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Characterization of invariant inner products

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Abstract. In several situations in differential geometry, one can be interested in determining all inner products on a vector space that are invariant under a given group action. For example, bi-invariant Riemannian metrics on a Lie group G are characterized by $\text{Ad}(G)$ -invariant inner products on the Lie algebra \mathfrak{g} . Analogously, G -invariant Riemannian metrics on a homogeneous space $\mathcal{M} = G/H$ are characterized by $\text{Ad}(H)$ -invariant inner products on the tangent space $T_H\mathcal{M}$. In addition, given a G -equivariant diffeomorphism between a manifold \mathcal{M} and a Euclidean space V , G -invariant log-Euclidean metrics can be defined on \mathcal{M} by pullback of G -invariant inner products on V . There exists a general procedure based on representation theory to find all invariant inner products on a completely reducible Hermitian space. It consists in changing the viewpoint from invariant inner products to equivariant automorphisms. The goal of this work is to diffuse this method to communities of applied mathematics which use differential geometry. Therefore, in this work, we recall this general method that we did not find elsewhere, along with an elementary presentation of the basics of representation theory.

Keywords: Invariant inner product · Invariant Riemannian metric · Group action · Representation theory

1 Introduction

When one looks for appropriate metrics on a given space representing some data, it is natural to require them to be invariant under a certain group action. For example, when data are represented by Symmetric Positive Definite (SPD) matrices, one can use Riemannian metrics that are invariant under the congruence action of the general linear group (affine-invariant metrics [15, 13, 9, 8, 12, 3]), the orthogonal group [18] (e.g. log-Euclidean [1], Bures-Wasserstein [4, 7, 16, 10], Bogoliubov-Kubo-Mori metrics [14, 11]), the group of positive diagonal matrices [5, 6, 17], the permutation group. In several situations, the question of finding all G -invariant Riemannian metrics on a manifold reduces to finding all H -invariant inner products on a vector space where H is a Lie subgroup of G . For example, bi-invariant metrics (or pseudo-metrics) on a Lie group G are characterized by $\text{Ad}(G)$ -invariant inner products (or non-degenerate symmetric bilinear forms)

on its Lie algebra \mathfrak{g} . On a homogeneous space $\mathcal{M} = G/H$, G -invariant metrics are characterized by $\text{Ad}(H)$ -invariant inner products on the tangent space $\mathfrak{m} = T_H\mathcal{M}$ at the equivalence class $H \in G/H$.

Another frequent case is when the manifold \mathcal{M} is diffeomorphic to a vector space V . For example, the cone of SPD matrices is diffeomorphic to the vector space of symmetric matrices by the symmetric matrix logarithm $\log : \text{Sym}^+(n) \rightarrow \text{Sym}(n)$. Similarly, the ellipsope of full-rank correlation matrices $\text{Cor}^+(n)$ is diffeomorphic to a vector space of dimension $\frac{n(n-1)}{2}$. Hence, given a G -equivariant diffeomorphism $\phi : \mathcal{M} \rightarrow V$ between the manifold \mathcal{M} and a vector space V , the G -invariant inner products on V provide natural flat G -invariant Riemannian metrics on \mathcal{M} by pullback. These three examples of bi-invariant metrics on Lie groups, Riemannian homogeneous manifolds and “Euclideanized” manifolds motivate to find all G -invariant inner products on a vector space V . Indeed, if one wants to impose an invariance on a space, this requirement defines a family of metrics in general, rarely a unique metric. Therefore, there is no reason a priori to distinguish between all the metrics that satisfy this requirement. Then, it is possible to reduce the choice by requiring other invariance or constraints, or to optimize within the family in function of the data for example.

