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Dimbihery Rabenoro, Xavier Pennec. The geometry of Riemannian submersions from compact Lie groups. Application to flag manifolds. 2023. hal-04232479

**HAL Id: hal-04232479**

**<https://inria.hal.science/hal-04232479>**

Preprint submitted on 8 Oct 2023

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# The geometry of Riemannian submersions from compact Lie groups. Application to flag manifolds

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March 1, 2023

## Abstract

We study the geometry of Riemannian manifolds of finite dimension which are isometric to a quotient of a compact Lie group endowed with the quotient of a bi-invariant metric. Then, we apply the results obtained to enlight the geometry of flag manifolds.

## 1 Introduction

In this paper, we study the geometry of Riemannian manifolds of finite dimension which are isometric to a quotient of a compact Lie group endowed with the quotient of a bi-invariant metric [5], [7]. Such a manifold is thus a Riemannian submersion from the compact Lie group, over which the latter is a principal bundle [6]. The initial motivation for our work was to enlight the geometry of flag manifolds.

Flags are properly nested sequences of linear subspaces [1], [9]. They can equivalently be described as sequences of mutually orthogonal incremental subspaces whose partial sums generates the nested sequence. Flag manifolds have recently been a rising subject of interest in statistical machine learning for generalizing Grassmannians and handling multiple subspaces at the same time, as in [11]. It was also recently shown that Principal Component Analysis (PCA) can be rephrased as an optimization on the Flag. One of the novelty of [14] is to realize that the nested sequence of linear subspaces actually optimizes a criterion on the flag manifold: the accumulated unexplained variance (AUV). On this basis, new robust variants like the minimal accumulated unexplained p-variance have been proposed, even for manifold-valued data [13].

The ubiquitous appearance of flags in other domains such as numerical analysis exemplified in [15] led these authors to develop the mathematical tools for Riemannian optimization algorithms on the flag manifold. This landmark paper describes flag spaces in multiple ways with different geometric structures. However, it is not always easy to disentangle what is due to each structure. In this paper, we clarify the structure of flag manifolds as Riemannian submersions from the orthogonal group. In that purpose, we develop the general setting described at the beginning of this introduction and prove that flag manifolds fall within this framework.

For an arbitrary compact Lie group  $\mathbb{G}$  endowed with a bi-invariant metric  $g^{\mathbb{G}}$ , the key point is the decomposition of smooth vector fields as linear combination of left-invariant vector fields with coefficients which are smooth functions on  $\mathbb{G}$ . Now, the manifold  $(\mathbb{B}, g^{\mathbb{B}})$  which we study is a Riemannian submersion from  $(\mathbb{G}, g^{\mathbb{G}})$ . Applying O’Neill’s formula [12], we derive notably the Levi-Civita connection and the curvature tensors of  $(\mathbb{B}, g^{\mathbb{B}})$ . In particular, we obtain closed-forms of these Riemannian objects in the case of flag manifolds.

In Section 2, we describe the general framework and the basic geometry of  $(\mathbb{B}, g^{\mathbb{B}})$ . Then, in Section 3, we prove that flag manifolds can be placed inside this setting. Finally, in Section 4, we establish the formulas for the Levi-Civita connection and the curvature tensors of  $(\mathbb{B}, g^{\mathbb{B}})$ .

## 2 The general framework

**Theorem 1.** *Let  $\mathbb{B}$  be a subset of a manifold  $M$ . Let  $\mathbb{G}$  be a compact lie group, endowed with a bi-invariant metric  $g^{\mathbb{G}}$ , which acts smoothly from the left on  $M$ . For  $p \in M$ , let  $\mathbb{K}(p)$  be its isotropy group. Assume that*

- (i)  $\mathbb{B}$  is the orbit of some  $p_0 \in M$ .
- (ii) The action of  $\mathbb{K}(p_0)$  on  $\mathbb{G}$  by right-multiplication is isometric.

- Let  $\psi : \mathbb{G} \rightarrow \mathbb{G}/\mathbb{K}(p_0)$  be the canonical map and  $\pi$  the map defined by  $\pi : \mathbb{G} \rightarrow \mathbb{B}$  and  $\pi(Q) = Q \cdot p_0$ . Then,
- (1)  $\mathbb{B}$  is an embedded submanifold of  $M$  and  $\pi$  induces a diffeomorphism  $\bar{\pi} : \mathbb{G}/\mathbb{K}(p_0) \simeq \mathbb{B}$  such that  $\bar{\pi} \circ \psi = \pi$ .
  - (2) The quotient metric  $\bar{g}^{\mathbb{G}}$  on  $\mathbb{G}/\mathbb{K}(p_0)$  is well-defined and  $\pi$  is a Riemannian submersion from  $(\mathbb{G}, g^{\mathbb{G}})$  onto  $(\mathbb{B}, g^{\mathbb{B}})$ , where  $g^{\mathbb{B}} := \bar{\pi}_* \bar{g}^{\mathbb{G}}$ .
  - (3)  $(\mathbb{G}, \pi, \mathbb{B}, \mathbb{K}(p_0))$  is a principal fiber bundle, i.e. with typical fiber the Lie group  $\mathbb{K}(p_0)$ .

This is represented by the following diagram.

$$\begin{array}{ccc} \mathbb{G} & & \\ \psi \downarrow & \searrow \pi & \\ \mathbb{G}/\mathbb{K}(p_0) & \xrightarrow{\sim \bar{\pi}} & \mathbb{B} \hookrightarrow M \end{array}$$

*Proof.* (1) is standard: See Proposition A.2. in [3]. For (2), by section 29.21 in [8], (ii) implies that  $\bar{g}^{\mathbb{G}}$  is well-defined and that  $\psi$  is a Riemannian submersion from  $(\mathbb{G}, g^{\mathbb{G}})$  onto  $(\mathbb{G}/\mathbb{K}(p_0), \bar{g}^{\mathbb{G}})$ . Finally, (3) is a direct consequence of Lemma 18.3 in [8]. Indeed,  $\mathbb{K}(p_0)$  acts freely on  $\mathbb{G}$  by right-multiplication and the orbits of this action are the fibers of  $\pi$ .  $\square$

**Remark 1.** It is not necessary in the proof of Theorem 1 to assume that  $g^{\mathbb{G}}$  is bi-invariant. However we use this condition later, so we suppose it from the beginning.

