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Optimal control of microalgae culture accounting for photoinhibition and light attenuation[★]

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Abstract: In a photobioreactor, due to the gradient of light, microalgae are successively exposed to conditions of low and high (inhibiting) light. This phenomenon can be captured by the Han model, which is a common mechanistic model of photoinhibition. Based on Han’s description, we introduce a dynamic system of microalgae growth involving two control variables: the light intensity and the dilution rate of the reactor. This model is derived from slow/fast dynamic considerations in a chemostat system accounting for the light gradient due to absorption and scattering following the Lambert-Beer’s law. Then, we formulate and study an optimal control problem in order to fully-characterize the optimal light supply and dilution strategies that maximize the harvested biomass. Our study, mainly based on Pontryagin’s maximum principle (PMP), shows that singular arcs and *turnpike*-like behaviors appear in the optimal solution. In particular, we prove that the optimal strategy maintains the biomass at a constant level along singular arcs, and we determine its static value. The theoretical results are illustrated throughout this paper using a direct optimization method.

Keywords: Optimal Control, Microalgae, Pontryagin’s principle, Light, Modelling, Han model.

1. INTRODUCTION

Microalgal based processes involve new challenges for modelling and control (Bernard, 2011). Microalgae are sensitive to different factors as, for example, pH, temperature, nutrients, dissolved CO₂. On top of them, light the most important factor influencing the growth rate of microalgae. Two conditions are not favourable for growth: *photolimitation*, when the perceived light is too low to trigger an optimal performance of the photosynthetic machinery, and *photoinhibition*, where a strong light can damage certain proteins in the cells and slow down the photon harvesting process.

We consider a reduced version of the Han model (Han, 2002) to represent the photoinhibition and photolimitation in the photobioreactor. This model has already been used and it has proven its efficiency for shape optimization of raceway ponds (Bernard et al., 2021a) and optimization of algal productivity (Bernard et al., 2021b). A similar three population model (Eilers and Peeters, 1988) for photosynthesis and photoinhibition was used to formulate and solve optimal control problems (OCPs) to maximize growth rate (Štěpán Papáček et al., 2008; Celikovsky et al., 2010). Usually, a reduction based on slow/fast manifolds is considered. In this work, we formulate an optimal control

problem where the objective is to maximize the harvested biomass in a photobioreactor.

Pontryagin’s maximum principle (PMP, (Pontryagin, 1987)) provides powerful tools to address the optimal control problems encountered in biotechnology, and turns out to be highly effective in solving optimization and control problems in photobioreactors (Bayen and Mairet, 2017; Djema et al., 2022; Mairet and Bayen, 2021). In several cases, it has been reported that the optimal solutions associated to these OCPs often exhibit a *turnpike*-like behaviour when the operation time is large enough (see, e.g., Caillaud et al. (2022); Djema et al. (2021)). Generally speaking, the *turnpike* feature states that the optimal solutions approach a static behaviour or stay close to a steady-state most of the time (Trélat and Zuazua, 2015; Porretta and Zuazua, 2013).

This paper is organized as follows. In Section 2 we introduce the biological model describing the evolution of the biomass in the photobioreactor. In Section 3, we state the optimal control problem (OCP) of interest. In Sections 4–5, we derive some properties of the optimal solution of the studied OCP, then we state our main result: the bioreactor must be operated at constant biomass describing turnpike properties. Next, in Section 6, we determine the value of the optimal biomass. Finally, in Section 7, we illustrate our results using direct optimization methods (implemented in BOCOP: optimal control solver).

