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# Robust Stability Analysis for Continuous-Time Parameter-Varying Persidskii Systems

Junfeng Zhang, Christophe Combastel, Denis Efimov, Ali Zolghadri

**Abstract**—The class of parameter-varying generalized Persidskii systems is introduced. For models within this class, characterized by nonlinearities satisfying the passivity property, the conditions for (integral) input-to-state stability are proposed. These conditions are established using both, parameter-dependent and parameter-independent, Lyapunov functions. To formulate these conditions, parameterized matrix inequalities are used, which can be reduced into linear ones under additional assumptions concerning the model's dependence on scheduling variables. The efficiency of these stability conditions is illustrated through a numerical example.

**Index Terms**—Persidskii systems, LPV, Input-to-state stability, LMIs

## I. INTRODUCTION

The analysis of robust stability and the design of controllers or estimators represent fundamental problems in the field of automatic control theory [1], [2]. The input-to-state stability framework [3], [4] has emerged as one of the most widely used concepts for examining stability in the presence of various forms of uncertainty. In the context of general nonlinear dynamical systems, the synthesis of standard control or estimation algorithms can be challenging, often due to the complexity involved in constructing a Lyapunov function for stability assessment. A common way to overpass this issue consists in using the canonical models: linear parameter-varying (LPV) systems [5], homogeneous dynamics [6], Lur'e models [7], [8]. An extension of the latter is given by Persidskii systems. This class of nonlinear models was first introduced for stability analysis in [9], where a linear combination of the integrals of the nonlinearities was used as a Lyapunov function. That result was extended in [10] by augmenting the Lyapunov function through a combination of the absolute values of the states. Furthermore, Persidskii systems were studied in the context of diagonal stability [11], [12], sliding mode control [13], [14], [15] and Lur'e systems [7], with applications to opinion dynamics [16], neural networks [17], [18], [19] and digital filters [20]. Following the foundational results [9], [10], one of the main advantages of Persidskii dynamics is the availability of a canonical form of Lyapunov function. Recently, several additional results have been proposed to enhance the existing theory, addressing various analysis problems [21], [22], [23], [24], [25], [26]. These developments have a common feature:

they lead to the verification of linear matrix inequalities (LMIs), which is an interesting and useful feature of this class of nonlinear systems.

In many cases, it is not always possible to transform the model of a process into a known canonical form. In such instances, various approximations can be employed, e.g., incorporating the residual terms into perturbations. In this way, the LPV framework is widely used, enabling the equivalent representation of a nonlinear system in a linear form with time-varying parameters. Subsequently, this opens the door to the application of the large spectrum of well-established methods and tools of linear system theory. Unfortunately, transformation of stabilizing nonlinearities to LPV form may introduce some conservatism due to a possible loss of stability property yielded by the nonlinear dynamics. In this work, an extension of the LPV framework to Persidskii systems is proposed, where the scheduling parameters can represent the additional time- or state-varying terms, parametric or signal uncertainty, etc., while the useful (passive or negative feedback) nonlinearities are kept in the model. Such a new development can help in analysis of a wide class of nonlinear systems being close to the Persidskii dynamics by using the related Lyapunov function, whose application usually results in LMI constructive stability conditions.

The paper is organized as follows: Section II provides the preliminaries on input-to-state stability and Persidskii systems. In Section III, the problem statement is introduced. Parameter-independent Lyapunov functions are investigated in Section IV, followed by the analysis of parameter-dependent Lyapunov functions in Section V. Section VI demonstrates efficiency of the obtained conditions on an example of a time-varying nonlinear mechanical system.

## Notation

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ , where  $\mathbb{R}$  is the set of real numbers.
- $|\cdot|$  denotes the absolute value in  $\mathbb{R}$ ,  $\|\cdot\|$  is used for the Euclidean norm on  $\mathbb{R}^n$ .
- For a (Lebesgue) measurable function  $d : \mathbb{R}_+ \rightarrow \mathbb{R}^m$  and  $[t_0, t_1] \subset \mathbb{R}_+$  define the norm  $\|d\|_{[t_0, t_1]} = \text{ess sup}_{t \in [t_0, t_1]} \|d(t)\|$ , then  $\|d\|_\infty = \|d\|_{[0, +\infty)}$  and the set of  $d$  with the property  $\|d\|_\infty < +\infty$  we further denote as  $\mathcal{L}_\infty^m$  (the set of essentially bounded measurable functions).
- The notation  $DV(x)f(x)$  stands for the directional derivative of a continuously differentiable function  $V$  with respect to the vector field  $f$  evaluated at the point  $x$ .

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- A finite series of integers  $1, 2, \dots, n$  is denoted by  $\overline{1, n}$ , and  $\{\overline{1, n}\} = \{1, 2, \dots, n\}$ .
- Denote the identity matrix of dimension  $n \times n$  by  $I_n$ , the vector of dimension  $n$  or the matrix of dimension  $n \times m$  with all elements equal 1 by  $\mathbf{1}_n$  and  $\mathbf{1}_{n \times m}$ , respectively.
- $\text{diag}\{g\} \in \mathbb{D}_+^n$  represents a diagonal matrix of dimension  $n \times n$  with a vector  $g \in \mathbb{R}_+^n$  on the main diagonal, where  $\mathbb{D}_+^n \subset \mathbb{R}^{n \times n}$  is the set of nonnegative diagonal matrices.
- For a matrix  $A \in \mathbb{R}^{n \times n}$ , denote its  $i^{\text{th}}$  row and column by  $A^{(i)}$  and  $A^{[i]}$ , respectively, for  $i = \overline{1, n}$ . The relation  $P \prec 0$  ( $P \preceq 0$ ) means that a symmetric matrix  $P \in \mathbb{R}^{n \times n}$  is negative (semi-)definite.

## II. PRELIMINARIES

In this paper, it is conventionally assumed that if the upper limit of a summation or a sequence is smaller than the lower one, then the corresponding terms (conditions) have to be omitted.

A continuous function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{K}$  if  $\alpha(0) = 0$  and the function is strictly increasing. The function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{K}_\infty$  if  $\alpha \in \mathcal{K}$  and it is increasing to infinity. A continuous function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{KL}$  if  $\beta(\cdot, t) \in \mathcal{K}$  for each fixed  $t \in \mathbb{R}_+$  and  $\beta(s, \cdot)$  is decreasing to zero for each fixed  $s > 0$ .