## 2.1 Vertical and horizontal parts

Denote by  $e$  the neutral element of  $\mathbb{G}$  and, for  $\tilde{Q} \in \mathbb{G}$ , by  $\mathcal{L}_{\tilde{Q}}^{\mathbb{G}}$  the left-translation by  $\tilde{Q}$  in  $\mathbb{G}$ : for all  $Q \in \mathbb{G}$ ,  $\mathcal{L}_{\tilde{Q}}^{\mathbb{G}}(Q) = \tilde{Q}Q$ . Let  $\mathfrak{g}$  be its Lie algebra. Set  $N := \dim(\mathfrak{g}) = \dim(\mathbb{G})$ . For all  $\Omega, \Omega' \in \mathfrak{g}$ , set

$$\llbracket \Omega, \Omega' \rrbracket := g_e(\Omega, \Omega').$$

When  $\mathbb{G}$  is a matrix Lie group, i.e.  $\mathbb{G} \subset GL_n(\mathbb{R})$  for  $n \geq 1$ ,  $\mathfrak{g} = M_n(\mathbb{R})$  and

$$\llbracket \Omega, \Omega' \rrbracket := \Omega\Omega' - \Omega'\Omega.$$

For all  $Q \in \mathbb{G}$ , the vertical and horizontal subspaces in  $T_Q\mathbb{G}$  are respectively  $V_Q^\pi := \ker(d\pi)_Q$  and  $H_Q^\pi := (V_Q^\pi)^\perp$ . We also introduce the linear projections  $\text{ver}_Q^\pi : T_Q\mathbb{G} \rightarrow V_Q^\pi$  and  $\text{hor}_Q^\pi : T_Q\mathbb{G} \rightarrow H_Q^\pi$ . The next lemma states that the vertical and horizontal parts are invariant wrt left-translations in  $\mathbb{G}$ .

**Lemma 1.** For  $Q, \tilde{Q} \in \mathbb{G}$  and  $\Delta \in T_Q\mathbb{G}$ , set  $\tilde{Q}_*(\Delta) := (d\mathcal{L}_{\tilde{Q}}^{\mathbb{G}})_Q(\Delta)$ . Then, for all  $\Omega \in \mathfrak{g}$ ,

$$\text{ver}_{\tilde{Q}Q}^\pi \left( \left( \tilde{Q}Q \right)_* \Omega \right) = \tilde{Q}_* \text{ver}_Q^\pi(Q_*\Omega) \quad (2.1)$$

and

$$\text{hor}_{\tilde{Q}Q}^\pi \left( \left( \tilde{Q}Q \right)_* \Omega \right) = \tilde{Q}_* \text{hor}_Q^\pi(Q_*\Omega). \quad (2.2)$$

*Proof.*  $\mathbb{G}$  acts from the left on  $\mathbb{B} \subset M$ , so that for all  $Q, \tilde{Q} \in \mathbb{G}$ ,  $\tilde{Q} \cdot (Q \cdot p_0) = (\tilde{Q}Q) \cdot p_0$ , i.e.  $\tilde{Q} \cdot \pi(Q) = \pi(\tilde{Q}Q)$ . This means that  $\mathcal{L}_{\tilde{Q}}^{\mathbb{B}} \circ \pi = \pi \circ \mathcal{L}_{\tilde{Q}}^{\mathbb{G}}$ , where, for any  $p \in \mathbb{B}$ ,  $\mathcal{L}_{\tilde{Q}}^{\mathbb{B}}(p) := \tilde{Q} \cdot p$ . This implies that for any  $Q_*\Omega \in T_Q\mathbb{G}$ ,

$$\left( d\mathcal{L}_{\tilde{Q}}^{\mathbb{B}} \right)_{\pi(Q)} \left( (d\pi)_Q(Q_*\Omega) \right) = (d\pi)_{\mathcal{L}_{\tilde{Q}}^{\mathbb{G}}(Q)} \left( \left( d\mathcal{L}_{\tilde{Q}}^{\mathbb{G}} \right)_Q(Q_*\Omega) \right) = (d\pi)_{\tilde{Q}Q} \left( \tilde{Q}_*(Q_*\Omega) \right).$$

Since  $\left( d\mathcal{L}_{\tilde{Q}}^{\mathbb{B}} \right)_{\pi(Q)}$  is an isomorphism, we deduce that:  $Q_*\Omega \in V_Q^\pi \iff \tilde{Q}_*(Q_*\Omega) \in V_{\tilde{Q}Q}^\pi$ . Since  $g^{\mathbb{G}}$  is a left-invariant metric, this latter equivalence implies the following one:  $Q_*\Omega \in H_Q^\pi \iff \tilde{Q}_*(Q_*\Omega) \in H_{\tilde{Q}Q}^\pi$ . Now,

$$Q_*\Omega = \text{ver}_Q^\pi(Q_*\Omega) + \text{hor}_Q^\pi(Q_*\Omega) \implies \left( \tilde{Q}Q \right)_* \Omega = \left( \tilde{Q} \right)_* \text{ver}_Q^\pi(Q_*\Omega) + \left( \tilde{Q} \right)_* \text{hor}_Q^\pi(Q_*\Omega). \quad (2.3)$$

By the preceding equivalences, (2.3) yields the decomposition of  $\left( \tilde{Q}Q \right)_* \Omega$  into its vertical and horizontal parts in  $T_{\tilde{Q}Q}\mathbb{G}$ , which concludes the proof.  $\square$

