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# Parametric Continued Fractions for $\pi^2$ , $\zeta(3)$ , and other Constants

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## Abstract

We give an extensive list of parametrized *families* of polynomial continued fractions of *smallest possible degrees* for  $\pi^2$  and  $\zeta(3)$ , and mention similar results for other constants.

## 1 Introduction

One can find in the literature a huge number of continued fractions (abbreviated CFs) for interesting constants and functions. The purpose of this paper is to give possibly *experimental* results showing that there often exist *families* of such CFs, usually depending on two, three, and even four parameters. In many cases these experimental results can be *proved*, usually because either  $p_n/q_n$  or  $p_n/q_n - p_{n-1}/q_{n-1}$  have an explicit expression. In other cases, I have not bothered to try to prove the experimental results, although it may not be difficult. Thus, when I speak of results, they should be considered experimental, but with considerable evidence. I would appreciate receiving any additional data of the same kind.

Notation: a CF will be written as a pair of sequences  $(a_n, b_n)_{n \geq 0}$  representing the CF  $a_0 + b_0/(a_1 + b_1/(a_2 + b_2/(a_3 + \dots)))$  (warning: note this precise convention, other authors have different notation). All the CFs that we will encounter will be *convergent*: if as usual  $p_n/q_n = a_0 + b_0/(a_1 + \dots + b_{n-1}/a_n)$ , this means that  $z = \lim_{n \rightarrow \infty} p_n/q_n$  exists, and when we speak of the speed of convergence, we mean the behavior of  $z - p_n/q_n$ .

Note that we always assume that  $b_n \neq 0$  for all  $n$ , but we allow a finite number of  $a_n$  to vanish.

- Definition 1.1** (1) Let  $(a_n)_{n \geq 0}$  be a sequence (always assumed to be of rational numbers, and usually of integers). We say that it is ultimately polynomial if there exists a polynomial denoted  $a(x)$  (by abuse of notation) with rational coefficients such that  $a_n = a(n)$  and  $a_n \neq 0$  for  $n$  sufficiently large.
- (2) We say that the CF  $(a_n, b_n)_{n \geq 0}$  is of polynomial type if the sequences  $a_n$  and  $b_n$  are ultimately polynomial.

As a practical notation, if  $a_n = a(n)$  for  $n \geq n_0$ , we will denote the sequence  $(a_n)_{n \geq 0}$  by the  $(n_0 + 1)$ -component vector  $A = (a_0, a_1, \dots, a_{n_0-1}, a(n))$ , where the last entry is a polynomial. Of course this is not unique: we also have for instance  $A = (a_0, a_1, \dots, a_{n_0-1}, a_{n_0}, a(n))$  with  $a_{n_0} = a(n_0)$ .

Thus a CF will be denoted by a pair of such vectors  $A$  and  $B$  representing  $(a_n)$  and  $(b_n)$ , and if the CF converges to some number  $z$ , we will write  $z = (a_n, b_n)$ .

**Definition 1.2** We recall that two CFs  $(a_n, b_n)$  and  $(a'_n, b'_n)$  are said to be equivalent if there exists an integer  $k$  such that  $(a'_n, b'_n) = (a_{n+k}, b_{n+k})$  for  $n$  sufficiently large, i.e., if the tails of the CFs are ultimately the same.

Since we assume everything rational, if we set  $z = (a_n, b_n)$  and  $z' = (a'_n, b'_n)$ , the CFs for  $z$  and  $z'$  are equivalent if and only if there exists a Möbius transformation with integer coefficients sending  $z$  to  $z'$ , in other words  $z' = (Az + B)/(Cz + D)$  for integers  $A, B, C, D$  with  $AD - BC \neq 0$ .

We can now give our final definition:

**Definition 1.3** We say that a pair of polynomials  $(A(x), B(x))$  with rational coefficients represents a constant  $z$  if there exists a CF  $(a_n, b_n)$  converging to  $z$  such that  $a_n = A(n)$  and  $b_n = B(n)$  for  $n$  sufficiently large, and in that case we simply write by abuse of notation  $z = (A(n), B(n))$ .

Because of what we have said above, this is almost, but not quite, equivalent to the fact that the CF  $(A(n), B(n))_{n \geq 0}$  converges to a Möbius transform of  $z$  (always assumed with integer coefficients): indeed, some values of  $A(n)$  and/or  $B(n)$  may vanish, which kills this property. But if we avoid these vanishing values by changing the corresponding coefficients of the CF to any nonzero value, this is indeed true.

Our goal is, given some interesting constant  $z$ , to find all (or at least many) pairs  $(A, B)$  representing  $z$ , subject to suitable restrictions. In particular, recall that if  $(a_n, b_n)_{n \geq 0}$  is a CF converging to  $z$  and  $r_n$  is any sequence with  $r_0 = 1$  and  $r_n \neq 0$  for all  $n$ , the CF  $(a_n r_n, b_n r_n r_{n+1})_{n \geq 0}$  also converges to  $z$ . We will thus restrict to polynomials  $(A, B)$  of smallest possible degrees.

To make this more precise, we introduce the following definitions:

**Definition 1.4** We say that  $z \in \mathbb{C}$  is a rational period of degree  $k$  if it is equal to the sum of a convergent series of the form  $\sum_{n \geq 1} \chi(n) f(n)$ , where  $\chi$  is a periodic arithmetic function taking rational values, and  $f \in \mathbb{Q}(x)$  is a rational function with rational coefficients whose denominator is of degree  $k$ .

Note that this definition is *not* compatible with the definition of periods as given by Kontsevitch–Zagier [3]. Note also that a  $\mathbb{Q}$ -linear combination of rational periods of degree  $\leq k$  is again a rational period of degree  $\leq k$ .

Examples of periods of degree 1:  $\pi$ ,  $\pi/\sqrt{d}$ ,  $\log(2)$ ,  $\sqrt{2}\log(1+\sqrt{2})$ ,  $\sqrt{3}\log(2+\sqrt{3})$ ,  $\sqrt{5}\log((1+\sqrt{5})/2)$ .

Examples of periods of degree 2:  $\pi^2$ ,  $\pi^2/\sqrt{d}$ ,  $G = L(\chi_{-4}, 2)$  (Catalan’s constant),  $G_3 = L(\chi_{-3}, 2)$ .

Examples of periods of degree 3:  $\pi^3$ ,  $\pi^3/\sqrt{d}$ ,  $\zeta(3)$ .

