibrium, we have that $x^\tau_n > 0$. Therefore, if $x^\tau_n = 0$, then $\forall m \in \sigma(n)$, $x^\tau_m = 0$. Now, assume that $x^\tau_n = 0$. Then, by induction, $\forall k \in \mathbb{N}$, $\forall m \in \sigma^k(n)$, $x^\tau_m = 0$. As $\mathcal{N}$ is a strategic relationship class, $\mathcal{N} = \sigma^{|\mathcal{N}|}(n)$. Therefore, $x^\tau_m = 0$. □

Let $\mathcal{U}^\dagger \overset{\text{def}}{=} \mathcal{U} \cap (\mathbb{R}^*_+)^{2N}$ and $\mathcal{U}^\dagger_n \overset{\text{def}}{=} \mathcal{U}_n \cap (\mathbb{R}^*_+)^2$. As a consequence of Proposition 2, there exists a trivial Nash Equilibrium which coincides with a “general and complete disarmament” strategy ($x = 0$). In the following sections, we prove the existence and uniqueness of interior points Nash Equilibria $x^* \in \mathcal{U}^\dagger$. An interior point Nash Equilibrium does not lie on the boundary of $(\mathbb{R}^*_+)^2$.

3.1 Existence of a Nash Equilibrium

It is well-known that every compact-concave game admits a Nash Equilibrium. As the utility functions of the game $\mathcal{G}$ are concave in their own strategy space, we can build a compact concave game $\mathcal{G}'$ by imposing constraints to $\mathcal{G}$, ensuring the existence of a Nash Equilibrium for $\mathcal{G}'$. Let $\mathcal{U}' \subset \mathcal{U}$ be a compact convex.

Lemma 2. *Assume that a Nash Equilibrium is reached at an interior point of $\mathcal{U}'$. Then, since the players' utility functions are concave in their own strategy space, it is also a Nash Equilibrium for the game with strategy space $\mathcal{U}$.*

Proof. The proof relies directly on the concavity of the objective functions.

Using this property, we can show under further assumptions on the asymptotic behavior of the utility functions that the boundedness of $\mathcal{U}$ is not required.

Definition 4. *Let $\Psi : \mathcal{U} \subset E \to \mathbb{R}$ be a continuous function. We say that $\Psi(\cdot)$ is uniformly coercive in x_n over E_n if and only if there exists $r_n > 0$ and $g : E_n \to \mathbb{R}$ such that:*

$$\begin{aligned} \forall x \in \mathcal{U}, &\qquad \nabla_n \cdot \Psi(x) \geq g(x_n), \\ \forall y \in E_n, &\qquad \|y\| \geq r_n \implies g(y) > 0. \end{aligned}$$

where $\nabla_n \cdot \Psi(x) = \frac{\langle \nabla_{x_n} \Psi(x), x_n \rangle}{\|x_n\|}$ is the directional derivative of $\Psi(\cdot)$ along x_n.

Using (11) and that $\phi_k(\cdot)$ are bounded, we check that the opposite of objective functions $-\bar{J}_n^k(\cdot), \forall n$ are uniformly coercive over their own strategy space.

Proposition 3. *Let $\widetilde{\mathcal{G}} = (\mathcal{N}, \mathcal{U}, (\Psi_n)_n)$ be a concave game. Assume $\mathcal{U} \subset E$ is closed and $-\Psi_n(\cdot)$ is uniformly coercive over its own strategy space $\mathcal{U}_n, \forall n$. Then, $\widetilde{\mathcal{G}}$ admits a Nash Equilibrium.*

