

Infrared: a declarative tree decomposition-powered framework for bioinformatics

Additional file 1

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Recall the following few definitions and notation from the main text. Given the cluster tree (T, χ, ϕ) and a node u of T . If u has the parent v , then $\text{diff}(u) = \chi(u) \setminus \chi(v)$ and $\text{sep}(u) = \chi(v) \cap \chi(u)$; otherwise, if u is the root, $\text{diff}(u) = \chi(u)$ and $\text{sep}(u) = \emptyset$. Moreover, we defined T_u as the subtree under node u , $\chi(T_u) = \bigcup_{u' \in T_u} \chi(u')$ and $\text{diff}(T_u) = \chi(T_u) \setminus \text{sep}(u)$. We use the following lemma.

Lemma S1. $\{\text{diff}(T_c) \mid c \text{ child of } u\}$ is a partition of $\chi(T_u) \setminus \chi(u) = \text{diff}(T_u) \setminus \text{diff}(u)$.

Proof. Let u be a node of the cluster tree (T, χ, ϕ) , c and c' be children of u , then $\text{diff}(T_c) \cap \text{diff}(T_{c'}) = \emptyset$, since—due to tree decomposition property (3)—any common element must be in $\chi(u)$ in contradiction to the definition of diff on subtrees; moreover,

$$\begin{aligned} \bigcup \{\text{diff}(T_c) \mid c \text{ child of } u\} &= \bigcup_{c \text{ child of } u} \chi(c) \setminus \text{sep}(c) \\ &= \bigcup_{c \text{ child of } u} \chi(c) \setminus \bigcup_{c \text{ child of } u} \text{sep}(c) \\ &= \bigcup_{c \text{ child of } u} \chi(c) \setminus \chi(u) = \text{diff}(T_u) \setminus \text{diff}(u). \end{aligned}$$

□

1 Correctness of optimization

Proof of Proposition 1. We are going to show the correctness by structural induction on the cluster tree. For the base case, let u be a leaf of the cluster tree. Since the subtree T_u is the node u itself, Eq. 3 for a leaf u is written as

$$m_{u \rightarrow v}(\bar{x}) = \max_{\substack{\bar{x}' \in \mathcal{A}_{\text{diff}(u)} \\ \bar{x} \cup \bar{x}' \text{ is valid}}} \sum_{f \in \phi(u)} \alpha_f f(\bar{x} \cup \bar{x}').$$

Thus, Proposition 1 holds for leaves u , since they do not have children.

For the induction step, let u be an internal node of the cluster tree and assume Proposition 1 holds for all children of u . Denote the children of u as $c_1 \dots, c_k$. We directly apply the induction hypothesis to the r.h.s of the claim.

$$\text{r.h.s. (4)} = \max_{\substack{\bar{x}' \in \mathcal{A}_{\text{diff}(u)} \\ \bar{x} \cup \bar{x}' \text{ is valid}}} \left[\sum_{f \in \phi(u)} \alpha_f f(\bar{x} \cup \bar{x}') + \sum_{c_i \text{ child of } u} \max_{\substack{\bar{y}_i \in \mathcal{A}_{\text{diff}(T_{c_i})} \\ \bar{x} \cup \bar{x}' \cup \bar{y}_i \text{ is valid}}} \sum_{f \in \phi(T_{c_i})} \alpha_f f(\bar{x} \cup \bar{x}' \cup \bar{y}_i) \right]$$

Our goal is to show equivalence to the r.h.s. of Eq. (3). By Lemma S1 set $\text{diff}(T_u) \setminus \text{diff}(u)$ is disjointly decomposed into $\text{diff}(T_{c_1}) \dots \text{diff}(T_{c_k})$; moreover, $\phi(T_u) \setminus \phi(u)$ is partitioned into $\phi(T_{c_i})$, $i = 1 \dots k$. It follows that for every fixed assignment $\bar{x} \cup \bar{x}'$, the maximizations over $\bar{y}_i \in \text{diff}(T_{c_i})$ are independent for different i ; as well the sum over the functions of bag u is independent of the maximizations. Thus, we can move the maximizations over the assignments \bar{y}_i in front of the summations.