To answer this question, the central notion is the reducibility or irreducibility of a vector space under a group action. We say that V is G -irreducible when there is no other subvector space than $\{0\}$ and V that is stable under the action of G . It can easily be proved that inner products on an irreducible space are positive scalings of one another. Hence, when V is completely reducible, i.e. V can be expressed into a direct sum of irreducible subspaces, then any positive linear combination of inner products on each irreducible subspace is an invariant inner product. For example, if $G = \{-1, 1\} \times \{-1, 1\}$ acts on $V = \mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$ component by component, then one can easily check that all G -invariant inner products are given by $\varphi((x, y), (x, y)) = \alpha x^2 + \beta y^2$ with $\alpha, \beta > 0$.

However, there are other invariant inner products in general. Take another example where $G = \{-1, 1\}$ acts on $V = \mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$ globally. It is clear that all the G -invariant inner products are given by $\varphi((x, y), (x, y)) = \alpha x^2 + 2\gamma xy + \beta y^2$ with $\alpha, \beta > 0$ and $\alpha\beta > \gamma^2$. In the first example, G acts differently on each copy of \mathbb{R} : for example $(1, -1) \cdot (x, y) = (x, -y)$. In the second example, G acts identically on each copy of \mathbb{R} , they are indistinguishable with respect to the action. This is why another coefficient is allowed between the two components. Therefore, in the irreducible decomposition, one has to group all the irreducible spaces on which G acts the same way to find all G -invariant inner products. That is what we is summarized in Theorem 1.

In this work, we assume that V is a complex vector space. We also assume that we know a G -invariant inner product on V and that V is completely reducible, i.e. there exists an irreducible decomposition of V . This is the case when G is finite [2, Maschke’s Theorem 10.2.10] and also when the group action is unitary and continuous [2, Corollary 10.3.5]. In particular, when G is finite or compact, it is well known that one can find *one* G -invariant inner product by taking any inner product and averaging over G with the counting measure or

the Haar measure $\mu: \langle x|y \rangle := \frac{1}{|G|} \sum_{a \in G} ax \cdot ay$ or $\int_G (ax \cdot ay) \mu(da)$. So the two hypotheses are automatically satisfied when G is finite or when G is compact and the action is continuous.

Our method relies on basic representation theory. Indeed, any G -invariant inner product on V is characterized, via the given inner product $\langle \cdot | \cdot \rangle$, by a G -invariant automorphism of V . Then, representation theory allows to find the general form of this automorphism, hence the general form of G -invariant inner products on V . Note that this characterization of invariant inner products is well known in algebra although we could not find it explicitly in the references we read. Yet, it seems to be much less known in the communities of applied mathematics which use differential geometry, although it seems quite useful. That is why we present it with the fewest details on representation theory.

2 Preliminaries on representation theory

2.1 Vocabulary of representation theory

The following terminology is neither unique in representation theory nor exclusive to this branch of mathematics. We use the term “module” for simplicity, in reference to “ G -module” which is frequent.

Definition 1 (Vocabulary of representation theory). *Let G be a group with neutral element e , and V be a complex vector space of finite dimension.*

- *A representation of G on V is a group homomorphism $\rho : G \longrightarrow \text{GL}(V)$. It is equivalent to a linear group action of G on V , i.e. a map $\rho : G \times V \longrightarrow V$ such that for all $a, b \in G$ and all $x \in V$, $\rho(a, \rho(b, x)) = \rho(ab, x)$, $\rho(e, x) = x$ and $\rho_a : x \in V \longmapsto \rho(a, x) \in V$ is linear.*
- *In this work, a (G -)module will designate a vector space V on which G acts linearly and continuously via $\rho : G \times V \longrightarrow V$.*
- *A submodule of V is a ρ -stable subvector space of V .*
- *A module homomorphism (resp. endomorphism, isomorphism, automorphism) is a ρ -equivariant linear map (resp. endomorphism, isomorphism, automorphism) between modules. Given modules V, V' , we denote $V \simeq V'$ when there exists a module isomorphism from V to V' and we say that V and V' are isomorphic modules.*
- *A module V is irreducible if it has no other submodules than $\{0\}$ and V . Otherwise, it is reducible.*
- *A module V is completely reducible if it is the direct sum of irreducible modules.*

From now on, let $(V, \langle \cdot | \cdot \rangle)$ be a Hermitian space (i.e. a complex vector space of finite dimension endowed with an inner product), $V \neq \{0\}$. We denote $\| \cdot \|$ the associated norm. The set $\text{U}(V) = \{f \in \text{GL}(V) | \forall x \in V, \|f(x)\| = \|x\|\}$ is the subgroup of isometries of $\text{GL}(V)$. Let $\rho : G \longrightarrow \text{U}(V)$ be a representation of a group G acting isometrically on V .

