

Appendix A. Position and mark data terms

Appendix A.1. Position term

If we consider a simple energy model with no interactions such that:

$$U(Y|X) = \sum_{y \in Y} V_{\text{pos}}(y_i, y_j|X) \quad (\text{A.1})$$

For any configuration of points Y we have:

$$\begin{aligned} p(y|Y, X) &= \frac{h(\{y\} \cup Y|X)}{h(Y|X)} \\ &= \frac{\exp\left(-\sum_{y' \in \{y\} \cup Y} V_{\text{pos}}(y'_i, y'_j|X)\right)}{\exp\left(-\sum_{y' \in Y} V_{\text{pos}}(y'_i, y'_j|X)\right)} \\ &= \exp(-V_{\text{pos}}(y_i, y_j|X)) \end{aligned} \quad (\text{A.2})$$

Here $p(y|Y, X)$ gives the probability of y being a point given points in the configuration Y . Without interactions between points in the model, we have $p(y|Y, X) = p(y|X)$. From $p(y_i, y_j|X) = \sigma(A_{\text{pos}}(X)[y_i, y_j])$, with σ the sigmoid function, we deduce:

$$V_{\text{pos}}(y_i, y_j|X) = \ln(1 + \exp(-A_{\text{pos}}(X)[y_i, y_j])) \quad (\text{A.3})$$

Appendix A.2. Mark terms

If we consider a simple energy model with no interactions such that:

$$U(Y|X) = \sum_{y \in Y} V_{\text{pos}}(y_i, y_j|X) + V_m(y_m|y_i, y_j, X) \quad (\text{A.4})$$

For any configuration of points Y we have:

$$\begin{aligned} p(y|Y, X) &= \frac{p(\{y\} \cup Y|X)}{p(Y|X)} \\ &= \frac{\exp\left(-\sum_{y' \in \{y\} \cup Y} V_{\text{pos}}(y'_i, y'_j|X) + V_m(y_m|y_i, y_j, X)\right)}{\exp\left(-\sum_{y' \in Y} V_{\text{pos}}(y'_i, y'_j|X) + V_m(y_m|y_i, y_j, X)\right)} \\ &= \exp(-V_{\text{pos}}(y_i, y_j|X) - V_m(y_m|y_i, y_j, X)) \end{aligned} \quad (\text{A.5})$$

Here $p(y|Y, X)$ gives the probability of y being a point, given points in the configuration Y . Without interactions between points in the model, we have $p(y|Y, X) = p(y|X)$.

$$\begin{aligned} p(y|X) &= \frac{1}{Z} \exp(-V_{\text{pos}}(y_i, y_j|X) - V_m(y_m|y_i, y_j, X)) \\ p(y_i, y_j|X)p(y_m|y_i, y_j, X) &= \frac{1}{Z} \exp(-V_{\text{pos}}(y_i, y_j|X)) \exp(-V_m(y_m|y_i, y_j, X)) \end{aligned} \quad (\text{A.6})$$

From the position energy definition, we have $p(y_i, y_j|X) \propto \exp(-V_{\text{pos}}(y_i, y_j|X))$, thus:

$$\begin{aligned} p(y_m|y_i, y_j, X) &= \frac{1}{Z'} \exp(-V_m(y_m|y_i, y_j, X)) \\ \frac{\exp(A_m(X, y_i, y_j)[y_m])}{\sum_{y'_m} \exp(A_m(X, y'_i, y'_j)[y'_m])} &= \frac{1}{Z'} \exp(-V_m(y_m|y_i, y_j, X)) \end{aligned} \quad (\text{A.7})$$

Appendix B. Parallelization: Cell size and computation

Appendix B.1. Determining cell size

For two points y, y' , we denote $y \underset{i}{\sim} y'$ when the two points are interacting, *i.e.* $V(y|\dots)$ depends on y' . This is a generalization of the distance based interaction where $y \underset{i}{\sim} y' \iff d(y, y') \leq d_{\text{max}}$. Then $\mathcal{N}_y^Y = \{y \in Y | y' \neq y, y' \underset{i}{\sim} y\}$.