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2. MATHEMATICAL MODEL

2.1 The reduced Han model for photobioreactors

We consider a photobioreactor in continuous mode operation. The biomass concentration of microalgae x (mgL^{-1}) evolves following the equation:

$$\dot{x} = \mu x - Rx - Dx \quad (1)$$

where R (s^{-1}) is the specific maintenance rate (Blanken et al., 2016), D (s^{-1}) is the dilution rate of the reactor (ratio of the influent flow rate over the volume of the reactor). The growth rate μ (s^{-1}) is derived from the reduced Han model (Fierro Ulloa et al., 2023; Han, 2002; Hartmann et al., 2014) as follows. Let us denote c (dimensionless) the fraction of photosynthetic units that are damaged due to excess of light on the photobioreactor and I_x ($\mu mol^{-1} \cdot m^{-1} \cdot s^{-1}$) the irradiance perceived by the culture, the dynamics of c can be derived from the Han model as,

$$\dot{c} = \frac{k_d \tau (\sigma I_x)^2}{1 + \tau \sigma I_x} (1 - c) - k_r c. \quad (2)$$

Here σ corresponds to the effective cross-section, τ is called turnover time (time needed to harvest one photon), k_d and k_r are the damage and repair rate respectively. The growth rate is then computed as (Fierro Ulloa et al., 2023),

$$\mu = \frac{K \sigma I_x (1 - c)}{1 + \tau \sigma I_x}. \quad (3)$$

2.2 Light attenuation due to biomass

Light availability decreases in the photobioreactor due to the self-shading and the light absorption of the chlorophyll content of the microalgae. We differentiate the irradiance that is applied to the culture, denoted by I ($\mu mol^{-1} \cdot m^{-1} \cdot s^{-1}$), which can be controlled, and the irradiance perceived by the culture I_x which affects the evolution of the state c . The perceived irradiance I_x ($\mu mol^{-1} \cdot m^{-1} \cdot s^{-1}$) follows the Beer-Lambert law,

$$I_x = I e^{-\xi x}, \quad (4)$$

where ξ ($mg^{-1} \cdot L$) is a coefficient derived from the light extinction rate and from the photobioreactor depth.

3. STATEMENT OF THE OPTIMAL CONTROL PROBLEM (OCP)

We define the *total biomass harvested* over the time window $[0, T_f]$ by,

$$J_{[0, T_f]}(D, I) = \int_0^{T_f} x(t) D(t) dt, \quad (5)$$

where x is the solution of the equation (1), here the controls are the irradiance I and the dilution rate D . The sets of admissible controls \mathcal{D} and \mathcal{I} are defined as,

$$\begin{aligned} \mathcal{D} &:= \{D : [0, +\infty) \rightarrow [0, D_{\max}] \ ; \ D(\cdot) \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+)\}, \\ \mathcal{I} &:= \{I : [0, +\infty) \rightarrow [0, I_{\max}] \ ; \ I(\cdot) \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+)\}, \end{aligned} \quad (6)$$

where $\mathcal{L}_{loc}^\infty(\mathbb{R}_+)$ is the space of locally integrable functions on every compact on \mathbb{R}_+ , $D_{\max} > 0$ is the maximum dilution rate given by the pump feeding capacity and I_{\max} is the maximum light intensity that can be applied.

For admissible non-negative initial conditions, we seek admissible controls, $D \in \mathcal{D}$ and $I \in \mathcal{I}$, over a fixed

time window $[0, T_f]$, that maximize the harvested biomass. Thus, the optimal control problem of interest reads as follows,

$$\begin{aligned} \max_{\substack{D \in \mathcal{D}, \\ I \in \mathcal{I}}} \quad & J_{[0, T_f]}(D, I) = \int_0^{T_f} x(t) D(t) dt, \\ \text{s.t.} \quad & \dot{x} = \frac{K \sigma I_x (1 - c)}{1 + \tau \sigma I_x} x - Dx - Rx, \\ & \dot{c} = \frac{k_d \tau (\sigma I_x)^2}{1 + \tau \sigma I_x} (1 - c) - k_r c, \end{aligned} \quad (\text{OCP})$$