By using Euler’s transformation of series it is clear that a period has a CF expansion where  $a(n)$  and  $b(n)$  are polynomials with rational coefficients for  $n$  large. However this is usually uninteresting, or at least not any more interesting than the series itself. Thus, really interesting CFs are those for which the degrees of the polynomials are reasonably small, specifically  $\deg(a(n)) \leq k$  and  $\deg(b(n)) \leq 2k$  for  $n$  sufficiently large (we will say that it has *bidegree*  $\leq (k, 2k)$ ). Trivial examples:  $S^+ = \sum_{n \geq 1} 1/P(n)$  and  $S^- = \sum_{n \geq 1} (-1)^{n-1}/P(n)$  with  $\deg(P) = k \geq 2$  have the CFs  $S^+ = ((0, P(n) + P(n-1)), (1, -P(n)^2))$  and  $S^- = ((0, P(n) - P(n-1)), (1, P(n)^2))$  by Euler, of the required degrees.

Note that if we apply Euler’s transformation to  $S = \sum_{n \geq 1} (n+1)/n^3 = \zeta(3) + \zeta(2)$  which is a period of degree 3, we obtain a CF of bidegree  $(4, 8)$ . However, there does exist a CF of bidegree  $(3, 6)$ , for instance we have the (easily proved) CF:

$$\zeta(3) + \zeta(2) = ((3, 2n^3 + 2n^2 + 3n + 1), (-1, -n(n+1)^5)).$$

For most of this paper, we will focus on the cases  $z = \pi^2$  (or equivalently  $z = \zeta(2) = \pi^2/6$ ) and  $z = \zeta(3)$ , and attempt to classify *all* representations of  $\pi^2$  with bidegree  $\leq (2, 4)$ , and of  $\zeta(3)$  with bidegree  $\leq (3, 6)$ . A completely analogous study can be made for other periods such as those mentioned above, we will briefly mention this in the last section.

Note that, contrary to [1], we are interested in CFs for “pure” periods such as those mentioned above, and not in linear combinations, which is considerably more restrictive since one needs to get rid of “parasitic” terms (for instance  $\log(2)$  for  $\pi^2$ , or  $\log(2)$  and  $\zeta(2)$  for  $\zeta(3)$ , see below).

## 2 CFs for $\pi^2$ : Convergence in $1/n^P$ , $P \in \mathbb{Z}_{\geq 1}$

As far as the author is aware, up to equivalence there exist exactly nine types of convergence of polynomial continued fractions for  $\pi^2$  with bidegree  $\leq (2, 4)$ : two families with convergence in  $1/n^P$  with  $P \in \mathbb{Z}$ , one with convergence in  $1/n^P$  with  $P \in 1/2 + \mathbb{Z}$ , one with convergence in  $(-1)^n/n^P$ , one with convergence  $1/(E^n n^P)$  for each of  $E = 4/3, 2, 4$ , and  $-8$ , and finally the unique CF of Apéry with convergence in  $1/E^n$  for  $E = -((1+\sqrt{5})/2)^{10}$ . We consider each of these families in turn.

## 2.1 First Family

Using Euler's transformation of series into CFs applied to  $\pi^2/6 = \sum_{n \geq 1} 1/n^2$ , we see that  $\pi^2/6$  is given by the CF  $((0, 2n^2 - 2n + 1), (1, -n^4))$ , with speed of convergence identical to that of the series, in  $1/n$ , which we write in abbreviated form as explained above where we do not give the initial values of  $a_n$  and  $b_n$ .

$$\pi^2 = (2n^2 - 2n + 1, -n^4).$$

Remark: even though this notation means that there can be a number of initial values of  $a_n$  and  $b_n$  which are not given by the general polynomial formulas, in most of the examples that we will see in this paper, only  $a_0$  and  $b_0$  would need to be specified (for instance here  $a_0 = 0$  and  $b_0 = 1$ ), the other  $a_n$  and  $b_n$  for  $n \geq 1$  are given by the polynomials.

If we apply the Bauer–Muir transformation, we check that for any nonnegative integer  $k \geq 0$  we also have  $\pi^2 = (2n^2 - 2n + k^2 + k + 1, -n^4)$ .

This naturally leads us to *search numerically* for other representations of the form  $\pi^2 = (A(n), B(n))$ , with  $A(n) = 2n^2 + a_1n + a_0$  and  $B(n)$  of degree 4 with leading coefficient  $-1$ . We speed up the search by assuming that  $B$  is a product of linear factors with integral coefficients (you are welcome to try other types of polynomials, but I am pretty sure that you will not find others), and since  $(A(n + j), B(n + j))$  is equivalent to  $(A(n), B(n))$ , we can assume that  $B(n) = -n(n + u_1)(n + u_2)(n + u_3)$  with  $0 \leq u_1 \leq u_2 \leq u_3$ .

We find a *very large* number of such representations, from which one can easily extract one and two-parameter families, less easily three-parameter families, and finally with considerable difficulty, a single four-parameter family to rule them all:  $\pi^2 = (A(n), B(n))$  with

$$\begin{aligned} A(n) &= 2n^2 + (3u + v + w - 2)n \\ &\quad + k^2 + u^2 + uk + vk + wk + uv + uw + vw + k - u + 1 \\ B(n) &= -n(n + u)(n + u + v)(n + u + w), \end{aligned}$$

where  $u, v, w$ , and  $k$  are nonnegative integer parameters, with speed of convergence in  $C/n^{u+v+w+2k+1}$  for some constant  $C$ .

An extensive additional search seems to show that there are no additional sporadic representations. However, we will see in Section 3 that it is also possible to choose  $v$  or  $w$  (but not both) in  $1/2 + \mathbb{Z}$ .

**Proof:** in the present case, it is possible, not only to prove that the four-parameter family above does represent  $\pi^2$ , but also to explain where it originates.

Consider the series  $S = \sum_{n \geq 1} 1/\prod_{0 \leq i \leq m-1} (n+i)^{e_i}$ , where  $e_i = 1$  if  $0 \leq i \leq m_1 - 1$  or  $m_1 + m_2 \leq i \leq m - 1$ , and  $e_i = 2$  if  $m_1 \leq i \leq m_1 + m_2 - 1$ , where  $m_1 \geq 0$ ,  $m_2 \geq 1$ , and  $m \geq m_1 + m_2$ . By expanding into partial fractions it is immediate to show that the sum  $S$  satisfies  $S = a\pi^2 + b$  with  $a, b$  rational and  $a \neq 0$ . On the other hand, applying Euler's transformation of  $S$  into a CF, and

simplifying by suitable factors, it is not difficult to show that  $S$  (hence  $\pi^2$ ) is represented by the CF  $(A(n), B(n))$ , with

$$\begin{aligned} A(n) &= (n-1)(n+m_1-1) + (n+m_1+m_2-1)(n+m-1) \quad \text{and} \\ B(n) &= -n(n+m_1)(n+m_1+m_2-1)(n+m-1) . \end{aligned}$$

This gives the three-parameter family above when  $k=0$ , and the case of general  $k$  is obtained by successive Bauer–Muir accelerations.