## 2.2 Basic geometry of the base

Recall that for  $p \in \mathbb{B}$  and  $Q \in \pi^{-1}(p)$ , the restriction of  $(d\pi)_Q$  to  $H_Q^\pi$  is an isomorphism between  $H_Q^\pi$  and  $T_p\mathbb{B}$ . For  $\Delta \in T_p\mathbb{B}$ , the horizontal lift wrt  $\pi$  of  $\Delta$  at  $Q$  is denoted by  $\Delta_Q^\sharp$ . It is the unique vector in  $H_Q^\pi$  whose image by  $(d\pi)_Q$  is  $\Delta$ . Then,  $g^\mathbb{B}$  is defined by

$$g_p^\mathbb{B}(\Delta, \tilde{\Delta}) = g_Q^\mathbb{G}(\Delta_Q^\sharp, \tilde{\Delta}_Q^\sharp). \quad (2.4)$$

**Proposition 1.**  $(\mathbb{B}, g^\mathbb{B})$  is a complete Riemannian manifold and the geodesic in  $\mathbb{B}$  through  $p$  in the direction  $\Delta$  is of the form

$$t \mapsto \pi \left( \mathcal{L}_Q^\mathbb{G} \left( \exp_{\mathfrak{g}} \left( t (Q^{-1})_* \Delta_Q^\sharp \right) \right) \right),$$

where  $Q \in \pi^{-1}(p)$  and  $\exp_{\mathfrak{g}} : \mathfrak{g} \rightarrow \mathbb{G}$  is the Lie group exponential map of  $\mathbb{G}$ .

*Proof.* Since  $\mathbb{G}$  is compact,  $\mathbb{B}$  is complete and by the Hopf-Rinow theorem,  $(\mathbb{B}, g^\mathbb{B})$  is complete. Since  $g^\mathbb{G}$  is bi-invariant, the geodesics of  $(\mathbb{G}, g^\mathbb{G})$  are left-translates of one-parameter subgroups of  $\mathbb{G}$ . Now, by Lemma 1,  $(Q^{-1})_* \Delta_Q^\sharp \in H_e^\pi$ . So, the curve  $t \mapsto \mathcal{L}_Q^\mathbb{G} \left( \exp_{\mathfrak{g}} \left( t (Q^{-1})_* \Delta_Q^\sharp \right) \right)$  is the horizontal geodesic in  $\mathbb{G}$  through  $Q$ , in the direction  $\Delta_Q^\sharp$ . Thus, its image by  $\pi$  is indeed the geodesic in  $\mathbb{B}$  through  $p$  in the direction  $\Delta$ .  $\square$

## 3 A new representation of flag manifolds

### 3.1 Some usual representations of flag manifolds

A flag of vector spaces in  $\mathbb{R}^n$  is a filtration  $\mathcal{V}$  of subspaces  $V_0 = \{0\} \subset V_1, \dots \subset V_r = \mathbb{R}^n$ . For all  $1 \leq i \leq r$ , let  $W_i$  be the orthogonal complement of  $V_{i-1}$  in  $V_i$ . Then,  $(W_i)_{1 \leq i \leq r}$  is a sequence of mutually orthogonal linear subspaces of  $\mathbb{R}^n$ , called the incremental subspaces representation of the flag  $\mathcal{V}$ . The sequence  $\mathbf{I} = (q_i)_{1 \leq i \leq r}$ , where  $q_i := \dim(W^i)$ , is called the type of  $\mathcal{V}$ . The set of all flags of type  $\mathbf{I}$  is denoted by  $\Phi^\mathbf{I}$ .

On one hand,  $\Phi^\mathbf{I}$  is represented as follows: see [15]. For  $1 \leq q \leq n$ , we denote by  $G_n^q$  the Grassmannian of all linear subspaces of dimension  $q$  of  $\mathbb{R}^n$  and we identify  $G_n^q$  with orthogonal projectors. Then, we define the product  $G^\mathbf{I}$  of Grassmannians associated to the type  $\mathbf{I}$  by  $G^\mathbf{I} := \prod_{i=1}^r G_n^{q_i}$ . In this projector perspective,  $\Phi^\mathbf{I}$  is identified with the submanifold  $\mathcal{F}^\mathbf{I}$  of  $G^\mathbf{I}$  defined by

$$\mathcal{F}^\mathbf{I} := \{ \mathcal{P} = (P_i)_{1 \leq i \leq r} \in G^\mathbf{I} : P_i^2 = P_i = P_i^T, P_i P_j = 0, i \neq j \}. \quad (3.1)$$

On the other hand, in [10], the following representation of  $\Phi^\mathbf{I}$  is considered. Let  $(\lambda_i)_{1 \leq i \leq r}$  be distinct real numbers. Consider the diagonal matrix  $\mathcal{D}_0 = \text{Diag}[\lambda_1 I_{q_1}, \dots, \lambda_r I_{q_r}]$ , where  $I_{q_i}$  is the identity matrix of order  $q_i$ , and the left-action of the orthogonal group  $O(n)$  on  $\text{Sym}_n$  defined by  $Q \cdot S := QSQ^T$ . Then,  $\Phi^\mathbf{I}$  is identified with  $O(n) \cdot \mathcal{D}_0 \subset \text{Sym}_n$ , i.e. the orbit of  $\mathcal{D}_0$ . Set

$$O(\mathbf{I}) := \prod_{i=1}^r O(q_i).$$

Then, through this representation,  $\Phi^\mathbf{I}$  is an embedded submanifold of  $\text{Sym}_n$  which is diffeomorphic to  $O(n)/O(\mathbf{I})$ .

A similar formalism is developed in [3] to study the geometry of  $G_n^q$ . Namely, consider the projector  $P_0 := \text{Diag}[I_q, 0_{n-q}] \in \text{Sym}_n$ . Then,  $G_n^q$  is the orbit of  $P_0$  under the same action of  $O(n)$  on  $\text{Sym}_n$  and the map  $\pi^q : O(n) \rightarrow G_n^q$  defined by  $\pi^q : Q \mapsto Q \cdot P_0$  is a Riemannian submersion. Therefore, the geometry of  $G_n^q$  is described in terms of that of  $O(n)$ .