We assume that V is completely reducible. For example, this is the case when G is finite [2, Maschke’s Theorem 10.2.10] or when ρ is unitary and continuous [2, Corollary 10.3.5].

Note that if W is a submodule of V , then W^\perp (where orthogonality refers to the inner product $\langle \cdot | \cdot \rangle$) is also a submodule of V . Indeed, if $x \in W^\perp$ and $a \in G$, then for all $y \in W$, $\rho(a^{-1})(y) \in W$ because W is a module so $\langle \rho(a)(x) | y \rangle = \langle x | \rho(a^{-1})(y) \rangle = 0$, so $\rho(a)(x) \in W^\perp$. Therefore, we can assume that the decomposition of V into irreducible submodules is orthogonal.

2.2 The big picture

We split the method into three steps:

1. Transform the problem of finding all ρ -invariant inner products on V into finding all ρ -equivariant automorphisms of V .

Let φ be an inner product on V . We recall the musical isomorphisms of φ :

- “Flat” $b_\varphi : x \in V \mapsto \varphi(x, \cdot) \in V^*$ which is a linear isomorphism by the Riesz representation theorem,
- “Sharp” $\#_\varphi = b_\varphi^{-1} : V^* \rightarrow V$ the inverse linear isomorphism.

We denote $b = b_{\langle \cdot | \cdot \rangle}$ and $\# = \#_{\langle \cdot | \cdot \rangle}$. Then φ is a ρ -invariant inner product on V if and only if $f_\varphi := \# \circ b_\varphi : V \rightarrow V$ is a ρ -equivariant automorphism of V . We retrieve φ from $f := f_\varphi$ by $\varphi(x, y) = \langle f(x) | y \rangle$ for all $x, y \in V$.

The core of the method is to determine the general form of f .

2. Find all ρ -equivariant automorphisms of V using representation theory.
3. Go back to the initial problem.

Before the general case (V completely reducible), it is natural to start with the particular case where V is irreducible (Section 3.1). Then, we explain why the general case does not reduce to the irreducible case but only to an intermediate case (Section 3.2). Therefore, we treat this intermediate case (Section 3.3) and we conclude with the general case (Section 3.4).

3 Find all equivariant automorphisms

3.1 The particular case: V irreducible

We start with the simplest case where V is irreducible. The key result to characterize all ρ -equivariant automorphisms of V is Schur’s lemma.

Lemma 1. [2, Schur’s Lemma 10.7.6] *Let V, W be irreducible modules.*

1. *A module homomorphism $f : V \rightarrow W$ is either null or a module isomorphism.*
2. *A module endomorphism $f : V \rightarrow V$ is a scaling, i.e. there exists $\alpha \in \mathbb{C}$ such that $f = \alpha \text{Id}_V$.*

Indeed, the first statement holds because $f(V)$ is a submodule of W so $f(V) = \{0\}$ or W and $\ker(f)$ is a submodule of V so $\ker(f) = V$ or $\{0\}$. The second statement holds because if $\alpha \in \mathbb{C}$ is an eigenvalue of f – it exists because \mathbb{C} is algebraically closed –, then $f - \alpha \text{Id}_V$ is not a module isomorphism so it is null by the first statement.

