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Observer Design via Non-Monotone Lyapunov Functions for Monitoring Heterotrophic Microalgae

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Abstract: We propose an observer to estimate the biomass of microalgae growing under heterotrophic conditions. In the heterotrophic microalgae modeling we have assumed that it is growth-limited by multiple substrate, acetate and oxygen. Then, we design a nonlinear observer through a combination of Lyapunov's direct method and exact convex representations, the design conditions are given in terms of linear matrix inequalities (LMIs). The observer is implemented in simulation, and additionally, we compare our proposal with existing conditions.

Keywords: Observer, Linear Matrix Inequalities, Microalgae, Non-Monotone Lyapunov

1. INTRODUCTION

Microalgae are emerging organisms in biotechnology which offer many new perspectives for various applications, such as food, feed cosmetic or pharmaceutical industries (Rizwan et al., 2018). Different strategies for cultivating them have been developed, depending on the sources of energy and carbon. These are *photoautotroph* when their growth is determined by the irradiance received, while consuming carbon dioxide; they are *heterotroph* when the growth is governed by the sugars assimilated. According to Xiao et al. (2020), the oxidation of these sugars provides the carbon and energy required for a faster cell growth compared with the photoautotrophic case.

One of the drivers of interest in microalgae research is their ability for lipid production. It was pointed out that their lipids percentage and their maximum biomass concentration can be improved when microalgae is cultivated under heterotrophic conditions. Other applications include wastewater treatment processes (Oviedo et al. (2022)) that require continuous monitoring to ensure their quality. The monitoring can be manual or via automatic sampling (Quevauviller et al. (2007)), where non-automatic measurements require clean flasks, representative sample volume, proper sampling and analysis methods; this task might be considered as time consuming, repetitive and cannot be performed by untrained personal, since the medium composition can be altered. Under this context, automatic monitoring has arise.

We could be interested for example, in monitoring biomass concentration inside a chemostat, which can be measured by physical sensors using methods such as optical density or dielectric spectroscopy. However, these sensors have a high cost, require calibration and maintenance. As an alternative, biomass can be estimated by means of software

sensors, known as observers. There are many strategies available to calibrate or synthesize observers, and depending on the approach chosen, we might succeed in this task or not, this limitation is known as *conservativeness*, Sala (2009). Some techniques are based on Lyapunov's direct method that consists in providing a Lyapunov function to prove that the estimation error is stable. The sources of conservativeness in Lyapunov-based methods are several. The converse theorem says that if a system is asymptotically stable there exists a Lyapunov function; but, there is not an unique way to find it. Here we focus on strategies based on linear matrix inequalities (LMIs).

The classical condition for discrete-time asymptotic stability is $V_{k+1} - V_k < 0$. The latter implies that the Lyapunov function V_k must decrease for each step, it is denoted as monotonically decreasing. Relaxed formulations for this condition have been presented in Ahmadi and Parrilo (2008) and Kruszewski et al. (2008) where the monotonicity requirement to certify stability was relaxed. Instead, non-monotonically decreasing Lyapunov functions are allowed. Later, these results were extended to design nonlinear observers by Derakhshan and Fatehi (2015). Their adaptation consists in applying the additive identity, a change of variables, together with Schur's complement, congruence and substitution. Even if that approach seems effective, those steps yield to monotone Lyapunov functions, and maybe there is another way to proceed.

The methodology for nonlinear observer design has been illustrated in Fig. 1. The contributions in this manuscript can be summarized as: 1) an observer is designed for monitoring biomass concentration of heterotrophic microalgae in a chemostat, and 2) new conditions for observer design via non-monotone Lyapunov functions in terms of LMIs.

The report is organized as follows: in section 2 the notation and lemmas for observer design via monotone and non-monotone Lyapunov functions are presented; then, in section 3 the model of heterotroph microalgae growth-limited by acetate and oxygen is introduced. Later, in

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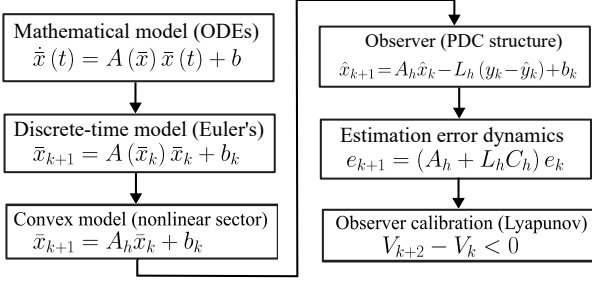


Fig. 1. Summary of the proposed approach.

section 4 new conditions for observer design based via non-monotone Lyapunov functions are established. In section 5 an observer for biomass estimation is designed and implemented in simulation; then, the new conditions are compared with the existing ones. Finally, in section 6 a conclusion and possible extensions of the results are given.

2. PRELIMINARIES

The notation, definitions and lemmas that will be used throughout this document are listed below:

First, our notation employed is: the superscript T in $(*)^T$ denotes the transpose of $(*)$. A positive (negative) definite matrix Q is denoted as $Q > 0$ ($Q < 0$). The sub-index $k \in \{0, 1, 2, \dots\}$ in $(*)_k$ denotes the sample value of $(*)$.

Lemma 1. (Finsler's, Oliveira and Skelton (2001)). Let $\mathcal{Q} \in \mathbb{S}^{n \times n}$, $x \in \mathbb{R}^n$, $\mathcal{B} \in \mathbb{R}^{m \times n}$ and $\text{rank}(\mathcal{B}) < n$. Then, the following expressions are equivalent: 1) $x^T \mathcal{Q} x < 0$, $\forall \mathcal{B}x = 0$, $x \neq 0$, and 2) $\exists \mathcal{X} \in \mathbb{R}^{n \times m}$: $\mathcal{Q} + \mathcal{X}\mathcal{B} + \mathcal{B}^T \mathcal{X}^T < 0$.

Lemma 2. (Congruence). Let \mathcal{R} and \mathcal{Q} be proper size symmetric and full rank column matrices, respectively; then, the following holds $\mathcal{R} < 0 \Rightarrow \mathcal{R}^T \mathcal{Q} \mathcal{R} < 0$.

Lemma 3. (Schur's complement). Let $Q = Q^T$, $R = R^T$ and S , then, these are equivalent $R < 0$, $Q - SR^{-1}S < 0$, $\Leftrightarrow \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} < 0$.

Lemma 4. (Liu&Zhang's, Xiaodong and Qingling (2003)).