4. NECESSARY CONDITIONS ON OPTIMAL CONTROLS

We apply the Pontryagin's Maximum Principle (PMP), (Vinter and Vinter, 2010; Clarke, 2013; Hocking, 1991; Trélat, 2005)) to derive necessary conditions satisfied by the optimal controls in (OCP). Let us denote by $X = (x, c)$ and $\lambda = (\lambda_x, \lambda_c)$ respectively the state and adjoint variables. Then, the Hamiltonian, $H(X, \lambda, D, I)$, associated with (OCP) is defined by,

$$H(X, \lambda, \lambda_0, D, I) = \lambda^T \dot{X} + \lambda_0 x D$$

We note that the Hamiltonian is linear w.r.t. the control D , then we rewrite,

$$H(X, \lambda, \lambda_0, D, I) = \Psi(X, \lambda, I) + \varphi(X, \lambda) D + \tilde{\varphi}(X, \lambda), \quad (7)$$

where,

$$\Psi(X, \lambda, I) = \frac{K \sigma I_x (1 - c)}{1 + \tau \sigma I_x} \lambda_x x \quad (8)$$

$$+ \frac{k_d \tau (\sigma I_x)^2 (1 - c)}{1 + \tau \sigma I_x} \lambda_c,$$

$$\varphi(X, \lambda) = [-\lambda_x + \lambda_0] x, \quad (9)$$

$$\tilde{\varphi}(X, \lambda) = -R \lambda_x x - k_r \lambda_c c. \quad (10)$$

The function φ is the so-called switching function w.r.t. the control D . Since (OCP) is stated as a maximization problem, we fix $\lambda_0 = 1$, assuming that the extremals are normal. Pontryagin's Maximum Principle states that if (X, D, I) is optimal, then the absolutely-continuous adjoint state $\lambda = (\lambda_x, \lambda_c)$, satisfies,

$$\dot{\lambda}_x = -\partial H / \partial x, \quad \dot{\lambda}_c = -\partial H / \partial c.$$

Therefore, it follows that,

$$\begin{cases} \dot{\lambda}_x = \lambda_x \left[\mu \left(\frac{\xi x}{1 + \tau \sigma I_x} - 1 \right) + (D + R) \right] - D, \\ \quad + \lambda_c \frac{k_d \xi \mu \tau \sigma I_x (2 + \tau \sigma I_x)}{K(1 + \tau \sigma I_x)} \\ \dot{\lambda}_c = \lambda_x \frac{K \sigma I_x}{1 + \tau \sigma I_x} x + \lambda_c \left[\frac{k_d \tau (\sigma I_x)^2}{1 + \tau \sigma I_x} + k_r \right]. \end{cases} \quad (11)$$

4.1 The transversality conditions

In (OCP), the final conditions $x(T_f)$ and $c(T_f)$ are free. Therefore, the transversality conditions are given by,

$$\lambda_x(T_f) = 0, \quad \lambda_c(T_f) = 0. \quad (12)$$

4.2 The PMP maximization condition

The Pontryagin's principle states that the controls D and I satisfy, for almost all $t \in [0, T_f]$, the maximization condition,

$$(D(t), I(t)) \in \underset{\substack{u \in [0, D_{\max}], \\ v \in [0, I_{\max}].}}{\operatorname{argmax}} H(X(t), \lambda(t), \lambda_0, u, v) \quad (13)$$

In addition, as the augmented system is *autonomous*, then $H(X, \lambda, \lambda_0, D, I)$ is constant along the optimal trajectories, *i.e.*, for almost all $t \in [0, T_f]$:

$$\frac{d}{dt} H(X(t), \lambda(t), \lambda_0, D(t), I(t)) = 0 \quad (14)$$

Using (7)-(10), we readily deduce (since Ψ and $\tilde{\varphi}$ do not depend on D , and φ and $\tilde{\varphi}$ do not depend on I) that the PMP maximization condition (13) leads to the following result.

Proposition 1. Let $T_f > 0$ be a fixed final time horizon. Then, for almost all $t \in [0, T_f]$, we get,

i) the optimal control $D(t)$ satisfies,

$$D(t) = \begin{cases} 0, & \text{if } \lambda_x > 1, \\ D_{\max}, & \text{if } \lambda_x < 1, \\ D_s(t), & \text{if } \lambda_x(t) = 1, \end{cases} \quad (15)$$

ii) the optimal control $I(t)$ satisfies,

$$I(t) = \begin{cases} I_{\max} & \text{if } \lambda_c > \frac{-\lambda_x x K}{k_d \tau \sigma I_x^{\max} (2 + \tau \sigma I_x^{\max})}, \\ I_s(t) & \text{otherwise} \end{cases} \quad (16)$$

where we denote $I_x^{\max} = I_{\max} e^{-\xi x}$, D_s is the so-called singular arc control (when its associated switching function φ is zero), and I_s is given by,

$$I_s(t) = \frac{-1 + \sqrt{1 - \frac{\lambda_x(t)x(t)K}{\lambda_c(t)k_d}}}{\tau \sigma e^{-\xi x(t)}}. \quad (17)$$