Proof. Let $\mathcal{B}_n(r)$ be the closed ball of radius $r > 0$ in E_n. Let $\mathcal{B}(r) \stackrel{\text{def}}{=} \prod_{n=1}^N \mathcal{B}_n(r)$. Hence, $\Gamma(r) \stackrel{\text{def}}{=} \mathcal{B}(0, r) \cap \mathcal{U} \subset \mathcal{U}$ is the intersection of two closed convex sets and is bounded, thus it is a compact convex. Then, $\widetilde{\mathcal{G}}$ is a compact concave game over $\Gamma(r)$ and admits a Nash Equilibrium, denoted by x^*. Now, for all $n \in \mathcal{N}$, $-\Psi_n(\cdot)$ is uniformly coercive over E_n. Hence, there exists $r_n > 0$ and $g_n(\cdot)$ such that

$$\begin{aligned} \forall x \in \mathcal{U}, &\qquad -\nabla_n \cdot \Psi_n(x) \geq g(x_n), \\ \forall y \in E_n, &\qquad \|y\| \geq r_n \implies g(y) > 0. \end{aligned}$$

Set $r > \max_n \{r_n\}$. If there existed $n \in \mathcal{N}$ such that $x_n \in \partial \mathcal{B}_n(r)$, then the optimality conditions would write:

$$\nabla_n \cdot \Psi_n(x) \geq 0 \quad \text{with } \|x_n\| = r,$$

Now, $\|x_n\| = r \geq r_n$ thus $\frac{\partial \Psi}{\partial n}(x) \leq -g_n(x_n) < 0$, which is a contradiction. Therefore, for $r > 0$ large enough, Nash Equilibria of $\widetilde{\mathcal{G}}$ are either interior points of $\Gamma(r) \subset \mathcal{U}$ or boundary points of $\mathcal{U}$. Notice that in the former case, by Lemma 2, they are also interior points Nash Equilibria for the game with strategy space $\mathcal{U}$. Therefore, the game $\widetilde{\mathcal{G}}$ admits a Nash Equilibrium. $\square$

Proposition 4. *There exists a Nash Equilibrium solution of $\mathcal{G}$*

Proof. From Proposition 3 that for all $k > 0$, the game $\mathcal{G}^k \overset{\text{def}}{=} \left(\mathcal{N}, \mathcal{U}, (\bar{J}_n^k)_n\right)$ admits a Nash Equilibrium x^{*k}. Thus, the directional derivative of $\bar{J}_n^k(\cdot)$ at x^{*k} must be non-negative. For all $\zeta > 0$, let $F(\zeta) = \widehat{\lambda} + \phi\left(\sqrt{2}\frac{\kappa_+}{\check{C}\hat{\xi}}\zeta\right)$. From (11), we infer that $z \leq F(z)$ where $z = \check{C}\left\|\widehat{x^{*k}}\right\|$. As $\phi(\cdot)$ is strictly increasing, for all $p \in \mathbb{N}$, $z \leq F^p(z)$. Now, the assumptions on ϕ imply that F has a unique fixed point z_0 over $\mathbb{R}_+^*$ that depends only on ϕ, $\hat{\lambda}$, $\frac{\kappa_+}{\check{C}\hat{\xi}}$ and that $F^p(z)$ converges to z_0. Therefore, $z \leq z_0$. Let $r > z_0/\check{C}$ and $k > \sqrt{2}\frac{\kappa_+}{\hat{\xi}}r$. Hence, one can check that the games $\tilde{\mathcal{G}}^k = \left(\mathcal{N}, \mathcal{U} \cap B(r), (\bar{J}_n^k)_n\right)$ and $\tilde{\mathcal{G}} = \left(\mathcal{N}, \mathcal{U} \cap B(r), (\bar{J}_n)_n\right)$ are identical. Now, let x^* a Nash Equilibrium of $\tilde{\mathcal{G}}$. As $\|x^*\| \leq z/\check{C} \leq z_0/\check{C} < r$, x^* does not lie on $\partial B(r)$ thus it is a Nash Equilibrium of $\mathcal{G}$. □