In this paper, we prove that  $\mathcal{F}^\mathbf{I}$  is the orbit a fixed flag  $\mathcal{P}_0$  under a suitable action of  $O(n)$  on  $G^\mathbf{I}$ . Analogously to the case of  $G_n^q$ , the combination of the projector perspective and the Riemannian submersion setting allows to derive results on the geometry of the flag manifold  $\mathcal{F}^\mathbf{I}$ .

## 3.2 Flag manifolds in the general framework

The compact Lie group  $O(n)$  is endowed with the *bi-invariant* metric  $g^O$  defined by

$$g_Q^O(\Delta, \tilde{\Delta}) := \frac{1}{2} \text{tr}(\Delta^T \tilde{\Delta}), \quad Q \in \mathbb{G}, \Delta, \tilde{\Delta} \in T_Q O(n).$$

The group  $O(n)$  acts smoothly on  $G^I$  by  $(Q, \mathcal{P}) \mapsto Q \cdot \mathcal{P} := (QP_1Q^T, \dots, QP_rQ^T)$ . Introduce the standard flag  $\mathcal{P}_0^I := (P_0^i)_{1 \leq i \leq r} \in \mathcal{F}^I$ , where

$$P_0^i := \text{Diag}[0_{q_1}, \dots, I_{q_i}, \dots, 0_{q_r}], \quad 1 \leq i \leq r.$$

Clearly, the isotropy group of  $\mathcal{P}_0^I$  is isomorphic to  $O(I)$ . Now, this action stabilizes  $\mathcal{F}^I$ , so that we may consider the map  $\pi^I : O(n) \rightarrow \mathcal{F}^I$  defined by

$$\pi^I(Q) = Q \cdot \mathcal{P}_0^I = (QP_i^0Q^T)_{1 \leq i \leq r}.$$

**Proposition 2.**  $\mathcal{F}^I$  satisfies the conditions of Theorem 1 with  $M = G^I$ ,  $\mathbb{G} = O(n)$ ,  $g^{\mathbb{G}} = g^O$ ,  $p_0 = \mathcal{P}_0$  and  $\mathbb{K}(p_0) = O(I)$ . So, the map  $\pi^I$  is a Riemannian submersion and induces a principal bundle structure over  $\mathcal{F}^I$ .

*Proof.* We need to prove that (i) and (ii) of Theorem 1 hold. For any  $\mathcal{P} = (P_i)_{1 \leq i \leq r} \in \mathcal{F}^I$ , let  $Q \in O(n)$  such that for all  $1 \leq i \leq r$ , its  $i$ -th block  $Q^{(i)}$  of columns is an orthonormal basis of  $\text{Im}(P^i)$ . Then, for all  $1 \leq i \leq r$ ,  $Q^{-1}P_iQ = P_0^i$  and  $(QP_0^1Q^T, \dots, QP_0^rQ^T) = \mathcal{P}$ . This proves that the orbit of  $\mathcal{P}_0$  is  $\mathcal{F}^I$ , i.e. (i) holds. Since the metric  $g^O$  is right-invariant, (ii) holds.  $\square$

## 3.3 Basic geometry of flag manifolds

### 3.3.1 Tangent spaces

For any  $\mathcal{P} = (P_i)_i \in \mathcal{F}^I$ , the tangent space  $T_{\mathcal{P}}\mathcal{F}^I$  is a linear subspace of  $T_{\mathcal{P}}G^I = \bigoplus_{i=1}^r T_{P_i}G^i$ . By differentiating the relation  $P_iP_j = 0$  in (3.1), we obtain that

$$T_{\mathcal{P}}\mathcal{F}^I = \{\Delta = (\Delta^i)_{1 \leq i \leq r} \in T_{\mathcal{P}}G^I : \forall i \neq j, \Delta^iP^j + P^i\Delta^j = 0\}. \quad (3.2)$$

**Lemma 2.** For all  $Q \in O(n)$  and  $Q\Omega \in T_QO(d)$  with  $\Omega \in \mathfrak{so}(n)$ ,

$$(d\pi^I)_Q(Q\Omega) = (Q[\Omega, P_0^i]Q^T)_i.$$

*Proof.* Let  $\gamma : [0, 1] \rightarrow O(n)$  be a smooth curve with  $\gamma(0) = Q$  and  $\gamma'(0) = Q\Omega$ . Then,

$$(d\pi^I)_Q(Q\Omega) = \left. \frac{d}{dt} \right|_{t=0} \pi^I(\gamma(t)) = \left( \left. \frac{d}{dt} \right|_{t=0} \gamma(t) P_0^i \gamma(t)^T \right)_i.$$

Now  $\Omega^T = -\Omega$ . So, for all  $1 \leq i \leq r$ ,

$$\left. \frac{d}{dt} \right|_{t=0} \gamma(t) P_0^i \gamma(t)^T = (Q\Omega) P_0^i Q^T + Q P_0^i (Q\Omega)^T = Q[\Omega, P_0^i]Q^T.$$

$\square$

By Lemma 2, for any  $Q \in O(n)$ ,  $V_Q^{\pi^I} = \{Q\Omega \in T_QO(d) : [\Omega, P_0^i] = 0, 1 \leq i \leq r\}$ . The computation of  $[\Omega, P_0^i]$  implies that

$$V_Q^{\pi^I} = \{Q \text{Diag}[\Omega_{ii}] : \Omega_{ii} \in \mathfrak{so}(q_i), 1 \leq i \leq r\}.$$

Therefore,

$$H_Q^{\pi^I} = \left\{ Q\Omega : \Omega \in \mathfrak{so}(n), \tilde{\Omega}_{ii} = 0, 1 \leq i \leq r \right\}.$$

In particular, we recover that  $\dim(\mathcal{F}^I) = \dim(H_Q^{\pi^I}) = \sum_{i>j} q_i q_j$ .