**Lemma** (Finsler's lemma). [27] *Let  $x \in \mathbb{R}^n \setminus \{0\}$  and  $P, R \in \mathbb{R}^{n \times n}$  are symmetric, then  $x^\top P x \prec 0$  whenever  $x^\top R x = 0$  if and only if there exists  $\rho \in \mathbb{R}$  such that  $P - \rho R \prec 0$ .*

### A. Input-to-state stability

Consider a nonlinear system:

$$\dot{x}(t) = f(x(t), d(t)), \quad t \geq 0, \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the state,  $d(t) \in \mathbb{R}^m$  is the external input,  $d \in \mathcal{L}_\infty^m$ , and  $f : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$  is a locally Lipschitz continuous function,  $f(0, 0) = 0$ . For an initial condition  $x_0 \in \mathbb{R}^n$  and input  $d \in \mathcal{L}_\infty^m$ , define the corresponding solutions by  $x(t, x_0, d)$  for any  $t \geq 0$  for which the solution exists.

In this work we will be interested in the following stability properties [3], [4]:

**Definition 1.** The system (1) is called *input-to-state practically stable* (ISpS), if there are functions  $\beta \in \mathcal{KL}$ ,  $\gamma \in \mathcal{K}$  and a constant  $c \geq 0$  such that

$$\|x(t, x_0, d)\| \leq \beta(\|x_0\|, t) + \gamma(\|d\|_{[0, t]}) + c \quad \forall t \geq 0$$

for any  $x_0 \in \mathbb{R}^n$  and  $d \in \mathcal{L}_\infty^m$ . The function  $\gamma$  is called *nonlinear asymptotic gain*. The system is called *input-to-state stable* (ISS) if  $c = 0$ .

**Definition 2.** The system (1) is called *integral ISS* (iISS), if there are functions  $\alpha \in \mathcal{K}_\infty$ ,  $\gamma \in \mathcal{K}$  and  $\beta \in \mathcal{KL}$  such that for any  $x_0 \in \mathbb{R}^n$  and  $d \in \mathcal{L}_\infty^m$  the estimate holds:

$$\alpha(\|x(t, x_0, d)\|) \leq \beta(\|x_0\|, t) + \int_0^t \gamma(\|d(s)\|) ds \quad \forall t \geq 0.$$

These properties have the following characterizations in terms of existence of Lyapunov functions:

**Definition 3.** A smooth function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$  is called *ISpS-Lyapunov function* for the system (1) if there are  $r \geq 0$ ,  $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$  and  $\theta \in \mathcal{K}$  such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \\ DV(x)f(x, d) \leq r + \theta(\|d\|) - \alpha_3(\|x\|)$$

for all  $x \in \mathbb{R}^n$  and all  $d \in \mathbb{R}^m$ . Such a function  $V$  is called *ISS-Lyapunov function* if  $r = 0$ , and it is *iISS-Lyapunov function* if additionally  $\alpha_3 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is just a positive definite function.

Note that an ISS-Lyapunov function can also satisfy the following equivalent condition for some  $\chi \in \mathcal{K}$ :

$$\|x\| > \chi(\|d\|) \Rightarrow DV(x)f(x, d) \leq -\alpha_3(\|x\|).$$

The relations between these Lyapunov characterizations and the robust stability properties are given below:

**Theorem 1.** *The system (1) is ISS (ISpS, iISS) if and only if it admits an ISS (ISpS, iISS)-Lyapunov function.*

A consequence of Theorem 1 and Definition 3 is that an ISS system (1) is also iISS.

### B. Parameter-varying Persidskii systems

Consider the following class of systems [28], [21]:

$$\dot{x}(t) = A_0(\theta(t))x(t) + \sum_{j=1}^M A_j(\theta(t))f^j(H_j x(t)) + d(t), \quad (2)$$

where  $x(t) = [x_1(t) \dots x_n(t)]^\top \in \mathbb{R}^n$  is the state vector,  $x(0) \in \mathbb{R}^n$ ;  $d(t) \in \mathbb{R}^n$  is the external disturbance,  $d \in \mathcal{L}_\infty^n$ ;  $\theta(t) \in \mathbb{R}^q$  is the vector of time-varying parameters,  $\theta \in \mathcal{L}_\infty^q$ ;  $f^j : \mathbb{R}^{k_j} \rightarrow \mathbb{R}^{k_j}$  with diagonal structure  $f^j(s) = [f_1^j(s_1) \dots f_{k_j}^j(s_{k_j})]^\top$ ,  $j = \overline{1, M}$  are continuous functions ensuring existence of solutions of the system (2) at least locally in the forward time; continuous matrix functions  $A_g : \mathbb{R}^q \rightarrow \mathbb{R}^{n \times k_g}$  for  $g = \overline{0, M}$  and matrices  $H_j \in \mathbb{R}^{k_j \times n}$  for  $j = \overline{1, M}$  are given. Further, for brevity and consistently with (2) we use the convention  $k_0 = n$  and  $H_0 = I_n$  with  $f^0(x) = x$ .

The model (2) belongs to the class of Persidskii system [10], [11] under the following passivity (or sector) condition imposed on the nonlinearities: for any  $j = \overline{1, M}$  and  $i = \overline{1, k_j}$ ,

$$s f_i^j(s) > 0 \quad \forall s \in \mathbb{R} \setminus \{0\}.$$

Consequently, all nonlinearities belong to a sector and may take zero values at zero only. If  $\theta(t) = \text{const}$ ,  $H_1 = I_n$  and  $A_r = 0$  for all  $r = \overline{2, M}$ , then we recover the system studied by Persidskii in the conventional framework [10]. In the case of  $M = 1$ , (2) belongs also to the class of Lur'e systems widely investigated in the absolute stability theory [8].

After a proper re-indexing and decomposition of  $f^j$ , there exists  $\nu \in \{0, M\}$  such that for all  $z = \overline{1, \nu}$  and  $i = \overline{1, k_z}$ :

$$\lim_{s \rightarrow \pm\infty} f_i^z(s) = \pm\infty;$$

and there exists  $\mu \in \{\nu, M\}$  such that for all  $j = \overline{1, \mu}$  and  $i = \overline{1, k_j}$ :

$$\lim_{s \rightarrow \pm\infty} \int_0^s f_i^j(\sigma) d\sigma = +\infty.$$

Thus, some of the nonlinearities are radially unbounded, and  $\nu = 0$  corresponds to the case when all nonlinearities are bounded (at least for negative or positive argument).

### III. PROBLEM STATEMENT

Consider a parameter-varying Persidskii (PVP) system (2). The following assumptions are first introduced to formulate/state the problem addressed in this paper.