$$= \max_{\substack{\bar{x}' \in \mathcal{A}_{\text{diff}(u)} \\ \bar{x} \cup \bar{x}' \text{ is valid}}} \max_{\substack{\bar{y}_1 \in \mathcal{A}_{\text{diff}(T_{c_1})} \\ \bar{x} \cup \bar{x}' \cup \bar{y}_1 \text{ is valid}}} \dots \max_{\substack{\bar{y}_k \in \mathcal{A}_{\text{diff}(T_{c_k})} \\ \bar{x} \cup \bar{x}' \cup \bar{y}_k \text{ is valid}}} \left[\sum_{f \in \phi(u)} \alpha_f f(\bar{x} \cup \bar{x}') + \sum_{c_i \text{ child of } u} \sum_{f \in \phi(T_{c_i})} \alpha_f f(\bar{x} \cup \bar{x}' \cup \bar{y}_i) \right]$$

Then, we combine the maximizations over the single partial assignments \bar{y}_i ; moreover we replace the sums with a single sum ranging over the combined assignments \bar{y} of $\text{diff}(T_u) \setminus \text{diff}(u)$.

$$= \max_{\substack{\bar{x}' \in \mathcal{A}_{\text{diff}(u)} \\ \bar{x} \cup \bar{x}' \text{ is valid}}} \max_{\substack{\bar{y} \in \mathcal{A}_{\text{diff}(T_u) \setminus \text{diff}(u)} \\ \bar{x} \cup \bar{x}' \cup \bar{y} \text{ is valid}}} \left[\sum_{f \in \phi(T_u)} \alpha_f f(\bar{x} \cup \bar{x}' \cup \bar{y}) \right]$$

The last term equals r.h.s. of Eq. (3), showing the induction step. \square

2 Correctness of partition function computation

Proof of Proposition 2. Again, we will show the correctness by induction. Starting with the base case where u is a leaf of the cluster tree, the subtree T_u is the node u and the child set of u is empty. Eq 5 is then written as

$$m_{u \rightarrow v}(\bar{x}) = \sum_{\substack{\bar{x}' \in \mathcal{A}_{\text{diff}(u)} \\ \bar{x} \cup \bar{x}' \text{ is valid}}} \prod_{f \in \phi(u)} \exp(\alpha_f f(\bar{x} \cup \bar{x}'))$$

which shows the correctness for leaves.

For the induction step, let u be an internal node of the cluster tree; we assume that the message $m_{c_i \rightarrow u}$ for all children c_i , $i \in \{1, \dots, k\}$ of u is correctly computed as in Proposition 2. Following the scheme as in the previous proof, we first rewrite the r.h.s of the claim applying the induction hypothesis,

r.h.s. (6) =

$$\sum_{\substack{\bar{x}' \in \mathcal{A}_{\text{diff}(u)} \\ \bar{x} \cup \bar{x}' \text{ is valid}}} \left[\prod_{f \in \phi(u)} \exp(\alpha_f f(\bar{x} \cup \bar{x}')) \times \prod_{c_i \text{ child of } u} \sum_{\substack{\bar{y}_i \in \mathcal{A}_{\text{diff}(T_{c_i})} \\ \bar{x} \cup \bar{x}' \cup \bar{y}_i \text{ is valid}}} \prod_{f \in \phi(T_{c_i})} \exp(\alpha_f f(\bar{x} \cup \bar{x}' \cup \bar{y}_i)) \right]$$