A sufficient condition for $Q_{hh} = \sum_{i=1}^r \sum_{j=1}^r h_i h_j Q_{ij} < 0$, to hold

$$\forall (h_i, h_j) \geq 0 \text{ is that } \exists \mathcal{Z}_{ij}: \begin{bmatrix} \mathcal{Z}_{11} (\mathcal{Z}_{21})^T \cdots (\mathcal{Z}_{r1})^T \\ \mathcal{Z}_{21} \quad \mathcal{Z}_{22} \quad \cdots (\mathcal{Z}_{r2})^T \\ \vdots \quad \vdots \quad \ddots \quad \vdots \\ \mathcal{Z}_{ij}^{r1} \quad \mathcal{Z}_{ij}^{r2} \quad \cdots \quad \mathcal{Z}_{ij}^{rr} \end{bmatrix} < 0,$$

$$Q_{ii} - \mathcal{Z}_{ii} \leq 0, \text{ and } Q_{ij} + Q_{ji} - \mathcal{Z}_{ij} - (\mathcal{Z}_{ij})^T \leq 0, \forall j < i.$$

Now, let us consider a convex system of the form

$$\bar{x}_{k+1} = A_h \bar{x}_k, \quad y_k = C_h \bar{x}_k, \quad (1)$$

and a parallel distributed compensation (PDC) observer

$$\hat{x}_k = A_h \hat{x}_k - L_h (y_k - \hat{y}_k), \quad \hat{y}_k = C_h \hat{x}_k. \quad (2)$$

where $A_h = \sum_{i=1}^r h_i A_i$, $C_h = \sum_{i=1}^r h_i C_i$, $L_h = \sum_{j=1}^r h_j L_j$, $\sum_{i=1}^r h_i = 1$ and $0 \leq h_i \leq 1$, with $(i, j) \in \{1, 2, \dots, r\}$. The state and the estimation vectors are \bar{x}_k and \hat{x}_k , respectively. More details about the convex system and the observer will be given in further sections.

Lemma 5. (Tanaka and Wang (2001)). The error associated to (1) and (2) is asymptotically stable with a monotonically decreasing Lyapunov function if $P > 0$, $Q_{ii} -$

$$\mathcal{Z}_{ii} \leq 0, \quad Q_{ij} + Q_{ji} - \mathcal{Z}_{ij} - (\mathcal{Z}_{ij})^T \leq 0, \quad \forall j < i, \text{ and } \begin{bmatrix} \mathcal{Z}_{11} (\mathcal{Z}_{21})^T \cdots (\mathcal{Z}_{r1})^T \\ \mathcal{Z}_{21} \quad \mathcal{Z}_{22} \quad \cdots (\mathcal{Z}_{r2})^T \\ \vdots \quad \vdots \quad \ddots \quad \vdots \\ \mathcal{Z}_{ij}^{r1} \quad \mathcal{Z}_{ij}^{r2} \quad \cdots \quad \mathcal{Z}_{ij}^{rr} \end{bmatrix} < 0, \quad Q_{ij} = \begin{bmatrix} -P & A_i^T P + C_i^T N_j^T \\ P A_i + N_j C_i & -P \end{bmatrix},$$

hold, $(i, j) \in \{1, 2, \dots, r\}$, where the gains are $L_j = P^{-1} N_j$.

Following the developments in Derakhshan and Fatehi (2015) to stabilize an estimation error associated to (1) and (2). First, consider a quadratic candidate Lyapunov function $V_k = e_k^T P e_k$ and non-monotonically convergence towards zero, that is, $V_{k+2} - V_k < 0$, it is equals to $e_k^T \bar{A}_h^T \bar{A}_h^T P \bar{A}_h e_k - e_k^T P e_k < 0$, where the error dynamics are $e_{k+1} = (A_h + L_h C_h) e_k = \bar{A}_h e_k$ and $e_{k+2} = (A_{h(k+1)} + L_{h(k+1)} C_{h(k+1)}) e_{k+1} = \bar{A}_{h(k+1)} e_{k+1}$. Using the additive identity with $e_k^T \bar{A}_h^T R \bar{A}_h e_k$ and substituting $y_k = \bar{A}_h e_k$, we obtain the following inequality $y_k^T \underbrace{(\bar{A}_h^T P \bar{A}_h - R)}_{\phi_1} y_k - e_k^T \underbrace{(\bar{A}_h^T R \bar{A}_h - P)}_{\phi_2} e_k < 0$.

Then, applying Schur's complement (Lemma 3), on $\phi_1 < 0$

$$\text{and } \phi_2 < 0, \text{ yields to } \begin{bmatrix} -P & C_h^T L_h^T + A_h^T \\ A_h + L_h C_h & -R^{-1} \end{bmatrix} < 0, \text{ and } \begin{bmatrix} -R & C_{h(k+1)}^T L_{h(k+1)}^T + A_{h(k+1)}^T \\ A_{h(k+1)} + L_{h(k+1)} C_{h(k+1)} & -P^{-1} \end{bmatrix} < 0. \text{ Us-}$$

ing congruence (Lemma 2) with $\begin{bmatrix} I & 0 \\ 0 & R \end{bmatrix}$ and $\begin{bmatrix} I & 0 \\ 0 & P \end{bmatrix}$, respec-

$$\text{tively; we get } \begin{bmatrix} -P & A_h^T R + C_h^T L_h^T R \\ R A_h + R L_h C_h & -R \end{bmatrix} < 0,$$

$$\begin{bmatrix} -R & C_{h(k+1)}^T L_{h(k+1)}^T P + A_{h(k+1)}^T P \\ P A_{h(k+1)} + P L_{h(k+1)} C_{h(k+1)} & -P \end{bmatrix} < 0.$$

The only way to obtain the observer gains from these inequalities is to consider $R = P$ and $N_h = P L_h$, it produces

$$\begin{bmatrix} -P & A_h^T P + C_h^T N_h^T \\ P A_h + N_h C_h & -P \end{bmatrix} < 0, \quad \begin{bmatrix} -P & C_{h(k+1)}^T N_{h(k+1)}^T + A_{h(k+1)}^T P \\ P A_{h(k+1)} + N_{h(k+1)} C_{h(k+1)} & -P \end{bmatrix} < 0.$$

Finally, after applying Liu & Zhang's relaxation it reduces to classical monotone conditions $V_{k+1} - V_k < 0$ in Lemma 5. Thus, using the additive identity, changing variables, and asking to fulfill $\phi_1 < 0$ and $\phi_2 < 0$ yields to monotone Lyapunov functions. Either the Lyapunov function is quadratic or not, as in Derakhshan and Fatehi (2015). Given this context, in this manuscript we provide sufficient conditions for nonlinear observer design with a non-monotonically decreasing Lyapunov function.