### 3.3.2 Horizontal lifts

**Lemma 3.** For all  $\Omega \in \mathfrak{so}(n)$ ,

$$\text{hor}_e^{\pi^1}(\Omega) = \frac{1}{2} \sum_{i=1}^r \llbracket [\Omega, P_0^i], P_0^i \rrbracket.$$

*Proof.* This is a consequence of computations of block-matrices in  $\mathfrak{so}(n)$ , partitioned wrt the type I.  $\square$

**Proposition 3.** For all  $\mathcal{P} = (P_i)_i \in \mathcal{F}^1$ ,  $\Delta = (\Delta_i)_i \in T_{\mathcal{P}}\mathcal{F}^1$  and  $Q \in (\pi^1)^{-1}(\mathcal{P})$ ,

$$\Delta_Q^\sharp = \frac{1}{2} \left( \sum_{i=1}^r \llbracket \Delta_i, P_i \rrbracket \right) Q.$$

*Proof.* This follows from the preceding Lemma and the description of the tangent spaces to  $\mathcal{F}^1$ .  $\square$

### 3.3.3 Metric, geodesics and Exponential map

For  $\mathcal{P} = (P_i)_i \in \mathcal{F}^1$  and  $\Delta, \tilde{\Delta} \in T_{\mathcal{P}}\mathcal{F}^1$ ,

$$g_{\mathcal{P}}^{\mathcal{F}^1}(\Delta, \tilde{\Delta}) = \frac{1}{4} \sum_{i,j} g_e^O(\llbracket \Delta_i, P_i \rrbracket, \llbracket \tilde{\Delta}_j, P_j \rrbracket)$$

**Remark 2.** This metric should be compared with that of [15], which is the restriction of that of  $G^1$ .

**Proposition 4.** For  $\mathcal{P} = (P_i)_i \in \mathcal{F}^1$  and  $\Delta = (\Delta_i)_i \in T_{\mathcal{P}}\mathcal{F}^1$ , the geodesic of  $\mathcal{F}^1$  starting at  $\mathcal{P}$  in the direction  $\Delta$  is of the form  $t \mapsto \pi^1(Q \exp_m(tQ^T \bar{\Omega} Q))$ , where  $\exp_m$  is the matrix exponential and

$$\bar{\Omega} = \frac{1}{2} \sum_{i=1}^r \llbracket \Delta_i, P_i \rrbracket.$$

Then,

$$\text{Exp}_{\mathcal{P}}^{\mathcal{F}^1}(\Delta) = (\exp_m(\bar{\Omega}) P_i \exp_m(-\bar{\Omega}))_i. \quad (3.3)$$

*Proof.* Since  $\bar{\Omega} = Q^T \bar{\Omega} Q$  and  $Q P_i^0 Q^T = P_i$ , the following computation implies that (3.3) holds:

$$\text{Exp}_{\mathcal{P}}^{\mathcal{F}^1}(\Delta) = \pi^1(Q \exp_m(Q^T \bar{\Omega} Q)) = \pi^1((\exp_m(\bar{\Omega}) Q)) = (\exp_m(\bar{\Omega}) Q P_i^0 Q^T \exp_m(\bar{\Omega}))_i.$$

$\square$

**Remark 3.** This should be compared with the Exponential map in [15].

**Remark 4.** There exist some algorithms for computing the Riemannian Logarithm of Riemannian submersions.

## 4 Levi-Civita connection and curvature

Throughout the sequel, in order to alleviate the notations, we assume that  $\mathbb{G}$  is a matrix Lie group, that is  $\mathbb{G} \subset GL_n(\mathbb{R})$ , for  $n \geq 1$ . This means that  $\mathcal{L}_{\tilde{Q}}^{\mathbb{G}}$  and  $(d\mathcal{L}_{\tilde{Q}}^{\mathbb{G}})_{\tilde{Q}}$  are both replaced by left-multiplication by  $\tilde{Q}$ . The results obtained under this assumption can easily be generalized to an arbitrary compact Lie group.

We shall make use, at times, of the following assumption.

**Assumption 1.** For all  $\Omega, \Omega' \in H_e^\pi$ ,  $\llbracket \Omega, \Omega' \rrbracket \in V_e^\pi$ .

## 4.1 Decomposition of vector fields

The idea is to decompose any smooth vector field on  $\mathbb{G}$  into a linear combination of left-invariant vector fields with coefficients which are smooth functions on  $\mathbb{G}$ . This is possible thanks to the following tool.

**Proposition 5.** *A 1-form  $\omega$  on a manifold  $M$  is smooth if and only if for every smooth vector field  $X$  on  $M$ , the function  $\omega(X)$  is smooth on  $M$ .*

*Proof.* We omit the proof. This is standard.  $\square$

**Corollary 1.** *Let  $(\epsilon_k)_{1 \leq k \leq N}$  be any basis of  $\mathfrak{g}$ . Let  $(E_k)_k$  be the left-invariant vector fields on  $\mathbb{G}$ , defined by  $E_k(Q) = Q_* \epsilon_k$ , for  $Q \in \mathbb{G}$ . Then, for any  $U \in \mathcal{X}(\mathbb{G})$  and  $Q \in \mathbb{G}$ ,*

$$U_Q = \sum_{k=1}^N u_k(Q) E_k(Q) = Q_* \left( \sum_{k=1}^N u_k(Q) \epsilon_k \right),$$

where the  $u_k$ 's are smooth functions on  $\mathbb{G}$ . Let  $\bar{U} : \mathbb{G} \rightarrow \mathfrak{g}$  be the map defined by

$$\bar{U}(Q) = (Q^{-1})_* U_Q = \sum_{k=1}^N u_k(Q) \epsilon_k.$$

Then,  $\bar{U}$  is a smooth map. Furthermore,

$$(d\bar{U})_Q = \sum_{k=1}^N (du_k)_Q \epsilon_k.$$

*Proof.* Apply Proposition 5.  $\square$

## 4.2 Levi-Civita connection

**Proposition 6.** *Let  $U, V \in \mathcal{X}(\mathbb{G})$ . Then,*

$$(\nabla_U^{\mathbb{G}} V)_Q = Q (d\bar{V})_Q (U_Q) + \frac{Q}{2} \llbracket \bar{U}(Q), \bar{V}(Q) \rrbracket. \quad (4.1)$$