As the set $\text{diff}(T_u) \setminus \text{diff}(u)$ is disjointly decomposed into $\text{diff}(T_{c_1}) \dots \text{diff}(T_{c_k})$, the second term in the bracket can be seen as the Cartesian product of k sets. We can move the summations in front of the production,

$$= \sum_{\substack{\bar{x}' \in \mathcal{A}_{\text{diff}(u)} \\ \bar{x} \cup \bar{x}' \text{ is valid}}} \left[\prod_{f \in \phi(u)} \exp(\alpha_f f(\bar{x} \cup \bar{x}')) \times \sum_{\substack{\bar{y}_1 \in \mathcal{A}_{\text{diff}(T_{c_1})} \\ \bar{x} \cup \bar{x}' \cup \bar{y}_1 \text{ is valid}}} \dots \sum_{\substack{\bar{y}_k \in \mathcal{A}_{\text{diff}(T_{c_k})} \\ \bar{x} \cup \bar{x}' \cup \bar{y}_k \text{ is valid}}} \prod_{c_i \text{ child of } u} \prod_{f \in \phi(T_{c_i})} \exp(\alpha_f f(\bar{x} \cup \bar{x}' \cup \bar{y}_i)) \right]$$

For every fixed assignment $\bar{x} \cup \bar{x}'$, the product over the functions of bag u is independent of the summations. Thus, we can move the summations over the assignments \bar{y}_i in front of the product,

$$= \sum_{\substack{\bar{x}' \in \mathcal{A}_{\text{diff}(u)} \\ \bar{x} \cup \bar{x}' \text{ is valid}}} \sum_{\substack{\bar{y}_1 \in \mathcal{A}_{\text{diff}(T_{c_1})} \\ \bar{x} \cup \bar{x}' \cup \bar{y}_1 \text{ is valid}}} \dots \sum_{\substack{\bar{y}_k \in \mathcal{A}_{\text{diff}(T_{c_k})} \\ \bar{x} \cup \bar{x}' \cup \bar{y}_k \text{ is valid}}} \left[\prod_{f \in \phi(u)} \exp(\alpha_f f(\bar{x} \cup \bar{x}')) \times \prod_{c_i \text{ child of } u} \prod_{f \in \phi(T_{c_i})} \exp(\alpha_f f(\bar{x} \cup \bar{x}' \cup \bar{y}_i)) \right]$$

Then, we combine the summations over the single partial assignments \bar{y}_i ; moreover we replace the products with a single product ranging over the combined assignments \bar{y} of $\text{diff}(T_u) \setminus \text{diff}(u)$ as $\phi(T_u) \setminus \phi(u)$ is partitioned into $\phi(T_{c_i})$, $i = 1 \dots k$.

$$= \sum_{\substack{\bar{x}' \in \mathcal{A}_{\text{diff}(u)} \\ \bar{x} \cup \bar{x}' \text{ is valid}}} \sum_{\substack{\bar{y} \in \mathcal{A}_{\text{diff}(T_u) \setminus \text{diff}(u)} \\ \bar{x} \cup \bar{x}' \cup \bar{y} \text{ is valid}}} \left[\prod_{f \in \phi(T_u)} \exp(\alpha_f f(\bar{x} \cup \bar{x}' \cup \bar{y})) \right]$$

The last term equals r.h.s. of Eq. (5), showing the induction step. \square

3 Correctness of sampling by stochastic traceback

Lemma S2. At edge $u \rightarrow v$, Alg. 3 choses a partial assignment $\bar{x} \in \mathcal{A}_{\text{diff}(u)}$ with probability

$$\mathbb{P}_{\text{alg}}(\bar{x} \mid x) = \frac{w_{u \rightarrow v}(\bar{x} \mid x)}{m_{u \rightarrow v}(x)}$$

where x is the partial assignment determined in the previous iterations and weight $w_{u \rightarrow v}(\bar{x} | x)$ is defined as

$$w_{u \rightarrow v}(\bar{x} | x) = \prod_{f \in \phi(u)} \exp(\alpha_f f(x \cup \bar{x})) \prod_{c \text{ child of } u} m_{c \rightarrow u}(x \cup \bar{x}) = \sum_{\substack{\bar{y} \in \mathcal{A}_{\text{diff}(T_u) \setminus \text{diff}(u)} \\ x \cup \bar{x} \cup \bar{y} \text{ is valid}}} \prod_{f \in \phi(T_u)} \exp(\alpha_f f(x \cup \bar{x} \cup \bar{y})). \quad (\text{S1})$$