## 2.2 Second Family

We also have the modified series obtained by removing the Euler factor at 2:  $\pi^2/8 = \sum_{n \geq 1} 1/(2n-1)^2$ , giving the CF  $((0, 1, 8n^2 - 16n + 10), (1, -(2n-1)^4))$  for  $\pi^2/8$ , that we write in abbreviated form

$$\pi^2 = (8n^2 - 16n + 10, -(2n-1)^4) .$$

All the work has been done for us in the first family: we simply replace  $n$  by  $n-1/2$  and multiply by suitable powers of 2 to simplify the formulas, giving the single four-parameter family  $\pi^2 = (A(n), B(n))$  with

$$\begin{aligned} A(n) &= 8n^2 + 4(3u + v + w - 4)n \\ &\quad + 4(k^2 + u^2 + uk + vk + wk + uv + uw + vw) + 4k - 10u - 2v - 2w + 10 \\ B(n) &= -(2n-1)(2n-1+2u)(2n-1+2u+2v)(2n-1+2u+2w) , \end{aligned}$$

with the same speed of convergence in  $C/n^{u+v+w+2k+1}$ .

However, inspection shows that while  $v$ ,  $w$ , and  $k$  must still be nonnegative, we can have  $u$  negative as long as  $u \geq -k - \min(v, w)$ . Note that here we *cannot* choose  $v$  or  $w$  in  $1/2 + \mathbb{Z}$ .

## 3 CFs for $\pi^2$ : Convergence in $1/n^P$ , $P \in 1/2 + \mathbb{Z}_{\geq 0}$

We recall the following formula due to Euler:

$$\sum_{n \geq 1} \frac{(2z)^{2n}}{n^2 \binom{2n}{n}} = 2 \operatorname{asin}^2(z) ,$$

which we will use several times. Here, we choose  $z=1$  to obtain the series

$$\sum_{n \geq 1} \frac{4^n}{n^2 \binom{2n}{n}} = \frac{\pi^2}{2} ,$$

and using Euler's transformation of series into CFs we see that  $\pi^2/2$  is given by the CF  $\pi^2/2 = ((0, 4n^2 - 5n + 2), (2, -2n^3(2n-1)))$  with convergence identical to that of the series in  $\sqrt{\pi}/n^{1/2}$ , which we write in abbreviated form

$$\pi^2 = (4n^2 - 5n + 2, -2n^3(2n-1)) .$$

If we apply the Bauer–Muir transformation, we check that for any nonnegative integer  $k \geq 0$  we also have  $\pi^2 = (4n^2 - 5n + 2k^2 + k + 2, -2n^3(2n - 1))$ .

Here, we could do a new search, but luckily we have already done the work: it is simply the first four-parameter family found above, but now with the parameter  $w$  being a half integer instead of an integer. We thus replace  $w$  by  $w - 1/2$  and simplify, so we obtain the four-parameter family  $\pi^2 = (A(n), B(n))$  with

$$\begin{aligned} A(n) &= 4n^2 + (6u + 2v + 2w - 5)n + 2k^2 \\ &\quad + (2u + 2v + 2w + 1)k + 2u^2 + (2v + 2w - 3)u + v(2w - 1) + 2 \\ B(n) &= -2n(n + u)(n + u + v)(2n + 2u + 2w - 1), \end{aligned}$$

where  $u, v$ , and  $k$  are nonnegative integer parameters and  $w \in \mathbb{Z}$ , with speed of convergence in  $C/n^{u+v+w+2k+1/2}$  for some constant  $C$ .

The *proof* of the validity of this family follows from the one given in Section 2.1.

## 4 CFs for $\pi^2$ : Convergence in $(-1)^n/n^P$

Using Euler’s transformation of series into CFs but now applied to  $\pi^2/12 = \sum_{n \geq 1} (-1)^{n-1}/n^2$ , we see that  $\pi^2/12$  is given by the CF  $((0, 2n - 1), (1, n^4))$ , with speed of convergence identical to that of the series, in  $(-1)^n/(2n^2)$ , which we write in abbreviated form

$$\pi^2 = (2n - 1, n^4).$$

If we apply the Bauer–Muir transformation, we immediately see that for any *odd* integer  $k \geq 1$  we also have  $\pi^2 = (k(2n - 1), n^4)$ . We easily check that this is also true for even integers, so we have found a one-parameter family. Now we search numerically for other representations of the form  $\pi^2 = (A(n), B(n))$ , with  $A(n)$  of degree 1 and  $B(n)$  monic of degree 4, as before assumed to be a product of linear factors with integral coefficients. Since  $(-A(n), B(n))$  is easily seen to be equivalent to  $(A(n), B(n))$ , we can also assume that the leading coefficient of  $A(n)$  is positive, so we set  $A(n) = a_1n + a_0$  with  $a_1 \geq 1$  and  $B(n) = n(n + u_1)(n + u_2)(n + u_3)$  with  $0 \leq u_1 \leq u_2 \leq u_3$ . We again find a large number of representations, and after some cleaning up, we obtain a three-parameter family and four one-parameter families, and no additional sporadic representations:

$$\begin{aligned} \pi^2 &= ((k + 1)(2n + 2u + 2v - 1), n(n + u)(n + u + 2v)(n + 2u + 2v)) \\ &= ((2u^2 + 8u + 7)n + (2u + 3)(u^2 + 3u + 1), n(n + u)(n + u + 1)(n + 2u + 4)) \\ &= ((2u^2 + 8u + 7)n + (2u + 3)(u + 2)(u + 3), n(n + u + 3)(n + u + 4)(n + 2u + 4)) \\ &= ((8u + 7)n + (24u^2 + 24u + 3), n(n + 2u)(n + 4u + 1)(n + 6u + 4)) \\ &= ((8u + 7)n + 6(u + 1)(4u + 3), n(n + 2u + 3)(n + 4u + 4)(n + 6u + 4)), \end{aligned}$$

where  $u$ ,  $v$ , and  $k$  are nonnegative integer parameters, with speeds of convergence  $(-1)^n C/n^{2k+2}$ ,  $(-1)^n C/n^{2u^2+8u+7}$ ,  $(-1)^n C/n^{2u^2+8u+7}$ ,  $(-1)^n C/n^{8u+7}$  and  $(-1)^n C/n^{8u+7}$  respectively, for some constants  $C$ .