Definition. [1] A point process of density h is Markovian under the neighborhood relationship $\underset{m}{\sim}$ if and only if $\forall Y \in \mathcal{Y}$ such that $h(Y) > 0$:

1. $\forall \tilde{Y} \subset Y, h(\tilde{Y}) > 0$
2. $\forall u \in S \times M, h(Y \cup \{u\})/h(Y)$ only depends on u and its neighbors $\{y \in Y, y \underset{m}{\sim} u\}$

Energy model. We consider a generic form for the energy of Y :

$$U(Y) = \sum_{y \in Y} V(y|\mathcal{N}_y^Y) \quad (\text{B.1})$$

where V is an arbitrary function that maps y and its neighborhood into \mathbb{R} .

Markovian relationship. We can show that the PP derived from U , is Markovian for any relation $\underset{m}{\sim}$ such that

$$\forall u \in S \times M, \{y \in S \times M, y \underset{m}{\sim} u\} \supseteq \{y \in S \times M, \exists \tilde{y} \in S \times M, y \underset{i}{\sim} \tilde{y} \underset{i}{\sim} u\} \quad (\text{B.2})$$

Then the relation $\underset{i}{\sim}^2$ (second degree neighbors for $\underset{i}{\sim}$ defined above), satisfies the Markovianity condition for this PP.

Perturbations in a Markovian PP. We try to find the minimum distance between two perturbations $Y \rightarrow Y'$ and $Y \rightarrow Y''$, so that acceptance rates (or Green ratio) for both moves are independent of each other.

We first will consider two births $Y' = Y \cup \{y_1\}$ and $Y'' = Y \cup \{y_2\}$. As acceptance ratio $r(Y \rightarrow Y')$ is function of $\exp\left(-\frac{U(Y')-U(Y)}{T}\right) = \frac{h(Y')}{h(Y)}$, it only depends on y_1 and its neighborhood according to relation $\underset{i}{\sim}^2$. Thus, for $d(y_1, y_2) > 2d_{\max}$ we have independence of $r(Y \rightarrow Y')$ and $r(Y \rightarrow Y'')$.

The demonstration is similar for a death move. A transform move ($Y' = Y \setminus \{y_1\} \cup \{y'_1\}$, with $y_1 \in Y$, similarly for Y'') can be split into successive death and birth moves; then the acceptance ratio is function of the neighborhood of y_1 and y'_1 for relation $\underset{i}{\sim}^2$. Under the condition $d(y_1, y'_1) \leq \delta_{\max}$, we need $d(y_1, y_2) > 2d_{\max} + 2\delta_{\max}$ to maintain independence.

Proof of Equation B.2. We look for the minimal subset of Y needed to compute $h(Y \cup \{u\})/h(Y) = \exp(U(Y) - U(Y \cup \{u\}))$. Unraveling, the energy difference $\Delta U = U(Y \cup \{u\}) - U(Y)$, we get:

$$\Delta U = V(u|\mathcal{N}_u^{Y \cup \{u\}}) + \sum_{\substack{y \in Y \\ y \in \mathcal{N}_u^Y}} \left(V(y|\mathcal{N}_y^{Y \cup \{u\}}) - V(y|\mathcal{N}_y^Y) \right) \quad (\text{B.3})$$

Thus, $h(Y \cup \{u\})/h(Y)$ is function of u , $\{y \in Y | y \in \mathcal{N}_u^Y\}$ and $\{y' \in Y | \exists y \in \mathcal{N}_u^Y, y' \in \mathcal{N}_y^Y\}$. Those are all encompassed in the set $\{y \in S \times M | \exists \tilde{y} \in S \times M, y \underset{i}{\sim} \tilde{y} \underset{i}{\sim} u\}$.