Theorem 1. *The game $\mathcal{G}$ admits an interior point Nash Equilibrium $x^* \in \mathcal{U}^\dagger$ under the sufficient conditions:*

$$C_n^{nuc}\xi_n < 2\kappa^-\alpha_n^{nuc}, \quad C_n^{conv}\xi_n < \kappa^-\alpha_n^{conv}.$$

Proof. Let $\varepsilon > 0$ and $\Gamma_\varepsilon = \bigcap_{n,\tau}\{x_n^\tau \geq \varepsilon\} \subset \mathcal{U}^\dagger$. From Proposition 3, as Γ_ε is a closed convex set, the game $\mathcal{G}^\sharp \overset{\text{def}}{=} \left(\mathcal{N}, \Gamma_\varepsilon, (\bar{J}_n)_n\right)$ admits a Nash equilibrium x^*. Reasoning by contradiction, suppose that $x^* \in \partial\Gamma_\varepsilon$. This means that there exists $n \in \mathcal{N}$ and $\tau \in \{\text{nuc}, \text{conv}\}$ such that $x_n^{*\tau} = \varepsilon$. Hence, using first order Taylor-Lagrange expansion, at least one of the following inequalities holds true:

$$\frac{\partial \bar{J}_n}{\partial x_n^{\text{conv}}}(x^*) \geq \left(\kappa^-\alpha^{\text{conv}} - C_n^{\text{conv}}\xi_n\right)\frac{\varepsilon}{\xi_n} + o(\varepsilon),$$
$$\frac{\partial \bar{J}_n}{\partial x_n^{\text{nuc}}}(x^*) \geq \left(2\kappa^-\alpha^{\text{nuc}} - C_n^{\text{nuc}}\xi_n\right)\frac{\varepsilon}{\xi_n} + o(\varepsilon).$$

If $\mathcal{G}$ parameters are chosen such that $\kappa^-\alpha^{\text{conv}} > C_n^{\text{conv}}\xi_n$, $2\kappa^-\alpha^{\text{nuc}} > C_n^{\text{nuc}}\xi_n$, for ε small enough, $\frac{\partial \bar{J}_n}{\partial x_n^\tau}(x^*) > 0$. It contradicts the necessary optimality condition $\frac{\partial}{\partial x_n^\tau}\bar{J}_n(x^*) \leq 0$ thus $x^* \in \mathcal{U}^\dagger$. Hence, $\mathcal{G}^\dagger \overset{\text{def}}{=} \left(\mathcal{N}, \mathcal{U}^\dagger, (\bar{J}_n)_n\right)$ admits the same Nash Equilibrium x^*, which is also an interior point Nash Equilibrium of $\mathcal{G}$. □

3.2 Uniqueness of the Interior Point Nash Equilibrium

Assumption 3 $\forall n \in \mathcal{N}, 1 - \frac{1}{x_{\sigma(n)}^{nuc}}\sum_{m \neq n}\frac{\partial x_{\sigma(n)}^{nuc}}{\partial x_m^{nuc}} \geq 0$ *and* $C_n^{nuc} - \frac{\alpha^{nuc}}{\xi_n}\sum_{m \neq n}\frac{\partial x_{\sigma(n)}^{nuc}}{\partial x_m^{nuc}} \geq 0.$

In the rest of the paper, we assume that Assumption 3 holds.