**Assumption 1.** Let  $\theta(t) \in \Theta$ , where  $\Theta \subset \mathbb{R}^q$  is a known compact set.

**Assumption 2.** Let  $\dot{\theta} \in \mathcal{L}_\infty^q$  and  $\|\dot{\theta}\|_\infty \leq \dot{\theta}_{\max}$  for a known constant  $\dot{\theta}_{\max} > 0$ .

Assumptions 1 and 2 formulate standard hypotheses for stability analysis of LPV systems: the first just restricts the set of admissible values for the vector of scheduling parameters, while the latter allows us to introduce in Lyapunov functions the dependence on  $\theta(t)$  [5]. In general, the dependence on  $\theta(t)$  should make the stability conditions less restrictive. However, this comes at the cost of requiring a more intricate numerical procedure for verification. In order to make the conditions more constructive, we will consider the case with linear dependence of the matrix functions in the vector of scheduling parameters:

**Assumption 3.** Let  $A_g(\theta) = q^{-1} \sum_{k=1}^q \theta_k A_{gk}$  for  $g = \overline{0, M}$ , where  $A_{gk} \in \mathbb{R}^{n \times k_g}$  are known matrices for  $k = \overline{1, q}$ .

The objective of this work is to propose conditions of IS(p)S and iISS for the system (2), using a parameter-independent Lyapunov function under Assumption 1, or parameter-dependent one introducing assumptions 1–3 (i.e., in both cases the information about the set  $\Theta$  and the velocity bound  $\dot{\theta}_{\max}$  should be used, but not the properties of a particular trajectory  $\theta(t)$ ). The conventional LPV framework transforms all nonlinearities, which may be difficult and restrictive, while the Persidskii or Lur'e system frameworks do

not consider uncertainty presented in the matrices describing the dynamics. In this work we are going to fill these gaps.

### IV. PARAMETER-INDEPENDENT LYAPUNOV FUNCTIONS

Recalling [21], [22], [24], consider the following structure of a candidate Lyapunov function for (2):

$$V(x) = x^\top P x + 2 \sum_{j=1}^M \sum_{i=1}^{k_j} \Lambda_i^j \int_0^{H_j^{(i)} x} f_i^j(s) ds, \quad (3)$$

where  $P \in \mathbb{R}^{n \times n}$  is a symmetric matrix and  $\Lambda^j = \text{diag}[\Lambda_1^j \dots \Lambda_{k_j}^j] \in \mathbb{D}_+^{k_j}$  are parameters to be tuned in order that (3) verifies the properties stated in Definition 3.

Our main result is stated in the following theorem:

**Theorem 2.** Let Assumption 1 be satisfied and there exist matrices  $P^\top = P \in \mathbb{R}^{n \times n}$ ,  $\Lambda^j \in \mathbb{D}_+^{k_j}$  for  $j = \overline{1, M}$ ,  $\Phi^\top = \Phi \in \mathbb{R}^{n \times n}$ ,  $\Psi \in \mathbb{R}^{n \times n}$  and matrix functions  $\Omega_g : \mathbb{R}^q \rightarrow \mathbb{R}^{n \times k_g}$ ,  $\Upsilon_{g,g} : \mathbb{R}^q \rightarrow \mathbb{D}_+^{k_g}$  ( $\tilde{\Upsilon}_{g,g}(\theta) = H_g^\top \Upsilon_{g,g}(\theta) H_g$ ) for  $g = \overline{0, M}$ ,  $\Upsilon_{0,j} : \mathbb{R}^q \rightarrow \mathbb{D}_+^{k_j}$  ( $\tilde{\Upsilon}_{0,j}(\theta) = H_j^\top \Upsilon_{0,j}(\theta) H_j$ ) for  $j = \overline{1, M}$ ,  $\Upsilon_{z,s} : \mathbb{R}^q \rightarrow \mathbb{D}_+^{k_s}$  ( $\tilde{\Upsilon}_{z,s}(\theta) = H_z^\top \Upsilon_{z,s}(\theta) H_s$ ) for  $z = \overline{1, M-1}$  and  $s = z+1, M$  such that the following matrix inequalities are verified for all  $\theta \in \Theta$ :

$$P \succeq 0, \Phi \succ 0, Q(\theta) \preceq 0,$$

$$P + \rho_1 \sum_{j=1}^\mu H_j^\top \Lambda^j H_j \succ 0,$$

$$\begin{aligned} & \sum_{r=0}^\nu \tilde{\Upsilon}_{r,r}(\theta) + \rho_2 \sum_{z=1}^{\nu-1} \sum_{s=z+1}^\nu H_z^\top \tilde{\Upsilon}_{z,s}(\theta) H_s \\ & + \rho_3 \sum_{j=1}^\nu H_j^\top \Upsilon_{0,j}(\theta) H_j \succ 0 \end{aligned}$$

for some  $\rho_1, \rho_2, \rho_3 \in \mathbb{R}$ , where<sup>1</sup>

$$Q = \begin{bmatrix} -\Psi^\top - \Psi & P + \Psi^\top A_0 - \Omega_0 & H_1^\top \Lambda^1 + \Psi^\top A_1 - \Omega_1 & \dots & H_M^\top \Lambda^M + \Psi^\top A_M - \Omega_M & \Psi^\top \\ P + A_0^\top \Psi - \Omega_0^\top & \Omega_0^\top A_0 + A_0^\top \Omega_0 + \Upsilon_{0,0} & \Omega_0^\top A_1 + A_0^\top \Omega_1 + \tilde{\Upsilon}_{0,1} & \dots & \Omega_0^\top A_M + A_0^\top \Omega_M + \tilde{\Upsilon}_{0,M} & \Omega_0^\top \\ \Lambda^1 H_1 + A_1^\top \Psi - \Omega_1^\top & A_1^\top \Omega_0 + \Omega_1^\top A_0 + \tilde{\Upsilon}_{0,1}^\top & \Omega_1^\top A_1 + A_1^\top \Omega_1 + \Upsilon_{1,1} & \dots & \Omega_1^\top A_M + A_1^\top \Omega_M + \tilde{\Upsilon}_{1,M} & \Omega_1^\top \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \Lambda^M H_M + A_M^\top \Psi - \Omega_M^\top & A_M^\top \Omega_0 + \Omega_M^\top A_0 + \tilde{\Upsilon}_{0,M}^\top & A_M^\top \Omega_1 + \Omega_M^\top A_1 + \tilde{\Upsilon}_{1,M} & \dots & \Omega_M^\top A_M + A_M^\top \Omega_M + \Upsilon_{M,M} & \Omega_M^\top \\ \Psi & \Omega_0 & \Omega_1 & \dots & \Omega_M & -\Phi \end{bmatrix},$$

then the system (2) is ISS. If the last inequality is replaced with

$$\begin{aligned} & \sum_{r=0}^M \tilde{\Upsilon}_{r,r}(\theta) + \rho_2 \sum_{z=1}^{M-1} \sum_{s=z+1}^M H_z^\top \tilde{\Upsilon}_{z,s}(\theta) H_s \\ & + \rho_3 \sum_{j=1}^M H_j^\top \Upsilon_{0,j}(\theta) H_j \succ 0 \end{aligned}$$

satisfying for all  $\theta \in \Theta$ , then the system (2) is iISS.