Proof. Let us order the partial assignments \bar{x}_i in $\mathcal{A}_{\text{diff}(u)}$ with i from 1 to $|\mathcal{A}_{\text{diff}(u)}|$, each is associated with weight $w_{u \rightarrow v}(\bar{x}_i | x) = \prod_{f \in \phi(u)} \exp(\alpha_f f(x \cup \bar{x}_i)) \prod_{c \text{ child of } u} m_{c \rightarrow u}(x \cup \bar{x}_i)$. Note that the sum of all weights $\sum_{i=1}^{|\mathcal{A}_{\text{diff}(u)}|} w_{u \rightarrow v}(\bar{x}_i | x)$ equals to $m_{u \rightarrow v}(x)$.

For a selected value t , Alg. 3 enumerates assignments \bar{x}_i starting from $i = 1$. In every iteration, it subtracts the weight $w_{u \rightarrow v}(\bar{x}_i | x)$. It is terminated as soon as t is negative, let this index be j , and assigns \bar{x}_j to $\text{diff}(u)$. In other words, \bar{x}_j is assigned to $\text{diff}(u)$ if

$$\sum_{i=0}^j w_{u \rightarrow v}(\bar{x}_i | x) \leq t < \sum_{i=0}^{j+1} w_{u \rightarrow v}(\bar{x}_i | x)$$

with $w_{u \rightarrow v}(\bar{x}_0 | x)$ is defined as 0 for $i = 0$. Since t is randomly and uniformly selected from 0 to $m_{u \rightarrow v}(x)$, the probability that Alg. 3 chooses \bar{x} is

$$\mathbb{P}_{\text{alg}}(\bar{x} | x) = \frac{w_{u \rightarrow v}(\bar{x} | x)}{m_{u \rightarrow v}(x)}.$$

Finally, the weight $w_{u \rightarrow v}(\bar{x} | x) = \prod_{f \in \phi(u)} \exp(\alpha_f f(x \cup \bar{x})) \prod_{c \text{ child of } u} m_{c \rightarrow u}(x \cup \bar{x})$ can be seen as a special case of Eq. 6 by replacing the partial assignment set $\mathcal{A}_{\text{diff}(u)}$ with a singleton $\{\bar{x}\}$. Applying Proposition 2 gives

$$w_{u \rightarrow v}(\bar{x} | x) = \sum_{\substack{\bar{y} \in \mathcal{A}_{\text{diff}(T_u) \setminus \text{diff}(u)} \\ x \cup \bar{x} \cup \bar{y} \text{ is valid}}} \prod_{f \in \phi(T_u)} \exp(\alpha_f f(x \cup \bar{x} \cup \bar{y})).$$

□

Proof of Proposition 3. Let us assume that the non-root nodes of the cluster tree are labeled in preorder as u_1, u_2, \dots, u_ℓ . Alg. 3 assigns values in the order of $\text{diff}(u_1), \dots, \text{diff}(u_\ell)$; it finally generates a total assignment since $\cup_{i=1}^\ell \text{diff}(u_i) = \mathcal{X}$.

Let $x^* = \bar{x}_1^* \cup \dots \cup \bar{x}_\ell^*$ be a total assignment where \bar{x}_i^* is a partial assignment of $\mathcal{A}_{\text{diff}(u_i)}$ for i from 1 to ℓ . The probability of x^* can be written as product of conditional probabilities

$$\mathbb{P}(x^*) = \mathbb{P}(\bar{x}_1^*) \mathbb{P}(\bar{x}_2^* | \bar{x}_1^*) \dots \mathbb{P}(\bar{x}_\ell^* | \cup_{k=1}^{\ell-1} \bar{x}_k^*) = \prod_{i=1}^\ell \mathbb{P}(\bar{x}_i^* | \cup_{k=1}^{i-1} \bar{x}_k^*)$$