Proposition 5. *The game $\mathcal{G}^\dagger$ is strongly monotone.*

Proof. Let $F_n \overset{\text{def}}{=} \left[\frac{\partial \bar{J}_n(x)}{\partial x_n^{\text{nuc}}} \; \frac{\partial \bar{J}_n(x)}{\partial x_n^{\text{conv}}}\right]^T$ be the gradient of player n's utility with respect to its own actions. We get: $\forall x_n \in \mathcal{U}$, $\frac{\partial \bar{J}_n(x)}{\partial x_n^{\text{conv}}} = \alpha^{\text{conv}} \phi(x_{\sigma(n)}^{\text{conv}}/\xi_n) - C_n^{\text{conv}} x_n^{\text{conv}}$, which implies that $\bar{J}_n(\cdot)$ reaches its maximum in the variable x_n^{conv} at $(x_n^{\text{conv}})^* = \frac{\alpha^{\text{conv}}}{C_n^{\text{conv}}} \phi\Big(\frac{x_{\sigma(n)}^{\text{conv}}}{\xi_n}\Big)$, if $(x_n^{\text{conv}})^*$ is admissible, or, else, at the border of the interval of definition of x_n^{conv}. Similarly, for Nation n's other decision variable, we get: $\frac{\partial \bar{J}_n(x)}{\partial x_n^{\text{nuc}}} = \alpha^{\text{nuc}} \phi\left(\frac{x_{\sigma(n)}^{\text{conv}} + x_{\sigma(n)}^{\text{nuc}}}{\xi_n}\right) + \lambda_n \sum_\tau \alpha^\tau \exp\Big(-\lambda_n \frac{x_n^{\text{nuc}}}{x_{\sigma(n)}^\tau}\Big) - C_n^{\text{nuc}} x_n^{\text{nuc}}$.

Consider the pseudo-Hessian matrix H^{nuc} of the players' utilities considering only nuclear weapons where each n^{th} row and m^{th} column component is given as $H_{n,m}^{\text{nuc}} = \frac{\partial F_n^{\text{nuc}}}{\partial x_m^{\text{nuc}}}, n, m \in \mathcal{N}$, i.e., in details:

$$H_{n,m}^{\text{nuc}} = \begin{cases} -\lambda_n^2 \sum_\tau \frac{\alpha^\tau}{x_{\sigma(n)}^\tau} \exp\Big(-\lambda_n \frac{x_n^{\text{nuc}}}{x_{\sigma(n)}^\tau}\Big) - C_n^{\text{nuc}} & \text{if } m = n, \\ \left[\frac{\alpha^{\text{nuc}}}{\xi_n} \phi'\left(\frac{x_{\sigma(n)}^{\text{conv}} + x_{\sigma(n)}^{\text{nuc}}}{\xi_n}\right) + \lambda_n^2 \frac{\alpha^{\text{nuc}} x_n^{\text{nuc}}}{(x_{\sigma(n)}^{\text{nuc}})^2} \exp\Big(-\lambda_n \frac{x_n^{\text{nuc}}}{x_{\sigma(n)}^{\text{nuc}}}\Big)\right] \frac{\partial x_{\sigma(n)}^{\text{nuc}}}{\partial x_m^{\text{nuc}}} & \text{if } m \neq n. \end{cases}$$

Thus $-H^{\text{nuc}}$ is a Z-matrix as its-off diagonal entries are negative. Under Assumption 3, we observe that $-H^{\text{nuc}}$ is strictly diagonally dominant thus $-H^{\text{nuc}}$ is an M-matrix, i.e., a Z-matrix with eigenvalues whose real parts are nonnegative [32]. Thus, H^{nuc} is negative definite. Furthermore, letting $F \overset{\text{def}}{=} \text{col}\big((F_n)_n\big)$, as $\frac{\partial F_n}{\partial x_n^{\text{conv}}}$ is constant for all $n \in \mathcal{N}$, the inequality $(y-x)^T \Big[F(y) - F(x)\Big] < 0, \forall x \neq y$ follows from [25, Theorem 6] and the game $\mathcal{G}^\dagger$ is strongly monotone. □

Theorem 2. *The game $\mathcal{G}$ has a unique interior point Nash Equilibrium $x^* \in \mathcal{U}^\dagger$.*

Proof. Let us consider that there exists two Nash Equilibria x^* and x^{**} solutions of the game $\mathcal{G}^\dagger$. As both x^* and x^{**} are Nash Equilibria they must satisfy the stationarity condition. Mutliplying the first order condition with $\big(y - x^*\big)$ at point x^* and $\big(y - x^{**}\big)$ at point x^{**}, we get:

$$(y - x^*)^T F(x^*) = 0, \quad \forall y \in \mathcal{U}^\dagger, \tag{12a}$$

$$(y - x^{**})^T F(x^{**}) = 0, \quad \forall y \in \mathcal{U}^\dagger. \tag{12b}$$