*Proof.* The relations  $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$  are valid for some  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  for all  $x \in \mathbb{R}^n$  due to imposed restrictions; according to Finsler's lemma, the condition  $P + \rho_1 \sum_{j=1}^\mu H_j^\top \Lambda^j H_j \succ 0$  satisfied for some  $\rho_1 \in \mathbb{R}$  implies that the unbounded terms in (3) ( $\mu$  characterizes the nonlinearities with unbounded integrals, as introduced at the end of Section II) are presented for all elements of the vector  $x$ . The time

<sup>1</sup>For brevity, the dependence on  $\theta$  may be omitted once the respective functions have been defined.

derivative of (3) for the system dynamics takes the form:

$$\begin{aligned} \dot{V} = & \dot{x}^\top P x + x^\top P \dot{x} + 2 \sum_{j=1}^M \dot{x}^\top H_j^\top \Lambda^j f^j(H_j x) \\ & + 2 \left[ \sum_{g=0}^M f^g(H_g x)^\top \Omega_g^\top(\theta) + \dot{x}^\top \Psi^\top \right] \left( \sum_{g=1}^M A_g(\theta) f^g(H_g x) + d - \dot{x} \right), \end{aligned}$$

where the last line is added by applying the descriptor approach [29] (this product equals to zero, while helping us in optimization of the resulted matrix inequalities). Opening the brackets, adding and subtracting the terms weighted by the matrices  $\Phi$  and  $\tilde{\Upsilon}_{g,s}$  with  $g = \overline{0, M}$  and  $s = \overline{g, M}$ , this expression can be rewritten as follows (due to Assumption 1,  $\theta \in \Theta$ ):

$$\begin{aligned} \dot{V} = & \zeta^\top Q(\theta) \zeta - \sum_{g=0}^M f^g(H_g x)^\top \Upsilon_{g,g}(\theta) f^g(H_g x) \\ & - 2 \sum_{k=0}^{M-1} \sum_{s=k+1}^M f^k(H_k x)^\top \tilde{\Upsilon}_{k,s}(\theta) f^s(H_s x) + d^\top \Phi d \\ & \leq - \sum_{g=0}^M f^g(H_g x)^\top \Upsilon_{g,g}(\theta) f^g(H_g x) \\ & - 2 \sum_{k=0}^{M-1} \sum_{s=k+1}^M f^k(H_k x)^\top \tilde{\Upsilon}_{k,s}(\theta) f^s(H_s x) + d^\top \Phi d, \end{aligned}$$

where

$$\zeta^\top = [\dot{x}^\top \ x^\top \ f^1(H_1 x)^\top \ \dots \ f^M(H_M x)^\top \ d^\top]^\top.$$

To get the desired ISS property we need further to substantiate the upper bound:

$$\dot{V} \leq -\alpha_3(\|x\|) + d^\top \Phi d$$

for some  $\alpha_3 \in \mathcal{K}_\infty$ . Thus, the following property has to be demonstrated,

$$\begin{aligned} \alpha_3(\|x\|) \leq & \sum_{g=0}^\nu f^g(H_g x)^\top \Upsilon_{g,g}(\theta) f^g(H_g x) \\ & + 2 \sum_{k=0}^{\nu-1} \sum_{s=k+1}^\nu f^k(H_k x)^\top \tilde{\Upsilon}_{k,s}(\theta) f^s(H_s x) \end{aligned}$$

for all  $x \in \mathbb{R}^n$  and  $\theta \in \Theta$  (only unbounded nonlinearities should be taken into account, hence, the summation until  $\nu$  is kept only). The right-hand side of the above inequality is an unbounded function of  $x$  if, according to Finsler's lemma, the fulfillment of the matrix inequality

$$\begin{aligned} & \sum_{r=0}^\nu \tilde{\Upsilon}_{r,r}(\theta) + \rho_2 \sum_{z=1}^{\nu-1} \sum_{s=z+1}^\nu H_z^\top \tilde{\Upsilon}_{z,s}(\theta) H_s \\ & + \rho_3 \sum_{j=1}^\nu H_j^\top \Upsilon_{0,j}(\theta) H_j \succ 0 \end{aligned}$$

is guaranteed for some  $\rho_2, \rho_3 \in \mathbb{R}$ , for all  $\theta \in \Theta$ . This ensures the existence of such a function  $\alpha_3$ , and that the system (2) is ISS. If the latter property is verified not for the first  $\nu$  nonlinearities, but for all  $M$ , then it implies existence of a positive definite function  $\alpha_3 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , which due to Theorem 1 is equivalent to iISS property for (2).  $\square$

**Remark 1.** A necessary condition for feasibility of the matrix inequality  $Q(\theta) \preceq 0$  required in the formulation of Theorem 2 is that all matrices on the main diagonal of  $Q$  are nonnegative definite, i.e.,  $\Omega_g^\top(\theta) A_g(\theta) + A_g^\top(\theta) \Omega_g(\theta) \preceq 0$  for  $g = \overline{0, M}$ , which can be easily checked separately. It may also provide a hint on the choice of the structure of the functions  $\Omega_g(\theta)$  for known shape of  $A_g(\theta)$ .

**Remark 2.** The Lyapunov function (3) is independent in  $\theta$ . However, utilization of the descriptor approach allowed us to introduce the analogues of parameter dependent matrix functions  $\Omega_g(\theta)$ ,  $g = \overline{0, M}$  to check stability of  $A_g(\theta)$ : indeed, on the main diagonal of  $Q(\theta)$  we have the elements  $\Omega_g^\top(\theta) A_g(\theta) + A_g^\top(\theta) \Omega_g(\theta)$  that assess stability of  $A_g(\theta)$ . The terms  $P, H_1^\top \Lambda^1, \dots, H_M^\top \Lambda^M$  and  $\Omega_0(\theta), \Omega_1(\theta), \dots, \Omega_M(\theta)$  appear with different signs, respectively, in some of the cells of the matrix  $Q(\theta)$ , then a possible choice is to equate them, which leads to a natural simplification  $\Psi = 0$ , and results in cancellation of the terms introduced by the descriptor approach (i.e., the obtained matrix inequality will be the same as after a direct substitution in  $\dot{V}$  the expression of  $\dot{x}$  from (2)). Thus, the descriptor approach allows us to get more general matrix inequalities, in some way combining advantages of parameter-dependent and parameter-independent Lyapunov functions.