with the conditional Boltzmann probability

$$\mathbb{P}(\bar{x}_i^* | \cup_{k=1}^{i-1} \bar{x}_k^*) = \frac{\sum_{\substack{\bar{x} \in \mathcal{A}_{\text{diff}(u_{i+1}) \cup \dots \cup \text{diff}(u_\ell)} \\ \bar{x} \text{ is valid}}} \prod_{f \in \phi_{\text{all}}} \exp(\alpha_f f(\cup_{k=1}^{i-1} \bar{x}_k^* \cup \bar{x}_i^* \cup \bar{x}))}{\sum_{\substack{\bar{x}' \in \mathcal{A}_{\text{diff}(u_i)} \\ \bar{x}' \text{ is valid}}} \sum_{\substack{\bar{x} \in \mathcal{A}_{\text{diff}(u_{i+1}) \cup \dots \cup \text{diff}(u_\ell)} \\ \bar{x} \text{ is valid}}} \prod_{f \in \phi_{\text{all}}} \exp(\alpha_f f(\cup_{k=1}^{i-1} \bar{x}_k^* \cup \bar{x}' \cup \bar{x}))} \quad (\text{S2})$$

where ϕ_{all} is the set of all network functions in the feature network. Note that for $i = 1$, the union $\cup_{k=1}^0 \bar{x}_k^*$ is empty.

At the edge $u_i \rightarrow v$, where the variables $\text{diff}(u_1), \dots, \text{diff}(u_{i-1})$ have been assigned by $\bar{x}_1^*, \dots, \bar{x}_{i-1}^*$, the probability that Alg. 3 chooses \bar{x}_i^* for $\text{diff}(u_i)$ is given by Lemma S2 as

$$\mathbb{P}_{\text{alg}}(\bar{x}_i^* | \cup_{k=1}^{i-1} \bar{x}_k^*) = \frac{w_{u_i \rightarrow v}(\bar{x}_i^* | \cup_{k=1}^{i-1} \bar{x}_k^*)}{m_{u_i \rightarrow v}(\cup_{k=1}^{i-1} \bar{x}_k^*)}. \quad (\text{S3})$$

Our goal is to show that two conditional probabilities $\mathbb{P}(\bar{x}_i^* | \cup_{k=1}^{i-1} \bar{x}_k^*)$ of Eq. S2 and $\mathbb{P}_{\text{alg}}(\bar{x}_i^* | \cup_{k=1}^{i-1} \bar{x}_k^*)$ of Eq. S3 are actually the same.

We rewrite \mathbb{P}_{alg} of Eq. S3 using Eq. S1 and the definition of conditional partition function as in Eq. 5

$$\begin{aligned}
& \mathbb{P}_{\text{alg}}(\bar{x}_i^* \mid \cup_{k=1}^{i-1} \bar{x}_k^*) \\
&= \frac{\sum_{\bar{y} \in \mathcal{A}_{\text{diff}(T_{u_i}) \setminus \text{diff}(u_i)}} \prod_{f \in \phi(T_{u_i})} \exp(\alpha_f f(\cup_{k=1}^{i-1} \bar{x}_k^* \cup \bar{x}_i^* \cup \bar{y}))}{x \cup \bar{x}_i^* \cup \bar{y} \text{ is valid}} \\
&= \frac{\sum_{\bar{x}' \in \mathcal{A}_{\text{diff}(u_i)}} \sum_{\bar{y} \in \mathcal{A}_{\text{diff}(T_{u_i}) \setminus \text{diff}(u_i)}} \prod_{f \in \phi(T_{u_i})} \exp(\alpha_f f(\cup_{k=1}^{i-1} \bar{x}_k^* \cup \bar{x}' \cup \bar{y}))}{\bar{x}' \text{ is valid} \quad x \cup \bar{x}' \cup \bar{y} \text{ is valid}}
\end{aligned}$$