3. HETEROTROPHIC MICROALGAE MODEL

The model for microalgae growth-limited by acetate in a chemostat can be found in Turon et al. (2015), where acetate uptake rate was described via a Monod function. According to Garcia-Ochoa et al. (2010) this type of function can also be employed to describe oxygen uptake rate. Thus, adapting both formulations to consider microalgae both nutrients (acetate and oxygen) as growth-limiting factors via Liebig's minimum law (De Baar (1994)) gives:

$$\begin{aligned}
\dot{s}(t) &= -\frac{\bar{\mu}}{\gamma_a} \phi(s, o) x(t) + d(s_{in} - s(t)), \\
\dot{x}(t) &= (\bar{\mu} \phi(s, o) - d) x(t), \\
\dot{o}(t) &= -\frac{\bar{\mu}}{\gamma_o} \phi(s, o) x(t) + d(o_{in} - o(t)) + k_L a (o^* - o(t)),
\end{aligned} \tag{3}$$

the acetate, biomass and oxygen concentrations are denoted by the variables s , x and o , respectively. We are assuming that the acetate and oxygen concentration can be measured. The apparent microalgae growth rate is $\bar{\mu}$, the dilution rate d is bounded by two constants $d \in [\underline{d}, \bar{d}]$, the yield constants for acetate and oxygen are γ_a and γ_o , the input concentration of acetate and oxygen are s_{in} and o_{in} , respectively. The substrates uptake rate can be described via Liebig's minimum law, which allow us to determine the most growth-limiting factor among acetate and oxygen, $\phi(s, o) = \min\left(\frac{s(t)}{k_a + s(t)}, \frac{o(t)}{k_o + o(t)}\right)$. Due to $\phi(s, o)$ there is a switching in the dynamics when a substrate is not longer the most-limiting. The half-saturation constants for acetate and oxygen are k_a and k_o . Finally, the oxygen saturation and the mass transfer coefficients are o^* and $k_L a$, respectively. Assuming $o_{in} = o^*$ our model (3) is:

$$\begin{aligned}
\begin{bmatrix} \dot{s}(t) \\ \dot{x}(t) \\ \dot{o}(t) \end{bmatrix} &= \underbrace{\begin{bmatrix} -d - \frac{\bar{\mu}}{\gamma_a} \phi(s, o) & 0 \\ 0 & \bar{\mu} \phi(s, o) - d \\ 0 & -\frac{\bar{\mu}}{\gamma_o} \phi(s, o) - (d + k_L a) \end{bmatrix}}_{A(\bar{x})} \underbrace{\begin{bmatrix} s(t) \\ x(t) \\ o(t) \end{bmatrix}}_{\bar{x}(t)} + \underbrace{\begin{bmatrix} ds_{in} \\ 0 \\ (d + k_L a) o_{in} \end{bmatrix}}_b. \tag{4} \\
\dot{\bar{x}}(t) &
\end{aligned}$$

Time dependency is dropped-out thereafter for simplicity.

3.1 Time discretization

The microalgae growth was described in continuous-time by means of ordinary differential equations; but, a real-time monitoring process is programmed in a computer or embedded device that operates in discrete-time. Thus, a discrete-time version of our model must be computed. We can approximate the time-derivative of \bar{x} at the instant k via Euler's explicit method, $\dot{\bar{x}}_k \approx \frac{\bar{x}(k+T_s) - \bar{x}(k)}{T_s}$, where the sampling period T_s must be small enough due to the switching occurring on the uptake function $\phi(s, o)$. It is well known that these systems are prone to numerical error if time discretization is not properly addressed. Thus, applying Euler's approximation for (3) gives:

$$\bar{x}_{k+1} = A(\bar{x}_k) \bar{x}_k + b_k, \tag{5}$$

where $\bar{x}_k = [s_k \ x_k \ o_k]^T$, $b_k = [ds_{in} T_s \ 0 \ (d + k_L a) o_{in} T_s]^T$ and $A(\bar{x}_k) = \begin{bmatrix} 1 - dT_s & -\frac{\bar{\mu}}{\gamma_a} \phi(s, o) T_s & 0 \\ 0 & 1 + (\bar{\mu} \phi(s, o) - d) T_s & 0 \\ 0 & -\frac{\bar{\mu}}{\gamma_o} \phi(s, o) T_s & 1 - (d + k_L a) T_s \end{bmatrix}$.

In further sections we design an observer for monitoring the biomass concentration using this discrete-time model.

4. NONLINEAR OBSERVER SYNTHESIS

In this section, we assume that oxygen and acetate are measured, and we want to estimate the biomass.

4.1 Exact convex representations

Via the nonlinear sector approach from Taniguchi et al. (2001) a nonlinear system of the form $x_{k+1} = A(z_k) x_k$,

$y_k = C(z_k) x_k$, can be algebraically equivalent recast as the convex sum of constant matrices within a region of interest Ω . This by interpolating between the extreme values of the non-constant terms or premise variables, denoted z_k . The interpolation or weight functions can be build using upper and lower bounds of z_k in the matrices $A(\cdot)$ and $C(\cdot)$, that is, given $z_k^{\mathbf{q}} \in [z_{\min}^{\mathbf{q}}, z_{\max}^{\mathbf{q}}]$, $\forall z_k^{\mathbf{q}} \in \Omega$, then, $z_k^{\mathbf{q}} = \underbrace{\left(\frac{z_{\max}^{\mathbf{q}} - z_k^{\mathbf{q}}}{z_{\max}^{\mathbf{q}} - z_{\min}^{\mathbf{q}}}\right)}_{w_{\mathbf{q}}^0} z_{\min}^{\mathbf{q}} + \underbrace{\left(\frac{z_k^{\mathbf{q}} - z_{\min}^{\mathbf{q}}}{z_{\max}^{\mathbf{q}} - z_{\min}^{\mathbf{q}}}\right)}_{w_{\mathbf{q}}^1} z_{\max}^{\mathbf{q}}$, with

$\mathbf{q} \in \{1, 2, \dots, \rho\}$. These weight functions hold convex sum properties, $0 \leq w_{\mathbf{q}}^0 \leq 1$, $0 \leq w_{\mathbf{q}}^1 \leq 1$ and $w_{\mathbf{q}}^0 + w_{\mathbf{q}}^1 = 1$. Due to these properties the weight functions can be stacked to build the so called membership functions $h_i(z_k)$ as $h_1(z_k) = w_1^0 w_2^0 \dots w_\rho^0$, $h_2(z_k) = w_1^1 w_2^0 \dots w_\rho^0$, $h_3(z_k) = w_1^0 w_2^1 \dots w_\rho^0$, and so on, until $h_r(z_k) = w_1^1 w_2^1 \dots w_\rho^1$, where $r = 2^\rho$. Finally, the constant matrices (A_i, C_i) are obtained evaluating the ρ non-constant terms at their extreme values such that $A_i = A(z_k)|_{h_i=1}$ and $C_i = C(z_k)|_{h_i=1}$ for $i \in \{1, 2, \dots, r\}$. After this, we assume that the matrices (A_i, C_i) obtained are observable, individually and their convex interpolations.