A drawback of the conditions formulated in Theorem 2 is that their verification for all  $\theta \in \Theta$  may be computationally costly. Imposing affinity of the system matrix functions  $A_g(\theta)$  in  $\theta$  and restricting the matrix functions  $\Omega_g(\theta)$  to be constant for all  $g = \overline{0, M}$ , these stability conditions can be reduced to LMIs:

**Corollary 1.** Let assumptions 1 and 3 be satisfied with  $\Theta \subset \mathbb{R}_+^q$ , and there exist matrices  $P^\top = P \in \mathbb{R}^{n \times n}$ ,  $\Lambda^j \in \mathbb{D}_+^{k_j}$  for  $j = \overline{1, M}$ ,  $\Phi^\top = \Phi \in \mathbb{R}^{n \times n}$ ,  $\Psi \in \mathbb{R}^{n \times n}$ ,  $\Omega_g \in \mathbb{R}^{n \times k_g}$ ,  $\Upsilon_{g,g} \in \mathbb{D}_+^{k_g}$  ( $\tilde{\Upsilon}_{g,g} = H_g^\top \Upsilon_{g,g} H_g$ ) for  $g = \overline{0, M}$ ,  $\Upsilon_{0,j} \in \mathbb{D}_+^{k_j}$  ( $\tilde{\Upsilon}_{0,j} = H_j^\top \Upsilon_{0,j}$ ) for  $j = \overline{1, M}$ ,  $\Upsilon_{z,s} \in \mathbb{D}_+^{n}$  ( $\tilde{\Upsilon}_{z,s} = H_z \Upsilon_{z,s} H_s^\top$ ) for  $z = \overline{1, M-1}$  and  $s = \overline{z+1, M}$  such that the following LMIs are verified:

$$P \succeq 0, \Phi \succ 0, \\ Q_k \preceq 0, \forall k = \overline{1, q},$$

$$P + \rho_1 \sum_{j=1}^\mu \mathcal{H}_j^\top \Lambda^j \mathcal{H}_j \succ 0,$$

$$\begin{aligned} & \sum_{r=0}^\nu \tilde{\Upsilon}_{r,r} + \rho_2 \sum_{z=1}^{\nu-1} \sum_{s=z+1}^\nu H_z^\top \tilde{\Upsilon}_{z,s} H_s + \rho_3 \sum_{j=1}^\nu H_j^\top \Upsilon_{0,j} H_j \succ 0 \\ & \text{for some } \rho_1, \rho_2, \rho_3 \in \mathbb{R}, \text{ where} \end{aligned}$$

$$Q_k = \begin{bmatrix} -\Psi^\top - \Psi & P + \Psi^\top A_{0k} - \Omega_0 & H_1^\top \Lambda^1 + \Psi^\top A_{1k} - \Omega_1 & \dots & H_M^\top \Lambda^M + \Psi^\top A_{Mk} - \Omega_M & \Psi^\top \\ P + A_{0k}^\top \Psi - \Omega_0^\top & \Omega_0^\top A_{0k} + A_{0k}^\top \Omega_0 + \Upsilon_{0,0} & \Omega_0^\top A_{1k} + A_{0k}^\top \Omega_1 + \tilde{\Upsilon}_{0,1} & \dots & \Omega_0^\top A_{Mk} + A_{0k}^\top \Omega_M + \tilde{\Upsilon}_{0,M} & \Omega_0^\top \\ \Lambda^1 H_1 + A_{1k}^\top \Psi - \Omega_1^\top & A_{1k}^\top \Omega_0 + \Omega_1^\top A_{0k} + \tilde{\Upsilon}_{0,1} & \Omega_1^\top A_{1k} + A_{1k}^\top \Omega_1 + \Upsilon_{1,1} & \dots & \Omega_1^\top A_{Mk} + A_{1k}^\top \Omega_M + \tilde{\Upsilon}_{1,M} & \Omega_1^\top \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \Lambda^M H_M + A_{Mk}^\top \Psi - \Omega_M^\top & A_{Mk}^\top \Omega_0 + \Omega_M^\top A_{0k} + \tilde{\Upsilon}_{0,M} & A_{Mk}^\top \Omega_1 + \Omega_M^\top A_{1k} + \tilde{\Upsilon}_{1,M} & \dots & \Omega_M^\top A_{Mk} + A_{Mk}^\top \Omega_M + \Upsilon_{M,M} & \Omega_M^\top \\ \Psi & \Omega_0 & \Omega_1 & \dots & \Omega_M & -\Phi \end{bmatrix},$$

then the system (2) is ISS. If the last LMI is replaced with

$$\begin{aligned} & \sum_{r=0}^M \tilde{\Upsilon}_{r,r} + \rho_2 \sum_{z=1}^{M-1} \sum_{s=z+1}^M H_z^\top \tilde{\Upsilon}_{z,s} H_s \\ & + \rho_3 \sum_{j=1}^M H_j^\top \Upsilon_{0,j} H_j \succ 0, \end{aligned}$$

then the system (2) is iISS.

*Proof.* The proof is a direct consequence of the result of Theorem 2 under application of Assumption 3 for decomposition of the matrix function  $Q$ . Indeed, in this case

$$\begin{aligned} \dot{V} &= q^{-1} \sum_{k=1}^q \theta_k \zeta^\top Q_k \zeta - \sum_{g=0}^M f^g(H_g x)^\top \Upsilon_{g,g} f^g(H_g x) \\ & - 2 \sum_{k=0}^{M-1} \sum_{s=k+1}^M f^k(H_k x)^\top \tilde{\Upsilon}_{k,s} f^s(H_s x) + d^\top \Phi d. \end{aligned}$$

and since  $\theta_k \in \mathbb{R}_+$  the property  $Q_k \preceq 0$  for all  $k = \overline{1, q}$  implies  $\sum_{k=1}^q \theta_k \zeta^\top Q_k \zeta \preceq 0$ . All other steps and arguments are the same.  $\square$

The ISpS stability conditions can be obtained from the inequalities derived above. For example, if the nonlinearities  $f^s$  are all globally bounded for  $s = \overline{\nu+1, M}$ , then  $-\Upsilon_{s,s}$  can be selected in  $\mathbb{D}_+^{k_s}$  (before it was  $\Upsilon_{s,s} \in \mathbb{D}_+^{k_s}$ ), and the terms  $\sum_{s=\nu+1}^M f^s(H_s x)^\top \Upsilon_{s,s} f^s(H_s x)$  in the expression of  $\dot{V}$  can be upper bounded by the constant.