As a consequence, if V is an irreducible module, then:

- Module automorphisms on V are the non null scalings.
- Module isomorphisms between V and an irreducible module W are unique up to scaling. Indeed, if $f, f_0 : V \rightarrow W$ are module isomorphisms, then $f_0^{-1} \circ f : V \rightarrow V$ is a scaling, i.e. there exists $\alpha \in \mathbb{C}^*$ such that $f = \alpha f_0$.
- The ρ -invariant inner products on V are positive multiples of $\langle \cdot | \cdot \rangle$.

3.2 The general case does not reduce to the previous case

We continue with the case where V is completely reducible. Therefore, let V_1, \dots, V_m be irreducible submodules of V such that $V = V_1 \oplus \dots \oplus V_m$. From the previous case, it is clear that the map $f : x = x_1 + \dots + x_m \in V \mapsto \alpha_1 x_1 + \dots + \alpha_m x_m \in V$ with $\alpha_1, \dots, \alpha_m \in \mathbb{C}^*$ is a module automorphism of V . The question is: are they the only ones? The answer is no and the following consequence of Schur's lemma allows to explain precisely why.

Lemma 2 (Consequence of Schur's lemma on the irreducible decomposition). *We group V_1, \dots, V_m by classes $\mathcal{C}^1, \dots, \mathcal{C}^p$ of isomorphic irreducible modules. The decomposition becomes $V = \mathcal{V}^1 \oplus \dots \oplus \mathcal{V}^p$ with $\mathcal{V}^k = \bigoplus_{V_i \in \mathcal{C}^k} V_i$. Let $f : V \rightarrow V$ be a module automorphism. Then $f(\mathcal{V}^k) = \mathcal{V}^k$ for all $k \in \{1, \dots, p\}$.*

Proof. For $i, j \in \{1, \dots, m\}$, let $f_{ij} = \text{proj}_{V_j} \circ f|_{V_i} : V_i \rightarrow V_j$. Let $k \in \{1, \dots, p\}$ and let $i \in \{1, \dots, n\}$ such that $V_i \in \mathcal{C}^k$. By Schur's lemma, for all $j \in \{1, \dots, n\}$, f_{ij} is null or it is an isomorphism. Since f is an isomorphism, there exists $j \in \{1, \dots, m\}$ such that f_{ij} is non null, thus an isomorphism. Hence $V_j \in \mathcal{C}^k$ so $V_j \subseteq \mathcal{V}^k$. Therefore, $f(V_i) = V_j \subseteq \mathcal{V}^k$. This inclusion is valid for all $V_i \in \mathcal{C}^k$ so $f(\mathcal{V}^k) \subseteq \mathcal{V}^k$ and $f(\mathcal{V}^k) = \mathcal{V}^k$ by equality of dimensions because f is bijective.

In other words, the study of ρ -invariant automorphisms of V cannot be reduced to the V_i 's but only to the \mathcal{V}^k 's: there is no reason that $f(V_i) = V_i$ for all $i \in \{1, \dots, m\}$ (unless all classes are singletons). So we need to study the case $V = \mathcal{V} \simeq mW$ with W irreducible, where mW is a notation for the direct sum of m irreducible modules isomorphic to W .

3.3 The intermediate case: $V \simeq mW$ with W irreducible

We assume that $V = V_1 \oplus \dots \oplus V_m$ where $V_1 \simeq \dots \simeq V_m$ are isomorphic irreducible modules. Let W be an irreducible module isomorphic to them, endowed with a G -invariant inner product $\langle \cdot | \cdot \rangle$. Let $\psi_i : V_i \rightarrow W$ be the unique module

isomorphism which is an isometry. Indeed, the module isomorphism is unique up to scaling and the pullback of the inner product $\langle \cdot | \cdot \rangle$ onto V_i is necessarily a scaling of the restriction of $\langle \cdot | \cdot \rangle$ to V_i so there is a unique choice of ψ_i such that it is an isometry. For example, W can be taken as one of the V_i 's.