This methodology has been widely employed in combination with Lyapunov's direct method to obtain conditions in terms of LMIs for nonlinear control and observer synthesis, stability analysis, and so on. Thus, after rewriting our model within a region Ω we obtain a system of the form:

$$\bar{x}_{k+1} = A_h \bar{x}_k + b_k, \quad y_k = C_h \bar{x}_k, \tag{6}$$

where $A_h = \sum_{i=1}^r h_i A_i$, $C_h = \sum_{i=1}^r h_i C_i$, $0 \leq h_i \leq 1$, $\sum_{i=1}^r h_i = 1$, $i \in \{1, 2, \dots, r\}$ and $r = 2^\rho$. In this manuscript we consider that the premise variables z^i are measured.

Following the nonlinear sector for (5); the non-constant terms are $z_1 = \phi(s, o) \in [\phi_{\min}, 1]$ and $z_2 = d \in [\underline{d}, \bar{d}]$. The weight functions are $w_1^0 = \frac{1 - \phi(s, o)}{1 - \phi_{\min}}$, $w_1^1 = \frac{\phi(s, o) - \phi_{\min}}{1 - \phi_{\min}}$, $w_2^0 = \frac{\bar{d} - d}{\bar{d} - \underline{d}}$ and $w_2^1 = \frac{d - \underline{d}}{\bar{d} - \underline{d}}$, these can be used to define the membership functions as $h_1 = w_1^0 w_2^0$, $h_2 = w_1^1 w_2^0$, $h_3 = w_1^0 w_2^1$ and $h_4 = w_1^1 w_2^1$. Finally, the constant matrices were:

$$\begin{aligned}
A_1 &= \begin{bmatrix} 1 - \underline{d}T_s & -\frac{\bar{\mu}}{\gamma_a} T_s \phi_{\min} & 0 \\ 0 & 1 + (\bar{\mu} \phi_{\min} - \underline{d}) T_s & 0 \\ 0 & -\frac{\bar{\mu}}{\gamma_o} T_s \phi_{\min} & 1 - (\underline{d} + k_L a) T_s \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 - \underline{d}T_s & -\frac{\bar{\mu}}{\gamma_a} T_s & 0 \\ 0 & 1 + (\bar{\mu} - \underline{d}) T_s & 0 \\ 0 & -\frac{\bar{\mu}}{\gamma_o} T_s & 1 - (\underline{d} + k_L a) T_s \end{bmatrix}, \\
A_3 &= \begin{bmatrix} 1 - \bar{d}T_s & -\frac{\bar{\mu}}{\gamma_a} T_s \phi_{\min} & 0 \\ 0 & 1 - (\bar{\mu} \phi_{\min} - \bar{d}) T_s & 0 \\ 0 & -\frac{\bar{\mu}}{\gamma_o} T_s \phi_{\min} & 1 - (\bar{d} + k_L a) T_s \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 - \bar{d}T_s & -\frac{\bar{\mu}}{\gamma_a} T_s & 0 \\ 0 & 1 + (\bar{\mu} - \bar{d}) T_s & 0 \\ 0 & -\frac{\bar{\mu}}{\gamma_o} T_s & 1 - (\bar{d} + k_L a) T_s \end{bmatrix},
\end{aligned}$$

and the output distribution matrix is $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

4.2 Designs via Non-monotone Lyapunov Functions

There are two main ideas to be considered when an observer with a parallel distributed compensation (PDC) structure is being designed. The first one is that its structure is based on the classical Luenberger's observer, which means that it mimics the model of the system and it allow us to incorporate the nonlinearities contained within the membership functions $h(z_k)$. The second, is that stabilization strategy consists in guaranteeing stability for each vertex individually (A_i, C_i), as well as the convex

interpolations of the subsystems. Thus, a PDC observer for a nonlinear system of the form (6) would be:

$$\hat{x}_{k+1} = A_h \hat{x}_k - L_h (y_k - \hat{y}_k) + b_k, \quad \hat{y}_k = C_h \hat{x}_k, \quad (7)$$

where \hat{x}_k is the state estimation, $L_h = \sum_{i=1}^r h_j L_j$ and L_j are the observer gains to be designed such that the vertices (A_i, C_i) and their convex interpolations are stable. The estimation error are $e_k = \bar{x}_k - \hat{x}_k$ and its dynamics are

$$e_{k+1} = (A_h + L_h C_h) e_k. \quad (8)$$

Now we will establish conditions for guaranteeing stability of the estimation error, while calibrating the observer.

Theorem 6. The origin of the estimation error dynamics (8) is asymptotically stable with a non-monotonically decreasing Lyapunov function, if there are proper size matrices $P > 0$, N_j , $Z_{ij}^{\mathbf{v}\mathbf{w}}$, Λ_{ij}^1 and Λ_{ij}^2 , such that:

$$\begin{aligned} & \mathbf{Z}_{ii} < 0, Q_{ii}^{\mathbf{v}\mathbf{w}} - Z_{ii}^{\mathbf{v}\mathbf{w}} - \Lambda_{ii}^1 \leq 0, \quad i = j, \quad \mathbf{v} = \mathbf{w}, \\ & Q_{ii}^{\mathbf{v}\mathbf{w}} + Q_{ii}^{\mathbf{w}\mathbf{v}} - Z_{ii}^{\mathbf{v}\mathbf{w}} - (Z_{ii}^{\mathbf{w}\mathbf{v}})^T - \Lambda_{ii}^2 \leq 0, \quad i = j, \quad \forall \mathbf{w} < \mathbf{v}, \\ & Q_{ij}^{\mathbf{v}\mathbf{v}} + Q_{ji}^{\mathbf{v}\mathbf{v}} - Z_{ij}^{\mathbf{v}\mathbf{v}} - Z_{ji}^{\mathbf{v}\mathbf{v}} - \Lambda_{ij}^1 - (\Lambda_{ij}^1)^T \leq 0, \quad \forall j < i, \quad \mathbf{v} = \mathbf{w}, \\ & Q_{ij}^{\mathbf{v}\mathbf{w}} + Q_{ij}^{\mathbf{w}\mathbf{v}} - Z_{ij}^{\mathbf{v}\mathbf{w}} - (Z_{ij}^{\mathbf{w}\mathbf{v}})^T - \Lambda_{ij}^2 - (\Lambda_{ij}^2)^T \leq 0, \quad \forall \mathbf{w} < \mathbf{v}, \quad j < i, \\ & \left(\frac{2}{r-1} \right) \mathbf{Z}_{ii} + \mathbf{Z}_{ij} + \mathbf{Z}_{ji} < 0, \quad \mathbf{Z}_{ij} = \begin{bmatrix} Z_{ij}^{11} & (Z_{ij}^{21})^T & \dots & (Z_{ij}^{r1})^T \\ Z_{ij}^{21} & Z_{ij}^{22} & \dots & (Z_{ij}^{r2})^T \\ \vdots & \vdots & \ddots & \vdots \\ Z_{ij}^{r1} & Z_{ij}^{r2} & \dots & Z_{ij}^{rr} \end{bmatrix}, \\ & \begin{bmatrix} \Lambda_{11}^1 & (\Lambda_{21}^1)^T & \dots & (\Lambda_{r1}^1)^T \\ \Lambda_{21}^1 & \Lambda_{22}^1 & \dots & (\Lambda_{r2}^1)^T \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda_{r1}^1 & \Lambda_{r2}^1 & \dots & \Lambda_{rr}^1 \end{bmatrix} < 0, \quad \begin{bmatrix} \Lambda_{11}^2 & (\Lambda_{21}^2)^T & \dots & (\Lambda_{r1}^2)^T \\ \Lambda_{21}^2 & \Lambda_{22}^2 & \dots & (\Lambda_{r2}^2)^T \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda_{r1}^2 & \Lambda_{r2}^2 & \dots & \Lambda_{rr}^2 \end{bmatrix} < 0, \end{aligned}$$