Proof. As (7) holds, finding the optimal light I in (13) is equivalent to solve the problem for almost every $t \in [0, T_f]$,

$$I(t) \in \underset{0 \leq I \leq I_{\max}}{\operatorname{argmax}} \Psi(X, \lambda, I), \quad (18)$$

where Ψ is defined in (8). For fixed $t \in [0, T_f]$, we define the Lagrangian associated to the problem (18) as,

$$\mathcal{L}(I, \theta) = \Psi(X, \lambda, I) + \theta(I_{\max} - I). \quad (19)$$

where if (I^*, θ^*) is a saddle point of \mathcal{L} with $\theta^* \geq 0$ then I^* is optimal for the problem (18). This condition is equivalent to the KKT conditions,

$$\frac{\partial \Psi(X, \lambda, I)}{\partial I} - \theta = 0, \quad (20)$$

$$\theta(I_{\max} - I) = 0, \quad (21)$$

$$\theta \geq 0. \quad (22)$$

Condition (20) gives,

$$\frac{\lambda_x x \mu}{1 + \tau \sigma I_x} + \frac{\lambda_c k_d \tau \sigma I_x \mu (2 + \tau \sigma I_x)}{K(1 + \tau \sigma I_x)} - \theta = 0.$$

As (22) holds, we deduce that,

$$\frac{\lambda_c k_d \tau \mu \sigma I_x (2 + \tau \sigma I_x)}{K(1 + \tau \sigma I_x)} \geq -\frac{\lambda_x x \mu}{1 + \tau \sigma I_x},$$

$$\lambda_c \geq -\frac{\lambda_x x K}{k_d \tau \sigma I_x (2 + \tau \sigma I_x)}, \quad (23)$$

where the equality holds if and only if $I < I_{\max}$, and the optimal light is given by,

$$(1 + \tau \sigma I_x)^2 - 1 = -\frac{\lambda_x x K}{\lambda_c k_d}.$$

As $(1 + \tau \sigma I_x)$ is always positive, it follows that,

$$\tau \sigma I_x = -1 + \sqrt{1 - \frac{\lambda_x x K}{\lambda_c k_d}}. \quad (24)$$

On the other hand, $I = I_{\max}$ if and only if the inequality in (23) is strict due to conditions (21) and (22).

For the dilution-control, we use the same argument, since D maximizes the function,

$$\varphi(X, \lambda) D = (-\lambda_x + 1) D, \quad (25)$$

and since it is linear in D , we deduce that $D = 0$ if $-\lambda_x + 1 < 0$ and $D = D_{\max}$ if $-\lambda_x - 1 > 0$. Next, when $-\lambda_x + 1 = 0$ (over a specific time interval $[t_1, t_2]$, where $t_2 > t_1$), we write $D(t) = D_s(t)$, $t \in [t_1, t_2]$, D_s is the singular control arc that may appear in the structure of the optimal control D . □

From *Proposition 1*, we can assert that the optimal control over $[0, T_f]$ for the dilution rate D can be a concatenation of arcs *Bang*(0), *Bang*(D_{\max}) and *Singular* (not necessarily in this order). On the other hand, the optimal control for light intensity I over $[0, T_f]$ can be a concatenation of arcs *Bang*(I_{\max}) and I_s .

5. PROPERTIES OF THE ADJOINT VARIABLES AND THE OPTIMAL CONTROLS

Proposition 2. The adjoint variables λ_x and λ_c are non-negative and non-positive, respectively.