*Proof.* First, for  $1 \leq k \leq \ell \leq N$ ,  $E_k$  and  $E_\ell$  are livf's. So, for any  $Q \in \mathbb{G}$ ,

$$(\nabla_{E_k}^{\mathbb{G}} E_\ell)_Q = \frac{1}{2} [E_k, E_\ell]_Q^{\mathbb{G}} = \frac{Q}{2} \llbracket \epsilon_k, \epsilon_\ell \rrbracket. \quad (4.2)$$

Now, in order to derive (4.1), decompose  $U$  and  $V$  as  $U = \sum_{k=1}^N u_k E_k$  and  $V = \sum_{\ell=1}^N v_\ell E_\ell$ . Then, apply the Leibniz rule and conclude by (4.2).  $\square$

**Proposition 7.** *Let  $X, Y \in \mathcal{X}(\mathbb{B})$  and  $b \in \mathbb{B}$ . Then, for any  $Q \in \pi^{-1}(b)$ ,*

$$(\nabla_X^{\mathbb{B}} Y)_b = (d\pi)_Q \left( Q (d\bar{Y}^\sharp)_Q (X^\sharp(Q)) \right) + \frac{1}{2} (d\pi)_Q \left( Q \llbracket \bar{X}^\sharp(Q), \bar{Y}^\sharp(Q) \rrbracket \right). \quad (4.3)$$

Suppose that Assumption 1 holds. Then,

$$(\nabla_X^{\mathbb{B}} Y)_b = (d\pi)_Q \left( Q (d\bar{Y}^\sharp)_Q (X^\sharp(Q)) \right). \quad (4.4)$$

*Proof.* Since  $\pi$  is a Riemannian submersion, we have that  $(\nabla_X^{\mathbb{B}} Y)_b = (d\pi)_Q \left( (\nabla_{X^\sharp}^{\mathbb{G}} Y^\sharp)_Q \right)$ . So (4.3) follows from (4.1). For the second part, Assumption 1 implies that for all  $Q \in \mathbb{G}$ ,  $\llbracket \bar{X}^\sharp(Q), \bar{Y}^\sharp(Q) \rrbracket \in \text{Ver}_e^\pi$ . Now, by Lemma 1, for all  $Q \in \mathbb{G}$ ,  $Q \llbracket \bar{X}^\sharp(Q), \bar{Y}^\sharp(Q) \rrbracket \in \text{Ver}_Q^\pi$ .  $\square$

### 4.3 Curvature

Throughout the sequel, we assume that  $\{\epsilon_k : 1 \leq k \leq N\}$  is an orthonormal basis of  $\mathfrak{g}$  such that  $\{\epsilon_k : 1 \leq k \leq N^h\}$  is an orthonormal basis of  $H_e^\pi$  and  $\{\epsilon_k : N^h + 1 \leq k \leq N\}$  is an orthonormal basis of  $V_e^\pi$ . Since  $g^\mathbb{G}$  is a left-invariant metric, for all  $Q \in \mathbb{G}$ ,  $\{E_k(Q) : 1 \leq k \leq N^h\}$  and  $\{E_k(Q) : N^h + 1 \leq k \leq N\}$  are respectively orthonormal basis of  $H_Q^\pi$  and  $V_Q^\pi$ .

#### 4.3.1 (0, 4) curvature tensor

**Lemma 4.** For any  $X, Y, Z, W \in \mathcal{X}(\mathbb{B})$  and  $Q \in \mathbb{G}$ ,

$$R_Q^\mathbb{G}(X^\sharp, Y^\sharp, Z^\sharp, W^\sharp) = \frac{1}{4}g_e^\mathbb{G} \left( \llbracket \overline{X^\sharp}(Q), \overline{Y^\sharp}(Q) \rrbracket, \llbracket \overline{Z^\sharp}(Q), \overline{W^\sharp}(Q) \rrbracket \right).$$

*Proof.* □

**Lemma 5.** For any  $X, Y \in \mathcal{X}(\mathbb{B})$  and  $Q \in \mathbb{G}$ ,

$$\text{ver}_Q^\pi \left( [X^\sharp, Y^\sharp]_Q^\mathbb{G} \right) = Q \text{ver}_e^\pi \left( \llbracket \overline{X^\sharp}(Q), \overline{Y^\sharp}(Q) \rrbracket \right).$$

*Proof.* For any  $U, V \in \mathcal{X}(\mathbb{G})$ ,

$$[U, V]^\mathbb{G} = \sum_{k, \ell=1}^N [u_k E_k, v_\ell E_\ell]^\mathbb{G} = \sum_{k, \ell=1}^N u_k (E_k \cdot v_\ell) E_\ell + u_\ell (E_\ell \cdot v_k) E_k + u_k v_\ell [E_k, E_\ell]^\mathbb{G}.$$

Since  $X^\sharp$  and  $Y^\sharp$  are horizontal vector fields, the choice of the basis  $\{\epsilon_k : 1 \leq k \leq N\}$  of  $\mathfrak{g}$  implies that

$$[X^\sharp, Y^\sharp]^\mathbb{G} = \sum_{k, \ell=1}^{N^h} x_k^\sharp(E_k \cdot y_\ell^\sharp) E_\ell + x_\ell^\sharp(E_\ell \cdot y_k^\sharp) E_k + x_k^\sharp y_\ell^\sharp [E_k, E_\ell]^\mathbb{G}.$$

So, for any  $Q \in \mathbb{G}$ ,

$$\text{ver}_Q^\pi \left( [X^\sharp, Y^\sharp]_Q^\mathbb{G} \right) = \sum_{k, \ell=1}^{N^h} x_k^\sharp(Q) y_\ell^\sharp(Q) \text{ver}_Q^\pi \left( [E_k, E_\ell]_Q^\mathbb{G} \right).$$

Now, by Lemma 1,

$$\text{ver}_Q^\pi \left( [E_k, E_\ell]_Q^\mathbb{G} \right) = \text{ver}_Q^\pi \left( Q \llbracket \epsilon_k, \epsilon_\ell \rrbracket \right) = Q \text{ver}_e^\pi \left( \llbracket \epsilon_k, \epsilon_\ell \rrbracket \right).$$