Comparing with $\mathbb{P}(\bar{x}_i^* \mid \cup_{k=1}^{i-1} \bar{x}_k^*)$ of Eq. S2, only a subset of network functions $\phi(T_{u_i})$ are taken into account. We are going to bring the rest into the expression. First, let us consider the functions assigned to the nodes that Alg. 3 has visited before edge $u_i \rightarrow v$, denoted by $\phi_{\text{visited}} := \cup_{k=1}^{i-1} \phi(u_k)$. Multiplying nominator and denominator by partial partition function $\prod_{f \in \phi_{\text{visited}}} \exp(\alpha_f f(\cup_{k=1}^{i-1} \bar{x}_k^*))$ restricted to ϕ_{visited} and the partial assignment $\cup_{k=1}^{i-1} \bar{x}_k^*$ gives

$$\begin{aligned}
&= \frac{\left(\sum_{\bar{y} \in \mathcal{A}_{\text{diff}(T_{u_i}) \setminus \text{diff}(u_i)}} \prod_{f \in \phi(T_{u_i})} \exp(\alpha_f f(\cup_{k=1}^{i-1} \bar{x}_k^* \cup \bar{x}_i^* \cup \bar{y})) \right) \left(\prod_{f \in \phi_{\text{visited}}} \exp(\alpha_f f(\cup_{k=1}^{i-1} \bar{x}_k^*)) \right)}{x \cup \bar{x}_i^* \cup \bar{y} \text{ is valid}} \\
&= \frac{\left(\sum_{\bar{x}' \in \mathcal{A}_{\text{diff}(u_i)}} \sum_{\bar{y} \in \mathcal{A}_{\text{diff}(T_{u_i}) \setminus \text{diff}(u_i)}} \prod_{f \in \phi(T_{u_i})} \exp(\alpha_f f(\cup_{k=1}^{i-1} \bar{x}_k^* \cup \bar{x}' \cup \bar{y})) \right) \left(\prod_{f \in \phi_{\text{visited}}} \exp(\alpha_f f(\cup_{k=1}^{i-1} \bar{x}_k^*)) \right)}{\bar{x}' \text{ is valid} \quad x \cup \bar{x}' \cup \bar{y} \text{ is valid}}
\end{aligned}$$

As $\text{diff}(T_{u_i})$ is disjoint to the dependency of any function of ϕ_{visited} , we can further group the products

$$\begin{aligned}
&= \frac{\sum_{\bar{y} \in \mathcal{A}_{\text{diff}(T_{u_i}) \setminus \text{diff}(u_i)}} \prod_{f \in \phi_{\text{visited}} \cup \phi(T_{u_i})} \exp(\alpha_f f(\cup_{k=1}^{i-1} \bar{x}_k^* \cup \bar{x}_i^* \cup \bar{y}))}{x \cup \bar{x}_i^* \cup \bar{y} \text{ is valid}} \\
&= \frac{\sum_{\bar{x}' \in \mathcal{A}_{\text{diff}(u_i)}} \sum_{\bar{y} \in \mathcal{A}_{\text{diff}(T_{u_i}) \setminus \text{diff}(u_i)}} \prod_{f \in \phi_{\text{visited}} \cup \phi(T_{u_i})} \exp(\alpha_f f(\cup_{k=1}^{i-1} \bar{x}_k^* \cup \bar{x}' \cup \bar{y}))}{\bar{x}' \text{ is valid} \quad x \cup \bar{x}' \cup \bar{y} \text{ is valid}}
\end{aligned}$$

We consider now the rest of the network functions, denoted by ϕ_{rest} . These functions are assigned to the cluster nodes, denoted by $\mathcal{U}_{\text{rest}}$, that have not been visited before the edge $u_i \rightarrow v$ or are not in the subtree of u_i . Note that $T_{u_i} \cup \mathcal{U}_{\text{rest}} = \{u_i, \dots, u_\ell\}$. The partial partition function restricted to ϕ_{rest} for fixed partial assignment $\cup_{k=1}^{i-1} \bar{x}_k^*$ is

$$\sum_{\substack{\bar{y}' \in \mathcal{A}_{\cup_u \in \mathcal{U}_{\text{rest}} \text{ diff}(u)}} \\ \bar{y}' \text{ is valid}} \prod_{f \in \phi_{\text{rest}}} \exp(\alpha_f f(\cup_{k=1}^{i-1} \bar{x}_k^* \cup \bar{y}')).$$