Taking (12a) in $y = x^*$ and (12b) in $y = x^{**}$ and summing the two above equations, we get: $(x^{**} - x^*)^T (F(x^{**}) - F(x^*)) = 0$, which contradicts the fact that the game $\mathcal{G}^\dagger$ is strongly monotone from Proposition 5. Therefore, the game $\mathcal{G}^\dagger$ admits a unique Nash Equilibrium. □

3.3 Distributed Nash Equilibrium Seeking

When more than two Nations are involved, algorithmic methods need to be developed to compute Nash Equilibria.

Proposition 6. *The pseudo-gradient $F(x)$ of the game $\mathcal{G}$ is Lipschitz continuous in $x \in \mathcal{U}^\dagger$.*

Proof. The proof relies on the Triangle Inequality and Mean Value Theorem. Due to space limit, it is omitted. □

We assume that the Nations are in a full information feedback setting.

Assumption 4 *Each player $n \in \mathcal{N}$ can get the decisions of all its opponents $x_{-n} \in \prod_{m \neq n} \mathcal{U}_m$.*

Under Propositions 5 and 6, and Assumption 4, various gradient-based algorithms can be proposed to compute the interior point Nash Equilibrium solution of $\mathcal{G}$ and are proved to converge [9]. We will implement a regularization algorithm for monotone game, called proximal point method (PPM). PPM is an alternative method to distributed gradient descent schemes, whose interest lies in the fact that it avoids having to coordinate the players in their steplength choice. At iteration l, x^l is the solution of a Variational Inequality (VI) of the type: $(y-x)^T F^l(x) \leq 0, \quad \forall y \in \mathcal{U}$, with $F^l(x) = F(x) - \theta(x - x^l)$, and $\theta > 0$ is a regularization parameter which implicitly determines iteration bounds to reach a prescribed error level [21]. In this setting, the regularization parameters are required to be the same for all the players. Extensions of iterative proximal point method for monotone games where each player can independently select and adapt its algorithm parameter after each iteration [13] exist, but will not be considered in the current version of the work. In practice, the PPM algorithm can be implemented by solving at each iteration l: $x_n^{l+1} = \arg\max_{u \in \mathcal{U}_n} \left(\bar{J}_n(u, x_{-n}^l) - \theta \|u - x_n^l\|^2 \right), \quad \forall n \in \mathcal{N}$.

4 Analysis and Simulations in Case of a Duopoly

4.1 Analytical Analysis

In the case of a duopoly ($N=2$), we provide an analytical characterization of the interior point Nash Equilibrium of $\mathcal{G}$. We define $\widetilde{\mathcal{N}} \stackrel{\text{def}}{=} \{(n,m) \in \mathcal{N}^2 \mid n \neq m\}$. For all $(n,m) \in \widetilde{\mathcal{N}}$, we let x_m denote x_{-n} and $\sigma(n) \stackrel{\text{def}}{=} \{m\}$. Then, the interior point Nash Equilibrium of $\mathcal{G}$ is obtained by solving the following system:

$$\max_{x \in (\mathbb{R}_+^*)^2} \bar{J}_n(x_n, x_m), \qquad \forall (n,m) \in \widetilde{\mathcal{N}},$$

leading to the following first order (necessary) optimality conditions:

$$\frac{\partial \bar{J}_n}{\partial x_n^{\text{conv}}}(x_n, x_m) = 0, \quad \frac{\partial \bar{J}_n}{\partial x_n^{\text{nuc}}}(x_n, x_m) = 0, \qquad \forall (n,m) \in \widetilde{\mathcal{N}},$$

which can be written explicitly as the following system of equations:

$$\forall (n,m) \in \widetilde{\mathcal{N}}, \qquad C_n^{\text{conv}} x_n^{\text{conv}} - \alpha^{\text{conv}} \phi\left(\frac{x_m^{\text{conv}}}{\xi_n} \right) = 0, \tag{13a}$$

$$C_n^{\text{nuc}} x_n^{\text{nuc}} - \lambda_n \sum_{\tau} \exp\left(-\frac{\lambda_n x_n^{\text{nuc}}}{x_m^{\tau}} \right) - \alpha^{\text{nuc}} \phi\left(\frac{x_m^{\text{nuc}} + x_m^{\text{conv}}}{\xi_n} \right) = 0. \tag{13b}$$

Let $(n,m) \in \widetilde{\mathcal{N}}$. Define $\mathcal{R}_n^{\text{conv}} : x \mapsto \frac{\alpha^{\text{conv}}}{C_n^{\text{conv}}}\phi\Big(\frac{x}{\xi_n}\Big)$. Then one can rewrite (13a) as $x_n^{\text{conv}} = \mathcal{R}_n^{\text{conv}}(x_m^{\text{conv}})$. Hence, when dealing with two players, the Nash Equilibrium components $(x_n^{\text{conv}})_n$ are found at the intersection (χ_2, χ_1) of curves $(u, \mathcal{R}_1^{\text{conv}}(u))_{u>0}$ and $(\mathcal{R}_2^{\text{conv}}(v), v)_{v>0}$. Consider the following functional equations with unknowns $\mathcal{R}_n^{\text{nuc}} : x \mapsto y$:

$$C_n^{\text{nuc}} y - \lambda_n \left[\exp\left(-\frac{\lambda_n y}{x}\right) + \exp\left(-\frac{\lambda_n y}{\chi_n}\right)\right] - \alpha^{\text{nuc}} \phi\left(\frac{x+\chi_n}{\xi_n}\right) = 0, \ \forall n \in \mathcal{N}. \quad (14)$$

Then (13b) is equivalent to $\mathcal{R}_n^{\text{nuc}}(x_m^{\text{nuc}}) = x_n^{\text{nuc}}$ by setting $x = x_m^{\text{nuc}}$ and $y = x_n^{\text{nuc}}$. Let $\mu_n = \frac{\chi_n}{\xi_n}$, $\rho_n = C_n^{\text{nuc}}\xi_n$ and the change of variables $(X,Y) = (\frac{x^{\text{nuc}}}{\xi_n}, C_n^{\text{nuc}} y^{\text{nuc}})$, then (14) recasts as:

$$Y - \lambda_n \left[\exp\left(-\frac{\lambda_n}{\rho_n}\frac{Y}{X}\right) + \exp\left(-\frac{\lambda_n}{\rho_n}\frac{Y}{\mu_n}\right)\right] - \alpha^{\text{nuc}} \phi(X + \mu_n) = 0, \ \forall n \in \mathcal{N}. \quad (15)$$

Proposition 7. *Equation* (15) *has a unique solution.*

Proof. Let $\rho, \mu, \lambda > 0$ and $\alpha \in (0,1)$. For all $X, Y \geq 0$, define

$$h(X,Y) = \begin{cases} Y - \lambda \exp\Big(-\frac{\lambda}{\rho}\frac{Y}{X}\Big) - \lambda \exp\Big(-\frac{\lambda}{\rho}\frac{Y}{\mu}\Big) - \alpha\phi(X+\mu) & \text{if } X \neq 0, \\ Y - \lambda \exp\Big(-\frac{\lambda}{\rho}\frac{Y}{\mu}\Big) - \alpha\phi(\mu) & \text{else.} \end{cases}$$