hold where $(i, j, \mathbf{v}, \mathbf{w}) \in \{1, 2, \dots, r\}$, $Q_{ij}^{\mathbf{v}\mathbf{w}}$ is defined in (12) and the observer gains are $L_j = P^{-1} N_j$.

Proof. First, consider a quadratic candidate Lyapunov function $V_k = e_k^T P e_k$ associated to the the estimation error dynamics (8) where $P > 0$. Then, let us impose the non-monotonically decreasing condition from Ahmadi and Parrilo (2008), that is, $V_{k+2} - V_k < 0$, it can be equivalently written via the extended vector $\bar{e}^T = [e_k^T \ e_{k+1}^T \ e_{k+2}^T]^T$ as

$$\bar{e}^T Q \bar{e} = \begin{bmatrix} e_k \\ e_{k+1} \\ e_{k+2} \end{bmatrix}^T \begin{bmatrix} -P & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & P \end{bmatrix} \begin{bmatrix} e_k \\ e_{k+1} \\ e_{k+2} \end{bmatrix} < 0, \quad (9)$$

which is subject to the algebraic constraint

$$\begin{bmatrix} A_h + L_h C_h & -I_n & 0 \\ 0 & A_{h(k+1)} + L_{h(k+1)} C_{h(k+1)} & -I_n \end{bmatrix} \begin{bmatrix} e_k \\ e_{k+1} \\ e_{k+2} \end{bmatrix} = B \bar{e} = 0, \quad (10)$$

where n is the size of the vector e_k and I_n is an identity matrix. Denoting $\bar{A}_h = A_h + L_h C_h$ and $\bar{A}_{h(k+1)} = A_{h(k+1)} + L_{h(k+1)} C_{h(k+1)}$, the inequality (9) subject to (10) can be

cast via Finsler's lemma with $\mathcal{X} = \begin{bmatrix} \mathcal{X}_{11} & \mathcal{X}_{12} \\ \mathcal{X}_{21} & \mathcal{X}_{22} \\ \mathcal{X}_{31} & \mathcal{X}_{32} \end{bmatrix}$ as (11).

There are two questions that must be addressed now:

- (1) How can we select the slack variable \mathcal{X} ? Our goal is to obtain LMIs such that a Lyapunov function and the observer gains can be computed from the solution.
- (2) How can we take into account the interaction of the membership functions $(h(z_k), h(z_{k+1}))$? In this case the premise variables involved are from two different

instants, z_k and z_{k+1} . Thus, the classical relaxations such as Liu&Zhang's is not a straightforward option.

A possible selection for the slack-matrix is $\mathcal{X} = \begin{bmatrix} O_n & \alpha_2 G \\ G & O_n \\ \alpha_1 G & G \end{bmatrix}$,

where $G = G^T > 0$, α_1 and α_2 are given scalars and O_n is a n -size zero matrix. Due to this selection we might be introducing conservativeness; but, for simplicity we will continue with it. Then, substituting $N_h = P L_h$, $N_{h(k+1)} = P L_{h(k+1)}$; and considering that $\Gamma_h = G A_h + N_h C_h$ and $\Gamma_{h(k+1)} = G A_{h(k+1)} + N_{h(k+1)} C_{h(k+1)}$ in (11) yields to

$$\underbrace{\begin{bmatrix} -P & \alpha_2 \Gamma_{h(k+1)} + \Gamma_h^T & \alpha_1 \Gamma_h^T - \alpha_2 G \\ \Gamma_h + \alpha_2 \Gamma_{h(k+1)}^T & -2G & \Gamma_{h(k+1)}^T - \alpha_1 G^T \\ \alpha_1 \Gamma_h - \alpha_2 G^T & \Gamma_{h(k+1)} - \alpha_1 G & P - 2G \end{bmatrix}}_{Q_{hh(k+1)}} < 0.$$

The observer gains are obtained as $L_h = G^{-1} N_h$. Now, let us denote the terms associated to z_k with the sub-indexes (i, j) , and the ones associated to z_{k+1} with the sub-indexes (\mathbf{v}, \mathbf{w}) . The membership functions can be stacked as

$$\begin{aligned} Q_{hh(k+1)} &= \sum_{i=1}^r \sum_{j=1}^r \sum_{\mathbf{v}=1}^r \sum_{\mathbf{w}=1}^r h_i(z_k) h_j(z_k) h_{\mathbf{v}}(z_{k+1}) h_{\mathbf{w}}(z_{k+1}) Q_{ij}^{\mathbf{v}\mathbf{w}}, \\ Q_{ij}^{\mathbf{v}\mathbf{w}} &= \begin{bmatrix} -P & \alpha_2 \Gamma_{\mathbf{v}\mathbf{w}} + \Gamma_{ij}^T & \alpha_1 \Gamma_{ij}^T - \alpha_2 G \\ \Gamma_{ij} + \alpha_2 \Gamma_{\mathbf{v}\mathbf{w}}^T & -2G & \Gamma_{\mathbf{v}\mathbf{w}}^T - \alpha_1 G^T \\ \alpha_1 \Gamma_{ij} - \alpha_2 G^T & \Gamma_{\mathbf{v}\mathbf{w}} - \alpha_1 G & P - 2G \end{bmatrix}. \end{aligned} \quad (12)$$