## V. PARAMETER-DEPENDENT LYAPUNOV FUNCTIONS

In this case consider for (2) the following modification of (3):

$$V(x, \theta) = x^\top P(\theta) x + 2 \sum_{j=1}^M \sum_{i=1}^{k_j} \Lambda_i^j(\theta) \int_0^{H_j^{(i)} x} f_i^j(s) ds, \quad (4)$$

where  $P : \mathbb{R}^q \rightarrow \mathbb{R}^{n \times n}$  is a symmetric matrix function and denote by  $\Lambda^j(\theta) = \text{diag}[\Lambda_1^j(\theta) \dots \Lambda_{k_j}^j(\theta)] \in \mathbb{D}_+^{k_j}$  a nonnegative diagonal matrix function. We assume that  $V$  is differentiable with respect to  $\theta$ . In such a case, since the coefficients  $\Lambda_i^j$  are assumed to be dependent on  $\theta$ , the derivative of (4) contains the integrals of the nonlinearities, which requires the introduction of additional hypotheses on their relations with nonlinearities of the system:

**Assumption 4.** Let assume that, for any  $j = \overline{1, M}$  and  $i = \overline{1, k_j}$ , there exist  $W_{g,g}^{i,j} \in \mathbb{D}_+^{k_g}$  ( $\tilde{W}_{g,g}^{i,j} = H_g^\top W_{g,g}^{i,j} H_g$ ) for  $g = \overline{0, M}$ ,  $W_{0,z}^{i,j} \in \mathbb{D}_+^{k_z}$  ( $\tilde{W}_{0,z}^{i,j} = H_z^\top W_{0,z}^{i,j}$ ) for  $z = \overline{1, M}$ ,  $W_{k,s}^{i,j} \in \mathbb{D}_+^n$  ( $\tilde{W}_{k,s}^{i,j} = H_k W_{k,s}^{i,j} H_s^\top$ ) for  $k = \overline{1, M-1}$  and  $s = k+1, M$  such that

$$\begin{aligned} & \int_0^{H_j^{(i)} x} f_i^j(s) ds \leq \sum_{g=0}^M f^g(H_g x)^\top W_{g,g}^{i,j} f^g(H_g x) \\ & + 2 \sum_{k=0}^{M-1} \sum_{s=k+1}^M f^k(H_k x)^\top \tilde{W}_{k,s}^{i,j} f^s(H_s x) \end{aligned}$$

for all  $x \in \mathbb{R}^n$ .

Assumption 4 is naturally satisfied for polynomial functions, for example: if  $f_i^j(s) = s^a$  with  $a > 0$ , then  $\int_0^{H_j^{(i)} x} f_i^j(s) ds = \frac{1}{1+a} \left( H_j^{(i)} x \right)^{1+a} \leq \left( H_j^{(i)} f^0(H_0 x) \right) \left( W_{0,j}^{i,j} \right)_{i,i} f_i^j(H_j^{(i)} x)$  for any  $\left( W_{0,j}^{i,j} \right)_{i,i} \geq \frac{1}{1+a}$  all other elements in the matrices  $W_{k,s}^{i,j}$  can be selected to be zero.

**Theorem 3.** Let assumptions 1, 2 and 4 be satisfied and there exist matrices  $\Phi^\top = \Phi \in \mathbb{R}^{n \times n}$ ,  $\Psi \in \mathbb{R}^{n \times n}$ , symmetric continuously differentiable matrix functions  $P : \mathbb{R}^q \rightarrow \mathbb{R}^{n \times n}$ ,  $\Lambda^j : \mathbb{R}^q \rightarrow \mathbb{D}_+^{k_j}$  for  $j = \overline{1, M}$ , and matrix functions  $\Omega_g : \mathbb{R}^q \rightarrow \mathbb{R}^{n \times k_g}$ ,  $\Upsilon_{g,g} : \mathbb{R}^q \rightarrow \mathbb{D}_+^{k_g}$  ( $\tilde{\Upsilon}_{g,g}(\theta) = H_g^\top \Upsilon_{g,g}(\theta) H_g$ ) for  $g = \overline{0, M}$ ,  $\Upsilon_{0,j} : \mathbb{R}^q \rightarrow \mathbb{D}_+^{k_j}$  ( $\tilde{\Upsilon}_{0,j}(\theta) = H_j^\top \Upsilon_{0,j}(\theta)$ ) for  $j = \overline{1, M}$ ,  $\Upsilon_{z,s} : \mathbb{R}^q \rightarrow \mathbb{D}_+^n$  ( $\tilde{\Upsilon}_{z,s}(\theta) = H_z \Upsilon_{z,s}(\theta) H_s^\top$ ) for  $z = \overline{1, M-1}$  and  $s = z+1, M$  such that the following matrix inequalities are verified for all  $\theta \in \Theta$  and  $\dot{\theta} \in [-\dot{\theta}_{\max}, \dot{\theta}_{\max}]^q$ :

$$P(\theta) \succeq 0, \Phi \succ 0, Q(\theta, \dot{\theta}) \preceq 0,$$

$$P(\theta) + \rho_1 \sum_{j=1}^M H_j^\top \Lambda^j(\theta) H_j \succ 0,$$

$$\begin{aligned} & \sum_{r=0}^\nu \tilde{\Upsilon}_{r,r}(\theta) + \rho_2 \sum_{z=1}^{\nu-1} \sum_{s=z+1}^\nu H_z^\top \tilde{\Upsilon}_{z,s}(\theta) H_s \\ & + \rho_3 \sum_{j=1}^\nu H_j^\top \Upsilon_{0,j}(\theta) H_j \succ 0 \end{aligned}$$