For all $X \geq 0$, $Y \mapsto h(X,Y)$ is continuous and strictly increasing over $\mathbb{R}_+$. As $h(X,0) \leq -\lambda < 0$ and $\lim\limits_{Y \to +\infty} h(X,Y) = +\infty$, it is one-to-one from $\mathbb{R}_+$ to a set containing 0. Thus it has a unique zero on $\mathbb{R}_+$ and (15) has a unique solution. □

Due to limited space and for the sake of simplicity, we assume in what follows that $\lambda_n \exp\Big(-\frac{\lambda_n}{\rho_n}\frac{Y}{\mu_n}\Big)$ is negligible, i.e., we assume that preemptive nuclear strikes onto conventional targets achieve little security gain when compared to nuclear targets. Setting it to 0 yields the following equation:

$$Y - \lambda_n \exp\left(-\frac{\lambda_n}{\rho_n}\frac{Y}{X}\right) - \alpha^{\text{nuc}}\phi(X + \mu_n) = 0, \quad \forall n \in \mathcal{N}. \quad (16)$$

In the following, we provide a closed-form expression of the solution of (16), which relies on the special function $\mathcal{W}(\cdot)$, defined as the principal branch of the Lambert $\mathcal{W}$-function [7] , solution to $we^w = r, \ r \geq 0$. $\mathcal{W}(\cdot)$ cannot be expressed in terms of elementary functions, although some approximations, bounds, and integral representations, e.g., $\mathcal{W}(x) = \frac{1}{\pi}\int_0^\pi \ln\Big(1 + x\frac{\sin t}{t}e^{t\cot t}\Big)dt$ are well-known. For all $X, \rho, \lambda, \mu > 0$, let

$$G(X; \rho, \lambda, \mu) \stackrel{\text{def}}{=} \frac{\alpha^{\text{nuc}}}{\rho}\phi(X+\mu) + \frac{1}{\lambda} X \mathcal{W}\left(\lambda \exp\left(-\alpha^{\text{nuc}}\frac{\lambda}{\rho}\frac{\phi(X+\mu)}{X}\right)\right).$$

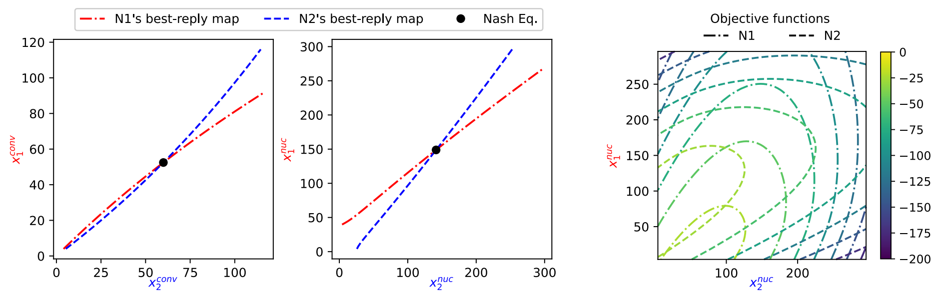

Fig. 1. In the special case of the duopoly, the interior point Nash Equilibrium is found graphically at the intersection point of the best-reply maps $(\mathcal{R}_n^{\text{nuc}})_{n=\text{N1,N2}}$ of each Nation as defined in (18). The model's parameters are given in Table 1.

Proposition 8. *The functional equation* (16) *with unknown* $X \mapsto Y(X)$ *admits a unique solution on* $\mathcal{C}^\infty(\mathbb{R}_+^*)$ *which satisfies :*

$$Y = \rho_n G(X;\, \rho_n, \lambda_n, \mu_n). \tag{17}$$

Proof. Let $n \in \mathcal{N}$. For all $X, Y > 0$, define $w = \frac{\lambda_n}{\rho_n X}(Y - \alpha^{\text{nuc}}\phi(X+\mu_n))$ and $r = \lambda_n \exp\left(-\alpha^{\text{nuc}}\frac{\lambda_n}{\rho_n}\frac{\phi(X+\mu_n)}{X}\right)$. One has the expression

$$w - r\exp(-w) = \frac{\lambda_n}{\rho_n X}\left[Y - \lambda_n \exp\left(-\frac{\lambda_n}{\rho_n}\frac{Y}{X}\right) - \alpha^{\text{nuc}}\phi(X+\mu_n)\right].$$