**Proof:** Although similar to the proof given in Section 2.1 the proof of the above is both more complicated and more interesting since we will need to generalize the construction. Here we consider the series

$$S = \sum_{n \geq 1} (-1)^{n-1} / \prod_{0 \leq i \leq m-1} (n+i)^{e_i},$$

with the same  $e_i$  as above. Once again we can simplify the result given by Euler's transformation of  $S$  into a CF, and we obtain that  $S$  is represented by the CF  $(A(n), B(n))$  with

$$\begin{aligned} A(n) &= -(n-1)(n+m_1-1) + (n+m_1+m_2-1)(n+m-1) \quad \text{and} \\ B(n) &= n(n+m_1)(n+m_1+m_2-1)(n+m-1). \end{aligned}$$

The problem is that the sum  $S$  is not anymore of the form  $a\pi^2 + b$ , but of the form  $a\pi^2 + c \log(2) + b$ . Thus, if we want a CF representing  $\pi^2$  itself, we need  $c = 0$ . It is not difficult to show that this is indeed the case when  $(m_1, m_2, m_3) := (m_1, m_2, m - m_1 - m_2)$  is of one of the following three shapes:  $(m_1, m_2, m_3) = (m_1, 2u+1, m_1)$  or  $(2u, 2u+2, 2u+3)$ , or  $(2u+3, 2u+2, 2u)$ . It should not be difficult to prove that there are no other cases, but I have not done so. In any event, the first shape followed by Bauer–Muir accelerations leads to the three-parameter family, and the other two shapes lead to the last two one-parameter families.

The first two one-parameter families do *not* come from this construction, but from a more general one. We first note the following result, whose easy proof is left to the reader

**Proposition 4.1** *Let  $c(x) \in \mathbb{Q}[x]$  be a polynomial with  $\deg(c) \geq 2$ , let  $z = \pm 1$ , let  $P \in \mathbb{Q}[x]$  be an irreducible polynomial which divides  $c(x)P(x+1) + zc(x-1)P(x-1)$ , let  $R(x) = (c(x)P(x+1) + zc(x-1)P(x-1))/P(x)$  and  $g(x) = \gcd(c(x), c(x-1))$ , and set*

$$S = \sum_{n \geq 1} \frac{z^n}{c(n)P(n)P(n+1)}.$$

*Assume that  $c(x)$ ,  $P(x)$ , and  $P(x+1)$  are pairwise coprime. We have the continued fraction  $S = [[0, R(n)/g(n)], [r_1, -zc(n)^2/(g(n)g(n+1))]]$  and the identity  $S = r_2 + \sum_{n \geq 1} z^n d(n)/c(n)$  for some rational numbers  $r_1$  and  $r_2$  and polynomial  $d$  with  $\deg(d) < \deg(c)$ .*

We also need the following conjecture:

**Conjecture 4.2** *Let  $u$  be a nonnegative integers. Up to a multiplicative constant there exists a unique polynomial  $P$  such that*

$$\begin{aligned} (x+2u+4)(x+u+1)P(x+1) - (x-1)(x+u-1)P(x-1) \\ = ((2u^2+8u+7)x + (2u+3)(u^2+3u+1))P(x), \end{aligned}$$



and we have  $\deg(P) = u(u + 3)$ . The same is true if we require

$$\begin{aligned} (x + 2u + 4)(x + u + 4)P(x + 1) - (x - 1)(x + u + 2)P(x - 1) \\ = ((2u^2 + 8u + 7)x + (2u + 3)(u + 2)(u + 3))P(x) . \end{aligned}$$

Assuming this conjecture, in Proposition 4.1 we choose  $z = -1$ ,  $c(x) = \prod_{0 \leq i \leq 2u+4} (n + i)^{e_i}$  as before, with  $m_1 = u$ ,  $m_2 = 2$ , and  $m_3 = u + 3$ , so  $m = 2u + 5$ , and  $P$  the polynomial satisfying the first identity above. We check that  $g(x) = (x + u) \prod_{0 \leq i \leq 2u+3} (x + i)$ , hence

$$-zc(n)^2/(g(n)g(n + 1)) = n(n + u)(n + u + 1)(n + 2u + 4) ,$$

and because of the identity for  $P$ ,

$$R(n)/g(n) = ((2u^2 + 8u + 7)n + (2u + 3)(u^2 + 3u + 1)) .$$

Finally, the identity for  $S$  given by Proposition 4.1 implies that  $S$  is a linear combination of  $\pi^2$ ,  $\log(2)$ , and 1. Similarly for the second identity. Thus, the other two one-parameter families follow from the above conjecture together with the additional conjecture that  $\log(2)$  does not occur in the linear combination for  $S$ .

**Remark:** Concerning the above conjecture, Peter Mueller from the Math-Overflow forum proves the following:

- (1) The two statements are equivalent.
- (2) If  $P$  exists it is indeed unique up to a multiplicative constant, and has degree  $u(u + 3)$ .

It is possible that there exist other, possibly more complicated, one-parameter families.

## 5 CFs for $\pi^2$ : Convergence in $1/((4/3)^n n^P)$

Here we use the  $\operatorname{asin}^2(z)$  series with  $z = \sqrt{3}/2$ , giving the series

$$\sum_{n \geq 1} \frac{3^n}{n^2 \binom{2n}{n}} = \frac{2\pi^2}{9} ,$$

and applying Euler's transformation shows that  $2\pi^2/9$  is given by the CF  $((0, 7n^2 - 8n + 3), (3, -6n^3(2n - 1)))$  with speed of convergence identical to that of the series, in  $3\sqrt{\pi}/((4/3)^n n^{3/2})$ , and as before we can write in abbreviated form

$$\pi^2 = (7n^2 - 8n + 3, -6n^3(2n - 1)) .$$

Applying Bauer–Muir does not give anything simple. So as before we search for representations  $\pi^2 = (A(n), B(n))$  with  $A(n) = 7n^2 + a_1n + a_0$  and  $B(n) =$

$-6n(n+u_1)(n+u_2)(2n-1+2u_3)$ . Note that we could try other types for  $B(n)$  with the same leading term  $-12n^4$  and completely factored, for instance  $B(n) = -n(2n+u_1)(2n+u_2)(3n+u_3)$ , but I have not found any new representations in this way.

Here, the situation is less favorable since I have not been able to classify completely the representations. I have found four two-parameter families:

$$\begin{aligned}
\pi^2 &= (7n^2 + (2u + 3v - 6)n - (5u^2 + 15vu + 18v^2 - 2), \\
&\quad - 6n(n+u)(n+u+3v)(2n+1-4u-6v)) \\
&= (7n^2 + (2u + 3v - 8)n - (5u^2 + (15v+2)u + 3(3v-1)(2v+1)), \\
&\quad - 6n(n+u)(n+u+3v)(2n-1-4u-6v)) \\
&= (7n^2 + (15u + 3v - 8)n - 3(2v+1)(3u+3v-1), \\
&\quad - 6n(n+u)(n+3u+3v)(2n-1-6v)) \\
&= (7n^2 + (15u + 3v - 3)n - 2(3v+4)(3u+3v+2), \\
&\quad - 6n(n+u)(n+3u+3v+3)(2n-5-6v))
\end{aligned}$$

with all parameters nonnegative, plus a number of apparently sporadic cases:

$$\begin{aligned}
\pi^2 &= (7n^2 - 16n + 3, -6n^3(2n-5)) \\
&= (7n^2 - 19n - 6, -6n^2(n+1)(2n-11)) \\
&= (7n^2 + 17n, -6n(n+3)^2(2n+1)) \\
&= (7n^2 + 14n - 36, -6n(n+3)(n+5)(2n-7)) \\
&= (7n^2 + 54n + 5, -6n(n+6)(n+10)(2n+1)) \\
&= (7n^2 + 31n - 60, -6n(n+1)(n+10)(2n-3)) ,
\end{aligned}$$

all with convergence speeds  $C/((4/3)^n n^P)$  for varying half-integral  $P$ .