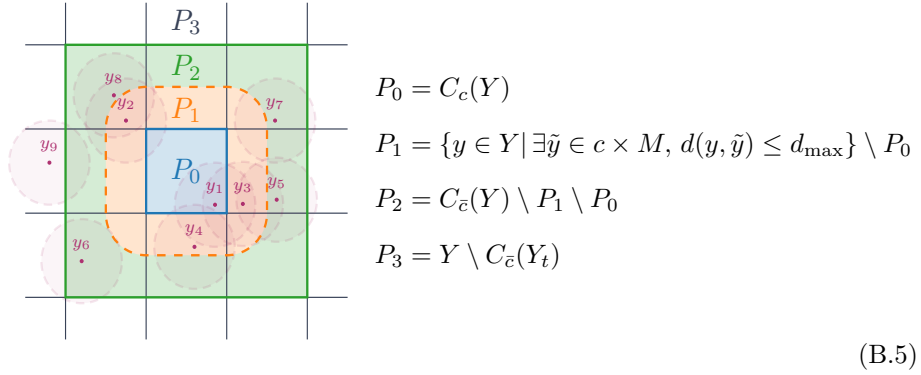
Appendix B.2. Computational shortcuts

In sampling Algorithm 2 we need to compute $\Delta U(Y \rightarrow Y')$ for a proposed move from Y to Y' in cell c . This computation is performed on multiple cells

at once. To reduce computational cost we show the equality bellow, where \bar{c} is the subset of S composed of c and its 8 neighbors.

$$\Delta U(Y \rightarrow Y') = \Delta U(C_{\bar{c}}(Y) \rightarrow C_{\bar{c}}(Y')) \quad (\text{B.4})$$

Proof. The proof relies on the partition of Y into subsets P_0, \dots, P_3 , and Y' into P'_0, \dots, P'_3



Considering a birth or death kernel in c (P_0), we have $P_1 = P'_1$, $P_2 = P'_2$, $P_3 = P'_3$.

By construction, with $|c| = 2d_{\max} + 2\delta_{\max}$, elements in P_0 can interact with points in P_0 and P_1 . Similarly, points in P_1 only interact with points in P_0, P_1, P_2 and so on for P_2 and P_3 .

We can split $\Delta U(Y \rightarrow Y') = \sum_{y \in Y} V(y|\mathcal{N}_y^Y) - \sum_{y \in Y'} V(y|\mathcal{N}_y^{Y'})$ into energy differences in each partition $\sum_{y \in P_k} V(y|\mathcal{N}_y^Y) - \sum_{y \in P'_k} V(y|\mathcal{N}_y^{Y'})$, $k = 0, \dots, 3$. The differences cancel out on P_3, P'_3 (since $P_3 = P'_3$ and $P_2 = P'_2$), and on P_2, P'_2 (since $P_3 = P'_3$, $P_2 = P'_2$ and $P_1 = P'_1$). Energy difference ΔU is then a function of points in P_0, P'_0, P_1, P'_1 with their neighborhood in $P_0, P'_0, \dots, P_2, P'_2$

The difference $\Delta U(C_{\bar{c}}(Y) \rightarrow C_{\bar{c}}(Y'))$, corresponds to the above $\Delta U(Y \rightarrow Y')$, setting $P_3 = P'_3 = \emptyset$. As shown above such a change does not alter the result, as no dependency on P_3, P'_3 remains.

For a translation kernel such that $Y' = Y \setminus \{y\} \cup \{y'\}$, with $y \in C_c(Y)$ and

$d(y, y') < \delta_{\max}$, we redefine

$$P_0 = \{y \in Y \mid \exists \tilde{y} \in c \times M, d(y, \tilde{y}) \leq \delta_{\max}\} \quad (\text{B.6})$$

$$P_1 = \{y \in Y \mid \exists \tilde{y} \in c \times M, d(y, \tilde{y}) \leq d_{\max} + \delta_{\max}\} \setminus P_0 \quad (\text{B.7})$$

and similarly for P'_0, P'_1 . Then we have $y' \in P'_0$, this ensures $P_1 = P'_1, P_2 = P'_2, P_3 = P'_3$. The rest of the proof for birth and death kernels can then apply.

References

- [1] F. Lafarge, G. Gimel'farb, X. Descombes, Geometric feature extraction by a multimarked point process, *IEEE Transactions on Pattern Analysis and Machine Intelligence* 32 (9) (2010) 1597–1609. doi:10.1109/TPAMI.2009.152.