A trivial choice to satisfy $Q_{hh(k+1)} < 0$ would be asking for $Q_{ij}^{\mathbf{v}\mathbf{w}} < 0$, $\forall (i, j, \mathbf{v}, \mathbf{w})$; but, this is too conservative. Thus, we proceed to adapt Liu&Zhang's relaxation for considering the positiveness of the convex functions and the interaction among the matrices. First, notice that $Q_{hh(k+1)} < 0$ is algebraically equivalent to:

$$\sum_{i,j=1}^r h_i h_j \begin{bmatrix} I_{3n} h_1(z_{k+1}) \\ I_{3n} h_2(z_{k+1}) \\ \vdots \\ I_{3n} h_r(z_{k+1}) \end{bmatrix}^T \begin{bmatrix} Q_{ij}^{11} & Q_{ij}^{12} & \dots & Q_{ij}^{1r} \\ Q_{ij}^{21} & Q_{ij}^{22} & \dots & Q_{ij}^{2r} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{ij}^{r1} & Q_{ij}^{r2} & \dots & Q_{ij}^{rr} \end{bmatrix} \begin{bmatrix} I_{3n} h_1(z_{k+1}) \\ I_{3n} h_2(z_{k+1}) \\ \vdots \\ I_{3n} h_r(z_{k+1}) \end{bmatrix} < 0.$$

Following the developments in Xiaodong and Qingling (2003); Arceo and Lauber (2020); the latter is satisfied

$$\forall h(z_{k+1}) \geq 0, \text{ if } \exists Z_{ij}^{\mathbf{v}\mathbf{w}}, \text{ such that: } \sum_{i,j=1}^r h_i h_j (Q_{ij}^{\mathbf{v}\mathbf{w}} - Z_{ij}^{\mathbf{v}\mathbf{w}}) \leq 0,$$

$$\sum_{i,j=1}^r h_i h_j \underbrace{(Q_{ij}^{\mathbf{v}\mathbf{w}} + Q_{ij}^{\mathbf{w}\mathbf{v}} - Z_{ij}^{\mathbf{v}\mathbf{w}} - (Z_{ij}^{\mathbf{w}\mathbf{v}})^T)}_{\Gamma_{ij}^{\mathbf{v}\mathbf{w}}} \leq 0, \quad \forall \mathbf{w} < \mathbf{v}, \quad \mathbf{Z}_{ii} < 0,$$

$$\text{and } \left(\frac{2}{r-1} \right) \mathbf{Z}_{ii} + \mathbf{Z}_{ij} + \mathbf{Z}_{ji} < 0, \quad \mathbf{Z}_{ij} = \begin{bmatrix} Z_{ij}^{11} & (Z_{ij}^{21})^T & \dots & (Z_{ij}^{r1})^T \\ Z_{ij}^{21} & Z_{ij}^{22} & \dots & (Z_{ij}^{r2})^T \\ \vdots & \vdots & \ddots & \vdots \\ Z_{ij}^{r1} & Z_{ij}^{r2} & \dots & Z_{ij}^{rr} \end{bmatrix},$$

hold. Then, we can rewrite the double-sum expressions as

$$\begin{aligned} & \begin{bmatrix} I_{3n} h_1(z_k) \\ I_{3n} h_2(z_k) \\ \vdots \\ I_{3n} h_r(z_k) \end{bmatrix}^T \begin{bmatrix} Q_{11}^{\mathbf{v}\mathbf{v}} - Z_{11}^{\mathbf{v}\mathbf{v}} & Q_{12}^{\mathbf{v}\mathbf{v}} - Z_{12}^{\mathbf{v}\mathbf{v}} & \dots & Q_{1r}^{\mathbf{v}\mathbf{v}} - Z_{1r}^{\mathbf{v}\mathbf{v}} \\ Q_{21}^{\mathbf{v}\mathbf{v}} - Z_{21}^{\mathbf{v}\mathbf{v}} & Q_{22}^{\mathbf{v}\mathbf{v}} - Z_{22}^{\mathbf{v}\mathbf{v}} & \dots & Q_{2r}^{\mathbf{v}\mathbf{v}} - Z_{2r}^{\mathbf{v}\mathbf{v}} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{r1}^{\mathbf{v}\mathbf{v}} - Z_{r1}^{\mathbf{v}\mathbf{v}} & Q_{r2}^{\mathbf{v}\mathbf{v}} - Z_{r2}^{\mathbf{v}\mathbf{v}} & \dots & Q_{rr}^{\mathbf{v}\mathbf{v}} - Z_{rr}^{\mathbf{v}\mathbf{v}} \end{bmatrix} \begin{bmatrix} I_{3n} h_1(z_k) \\ I_{3n} h_2(z_k) \\ \vdots \\ I_{3n} h_r(z_k) \end{bmatrix} \leq 0, \\ & \begin{bmatrix} I_{3n} h_1(z_k) \\ I_{3n} h_2(z_k) \\ \vdots \\ I_{3n} h_r(z_k) \end{bmatrix}^T \begin{bmatrix} \Gamma_{11}^{\mathbf{v}\mathbf{w}} & \Gamma_{12}^{\mathbf{v}\mathbf{w}} & \dots & \Gamma_{1r}^{\mathbf{v}\mathbf{w}} \\ \Gamma_{21}^{\mathbf{v}\mathbf{w}} & \Gamma_{22}^{\mathbf{v}\mathbf{w}} & \dots & \Gamma_{2r}^{\mathbf{v}\mathbf{w}} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{r1}^{\mathbf{v}\mathbf{w}} & \Gamma_{r2}^{\mathbf{v}\mathbf{w}} & \dots & \Gamma_{rr}^{\mathbf{v}\mathbf{w}} \end{bmatrix} \begin{bmatrix} I_{3n} h_1(z_k) \\ I_{3n} h_2(z_k) \\ \vdots \\ I_{3n} h_r(z_k) \end{bmatrix} \leq 0, \quad \forall \mathbf{w} < \mathbf{v}, \end{aligned}$$