I have not tried to prove the validity of the above CFs.

The upshot of this case is that first, that I have probably not found all possible cases, and second, that I have not seen the patterns leading either to additional two-parameter families, or even possibly to a three-parameter family.

Note, however, that in Section 7 we will meet the same *exact* situation (four two-parameter families plus six sporadic cases), which makes me believe that there is a relation between the CFs of that section and those of the present section.

## 6 CFs for $\pi^2$ : Convergence in $1/(2^n n^P)$

Here we use the  $\text{asin}^2(z)$  series with  $z = \sqrt{2}/2$ , giving the series

$$\sum_{n \geq 1} \frac{2^n}{n^2 \binom{2n}{n}} = \frac{\pi^2}{8},$$

and applying Euler's transformation shows that  $\pi^2/8$  is given by the CF  $((0, 3n^2 - 3n + 1), (1, -n^3(2n - 1)))$  with speed of convergence identical to that of the series, in  $\sqrt{\pi}/(2^n n^{3/2})$ , and as before we can write in abbreviated form

$$\pi^2 = (3n^2 - 3n + 1, -n^3(2n - 1)).$$

Here, applying Bauer–Muir gives the one-parameter family  $\pi^2 = (3n^2 - 3n + 1, -n^3(2n - 1 - 2k))$ , and we search for representations  $\pi^2 = (A(n), B(n))$  with  $A(n) = 3n^2 + a_1 n + a_0$  and  $B(n) = -n(n + u_1)(n + u_2)(2n - 1 + 2u_3)$ . After some work, we find the three-parameter family:

$$\begin{aligned} \pi^2 = & (3n^2 + (8u + 4v - 3)n + (2u - 1)(2u + 2v - 1), \\ & -n(n + 2u)(n + 2u + 2v)(2n + 2u - 2k - 1)), \end{aligned}$$

where all the parameters are nonnegative, with speed of convergence

$$C/(2^n n^{u+2v+3/2+3k})$$

for some constant  $C$ .

An extensive additional search seems to show that there are no additional sporadic representations.

It is easy to give a *proof* of the validity of the above family, which we leave to the reader.

## 7 CFs for $\pi^2$ : Convergence in $1/(4^n n^P)$

Here we use the  $\text{asin}^2(z)$  series with  $z = 1/2$ , giving the series

$$\sum_{n \geq 1} \frac{1}{n^2 \binom{2n}{n}} = \frac{\pi^2}{18},$$

and applying Euler's transformation shows that  $\pi^2/18$  is given by the CF  $((0, 5n^2 - 4n + 1), (1, -2n^3(2n - 1)))$  with speed of convergence identical to that of the series, in  $\sqrt{\pi}/3/(4^n n^{3/2})$ , and as before we can write in abbreviated form

$$\pi^2 = (5n^2 - 4n + 1, -2n^3(2n - 1)).$$

Applying Bauer–Muir leads to an explicit complicated one-parameter family which will be a special case of the ones below, and the situation is similar to that of Section 5: I have found four two-parameter families:

$$\begin{aligned}
\pi^2 &= (5n^2 + (14u + 21v - 4)n + (3u + 3v - 1)(3u + 6v - 1), \\
&\quad - 2n(n + u)(n + u + 3v)(2n - 1)) \\
&= (5n^2 + (14u + 21v - 6)n + 9u^2 + (27v - 8)u + 2(3v - 1)^2, \\
&\quad - 2n(n + u)(n + u + 3v)(2n + 1)) \\
&= (5n^2 + (17u + 21v + 15)n + 2(2u + 3v + 2)(3u + 3v + 2), \\
&\quad - 2n(n + u)(n + 3u + 3v + 3)(2n + 2u + 1)) \\
&= (5n^2 + (17u + 21v - 4)n + (3u + 3v - 1)(4u + 6v - 1), \\
&\quad - 2n(n + u)(n + 3u + 3v)(2n + 2u - 1)) ,
\end{aligned}$$

with all parameters nonnegative, plus a number of apparently sporadic cases:

$$\begin{aligned}
\pi^2 &= (5n^2 + 4n + 1, -2n^3(2n + 3)) \\
&= (5n^2 + 15n + 6, -2n^2(n + 1)(2n - 1)) \\
&= (5n^2 + 19n + 16, -2n(n + 3)^2(2n + 3)) \\
&= (5n^2 + 78n + 261, -2n(n + 3)(n + 7)(2n - 1)) \\
&= (5n^2 + 38n + 68, -2n(n + 3)(n + 5)(2n + 1)) \\
&= (5n^2 + 45n + 60, -2n(n + 1)(n + 10)(2n + 3)) ,
\end{aligned}$$

all with convergence speeds  $C/(4^n n^P)$  for varying half-integral  $P$ .

As in Section 5, I have not tried to prove the validity of these CFs, I have also probably not found all possible cases, and I have not seen the patterns leading either to additional one- or two-parameter families, or even possibly to a three-parameter family.

Nonetheless, the reader has certainly noticed that there is a striking resemblance between the families of this section and those of Section 5, and this should probably be investigated.

## 8 CFs for $\pi^2$ : Convergence in $(-1)^n/(8^n n^P)$

Let

$$F_n = \sum_{0 \leq k \leq n} \binom{n}{k}^3$$

be the  $n$ th *Franel number*. From the recursion found by Franel for these numbers and the computation of so-called Apéry limits (see for instance [2]), we have the CF  $\pi^2/3 = ((0, 7n^2 - 7n + 2), (8, 8n^4))$  with speed of convergence  $C/(-8)^n$  and whose  $n$ th denominator  $q(n)$  is given by  $q(n) = n!^2 F_n$ , giving the formula

$$\frac{\pi^2}{3} = \sum_{n \geq 1} (-1)^{n-1} \frac{8^n}{n^2 F_n F_{n-1}} .$$

As usual we write the CF in abbreviated form

$$\pi^2 = (7n^2 - 7n + 2, 8n^4) .$$

Bauer–Muir acceleration gives something complicated, so as usual we search. The situation is even worse here: I have found experimentally a large number of representations, for which I was able to extract three two-parameter families and a small number of one-parameter families, as follows:

$$\begin{aligned} \pi^2 &= (7n^2 + ((48 - 20v)u - 7)n + 2(4u - 1)((8 - 5v)u - 1), \\ &\quad 8n(n + uv)(n + u(v + 2))(n + 4u)) \\ &= (7n^2 + (24v - 20u - 7)n + (4v - 2)(4v - 5u - 1), \\ &\quad 8n(n + u)(n + 2v)(n + u + v)) \\ &= (7n^2 + (12u + 20v - 7)n + (4v - 2)(3u + 3v - 1), \\ &\quad 8n(n + 2v)(n + 3u + 2v)(n + 3u + 3v)) \\ &= (7n^2 + (30u - 7)n + (3u - 1)(9u - 2), 8n^2(n + 3u)^2) \\ &= (7n^2 + (34u - 7)n + 27u^2 - 17u + 2, 8n^2(n + u)^2) \\ &= (7n^2 + (22u - 7)n + (3u - 1)(5u - 2), 8n(n + u)^2(n + 3u)) \\ &= (7n^2 + (58u - 7)n + 87u^2 - 29u + 2, 8n(n + u)^2(n + 3u)) \\ &= (7n^2 + (48u - 23)n + 2(4u + 1)(8u - 7), 8n(n + 2)(n + 2u + 3)(n + 4u + 2)) \\ &= (7n^2 + (34u + 17)n + 3(3u + 1)(3u + 2), 8n(n + 1)(n + u)(n + u + 2)) , \end{aligned}$$

where  $u$  and  $v$  are nonnegative integers, except in the third two-parameter representation where one can have  $u \geq -\lfloor 2v/3 \rfloor$ , all with convergence speeds  $C/((-8)^n n^P)$  for varying integral  $P$ .

I have found many (for now more than 60) additional individual CFs, from which one can probably extract more one-parameter families. Here is a small sample:

$$\begin{aligned} \pi^2 &= (7n^2 + 26n + 8, 8n^3(n + 3)) \\ &= (7n^2 + 80n + 105, 8n^2(n + 1)(n + 2)) \\ &= (7n^2 + 24n + 9, 8n^2(n + 1)(n + 4)) \\ &= (7n^2 + 65n + 70, 8n^2(n + 1)(n + 8)) \\ &= (7n^2 - 15n - 12, 8n^2(n + 4)^2) \\ &= (7n^2 + 57n + 96, 8n^2(n + 4)^2) . \end{aligned}$$

## 9 CFs for $\pi^2$ : Convergence in $1/(-(1 + \sqrt{5})/2)^{10})^n$

This is the famous CF of Apéry:

$$\pi^2 = ((0, 11n^2 - 11n + 3), (30, n^4))$$

with convergence in  $24\pi^2/(1 + \sqrt{2})^{10n+5}$ .

This CF seems to be unique of its kind, i.e. of the form  $\pi^2 = (11n^2 + a_1n + a_0, n(n + u_1)(n + u_2)(n + u_3))$ .

## 10 CFs for $\pi^2$ : Other Speeds of Convergence

I have not found any convergent CF of the form  $\pi^2 = (A(n), B(n))$  with  $\deg(A) \leq 2$ ,  $\deg(B) \leq 4$  and  $A$  and  $B$  with integral coefficients other than the families mentioned above, up to trivial equivalence such as changing  $n$  into  $n + j$  or multiplying  $(A, B)$  by  $(c, c^2)$  for a nonzero constant  $c$ . Of course, there exist infinitely many other CFs for  $\pi^2$ , but of larger bidegrees.

## 11 CFs for $\zeta(3)$ : Convergence in $1/n^P$

We now study CFs for  $\zeta(3)$ . Similarly to  $\pi^2$ , as far as the author is aware, up to equivalence there exist exactly eight types of convergence of polynomial continued fractions for  $\zeta(3)$  with integral coefficients and bidegree  $\leq (3, 6)$ : three families with convergence in  $1/n^P$  with  $P \in \mathbb{Z}$ , a single CF with convergence in  $(-1)^n/n^P$ , a single CF with convergence in  $1/(4^n n^P)$ , a family of CFs with convergence in  $1/((-4)^n n^P)$ , a single CF with convergence in  $1/E^n$  with  $(1 + \sqrt{2})^4$ , and finally the unique CF of Apéry with convergence in  $1/E^n$  for  $E = (1 + \sqrt{2})^8$ . We consider each of these cases in turn.

### 11.1 First Family

The natural series giving  $\zeta(3)$  is  $\zeta(3) = \sum_{n \geq 1} 1/n^3$ , so Euler's transformation of series gives the CF  $\zeta(3) = ((0, (2n - 1)(n^2 - n + 1)), (1, -n^6))$ , abbreviated as usual as

$$\zeta(3) = ((2n - 1)(n^2 - n + 1), -n^6).$$

Bauer–Muir acceleration leads to the family  $\zeta(3) = ((2n - 1)(n^2 - n + 2k^2 + 2k + 1), -n^6)$  for nonnegative integer  $k$ .

Instead of doing a search, we generalize the proof given in Section 2.1: using the same notation, we choose  $e_i = 1, 2, 3, 2$ , or  $1$  according to  $0 \leq i < m_1$ ,  $m_1 \leq i < m_1 + m_2$ ,  $m_1 + m_2 \leq i < m_1 + m_2 + m_3$ ,  $m_1 + m_2 + m_3 \leq i < m_1 + m_2 + m_3 + m_4$ , or  $m_1 + m_2 + m_3 + m_4 \leq i < m$  respectively, and this guarantees that the CF  $(A(n), B(n))$  will have bidegree  $\leq (3, 6)$ , with

$$\begin{aligned} A(n) &= (n + m_1 + m_2 + m_3 - 1)(n + m_1 + m_2 + m_3 + m_4 - 1)(n + m - 1) \\ &\quad + (n - 1)(n + m_1 - 1)(n + m_1 + m_2 - 1) \quad \text{and} \\ B(n) &= -n(n + m_1)(n + m_1 + m_2)(n + m_1 + m_2 + m_3 - 1) \\ &\quad (n + m_1 + m_2 + m_3 + m_4 - 1)(n + m - 1). \end{aligned}$$

However the sum  $S$  will be a linear combination of  $1$ ,  $\pi^2$ , and  $\zeta(3)$ , and we do not want any  $\pi^2$ . For this, a *sufficient* condition is that  $m_3$  is odd and  $m_4 = m_2$  and  $m_5 = m_1$ , giving the parametric family  $(A, B)$  with

$$\begin{aligned} A(n) &= (n+u+v+2w)(n+u+2v+2w)(n+2u+2v+2w) \\ &\quad + (n-1)(n+u-1)(n+u+v-1) \quad \text{and} \\ B(n) &= -n(n+u)(n+u+v)(n+u+v+2w)(n+u+2v+2w)(n+2u+2v+2w), \end{aligned}$$

and note that

$$\begin{aligned} A(n) &= (2n+2u+2v+2w-1)(n^2 + (2u+2v+2w-1)n \\ &\quad + (u^2 + (3v+4w)u + 2v^2 + (6w+1)v + 4w^2 + 2w + 1)). \end{aligned}$$