Therefore,

$$\text{ver}_Q^\pi \left( [X^\sharp, Y^\sharp]_Q^\mathbb{G} \right) = Q \sum_{k, \ell=1}^{N^h} x_k^\sharp(Q) y_\ell^\sharp(Q) \text{ver}_e^\pi \left( \llbracket \epsilon_k, \epsilon_\ell \rrbracket \right) = Q \text{ver}_e^\pi \left( \llbracket \overline{X^\sharp}(Q), \overline{Y^\sharp}(Q) \rrbracket \right).$$

□

**Proposition 8.** For any  $X, Y, Z, W \in \mathcal{X}(\mathbb{B})$ , set

$$\mathfrak{R}^\mathbb{B}(X, Y, Z, W) := g^\mathbb{G} \left( \text{ver}^\pi \left( [X^\sharp, Y^\sharp]^\mathbb{G} \right), \text{ver}^\pi \left( [Z^\sharp, W^\sharp]^\mathbb{G} \right) \right).$$

Then, for all  $b \in \mathbb{B}$  and any  $Q \in \pi^{-1}(b)$ ,

$$R_b^\mathbb{B}(X, Y, Z, W) = \frac{1}{4}g_e^\mathbb{G} \left( \llbracket \overline{X^\sharp}(Q), \overline{Y^\sharp}(Q) \rrbracket, \llbracket \overline{Z^\sharp}(Q), \overline{W^\sharp}(Q) \rrbracket \right) + \frac{1}{2}\mathfrak{R}_b^\mathbb{B}(X, Y, Z, W) - \frac{1}{4}\mathfrak{R}_b^\mathbb{B}(Y, Z, X, W) - \frac{1}{4}\mathfrak{R}_b^\mathbb{B}(Z, X, Y, W).$$

where

$$\mathfrak{R}_b^\mathbb{B}(X, Y, Z, W) := g_e^\mathbb{G} \left( \text{ver}_e^\pi \left( \llbracket \overline{X^\sharp}(Q), \overline{Y^\sharp}(Q) \rrbracket \right), \text{ver}_e^\pi \left( \llbracket \overline{Z^\sharp}(Q), \overline{W^\sharp}(Q) \rrbracket \right) \right).$$

*Proof.* By O'Neill's formula,

$$R^\mathbb{B}(X, Y, Z, W) = R^\mathbb{G}(X^\sharp, Y^\sharp, Z^\sharp, W^\sharp) + \frac{1}{2}\mathfrak{R}^\mathbb{B}(X, Y, Z, W) - \frac{1}{4}\mathfrak{R}^\mathbb{B}(Y, Z, X, W) - \frac{1}{4}\mathfrak{R}^\mathbb{B}(Z, X, Y, W). \quad (4.5)$$

The first term of (4.5) is provided by Lemma 4. By Lemma 5, the other terms of (4.5) are given by

$$\mathfrak{R}_b^\mathbb{B}(X, Y, Z, W) = g_e^\mathbb{G} \left( \text{ver}_e^\pi \left( \llbracket \overline{X^\sharp}(Q), \overline{Y^\sharp}(Q) \rrbracket \right), \text{ver}_e^\pi \left( \llbracket \overline{Z^\sharp}(Q), \overline{W^\sharp}(Q) \rrbracket \right) \right).$$

□



### 4.3.2 (1, 3) curvature tensor

Recall that  $\{\epsilon_k : 1 \leq k \leq N^h\}$  is an orthonormal basis of  $H_e^\pi \mathbb{G}$ . Now, for any  $b \in \mathbb{B}$  and fixed  $Q \in \pi^{-1}(b)$ , set

$$w_k(b) := (d\pi)_Q(Q\epsilon_k) \in T_b\mathbb{B}.$$

Since the metric  $g^\mathbb{G}$  is left-invariant and  $\pi$  is a Riemannian submersion,  $\{w_k(b) : 1 \leq k \leq N^h\}$  is an orthonormal basis of  $T_b\mathbb{B}$ . So, for the (3, 1) curvature tensor  $R^\mathbb{B}(X, Y)Z$ , for any  $b \in \mathbb{B}$ ,

$$\begin{aligned} R_b^\mathbb{B}(X, Y)Z &= \sum_{k=1}^{N^h} g_b^\mathbb{B}(R_b^\mathbb{B}(X, Y)Z, w_k(b)) w_k(b) \\ &= \sum_{k=1}^{N^h} R_b^\mathbb{B}(X, Y, Z, W_k) w_k(b), \end{aligned}$$

where  $W_k$  is any vector field such that  $(W_k)_b = w_k(b)$ . For any  $b \in \mathbb{B}$  and fixed  $Q \in \pi^{-1}(b)$ , Proposition 8 provide the value at  $b$  of the tensor  $R^\mathbb{B}$  in function of that of the vector fields:

$$R_b^\mathbb{B}(X, Y, Z, W_k) = \rho_b(X_b, Y_b, Z_b, (W_k)_b).$$

Therefore,

$$R_b^\mathbb{B}(X, Y)Z = \sum_{k=1}^{N^h} \rho_b(X_b, Y_b, Z_b, w_k(b)) w_k(b).$$

### 4.3.3 Sectional curvature

**Proposition 9.** *Let  $X, Y \in \mathcal{X}(\mathbb{B})$  be orthonormal vector fields. Then, for all  $b \in \mathbb{B}$  and  $Q \in \pi^{-1}(b)$ ,*

$$K_b^\mathbb{B}(X, Y) = \frac{1}{4} \left\| \llbracket \overline{X^\sharp}(Q), \overline{Y^\sharp}(Q) \rrbracket \right\|_e^2 + \frac{3}{4} \left\| \text{ver}_e^\pi \left( \llbracket \overline{X^\sharp}(Q), \overline{Y^\sharp}(Q) \rrbracket \right) \right\|_e^2. \quad (4.6)$$