Proof. Inequality (23) can be rewritten as

$$\lambda_x \geq -\lambda_c \frac{k_d \tau \sigma I_x (2 + \tau \sigma I_x)}{x K} \quad (26)$$

and replacing the above in the dynamics of λ_c given by (11) we get,

$$\begin{aligned} \dot{\lambda}_c &\geq -\frac{\lambda_c k_d \tau (\sigma I_x)^2 (2 + \tau \sigma I_x)}{1 + \tau \sigma I_x} + \lambda_c \left[\frac{k_d \tau (\sigma I_x)^2}{1 + \tau I_x} + k_r \right] \\ &\geq \lambda_c [k_r - k_d \tau (\sigma I_x)^2]. \end{aligned}$$

For simplicity, let us denote $\psi = k_r - k_d \tau (\sigma I_x)^2$ and let be $t_0, t, s \in [0, T_f]$ with $t_0 \leq s \leq t$, we have,

$$\begin{aligned} \dot{\lambda}_c(s) &\geq \lambda_c(s) \psi(s) \\ \frac{d}{ds} \left(\lambda_c e^{-\int_{t_0}^s \psi(w) dw} \right) &\geq 0 \\ \lambda_c(t) e^{-\int_{t_0}^t \psi(w) dw} - \lambda_c(t_0) &\geq 0. \end{aligned}$$

This is valid for every $t_0 \leq t \leq T_f$, replacing t by T_f and using the transversality condition (12) we get,

$$0 \geq \lambda_c(t_0) \quad \forall t \in [0, T_f].$$

Then λ_c is non-positive, and by (26) we conclude that λ_x is non-negative. □

Proposition 3. There exists an interval $[T_f - \varepsilon, T_f]$ where $D(t) = D_{\max}$. If $\lambda_x(T_f - \varepsilon) = 1$, and let $\bar{\mu}$ be an upper bound of μ satisfying $\bar{\mu} < D_{\max} + R$, then,

$$\varepsilon \geq \frac{1}{D_{\max} + R - \bar{\mu}} \ln \left(\frac{D_{\max}}{\bar{\mu} - R} \right). \quad (27)$$

Proof. The existence of the time interval $[T_f - \varepsilon, T_f]$ is straightforward since λ_x is continuous, non-negative and satisfies (12). Now, to prove the inequality (27), we substitute (23) in the λ_x -dynamics (11), then we get,

$$\dot{\lambda}_x \geq \lambda_x \left[\mu \left(\frac{\xi x}{1 + \tau \sigma I_x} - 1 \right) + (D + R) \right] - D - \frac{\lambda_x \xi x \mu}{1 + \tau \sigma I_x}.$$

Denoting $\phi = -\mu + D + R$, let be $t_0, t, s \in [0, T_f]$ with $t_0 \leq s \leq t$, multiplying (28) by $e^{-\int_{t_0}^s \phi(w) dw}$ and integrating,

$$\frac{d}{ds} \left(\lambda_x(s) e^{-\int_{t_0}^s \phi(w) dw} \right) \geq -D(s) e^{-\int_{t_0}^s \phi(w) dw},$$

$$\lambda_x(t) e^{-\int_{t_0}^t \phi(w) dw} - \lambda_x(t_0) \geq - \int_{t_0}^t D(s) e^{-\int_{t_0}^s \phi(w) dw} ds.$$

Then, we get (27) by evaluating t at T_f and t_0 at $T_f - \varepsilon$. \square

5.1 An assumption over the growth rate

Function μ is continuous and non-negative in $[0, T_f]$, then it is bounded. We consider the following assumption.

Hypothesis 4. There exists a strictly positive lower bound $\underline{\mu}$ of the growth rate μ that satisfies

$$\underline{\mu} > R. \quad (28)$$

This condition is common in photobioreactor optimization (Masci et al., 2010; Bernard and Lu, 2022).

Under this assumption, we can get the value of the light I and the dilution D , in the singular arc. In the following, let us denote $[t_1, t_2]$ the interval of the singular arc.