Applying Bauer–Muir, we thus find the four-parameter family  $\zeta(3) = (A(n), B(n))$  with

$$\begin{aligned} A(n) &= (2n+2(u+v+w)-1)(n^2 + (2(u+v+w)-1)n + u^2 + 2v^2 + 4w^2 \\ &\quad + 3uv + 4uw + 6vw + v + 2w + 1 + 2k(u+2v+3w) + 2k^2 + 2k), \\ B(n) &= -n(n+u)(n+u+v)(n+u+v+2w)(n+u+2v+2w)(n+2u+2v+2w), \end{aligned}$$

where  $u, v, w$ , and  $k$  are nonnegative integer parameters, with speed of convergence in  $C/n^{2u+4v+6w+4k+2}$  for some constant  $C$ .

However, the sufficient condition (symmetry of  $(m_1, m_2, m_3, m_4, m_5)$ ) above is not necessary: it is easy to find as many nonsymmetrical examples as we like, the simplest being  $(m_1, m_2, m_3, m_4, m_5) = (0, 0, 2, 2, 0)$ . I have not yet been able to give a reasonable classification of these additional examples, here is a small sample:

$$\begin{aligned} \zeta(3) &= (2n^3 + 4n^2 + 18n + 8, -n^3(n+1)(n+3)^2) \\ &= (2n^3 + 8n^2 + 26n + 28, -n^2(n+2)(n+3)^3) \\ &= (2n^3 + 11n^2 + 53n + 60, -n^2(n+1)(n+2)(n+5)(n+6)) \\ &= (2n^3 + 17n^2 + 78n + 119, -n(n+2)^2(n+3)(n+5)(n+8)) \\ &= (2n^3 + 19n^2 + 93n + 180, -n(n+1)(n+4)(n+5)(n+6)^2) \\ &= (2n^3 + 20n^2 + 163n + 299, -n^3(n+3)(n+10)^2) \\ &= (2n^3 + 22n^2 + 156n + 360, -n(n+1)(n+2)(n+5)(n+8)(n+9)) \\ &= (2n^3 + 23n^2 + 201n + 500, -n^2(n+1)(n+5)(n+10)^2). \end{aligned}$$

## 11.2 Second Family

We also have the simple modification obtained by removing the Euler factor at 2:  $\zeta(3) = (8/7) \sum_{n \geq 1} 1/(2n-1)^3$ , so Euler's transformation gives the CF

$\zeta(3) = ((0, 1, 4(n-1)(4n^2 - 8n + 7)), (8/7, -(2n-1)^6))$ , abbreviated as usual as

$$\zeta(3) = (4(n-1)(4n^2 - 8n + 7), -(2n-1)^6) .$$

Bauer–Muir acceleration leads to the family  $\zeta(3) = (4(n-1)(4n^2 - 8n + 8k^2 + 8k + 7), -(2n-1)^6)$  for nonnegative integer  $k$ . As in the case of  $\pi^2$ , to find parametric families we simply replace  $n$  by  $n - 1/2$  and multiply by suitable powers of 2 in the first family found above, giving the four-parameter family  $\zeta(3) = (A(n), B(n))$  with

$$\begin{aligned} A(n) &= 4(n+u+v+w-1)(4n^2 + 8(u+v+w-1)n + 4u^2 + 8v^2 + 16w^2 \\ &\quad + 12uv + 16uw + 24vw - 4u + 4w + 8k(u+2v+3w) + 8k^2 + 8k + 7), \\ B(n) &= -(2n-1)(2n+u-1)(2n+u+v-1)(2n+u+v+2w-1) \cdot \\ &\quad \cdot (2n+u+2v+2w-1)(2n+2u+2v+2w-1) , \end{aligned}$$

where  $u, v, w$ , and  $k$  are nonnegative integer parameters, with speed of convergence in  $C/n^{2u+4v+6w+4k+2}$  for some constant  $C$ , and of course as in the first family, one also has the numerous additional apparently sporadic cases, such as

$$\zeta(3) = (16n^3 + 8n^2 + 124n - 2, -(2n-1)^3(2n+1)(2n+5)^2) .$$

### 11.3 Third Family

From work of Y. Yang [4], one finds the CF

$$\zeta(3) = ((0, (2n-1)(2n^2 - 2n + 1)), (2/7, -4n^6))$$

with extremely slow convergence in  $C/\log(n)$ , which as usual we abbreviate to  $\zeta(3) = ((2n-1)(2n^2 - 2n + 1), -4n^6)$ . Modifying his construction and applying Bauer–Muir, we find the two-parameter family  $\zeta(3) = (A(n), B(n))$  with

$$\begin{aligned} A(n) &= (2n+2u-1)(2n^2 + (4u-2)n + (2k+2)u + k^2 + 4k + 5), \\ B(n) &= -4n^3(n+2u)^3 , \end{aligned}$$

with  $u$  and  $k$  nonnegative, with speed of convergence in  $C/n^{2u+2k+4}$ .

It is interesting to compare this with the very similar two-parameter family  $\zeta(3) = (A(n), B(n))$  obtained by setting  $u = v = 0$  and replacing  $w$  by  $u$  in Section 11.1:

$$\begin{aligned} A(n) &= (2n+2u-1)(2n^2 + (4u-2)n + 8u^2 + (12k+4)u + 4k^2 + 4k + 2), \\ B(n) &= -4n^3(n+2u)^3 , \end{aligned}$$

with  $u$  and  $k$  nonnegative, with speed of convergence in  $C/n^{6u+4k+2}$ .



## 12 CFs for $\zeta(3)$ : Convergence in $(-1)^n/n^P$

Using Euler's transformation of series into CFs but now applied to  $(3/4)\zeta(3) = \sum_{n \geq 1} (-1)^{n-1}/n^3$ , we see that  $\zeta(3)$  is given by the CF  $((0, 3n^2 - 3n + 1), (4/3, n^6))$ , with speed of convergence identical to that of the series, in  $(-1)^n 2/(3n^3)$ , which we write in abbreviated form

$$\zeta(3) = (3n^2 - 3n + 1, n^6).$$

There is apparently no other CF of this type with convergence in  $(-1)^n/n^P$ . The main reason is probably that, if we try to apply similar constructions to those that we used for  $\sum_{n \geq 1} (-1)^{n-1}/n^2$ , we are now going to obtain linear combinations of 1,  $\zeta(3)$ , and the two additional "parasitic" terms  $\zeta(2)$  and  $\log(2)$ , instead of simply  $\log(2)$  alone.