for some  $\rho_1, \rho_2, \rho_3 \in \mathbb{R}$ , where

$$Q = \begin{bmatrix} -\Psi^\top - \Psi & P + \Psi^\top A_0 - \Omega_0 & H_1^\top \Lambda^1 + \Psi^\top A_1 - \Omega_1 & \dots & H_M^\top \Lambda^M + \Psi^\top A_M - \Omega_M & \Psi^\top \\ P + A_0^\top \Psi - \Omega_0^\top & \Omega_0^\top A_0 + A_0^\top \Omega_0 + \Gamma_{0,0} & \Omega_0^\top A_1 + A_0^\top \Omega_1 + \Gamma_{0,1} & \dots & \Omega_0^\top A_M + A_0^\top \Omega_M + \Gamma_{0,M} & \Omega_0^\top \\ \Lambda^1 H_1 + A_1^\top \Psi - \Omega_1^\top & A_1^\top \Omega_0 + \Omega_1^\top A_0 + \Gamma_{0,1} & \Omega_1^\top A_1 + A_1^\top \Omega_1 + \Gamma_{1,1} & \dots & \Omega_1^\top A_M + A_1^\top \Omega_M + \Gamma_{1,M} & \Omega_1^\top \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \Lambda^M H_M + A_M^\top \Psi - \Omega_M^\top & A_M^\top \Omega_0 + \Omega_M^\top A_0 + \Gamma_{0,M} & A_M^\top \Omega_1 + \Omega_M^\top A_1 + \Gamma_{1,M} & \dots & \Omega_M^\top A_M + A_M^\top \Omega_M + \Gamma_{M,M} & \Omega_M^\top \\ & \Psi & \Omega_0 & \dots & \Omega_M & -\Phi \end{bmatrix},$$

$$\Gamma_{0,0}(\theta, \dot{\theta}) = \sum_{z=1}^q \dot{\theta}_z \frac{\partial P(\theta)}{\partial \theta_z} + 2 \sum_{j=1}^M \sum_{i=1}^{k_j} W_{g,g}^{i,j} \sum_{z=1}^q \max \left\{ 0, \dot{\theta}_z \frac{\partial \Lambda_i^j(\theta)}{\partial \theta_z} \right\} + \Upsilon_{0,0}(\theta),$$

$$\Gamma_{j,j}(\theta, \dot{\theta}) = 2 \sum_{j=1}^M \sum_{i=1}^{k_j} W_{j,j}^{i,j} \sum_{z=1}^q \max \left\{ 0, \dot{\theta}_z \frac{\partial \Lambda_i^j(\theta)}{\partial \theta_z} \right\} + \Upsilon_{j,j}(\theta), \quad j = \overline{1, M},$$

$$\Gamma_{k,s}(\theta, \dot{\theta}) = 2 \sum_{j=1}^M \sum_{i=1}^{k_j} \tilde{W}_{k,s}^{i,j} \sum_{z=1}^q \max \left\{ 0, \dot{\theta}_z \frac{\partial \Lambda_i^j(\theta)}{\partial \theta_z} \right\} + \tilde{\Upsilon}_{k,s}$$

then the system (2) is ISS. If the last inequality is replaced with

$$\begin{aligned} & \sum_{r=0}^M \tilde{\Upsilon}_{r,r}(\theta) + \rho_2 \sum_{z=1}^{M-1} \sum_{s=z+1}^M H_z^\top \tilde{\Upsilon}_{z,s}(\theta) H_s \\ & + \rho_3 \sum_{j=1}^M H_j^\top \Upsilon_{0,j}(\theta) H_j \succ 0 \end{aligned}$$

satisfying for all  $\theta \in \Theta$ , then the system (2) is iISS.

*Proof.* The relations  $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$  are verified for some  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  for all  $x \in \mathbb{R}^n$  according to Finsler's lemma and the introduced matrix inequalities: 1) the conditions  $P(\theta) \succeq 0$ ,  $\Lambda^j(\theta) \succeq 0$  for  $j = \overline{1, M}$  and  $P(\theta) + \rho_1 \sum_{j=1}^M H_j^\top \Lambda^j(\theta) H_j \succ 0$  are verified for some  $\rho_1 \in \mathbb{R}$  for all  $\theta \in \Theta$ , where  $\Theta$  is a compact set (see Assumption 1), which implies that the unbounded terms in (4) ( $\mu$  characterizes the nonlinearities with unbounded integrals, see end of Section II) are presented for all elements of the vector  $x$ , this justifies the existence of  $\alpha_1$ ; 2) since  $P(\theta)$  and  $\Lambda^j(\theta)$  for  $j = \overline{1, M}$  are at least continuous functions, they take bounded values on a compact set  $\Theta$ , hence, the function  $\alpha_2$  exists. The time derivative of (4) for the system dynamics takes the form:

$$\begin{aligned} \dot{V} &= \dot{x}^\top P(\theta)x + x^\top P(\theta)\dot{x} + 2 \sum_{j=1}^M \dot{x}^\top H_j^\top \Lambda^j(\theta) f^j(H_j x) \\ &+ x^\top \dot{P}(\theta)x + 2 \sum_{j=1}^M \sum_{i=1}^{k_j} \dot{\Lambda}_i^j(\theta) \int_0^{H_j^{(i)} x} f_i^j(s) ds \\ &+ 2 \left[ \sum_{g=0}^M f^g(H_g x)^\top \Omega_g^\top(\theta) + \dot{x}^\top \Psi^\top \right] \left( \sum_{g=1}^M A_g(\theta) f^g(H_g x) + d - \dot{x} \right), \end{aligned}$$

where the last line is again added by applying the descriptor approach [29] being equal zero. In comparison to Theorem 2, here there are two terms in the second line:

$$\begin{aligned} x^\top \dot{P}(\theta)x &= \sum_{z=1}^q \dot{\theta}_z x^\top \frac{\partial P(\theta)}{\partial \theta_z} x, \\ \dot{\Lambda}_i^j(\theta) \int_0^{H_j^{(i)} x} f_i^j(s) ds &= \sum_{z=1}^q \dot{\theta}_z \frac{\partial \Lambda_i^j(\theta)}{\partial \theta_z} \int_0^{H_j^{(i)} x} f_i^j(s) ds \\ &\leq \sum_{z=1}^q \max \left\{ 0, \dot{\theta}_z \frac{\partial \Lambda_i^j(\theta)}{\partial \theta_z} \right\} \int_0^{H_j^{(i)} x} f_i^j(s) ds \\ &\leq \sum_{z=1}^q \max \left\{ 0, \dot{\theta}_z \frac{\partial \Lambda_i^j(\theta)}{\partial \theta_z} \right\} \left[ \sum_{g=0}^M f^g(H_g x)^\top W_{g,g}^{i,j} f^g(H_g x) \right. \\ &\quad \left. + 2 \sum_{k=0}^{M-1} \sum_{s=k+1}^M f^k(H_k x)^\top \tilde{W}_{k,s}^{i,j} f^s(H_s x) \right], \end{aligned}$$

where we utilized Assumption 4. Using these inequalities, opening the brackets, adding and subtracting the terms weighted by the matrices  $\Phi$  and  $\tilde{\Upsilon}_{g,s}$  with  $g = \overline{0, M}$  and

