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Remote State Estimation of Steered Systems with Limited Communications: an Event-Triggered Approach

Quentin Voortman, Denis Efimov *Senior Member, IEEE*, Alexander Pogromsky, Jean-Pierre Richard *Senior Member, IEEE*, and Henk Nijmeijer *Fellow, IEEE*

Abstract—In this paper, an approach is proposed for the remote observation of a dynamical system through a data-rate constrained communication channel. The focus is put on discrete-time systems with a Lipschitz nonlinearity, driven by an external signal, and subject to bounded state perturbation and measurement error. The problem at hand is providing estimates of system's state at a remote location, which is connected via a channel, which can only sent limited numbers of bits per unit of time. A solution, named observation scheme, is proposed in the form of several interacting agents. This solution is designed such that the maximum observation error is upper-bounded by a computable quantity dependent on system constants and selectable parameters. The scheme is designed in an event-triggered fashion, such that the actual communication rate is sometimes much lower than the theoretically evaluated maximum one, as it is demonstrated through simulations.

Index Terms—Event Detection; Nonlinear control systems; Observers

I. INTRODUCTION

Whether it is collective cruise control of connected cars, formation control for drones through Wi-Fi or Bluetooth, connected smart sensors, or another form of cyber-physical system technology, wireless communications are omnipresent in the modern industry. Since it is a booming application domain, new performance requirements are imposed, and the field of dynamics and control has to come up with new solutions to deal with these new problems and challenges. The problems related to the interactions between dynamical

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systems and communication technologies are numerous, and have led to the creation of an entire sub-field within the topic: control and estimation over communication channels. All problems share some common features: one or several dynamical systems, sometimes paired with sensors, actuators and controllers, are placed at remote locations from one another. To communicate, they have to employ communication channels which are limited, either in terms of frequency of communications, size of the messages, or are subject to losses and lags. When this setting is combined with noise, parametric uncertainty, perturbations, or deviations in initial conditions, there is a need to design efficient communication strategies to deal with these uncertainties, which can be understood as additional sources of information in the sense of Shannon's information theory [1].

Among the earliest works in this sub-field, one finds: [2], which considered state estimation under data-rate constraints for linear noisy systems and [3], which considered stabilization of a linear system under quantized state feedback. Many more results on observation, state estimation and control have been obtained for linear systems and broad overviews of these results can be found in [4], [5] and [6].

For nonlinear systems, important results are [7], [8] and [9], which are based on entropy concepts. Since those three papers, many different notions of entropy have been used to provide bounds on the sufficient and/or necessary data-rates to observe and/or control nonlinear dynamical systems over constrained channels (see [10], [11], [12], [13], [14], [15], [16], [17] and [18]).

Approximately at the same time as the research on control with data-rate constraints started, the topic of event-triggered control appeared as well in the dynamics and control community. Two early papers are: [19], where an event-triggered PID controller is presented and [20], where the effects of event-based sampling are compared to periodic sampling. For an introduction to event-based control, one can refer to [21]. For an overview of many sampling-related results, one can refer to [22].

Both control with data-rate constraints and event-based control have been used together for control and observation purposes. These works include [23], which uses an event-triggered sensor schedule for remote estimation for a linear system, [24], designing a remote estimator for a linear system

with unknown exogenous inputs, [25], where a remote estimator for a system with an energy harvesting sensor is developed, [26], which tackles distributed state estimation with data-rate constraints, [27], which considers networked state estimation with a shared communication medium and, [28], where an LMI approach is used for the networked state estimation problem over a shared communication medium.

A particular class of systems is constituted by Lipschitz-nonlinear dynamics, which are typically found when modeling mechanical systems. This class includes systems with trigonometric nonlinearities, which are globally Lipschitz. Square or cubic nonlinearities which are also sometimes encountered with mechanical systems are locally Lipschitz and, since they often occur on physical systems, they are often paired with a saturation function (due to having restricted movement in space), which makes them globally Lipschitz. Many results have been obtained for Lipschitz-nonlinear systems, among which we note [29] and [30] which both develop observers for Lipschitz-nonlinear systems.