Proposition 5. Under hypothesis 4, $I = I_{\max}$ in the singular arc

Proof. As $\lambda_x = 1$ in $[t_1, t_2]$, $\dot{\lambda}_x = 0$ in $]t_1, t_2[$, then in this interval, from (11) we get,

$$0 = \mu \left(\frac{\xi x}{1 + \tau \sigma I_x} - 1 \right) + R + \lambda_c \frac{k_d \xi K}{\mu} \left[(1 + \tau \sigma I_x)^2 - 1 \right],$$

developing this equation leads to,

$$\lambda_c \left[(1 + \tau \sigma I_x)^2 - 1 \right] = -\frac{Kx}{k_d} + \frac{K(1 + \tau \sigma I_x)}{k_d a} \left[1 - \frac{R}{\mu} \right].$$

As $I \leq I_{\max}$, using (28) we deduce that,

$$\begin{aligned} \lambda_c \left[(1 + \tau \sigma I_{\max} e^{-\xi x})^2 - 1 \right] &\geq \lambda_c \left[(1 + \tau \sigma I_x)^2 - 1 \right] \\ &> \frac{-Kx}{k_d} \end{aligned}$$

Again, using $\lambda_x = 1$, condition (16) is fulfilled. \square

Proposition 6. Under Hypothesis 4 the dilution in the singular arc satisfies,

$$D_s = \mu - R. \quad (29)$$

Proof. From proposition 5 the light in $]t_1, t_2[$ remains constant and equal to I_{\max} . Then, over $]t_1, t_2[$, we get,

$$H = (\mu - R)x + [(-\gamma + k_r)c + \gamma]\lambda_c, \quad (30)$$

where,

$$\gamma = \frac{k_d \tau (\sigma I_x^{\max})^2}{1 + \tau \sigma I_x^{\max}}. \quad (31)$$

As H is constant in time, taking the derivative in (30) we get:

$$0 = \dot{\mu}x + (\mu - R)\dot{x} + \dot{\lambda}_c [-(\gamma + k_r)c + \gamma] + \lambda_c [-\dot{\gamma}c + (\gamma + k_r)\dot{c} + \dot{\gamma}] \quad (32)$$

the derivative of μ in $]t_1, t_2[$ is,

$$\dot{\mu} = -\frac{\xi \mu \dot{x}}{1 + \tau \sigma I_x^{\max}} - \frac{\mu k_d \tau (\sigma I_x^{\max})^2}{1 + \tau \sigma I_x^{\max}} + \frac{K \sigma I_x^{\max} k_r c}{1 + \tau \sigma I_x^{\max}} \quad (33)$$

and the derivative of γ is given by,

$$\dot{\gamma} = \frac{k_d \tau \xi (\sigma I_x^{\max})^2 (2 + \tau \sigma I_x^{\max})}{(1 + \tau \sigma I_x^{\max})^2} \dot{x} \quad (34)$$

By replacing (33)-(34) in (32) we get,

$$(\mu - R)(\mu - R - D)x = 0,$$

then, as $\underline{\mu} > R$, we have that $\mu - R \neq 0$. Also, unless the initial condition of x is zero, $x(t) \neq 0 \forall t \in [0, T_f]$. Then, we conclude that $D = \mu - R$. \square

The next result is immediately deduced:

Corollary 7. Let us assume that a singular arc occurs in an interval $[t_1, t_2]$ with $t_1 < t_2$, under Hypothesis 4 the biomass concentration is constant in $[t_1, t_2]$.

6. ESTIMATING BIOMASS IN THE SINGULAR ARC

We saw that biomass in the photobioreactor x and the light intensity are constant in the singular arc, but this value must be determined. We assume that the singular arc $[t_1, t_2]$ is long enough for the state c to reach its equilibrium, this is because the equation (2) becomes *autonomous* in $[t_1, t_2]$ and c converges to the unique asymptotically stable steady state given by:

$$c_s = \frac{k_d \tau (\sigma I_x^{\max})^2}{k_r (1 + \tau \sigma I_x^{\max}) + k_d \tau (\sigma I_x^{\max})^2}.$$

Then the growth rate μ is constant and equal to

$$\mu_s = \frac{K \sigma I_x^{\max}}{1 + \tau \sigma I_x^{\max} + \frac{k_d}{k_r} (\sigma I_x^{\max})^2}. \quad (35)$$

This is the Haldane description corresponding to the growth rate of the Han model at steady state. This function reaches its maximum at the value

$$I_x^{\max} = I_{\text{opt}} \triangleq \frac{1}{\sigma \sqrt{\frac{k_d}{k_r} \tau}} \quad (36)$$