Let f be a module automorphism of V . We define the module endomorphisms $f_{ij} = \psi_j \circ \text{proj}_{V_j} \circ f \circ \psi_i^{-1} : W \rightarrow W$. By Schur's lemma, they are scalings: there exists $S_{ij} \in \mathbb{C}$ such that $f_{ij} = S_{ij} \text{Id}_W$. This defines a matrix $S = (S_{ij})_{1 \leq i, j \leq m} \in \text{Mat}_m(\mathbb{C})$. Then, f writes:

$$\begin{aligned} f(x) &= (\text{Id}_V \circ f)(x) = \left(\sum_{j=1}^m \text{proj}_{V_j} \right) \circ f \left(\sum_{i=1}^m x_i \right) = \sum_{i=1}^m \sum_{j=1}^m (\text{proj}_{V_j} \circ f)(x_i) \\ &= \sum_{i=1}^m \sum_{j=1}^m (\psi_j^{-1} \circ f_{ij} \circ \psi_i)(x_i) = \sum_{i=1}^m \sum_{j=1}^m S_{ij} \psi_j^{-1} \circ \psi_i(x_i). \end{aligned} \quad (1)$$

When f comes from an inner product φ as explained in Section 2.2, we have for all $x, y \in V$:

$$\varphi(x, y) = \langle f(x) | y \rangle = \sum_{i=1}^m \sum_{j=1}^m S_{ij} \langle \psi_j^{-1} \circ \psi_i(x_i) | y_j \rangle = \sum_{i=1}^m \sum_{j=1}^m S_{ij} (\psi_i(x_i) | \psi_j(y_j)),$$

because the V_i 's are orthogonal and $\psi_j : (V_j, \langle \cdot | \cdot \rangle) \rightarrow (W, (\cdot | \cdot))$ is an isometry. This implies that S is Hermitian Positive Definite (HPD). Indeed, let $w \in W \setminus \{0\}$, $a \in \mathbb{R}^m$ and $x = \sum_{i=1}^k a_i \psi_i^{-1}(w)$. Then:

- if $a = (1, \dots, 1)$, then $\varphi(x, x_j) = S_{ij} \|w\|^2$ and by conjugate symmetry of φ , we have $\varphi(x_i, x_j) = \varphi(x_j, x_i) = \overline{S_{ji}} \|w\|^2$ so $S_{ij} = \overline{S_{ji}}$,
- we have $\varphi(x, x) = \sum_{i,j} S_{ij} \overline{a_i} a_j \|w\|^2$ so for all $a \in \mathbb{R}^m \setminus \{0\}$, we have $\sum_{i,j} S_{ij} \overline{a_i} a_j > 0$.

Conversely, if S is Hermitian positive definite, then $\varphi(x, y) = \langle f(x) | y \rangle$ defines an inner product. It is clearly conjugate symmetric and if $x \neq 0$, there exists $w \in W \setminus \{0\}$ and $a \in \mathbb{R}^m \setminus \{0\}$ such that $x = \sum_{i=1}^k a_i \psi_i^{-1}(w)$ so the equality above proves that $\varphi(x, x) > 0$ (for the existence of w and a , take $i \in \{1, \dots, p\}$ such that $x_i \neq 0$ and define $w = \psi_i(x_i)$ and $a_j = \frac{\psi_j(x_j)}{\psi_i(x_i)}$ for $j \in \{1, \dots, m\}$).

So f is a module isomorphism of $V = V_1 \oplus \dots \oplus V_m$ if and only if there exists a Hermitian positive definite matrix $S \in \text{Mat}_m(\mathbb{C})$ such that f writes as in Equation (1). Now we have all the ingredients to state the global result.