## 13 CFs for $\zeta(3)$ : Convergence in $1/(4^n n^P)$

Let

$$D_n = \sum_{0 \leq k \leq n} \binom{n}{k}^2 \binom{2k}{k} \binom{2n-2k}{n-k}$$

be the  $n$ th so-called *Domb number*. From the recursion that one can find for these numbers and the computation of Apéry limits (see for instance [2]), we have the CF  $\zeta(3) = ((0, (2n-1)(5n^2-5n+2)), (12/7, -16n^6))$ , whose  $n$ th denominator  $q(n)$  is given by  $q(n) = n!^3 D_n / 2^n$ , giving the formula

$$\frac{56\zeta(3)}{3} = \sum_{n \geq 1} \frac{64^n}{n^3 D_n D_{n-1}}.$$

As usual we write the CF in abbreviated form

$$\zeta(3) = ((2n-1)(5n^2-5n+2), -16n^6).$$

Bauer–Muir acceleration gives something complicated, and as above I have not found any other CF of this type with convergence in  $1/(4^n n^P)$ .

## 14 CFs for $\zeta(3)$ : Convergence in $(-1)^n/(4^n n^P)$

From the famous series mentioned by Apéry but known before him

$$\zeta(3) = \frac{5}{2} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}},$$

one deduces the CF  $\zeta(3) = ((0, 3n^3 + n^2 - 3n + 1), (5/2, 2n^5(2n-1)))$  with the same speed of convergence as the series in  $\sqrt{\pi}/2/((-4)^n n^{5/2})$ , which we write as

$$\zeta(3) = (3n^3 + n^2 - 3n + 1, 2n^5(2n-1)).$$

Here, a variant of Bauer–Muir gives the family

$$\zeta(3) = (3n^3 + (11k + 1)n^2 - (11k + 3)n + 3k + 1, n^4(2n - k)(2n - k - 1))$$

with convergence in  $C/((-4)^n n^{5k+5/2})$ . A search apparently does not give any other CF.

## 15 CFs for $\zeta(3)$ : Convergence in $1/(1 + \sqrt{2})^{4n}$

Let

$$C_n = \sum_{n/2 \leq k \leq n} \binom{n}{k}^2 \binom{2k}{n}^2$$

(these numbers do not seem to have a name). From the recursion that one can find for these numbers and the computation of Apéry limits (see for instance [2]), we have the CF  $\zeta(3) = ((0, (2n - 1)(3n^2 - 3n + 1)), (8/7, -n^6))$  with speed of convergence  $4\pi^3/7/(1 + \sqrt{2})^{4n+2}$ , whose  $n$ th denominator  $q(n)$  is given by  $q(n) = n!^3 C_n / 4^n$ , giving the formula

$$\frac{7\zeta(3)}{2} = \sum_{n \geq 1} \frac{16^n}{n^3 C_n C_{n-1}}.$$

As usual we write the CF in abbreviated form

$$\zeta(3) = ((2n - 1)(3n^2 - 3n + 1), -n^6).$$

Since the exponent of the speed of convergence is irrational, Bauer–Muir acceleration would give irrational coefficients so is not applicable, and I have not been able to find other CFs of this type.

## 16 CFs for $\zeta(3)$ : Convergence in $1/(1 + \sqrt{2})^{8n}$

This is the famous CF of Apéry:

$$\zeta(3) = ((0, (2n - 1)(17n^2 - 17n + 5)), (6, -n^6))$$

with convergence in  $4\pi^3/(1 + \sqrt{2})^{8n+4}$ .

This CF seems to be unique of its kind, i.e. of the form  $\zeta(3) = (34n^3 + a_2 n^2 + a_1 n + a_0, -n(n + u_1)(n + u_2)(n + u_3)(n + u_4)(n + u_5)(n + u_6))$ .

## 17 CFs for $\zeta(3)$ : Other Speeds of Convergence

I have not found any convergent CF of the form  $\zeta(3) = (A(n), B(n))$  with  $\deg(A) \leq 3$ ,  $\deg(B) \leq 6$  and  $A$  and  $B$  with integral coefficients other than the families mentioned above, up to trivial equivalence such as changing  $n$  into  $n + j$  or multiplying  $(A, B)$  by  $(c, c^2)$  for a nonzero constant  $c$ . Of course, there exist infinitely many other CFs for  $\zeta(3)$ , but of larger bidegrees.

## 18 Other Periods

All the preceding sections dealt with CFs for  $\pi^2$  and  $\zeta(3)$ . A similar study can be done for other periods. I briefly summarize what I have found, restricting as usual to CFs of bidegrees  $\leq (k, 2k)$  for periods of degree  $k$ .

### 18.1 Periods of Degree 1

As just mentioned, we restrict to bidegrees  $\leq (1, 2)$ .

- $z = \pi$ : one finds parametric families of CFs with convergence in  $(-1)^n/n^P$ ,  $1/(2^n n^P)$ , and  $(-1)^n/(1 + \sqrt{2})^{2n}$ .

- $z = \pi/\sqrt{3}$ : one finds parametric families of CFs with convergence in  $1/((4/3)^n n^P)$ ,  $(-1)^n/(3^n n^P)$ ,  $1/(4^n n^P)$ , and  $(-1)^n/(2 + \sqrt{3})^{2n}$ .

- $z = \log(2)$ : one finds parametric families of CFs with convergence in  $(-1)^n/n^P$ ,  $1/(2^n n^P)$ ,  $(-1)^n/(8^n n^P)$ ,  $1/(9^n n^P)$ , and  $1/(1 + \sqrt{2})^{4n}$ .

Note that one can also study CFs for  $z = \log(m)$  for other rational values of  $m$ .

- $z = \sqrt{2}\log(1 + \sqrt{2})$ : one finds parametric families of CFs with convergence in  $1/(1 + \sqrt{2})^{2n}$ .

### 18.2 Periods of Degree 2

We restrict to bidegrees  $\leq (2, 4)$ .

- $z = \pi^2$ : see the extensive lists given in the preceding sections.

- $z = L(\chi_{-3}, 2)$ : one finds parametric families of CFs with convergence  $1/((9/8)^n n^P)$  and  $1/(9^n n^P)$ .

- $z = L(\chi_{-4}, 2) = G$ , Catalan's constant: one finds parametric families of CFs with convergence  $1/n^P$ ,  $(-1)^n/n^P$ , and  $1/(2^n n^P)$ .

### 18.3 Periods of Degree 3

We restrict to bidegrees  $\leq (3, 6)$ .

- $z = \zeta(3)$ : see the extensive lists given in the preceding sections.

- For other periods of degree 3 such as  $\pi^3$  and  $\pi^3/\sqrt{3}$ , either the CFs are isolated, or are one-parameter families coming from Bauer–Muir acceleration.

I would evidently welcome any additional data and comments on the subject of this paper.

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