In this static condition, λ_c is constant, and we can find its value from (11),

$$\lambda_c = -\frac{\mu_s x}{k_r}.$$

Replacing this value in the first equation, we get,

$$\frac{R}{\mu_s} = 1 - \xi x \frac{k_r - k_d \tau (\sigma I_x^{\max})^2}{k_r (1 + \tau \sigma I_x^{\max}) + k_d \tau (\sigma I_x^{\max})^2}. \quad (37)$$

We denote x_s the biomass concentration satisfying (37). It is important to note that this value does not depend on either D_{\max} or the initial conditions of X .

7. DIRECT OPTIMIZATION

We solve numerically the problem (OCP), with a direct method implemented in the software BOCOP (Bonnans

Table 1. Parameters values for the biological model.

Parameter	Value			Unit
	Ex. 1	Ex. 2	Ex. 3	
T_f	$2 \cdot 10^4$	$3 \cdot 10^4$	$1.5 \cdot 10^4$	s
$x(0)$	20	10		$mg \cdot L^{-1}$
$c(0)$	0.5	0.15		-
K	$8.70 \cdot 10^{-6}$			-
k_d	$2.99 \cdot 10^{-4}$			-
k_r	$6.80 \cdot 10^{-3}$			s^{-1}
τ	$0.25 \cdot 10^0$			s
σ	$4.70 \cdot 10^{-2}$			$m^2 \cdot \mu mol^{-1}$
D_{\max}	$4.00 \cdot 10^{-4}$			s^{-1}
I_{\max}	$1.00 \cdot 10^3$			$\mu mol \cdot m^{-2} s^{-1}$
R	$8.00 \cdot 10^{-6}$			s^{-1}
ξ	0.20			$mg^{-1} \cdot L$

et al., 2017). The problem is discretized by a two stage Gauss-Legendre method with 15000 time steps, and the tolerance for the IPOPT NLP solver is set at 10^{-25} . We consider the parameters of the Han model from (Grenier et al., 2020). We consider three different examples with the listed parameter values in Ex. 1-3.

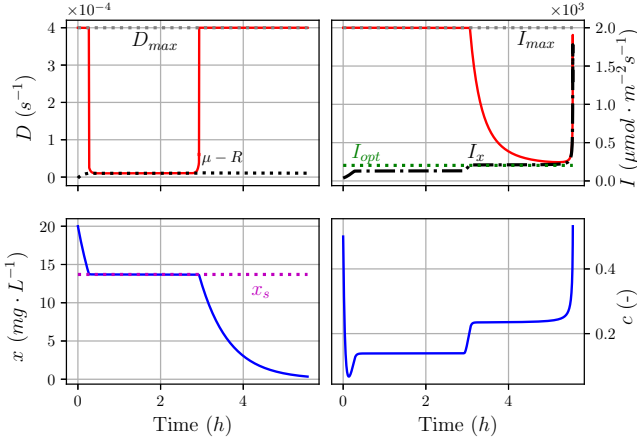


Fig. 1. **Example 1.** The optimal controls (in red), and the states (in blue). In the dilution plot, we also represent $\mu - R$ (black dotted line) to illustrate the Proposition 6. In the light case, we also represent I_{opt} defined in (36), and the value of the perceived light I_x (the black dotted line). The magenta dashed line in the biomass concentration plot corresponds to x_s defined in Section 6.

Figure 1, 2 and 3 show the simulation results. In the first example, the control structure of the dilution rate is *Bang-Singular-Bang* and for the light it is *Bang-Singular*. The initial biomass is higher than the concentration x_s predicted in the section 6, then the dilution is settled at maximum until the biomass reaches x_s , subsequently, the singular arc of dilution is activated, and the biomass x_s is maintained during this period. Finally, the dilution is settled at maximum, as was demonstrated in Proposition 3. On the other hand, in Figure 2, the behavior of the controls are similar, but, the initial biomass is lower than x_s , then the dilution rate is zero to allow the system to reach x_s .