$s = \overline{g, M}$ , the expression for  $\dot{V}$  can be rewritten as follows:

$$\begin{aligned} \dot{V} &= \zeta^\top Q(\theta, \dot{\theta}) \zeta - \sum_{g=0}^M f^g(H_g x)^\top \Upsilon_{g,g}(\theta) f^g(H_g x) \\ &- 2 \sum_{k=0}^{M-1} \sum_{s=k+1}^M f^k(H_k x)^\top \tilde{\Upsilon}_{k,s}(\theta) f^s(H_s x) + d^\top \Phi d \\ &\leq - \sum_{g=0}^M f^g(H_g x)^\top \Upsilon_{g,g}(\theta) f^g(H_g x) \\ &- 2 \sum_{k=0}^{M-1} \sum_{s=k+1}^M f^k(H_k x)^\top \tilde{\Upsilon}_{k,s}(\theta) f^s(H_s x) + d^\top \Phi d \end{aligned}$$

since  $Q(\theta, \dot{\theta}) \preceq 0$  for all  $\theta \in \Theta$  and  $\dot{\theta} \in [-\dot{\theta}_{\max}, \dot{\theta}_{\max}]^q$ , which are the sets of admissible values for these variables due to assumptions 1 and 2, where as before

$$\zeta^\top = [\dot{x}^\top \ x^\top \ f^1(H_1 x)^\top \dots f^M(H_M x)^\top \ d^\top]^\top.$$

The rest of the proof is the same as in Theorem 2.  $\square$

Verification of matrix inequalities formulated in Theorem 3 is more complicated since they depend on two independent vector variables  $\theta$  and  $\dot{\theta}$ . However, using Assumption 3 and additionally requiring a linear dependence of  $P$  and  $\Lambda^j$  in  $\theta$ , these conditions can be reduced to verification of LMIs.

## VI. EXAMPLE

Consider a mechanical system with cubic velocity friction term:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -a_1(t)x_1(t) - a_2(t)x_2(t) - a_3(t)x_2^3(t) + d(t), \end{aligned}$$

where  $x_1(t), x_2(t) \in \mathbb{R}$  are the system position and velocity, respectively,  $d(t) \in \mathbb{R}$  is a bounded disturbance;  $a_i(t)$  for  $i = 1, 2, 3$  are positive time-varying parameters, whose instantaneous values are unknown, but the sets of admissible deviations have been identified:  $a_{i \min} \leq a_i(t) \leq a_{i \max}$  for all  $t \geq 0$  with some  $0 < a_{i \min} \leq a_{i \max} < +\infty$  being the minimal and maximal possible values, respectively, for these parameters. Our goal is to verify robust stability of this system using the proposed conditions. It is easy to check that this model can be rewritten in the form (2) for  $M = 1$  and  $f^1(s) = s^3$ ,  $H_1 = [0 \ 1]$ ,  $k_1 = 1$  satisfying passivity condition, with

$$\begin{aligned} A_g(\theta) &= \sum_{k=1}^q \theta_k A_{gk}, \quad g = 0, 1, \quad q = 8, \\ A_{01} = A_{05} &= \begin{bmatrix} 0 & 1 \\ -a_{1 \min} & -a_{2 \min} \end{bmatrix}, \\ A_{02} = A_{06} &= \begin{bmatrix} 0 & 1 \\ -a_{1 \min} & -a_{2 \max} \end{bmatrix}, \\ A_{03} = A_{07} &= \begin{bmatrix} 0 & 1 \\ -a_{1 \max} & -a_{2 \min} \end{bmatrix}, \end{aligned}$$

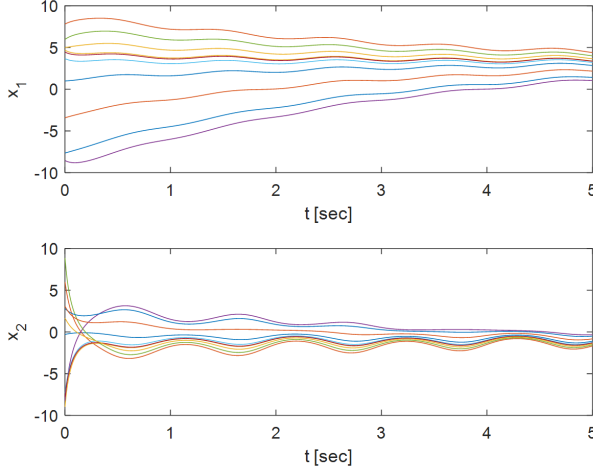


Figure 1. The results of simulation for mechanical system:  $x(t)$  versus time  $t$  [sec]

$$A_{04} = A_{08} = \begin{bmatrix} 0 & 1 \\ -a_{1 \max} & -a_{2 \max} \end{bmatrix},$$

$$A_{11} = \dots = A_{14} = \begin{bmatrix} 0 \\ -a_{3 \min} \end{bmatrix},$$

$$A_{15} = \dots = A_{18} = \begin{bmatrix} 0 \\ -a_{3 \max} \end{bmatrix},$$

where  $\sum_{k=1}^q \theta_k = 1$  and  $\theta_k \in [0, 1]$ . The compact set  $\Theta \subset \mathbb{R}_+^q$  can be straightforwardly computed, then Assumption 1 is verified. Moreover, it is clear that Assumption 3 is also satisfied, therefore, the LMIs of Corollary 1 can be used to check ISS property of this time-varying nonlinear system.

**Example.** Let

$$a_{1 \min} = 2, a_{2 \min} = 5, a_{3 \min} = 0.1,$$

$$a_{1 \max} = 3, a_{2 \max} = 10, a_{3 \max} = 0.2,$$

then the LMIs of Corollary 1 are (marginally) verified. Select

$$d(t) = \begin{bmatrix} 1 + \sin(6t) \\ 1 + \cos(3t) \end{bmatrix}, a_1(t) = 2 + \sin^2(3t),$$

$$a_2(t) = 5 + 5 \cos^2(3t), a_3(t) = 0.1 + 0.1 \sin^2(3t),$$

then in Fig. 1, the results of simulation are shown for ten different initial conditions chosen randomly in the set  $[-10, 10]^2$ . All trajectories converge to a vicinity of the origin proportional to the amplitude of the perturbation  $d$ , which is an illustration of ISS behavior.

## VII. CONCLUSION

In this paper, for the first time, the class of Parameter-Varying Persidskii (PVP) systems is introduced. This class serves as a framework to model complex dynamics involving multiple nonlinearities and various uncertainties. Within this framework, we present conditions for Input-to-State Stability (ISS) and integral Input-to-State Stability (iISS) in the form of matrix inequalities. These conditions are derived using either parameter-independent or parameter-dependent Lyapunov functions. Future directions for research include the design of stabilizing controls and observers for PVP systems.

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