Consequently, $(X,Y) \in (\mathbb{R}_+^*)^2$ satisfies (16) if and only if $we^w = r$, i.e., if and only if $w = \mathcal{W}(r)$, which writes as (17). □

From Proposition 8, we infer the closed-form expressions of the best-replies:

$$x_n^{\text{nuc}} = \mathcal{R}_n^{\text{nuc}}(x_m^{\text{nuc}}) = \xi_n G\left(\frac{x_m^{\text{nuc}}}{\xi_n};\, C_n^{\text{nuc}}\xi_n,\, \frac{\lambda_n}{C_n^{\text{nuc}}\xi_n}\right), \quad \forall (n,m) \in \widetilde{\mathcal{N}}. \tag{18}$$

4.2 Numerical Results and Discussions

As shown in Figure 1, Nash Equilibria can be numerically obtained when computing the intersection points of the curves drawn by the best-reply maps $\mathcal{R}_n^{\text{nuc}}$ computed as with (18) and $\alpha^{\text{conv}} = \alpha^{\text{nuc}} = \frac{1}{2}$. Note that the security and the utility of both Nations with mixed armaments in a mutual deterrence relationship are greater than with only conventional weapons. Indeed, from Table 1 we can compute the efficiency gap between the social optimum of disarmament and the interior point Nash Equilibrium. With only conventional armaments the efficiency gap is of 108 whereas with mixed armaments it drops to 74, representing an inefficiency reduction of 31%. This can be interpreted as a stabilizing effect of nuclear armaments.

Depending on the observed efficiency loss, it might be interesting, in an extension of our work, to provide a method for specifying a Pareto dominating solution that depends on the Nations' threat thresholds.

	Nash Equilibrium Strategy $(x_n^{\text{conv}}, x_n^{\text{nuc}})$	Conventional-only		Mixed armaments	
		Utility	*Security*	**Utility**	*Security*
Nation 1	$(52, 149)$	$\mathbf{-53}$	*6*	$\mathbf{-45}$	*14*
Nation 2	$(60, 141)$	$\mathbf{-55}$	*7*	$\mathbf{-29}$	*33*

Table 1. Utility (bold-red) and security (italic-black) functions at Nash Equilibrium. Parameters: $\xi_{\mathrm{N1}} = 200$, $\xi_{\mathrm{N2}} = 150$, $C_n^{\text{conv}} = C_n^{\text{nuc}} = 5 \times 10^{-3}$, $\lambda_n = 0.7$, $\forall n \in \{\mathrm{N1}, \mathrm{N2}\}$.

5 Conclusion

We formulate a security game in a context of mixed armament acquisition, involving a finite set of Nations in strategic relationship. Our model incorporates key elements such as resource allocation to R&D and storage, defense capabilities, and geopolitical considerations, to provide a realistic representation of the arms procurement process. To analyze the long-term solution of the dynamical model, we study the steady states of the dynamical game by solving the associated one-shot game. On the theoretical side, we provide two important contributions to the state of the art. First, we extend classical results about compact-convex games to unbounded convex games relying on the coercivity property of the utility functions. Second, our results showcase the stabilizing effect of nuclear armaments and provide possible guidelines for current treaties refinements or for the understanding of various arms competition scenarios, including multilateral arms competitions and asymmetric weaponry.

Future research could focus on studying other identified factors such as alliances and the role of emerging technologies. Additionally, empirical studies and use case analyses would be valuable to validate the model outcomes and assess its real-world applicability. Finally, we anticipate that our contributions will serve as a valuable tool for policymakers, analysts, and researchers, assisting in the development of effective arms control measures and promoting stability in international relations.

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