An important remark is that the curve $\mu - R$ takes negative values (see Figure 1), this is because the initial biomass

is too high leading to a lack of light in the culture and leading to growth rate close to zero. The Hypothesis 4, then, is not fulfilled in the entire interval. But, it is in the singular arc of dilution $[t_1, t_2]$. Actually, the results proved in Section 5 are still valid if we relax the inequality (28) for the condition $\mu(t) \geq R \forall t$ in the singular arc of dilution.

Observing the average light perceived by the algae culture I_x together with the light intensity I , it appears, coherently with Section 6, I_x is lower than the optimal light intensity I_{opt} , and in the singular arc of light, the value of I_x is equal to the value of I_{opt} .

In Figure 3 we present the results for Example 3 which has the same parameters as in the Example 2, but considering a shorter final time $T_f = 1.5 \cdot 10^4$. In this case, the structure of the dilution rate is *Bang-Bang* and the structure of the light is the same as in the Example 2. Then existence of the singular arc for dilution is not always ensured. In this case, there is not enough time for the biomass to reach the value estimated in section 6 and let a time window for harvesting the biomass at the end. In proposition 3 we discuss the length of this time window.

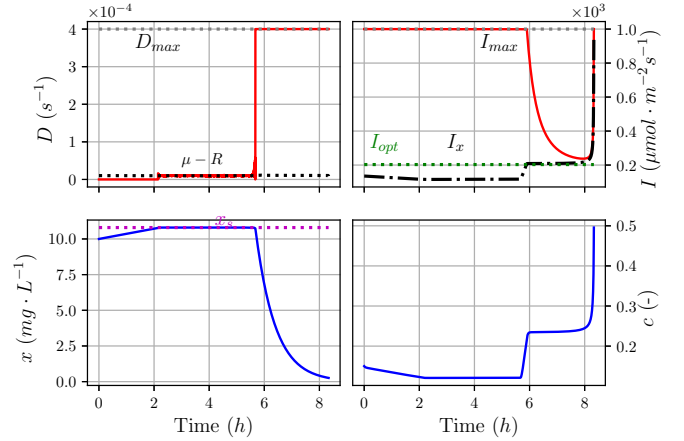


Fig. 2. **Example 2.** The optimal controls, trajectories and co-states, given by BOCOP. For legend and colors see Figure 1.

8. CONCLUSION

The singular arc for the dilution rate is key in the optimal problem (OCP). We prove that in this arc the biomass remains constant, at a concentration independent of the value of the maximum dilution rate and the final time for harvesting depends on the maximal dilution rate. The solution of this equation cannot be given explicitly. It is also interesting to note that the associated light in the singular arc of dilution is lower than the optimal light for the Han model at steady state. On the other hand, numerical results show that, when $I = I_s$, the optimal irradiance corresponds to the optimum of the Han model at steady state.

Larger values of the maximum light I_{max} allow operating the photobioreactor at higher biomasses. This fact is in agreement with experimental observations, for high density cultures (Gerla et al., 2011), where photoinhibition is negligible due to the self shading of the cells.

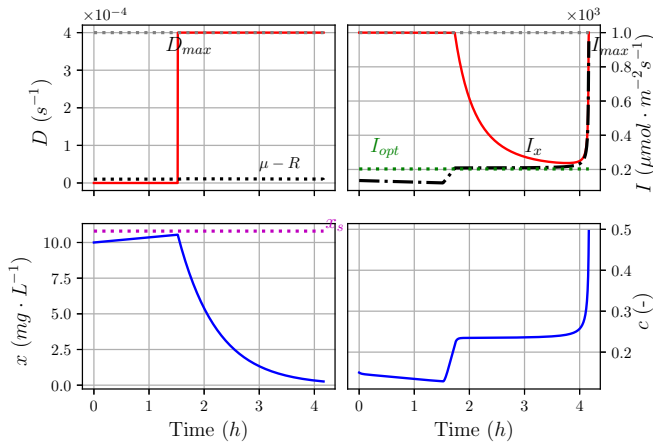


Fig. 3. **Example 3.** The optimal controls. The parameters are the same as in the Example 2, with a shorter final time T_f .

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