$$\begin{bmatrix} -P + \bar{A}_h \mathcal{X}_{11} + \bar{A}_h^T \mathcal{X}_{11}^T & \mathcal{X}_{12} \bar{A}_{h(k+1)} + \bar{A}_h^T \mathcal{X}_{21}^T - \mathcal{X}_{11} & \bar{A}_h^T \mathcal{X}_{31}^T - \mathcal{X}_{12} \\ \mathcal{X}_{21} \bar{A}_h + \bar{A}_{h(k+1)}^T \mathcal{X}_{12}^T - \mathcal{X}_{11}^T & \mathcal{X}_{22} \bar{A}_{h(k+1)} + \bar{A}_{h(k+1)}^T \mathcal{X}_{22}^T - \mathcal{X}_{21} - \mathcal{X}_{21}^T & \bar{A}_{h(k+1)}^T \mathcal{X}_{32}^T - \mathcal{X}_{31}^T - \mathcal{X}_{22} \\ \mathcal{X}_{31} \bar{A}_h - \mathcal{X}_{12}^T & \mathcal{X}_{32} \bar{A}_{h(k+1)} - \mathcal{X}_{31} - \mathcal{X}_{22}^T & P - \mathcal{X}_{32} - \mathcal{X}_{32}^T \end{bmatrix} < 0, \quad (11)$$

these will hold for all $h(z_k) \geq 0$, if $\exists (\Lambda_{ij}^1, \Lambda_{ij}^2)$ such that:

$$\begin{bmatrix} \Lambda_{11}^1 & (\Lambda_{21}^1)^T & \cdots & (\Lambda_{r1}^1)^T \\ \Lambda_{21}^1 & \Lambda_{22}^1 & \cdots & (\Lambda_{r2}^1)^T \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda_{r1}^1 & \Lambda_{r2}^1 & \cdots & \Lambda_{rr}^1 \end{bmatrix} < 0, \quad \begin{bmatrix} \Lambda_{11}^2 & (\Lambda_{21}^2)^T & \cdots & (\Lambda_{r1}^2)^T \\ \Lambda_{21}^2 & \Lambda_{22}^2 & \cdots & (\Lambda_{r2}^2)^T \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda_{r1}^2 & \Lambda_{r2}^2 & \cdots & \Lambda_{rr}^2 \end{bmatrix} < 0,$$

$$Q_{ii}^{\mathbf{v}\mathbf{v}} - \mathcal{Z}_{ii}^{\mathbf{v}\mathbf{v}} - \Lambda_{ii}^1 \leq 0,$$

$$Q_{ii}^{\mathbf{v}\mathbf{w}} + Q_{ii}^{\mathbf{w}\mathbf{v}} - \mathcal{Z}_{ii}^{\mathbf{v}\mathbf{w}} - (\mathcal{Z}_{ii}^{\mathbf{w}\mathbf{v}})^T - \Lambda_{ii}^2 \leq 0, \quad \forall \mathbf{w} < \mathbf{v},$$

$$Q_{ij}^{\mathbf{v}\mathbf{v}} + Q_{ji}^{\mathbf{v}\mathbf{v}} - \mathcal{Z}_{ij}^{\mathbf{v}\mathbf{v}} - \mathcal{Z}_{ji}^{\mathbf{v}\mathbf{v}} - \Lambda_{ij}^1 - (\Lambda_{ij}^1)^T \leq 0, \quad \forall j < i,$$

$$Q_{ij}^{\mathbf{v}\mathbf{w}} + Q_{ij}^{\mathbf{w}\mathbf{v}} - \mathcal{Z}_{ij}^{\mathbf{v}\mathbf{w}} - (\mathcal{Z}_{ij}^{\mathbf{w}\mathbf{v}})^T - \Lambda_{ij}^2 - (\Lambda_{ij}^2)^T \leq 0, \quad \forall \mathbf{w} < \mathbf{v}, j < i,$$

hold. This concludes the proof.

In the following theorem we present the conditions for stabilizing the estimation error with a constant gain L .

Theorem 7. The origin of the estimation error dynamics $e_{k+1} = (A_h + LC_h) e_k$ is asymptotically stable with a non-monotonically decreasing Lyapunov function, if there are proper size matrices P , G , N and \mathcal{Z}_{ij} , such that $P > 0$, $G + G^T > 0$, $\xi_i^i - \mathcal{Z}_{ii} \leq 0$, $\xi_i^i + \xi_i^{\mathbf{v}} + \xi_{\mathbf{v}}^i - \mathcal{Z}_{ij} - (\mathcal{Z}_{ij})^T \leq 0$,

$$\forall j < i, \text{ and } \begin{bmatrix} \mathcal{Z}_{11} & (\mathcal{Z}_{21})^T & \cdots & (\mathcal{Z}_{r1})^T \\ \mathcal{Z}_{21} & \mathcal{Z}_{22} & \cdots & (\mathcal{Z}_{r2})^T \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{Z}_{r1}^T & \mathcal{Z}_{r2}^T & \cdots & \mathcal{Z}_{rr}^T \end{bmatrix} < 0, \text{ hold, with } \xi_i^{\mathbf{v}} \text{ in}$$

$$(13), (i, \mathbf{v}) \in \{1, 2, \dots, r\}, \text{ and } L = G^{-1}N.$$

Proof. First, consider that the observer has the structure $\hat{x}_{k+1} = A_h \hat{x}_k - L(y_k - \hat{y}_k) + b_k$, $\hat{y}_k = C_h \hat{x}_k$. Its error dynamics are $e_{k+1} = (A_h + LC_h) e_k$. Now, consider a candidate Lyapunov function $V_k = e_k^T P e_k$, $P > 0$, that decreases non-monotonically $V_{k+2} - V_k < 0$. Following the previous devel-

opments, applying Finsler's Lemma with $\mathcal{X} = \begin{bmatrix} O_n & \alpha_2 G \\ G & O_n \\ \alpha_1 G & G \end{bmatrix}$,

where $G = G^T > 0$, α_1 and α_2 are two given real scalars; then, substituting $N = GL$, and stacking the convex func-

tions gives $\xi_{hh(k+1)} = \sum_{i=1}^r \sum_{\mathbf{v}=1}^r h_i(z_k) h_{\mathbf{v}}(z_{k+1}) \xi_i^{\mathbf{v}} < 0$, with

$$\xi_i^{\mathbf{v}} = \begin{bmatrix} -P & \alpha_2 \Gamma_{\mathbf{v}} + \Gamma_i^T & \alpha_1 \Gamma_i^T - \alpha_2 G \\ \Gamma_i + \alpha_2 \Gamma_{\mathbf{v}}^T & -2G & \Gamma_{\mathbf{v}}^T - \alpha_1 G^T \\ \alpha_1 \Gamma_i - \alpha_2 G^T & \Gamma_{\mathbf{v}} - \alpha_1 G & P - 2G \end{bmatrix}. \quad (13)$$

and $\Gamma_i = GA_i + NC_i$. Finally, $\xi_{hh(k+1)} < 0$ is guaranteed via Liu & Zhang's relaxation (Lemma 4). It yields to the LMIs given in this theorem, and concludes the proof.

Now we will use the previous results for observer design.