Multiplying again the partial partition function on nominator and denominator gives

$$\begin{aligned}
&= \frac{\left(\sum_{\bar{y} \in \mathcal{A}_{\text{diff}(T_{u_i}) \setminus \text{diff}(u_i)}} \prod_{f \in \phi_{\text{visited}} \cup \phi(T_{u_i})} \exp(\alpha_f f(\cup_{k=1}^{i-1} \bar{x}_k^* \cup \bar{x}_i^* \cup \bar{y})) \right)}{x \cup \bar{x}_i^* \cup \bar{y} \text{ is valid}} \\
&= \frac{\left(\sum_{\bar{y}' \in \mathcal{A}_{\cup_u \in \mathcal{U}_{\text{rest}} \text{ diff}(u)}} \prod_{f \in \phi_{\text{rest}}} \exp(\alpha_f f(\cup_{k=1}^{i-1} \bar{x}_k^* \cup \bar{y}')) \right)}{\bar{y}' \text{ is valid}} \\
&= \frac{\left(\sum_{\bar{x}' \in \mathcal{A}_{\text{diff}(u_i)}} \sum_{\bar{y} \in \mathcal{A}_{\text{diff}(T_{u_i}) \setminus \text{diff}(u_i)}} \prod_{f \in \phi_{\text{visited}} \cup \phi(T_{u_i})} \exp(\alpha_f f(\cup_{k=1}^{i-1} \bar{x}_k^* \cup \bar{x}' \cup \bar{y})) \right)}{\bar{x}' \text{ is valid} \quad x \cup \bar{x}' \cup \bar{y} \text{ is valid}} \\
&= \frac{\left(\sum_{\bar{y}' \in \mathcal{A}_{\cup_u \in \mathcal{U}_{\text{rest}} \text{ diff}(u)}} \prod_{f \in \phi_{\text{rest}}} \exp(\alpha_f f(\cup_{k=1}^{i-1} \bar{x}_k^* \cup \bar{y}')) \right)}{\bar{y}' \text{ is valid}}
\end{aligned}$$

Because of the disjointness between $\text{diff}(T_{u_i}) \setminus \text{diff}(u_i)$ and $\cup_u \in \mathcal{U}_{\text{rest}} \text{ diff}(u)$, we can group the products and combine the partial assignments into one $\bar{x} = \bar{y} \cup \bar{y}'$ as

$$\begin{aligned}
&= \frac{\sum_{\bar{x} \in \mathcal{A}_{\text{diff}(u_{i+1}) \cup \dots \cup \text{diff}(u_\ell)}} \prod_{f \in \phi_{\text{all}}} \exp(\alpha_f f(\cup_{k=1}^{i-1} \bar{x}_k^* \cup \bar{x}_i^* \cup \bar{x}))}{\bar{x} \text{ is valid}} \\
&= \frac{\sum_{\bar{x}' \in \mathcal{A}_{\text{diff}(u_i)}} \sum_{\bar{x} \in \mathcal{A}_{\text{diff}(u_{i+1}) \cup \dots \cup \text{diff}(u_\ell)}} \prod_{f \in \phi_{\text{all}}} \exp(\alpha_f f(\cup_{k=1}^{i-1} \bar{x}_k^* \cup \bar{x}' \cup \bar{x}))}{\bar{x}' \text{ is valid} \quad \bar{x} \text{ is valid}}.
\end{aligned}$$

Thus, $\mathbb{P}_{\text{alg}}(\bar{x}_i^* \mid \cup_{k=1}^{i-1} \bar{x}_k^*) = \mathbb{P}(\bar{x}_i^* \mid \cup_{k=1}^{i-1} \bar{x}_k^*)$. \square