3.4 The general case: V completely reducible

Theorem 1 (General form of a ρ -invariant inner product on V). *Let $V = \bigoplus_{k=1}^p \mathcal{V}^k$, with $\mathcal{V}^k = \bigoplus_{i=1}^{m_k} V_i^k$, be an orthogonal decomposition where $V_1^k \simeq \dots \simeq V_{m_k}^k$ are irreducible modules. For all $k \in \{1, \dots, p\}$, let $(W^k, (\cdot | \cdot)_k)$ be a Hermitian space and $\psi_i^k : V_i^k \rightarrow W^k$ be the unique module isomorphism which*

is an isometry. Then, an inner product φ on V is ρ -invariant if and only if there exist p HPD matrices $S^k \in \text{Mat}_{m_k}(\mathbb{C})$ for $k \in \{1, \dots, p\}$ such that for all $x = \sum_{k=1}^p \sum_{i=1}^{m_k} x_i^k \in V$ and $y = \sum_{k=1}^p \sum_{i=1}^{m_k} y_i^k \in V$:

$$\varphi(x, y) = \sum_{k=1}^p \sum_{1 \leq i, j \leq m_k} S_{ij}^k (\psi_i^k(x_i^k) | \psi_j^k(y_j^k))_k. \quad (2)$$

The number of parameters is $\sum_{k=1}^p m_k^2$ and the number of positivity constraints is $\sum_{k=1}^p m_k$.

Proof. We assemble the pieces of demonstration of the previous sections together. Let φ be a ρ -invariant inner product on V . Then, the map $f = \# \circ \flat_\varphi : V \rightarrow V$ is a module automorphism (Section 2.2). Hence for all $k \in \{1, \dots, p\}$, $f(\mathcal{V}^k) = \mathcal{V}^k$ (Section 3.2). Therefore, there exists an HPD matrix $S^k \in \text{Mat}_{m_k}(\mathbb{C})$ such that for all $x^k, y^k \in \mathcal{V}^k$, $\langle f(x^k) | y^k \rangle = \sum_{i=1}^{m_k} \sum_{j=1}^{m_k} S_{ij}^k (\psi_i^k(x_i^k) | \psi_j^k(y_j^k))_k$ (Section 3.3). Hence, the inner product φ writes:

$$\varphi(x, y) = \langle f(x) | y \rangle = \sum_{k=1}^p \langle f(x^k) | y^k \rangle = \sum_{k=1}^p \sum_{i=1}^{m_k} \sum_{j=1}^{m_k} S_{ij}^k (\psi_i^k(x_i^k) | \psi_j^k(y_j^k))_k. \quad (3)$$

Conversely, this bilinear form is an inner product since it is the sum of inner products on the \mathcal{V}^k 's (Section 3.3) which are supplementary. It is ρ -invariant because the orthogonal projections are equivariant (because $\langle \cdot | \cdot \rangle$ is invariant), the ψ_i 's are equivariant and the inner products $(\cdot | \cdot)_k$ are invariant. So this bilinear form is a ρ -invariant inner product on V .

Note that the choice of the inner product $(\cdot | \cdot)_k$ on W^k instead of $\lambda(\cdot | \cdot)_k$ with $\lambda > 0$ does not affect the general form of the inner product: it suffices to replace S^k by λS^k . Neither does the choice of the isometric parameterization $\psi_i^k : (V_i^k, \langle \cdot | \cdot \rangle) \rightarrow (W^k, (\cdot | \cdot)_k)$ instead of $\lambda_i \psi_i^k$ with $\lambda_i > 0$ for all $i \in \{1, \dots, m_k\}$: it suffices to replace S^k by $\Lambda S^k \Lambda$ where $\Lambda = \text{Diag}(\lambda_1, \dots, \lambda_{m_k})$.

4 Conclusion

We formalized a general method to determine all G -invariant inner products on a completely reducible Hermitian space V . Beyond linear algebra, this characterization is interesting when one wants to characterize invariant Riemannian metrics on Lie groups and homogeneous spaces. The characterization is also interesting when there exists a global diffeomorphism from a manifold to a vector space, such as for symmetric positive definite matrices or full-rank correlation matrices.

An important challenge is to adapt this method to real vector spaces. Moreover, since this method is based on representation theory, it would be interesting to investigate non-linear methods to characterize invariant Riemannian metrics on manifolds when this problem does not reduce to a linear problem.

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