5. NUMERICAL IMPLEMENTATION

In this section we design a nonlinear observer for monitoring biomass concentration. Then, we compare the conditions based on monotone Lyapunov functions (Lemma 5) with the new based on non-monotone ones (Theorem 6).

5.1 Observer synthesis for microalgae monitoring

The parameters for *Chlorella sorokiniana* growth-limited by acetate are $\bar{\mu} = 2.23$ per day, $k_a = 0.002gC/L$ and $\gamma_a = 0.84g/gC$, from Turon et al. (2015). The oxygen yield is $\gamma_o = 0.58g/gO$, in the heterotroph cultive of chlorella-like microalgae. The oxygen concentration at the input was equals to its saturation value, $o_{in} = o^* = 6.2202mgO/L$. Now assume an acetate concentration at the input $s_{in} = 0.4g/L$, a mass transfer coefficient $k_L a = 20$ per day, the oxygen half-saturation constant $k_o = 0.003gO/L$, a minimum uptake $\phi_{min} = 0.01$, a periodic dilution $d_t = 0.3 + 0.14 \sin(0.2t)$ and a sampling period $T_s = \frac{30}{1440}$ days. Thus, the LMIs in Theorem 7 were programmed in MATLAB using YALMIP and solved with MOSEK, the matrices obtained with $\alpha_1 = \alpha_2 = 0.5$ were

$$P = \begin{bmatrix} 1.2569 & 0.5772 & 0.3762 \\ 0.5772 & 1.4344 & 0.8359 \\ 0.3762 & 0.8359 & 1.5419 \end{bmatrix}, \quad L = \begin{bmatrix} -0.5775 & -0.0040 \\ 0.2384 & 0.3453 \\ -0.0040 & -0.1638 \end{bmatrix},$$

$$G = \begin{bmatrix} 1.0722 & 0.3783 & 0.2504 \\ 0.3783 & 0.9577 & 0.5480 \\ 0.2504 & 0.5480 & 1.2619 \end{bmatrix}.$$

The $\mathcal{Z}_{i\mathbf{v}}$ matrices were omitted for brevity. Consider as initial conditions $\bar{x}_0 = [0.05 \ 0.1 \ 0.006]^T$ for the system, and $\hat{x}_0 = [0.05 \ 0.3 \ 0.006]^T$ for the observer. The obtained signals for biomass concentration in simulation are shown in Fig. 2a and the time-evolution of the Lyapunov function is in Fig. 2b; as we can see, the trajectory of the Lyapunov function is monotonically decreasing, this is possible, since the monotone case is included within the non-monotone.

5.2 Comparison with monotone Lyapunov functions

We are interested now in performing a numerical comparison of the feasible solutions provided by the LMIs for observer design with monotone Lyapunov functions (Lemma 5) and the ones based on non-monotone Lyapunov functions (Theorem 6). Thus, consider the matrices:

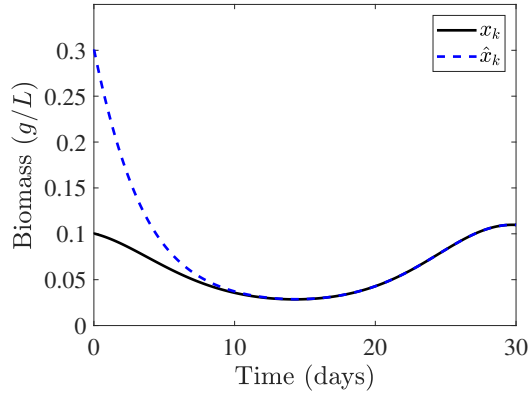
$$A_1 = \begin{bmatrix} 2 & -10 & 0.5 \\ 1 & 0 & -1 \\ 0.1 & 0.5 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} a & -10 & 0.5 \\ 1 & 0.65 & -1 \\ 0.1 & 0.5 & 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}^T, \quad C_2 = \begin{bmatrix} b \\ 0 \\ -1 \end{bmatrix}^T,$$

where $a \in [-10, 10]$ and $b \in [1.2, 1.8]$. The results of the feasibility tests for Lemma 5 and Theorem 6 with $\alpha_1 = \alpha_2 = 0.05$ are shown in Fig. 3, as expected the new conditions are able to overcome the ones in Lemma 5.

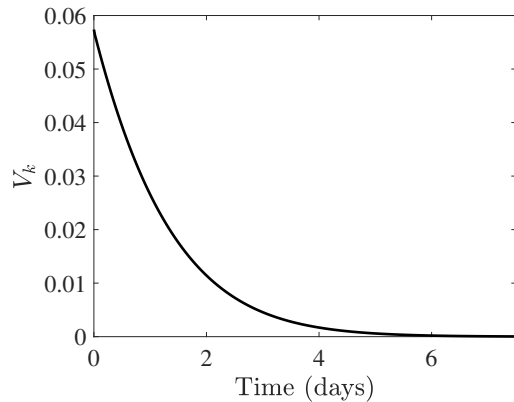
6. CONCLUSION

A new approach for monitoring heterotrophic microalgae was presented, it was illustrated in simulation where we use it to estimate biomass concentration.

New conditions for discrete-time nonlinear observer design via non-monotone Lyapunov functions were introduced, these were given in terms of LMIs. The new conditions include the monotone case as illustrated in the microalgae model. Previous approaches that yield to classical conditions have been pointed out, these classical conditions



(a) Evolution of the unmeasured (x_k) and estimated \hat{x}_k biomass.



(b) Time-evolution of the Lyapunov function.

Fig. 2. Estimation of the biomass and Lyapunov function.

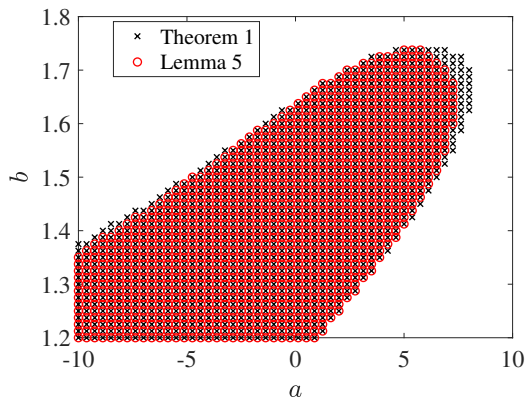


Fig. 3. Feasible points for LMIs in Lemma 5 (\circ) and the conditions in Theorem 6 (\times) with $\alpha_1 = \alpha_2 = 0.05$.

can be avoided if Finsler's Lemma is applied instead. A comparison between the new LMIs and classical monotone Lyapunov functions was carried out by means of a second example, our conditions were less conservative. However, the selection of the slack-variable \mathcal{X} needs more study.

Possible extensions of this work include non-monotone observers with unmeasured premise variables, since we assumed that the premise variables can be measured.

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