Suppose that Assumption 1 holds. Then, for all  $b \in \mathbb{B}$  and  $Q \in \pi^{-1}(b)$ ,

$$K_b^\mathbb{B}(X, Y) = \left\| \llbracket \overline{X^\sharp}(Q), \overline{Y^\sharp}(Q) \rrbracket \right\|_e^2. \quad (4.7)$$

*Proof.* By O'Neill's formula,

$$K_b^\mathbb{B}(X, Y) = K_P^\mathbb{G}(X, Y) + \frac{3}{4} \left\| \text{ver}_Q^\pi \left( \llbracket X^\sharp, Y^\sharp \rrbracket_Q^\mathbb{G} \right) \right\|_Q^2.$$

Then, Lemmas 5 and 4 combined to the left-invariance of the metric  $g^\mathbb{G}$  imply (4.6), and (4.7) follows readily.  $\square$

## 4.4 The case of Grassmannians and flag manifolds

First, notice that the Frobenius metric is bi-invariant, so that the metric considered on  $O(d)$  is.

Let  $X, Y \in \mathcal{X}(\mathcal{F}^1)$ . For any  $\mathcal{P} \in \mathcal{F}^1$  and  $Q \in (\pi^1)^{-1}(\mathcal{P})$ ,

$$\llbracket \overline{X^\sharp}(Q), \overline{Y^\sharp}(Q) \rrbracket = \frac{1}{4} Q^T \Omega_{\mathcal{P}}^{X, Y} Q,$$

where

$$\Omega_{\mathcal{P}}^{X, Y} := \left( \sum_{i, j} \llbracket [X_i(\mathcal{P}), P_i], [Y_j(\mathcal{P}), P_j] \rrbracket \right) \in \mathfrak{so}(n).$$

**Lemma 6.** *Assumption 1 holds for  $G_n^q$  but not for  $\mathcal{F}^1$ .*

*Proof.* This follows from simple computations in  $\mathfrak{so}(n)$ .  $\square$

#### 4.4.1 Levi-Civita connection of Grassmannians and flag manifolds

**Proposition 10.** For any  $X, Y \in \mathcal{X}(G_n^q)$  and  $P \in G_n^q$ ,  $Q \in (\pi^m)^{-1}(P)$ ,

$$\left( \nabla_X^{G_n^q} Y \right)_P = (d\pi^m)_Q \left( Q \left( d\overline{Y^\sharp} \right)_Q (X^\sharp(Q)) \right),$$

where  $X^\sharp(Q) = \llbracket X(\pi^m(Q)), \pi^m(Q) \rrbracket Q$  and  $\overline{Y^\sharp}(Q) = Q^T \llbracket Y(\pi^m(Q)), \pi^m(Q) \rrbracket Q \in \mathfrak{so}(n)$ .

*Proof.* Since Assumption 1 holds for  $G_n^q$ , the result follows readily from (4.4).  $\square$

We obtain an expression of the latter when  $X$  and  $Y$  are defined through charts of  $G_n^q$ .

**Proposition 11.** For any  $X, Y \in \mathcal{X}(\mathcal{F}^I)$  and  $\mathcal{P} \in \mathcal{F}^I$ ,  $Q \in (\pi^I)^{-1}(\mathcal{P})$ ,

$$\left( \nabla_X^{\mathcal{F}^I} Y \right)_\mathcal{P} = (d\pi^I)_Q \left( Q \left( d\overline{Y^\sharp} \right)_Q (X^\sharp(Q)) \right) + \frac{1}{8} (d\pi^I)_Q \left( \Omega_{\mathcal{P}}^{X,Y} Q \right).$$

where  $X^\sharp(Q) = \frac{1}{2} \left( \sum_{i=1}^r \llbracket X_i(\pi^I(Q)), \pi^I(Q) \rrbracket \right) Q$  and  $\overline{Y^\sharp}(Q) = \frac{1}{2} Q^T \left( \sum_{i=1}^r \llbracket Y_i(\pi^I(Q)), \pi^I(Q) \rrbracket \right) Q \in \mathfrak{so}(n)$ .

*Proof.* Apply Proposition 7.  $\square$

#### 4.4.2 Curvature of Grassmannians and flag manifolds

This general framework allows us to recover the sectional curvature of Grassmannians, given in Proposition 4.1. of [3].

**Proposition 12.** Let  $X, Y \in \mathcal{X}(G_n^q)$  be orthonormal vector fields. Then, for all  $P \in G_n^q$  and  $Q \in \pi^{-1}(P)$ ,

$$K_P^{G_n^q}(X, Y) = \left\| \llbracket X_P, Y_P \rrbracket \right\|_e^2.$$

*Proof.* First, notice that the statement makes sense. Indeed, for all  $P \in G_n^q$ ,  $\llbracket X_P, Y_P \rrbracket \in \mathfrak{so}(n)$ . Now, the result follows from (4.7), since Assumption 1 holds for Grassmannians.  $\square$

**Proposition 13.** Let  $X, Y \in \mathcal{X}(\mathcal{F}^I)$ . Then, for all  $\mathcal{P} \in \mathcal{F}^I$ ,

$$K_{\mathcal{P}}^{\mathcal{F}^I}(X, Y) = \frac{1}{16} \left( \left\| \Omega_{\mathcal{P}}^{X,Y} \right\|_e^2 + 3 \left\| \text{ver}_e^{\pi^I} \left( \Omega_{\mathcal{P}}^{X,Y} \right) \right\|_e^2 \right),$$

where for any  $\Omega \in \mathfrak{so}(n)$ ,

$$\text{ver}_e^{\pi^I}(\Omega) = \Omega - \frac{1}{2} \sum_{i=1}^r \llbracket \llbracket \Omega, P_0^i \rrbracket, P_0^i \rrbracket.$$

*Proof.* This is a consequence of (4.6).  $\square$

## Acknowledgements

The authors have received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (grant agreement G-Statistics No 786854). It was also supported by the French government through the 3IA Côte d'Azur Investments ANR-19-P3IA-0002 managed by the National Research Agency.

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