In this paper, we develop a communication scheme to remotely observe a discrete-time Lipschitz-nonlinear system with state perturbations and bounded measurement error. The system is connected to a remote location by means of a data-rate constrained communication channel. It is also steered by a driving signal, which is not measured at the remote location. The challenge is to design the communication protocol such that through the messages that are received from the system, it is possible to reconstruct estimates of the state at the remote location. Moreover, this should be achieved while using limited communication data-rates. The novelty of this paper in comparison with the aforementioned works, is that we consider an event-triggered communication protocol, which often requires much less than the theoretical maximum channel rate. A preliminary version of this paper [31] only considered linear systems and the communicated input did not contain the output error injection term. This paper also extends the results of [31] to the class of Lipschitz-nonlinear systems.

The paper is structured as follows. In Section III, we expose the details of the problem statement. Next, in Section IV, we develop the communication scheme. Section V is then dedicated to analytical bounds on the maximum observation error and communication rate. Finally, we conclude with simulations in Section VI, to insight on how the communication scheme functions and performs in practice.

II. NOTATIONS

- I_n : an $n \times n$ identity matrix;
- \star : a zero matrix of appropriate dimension;
- $\|v\|_2$, v is a vector: the Euclidean norm;
- $\|M\|_2$, M is a matrix: the operator norm induced by the pair $(\|\cdot\|_2, \|\cdot\|_2)$;
- $\sigma_i(M)$, M is a matrix: the singular values of M ranked in non-increasing order ($\sigma_1(M) = \|M\|_2$);
- $\text{vec}(M)$, M is a matrix: the vectorization of M ;
- $|S|$, S is a set: the cardinality of S ;
- $\mathfrak{B}_\varepsilon(x)$: the ball of radius ε in the norm $\|\cdot\|_2$, centered in x .

III. PROBLEM STATEMENT

We consider discrete-time systems of the following form

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + \varphi(Hx(k), u(k)) + d(k), \\ y(k) &= x(k) + w(k), \end{aligned} \quad (1)$$

where $x(k) \in \mathbb{R}^n$ is the state, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $u(k) \in \mathbb{R}^m$ is the driving signal, $\varphi: \mathbb{R}^p \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a mapping, $H \in \mathbb{R}^{p \times n}$, $d(k) \in \mathbb{R}^n$ is an unknown state perturbation, $y(k) \in \mathbb{R}^p$ is the output, and $w(k) \in \mathbb{R}^n$ is a measurement error.

Remark 1: The case of full state measurements is considered as this is equivalent to assuming existence of a local (not remote) state observer for the system with bounded estimation error. Since the main objective of the paper is to design a remote observer, this technicality is left out. We assume the following about the driving signal.

Assumption 1: The driving signal $u(k)$ is measured exactly and $u_i(k) \in [\underline{u}_i, \bar{u}_i]$, $i \in \{1, \dots, m\}$ where $[u_1(k) \dots u_m(k)]^\top = u(k) \in U := \prod_{i=1}^m [\underline{u}_i, \bar{u}_i]$.

The considered class of systems will be restricted to Lipschitz-nonlinear systems, which are commonly found structures of robotic and/or mechanical systems. We thus make the following assumption about the mapping φ .

Assumption 2: There exist L such that

$$\|\varphi(x_1, u) - \varphi(x_2, u)\| \leq L \|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathbb{R}^p,$$

uniformly in $u \in \mathbb{R}^m$ and there exists $\bar{\varphi}$ such that

$$\|\varphi(x, u_1) - \varphi(x, u_2)\| \leq \bar{\varphi} \|u_1 - u_2\|, \quad \forall u_1, u_2 \in U,$$

uniformly in $x \in \mathbb{R}^n$ (i.e., the mapping φ is uniformly globally Lipschitz with respect to both arguments).

Note that the previous assumption can be substituted, if the system is bounded-input bounded-state stable, by an assumption that the system is locally Lipschitz with respect to the state.

We make the following assumptions about the perturbations $d(k)$ and the errors $w(k)$.

Assumption 3: There exists a maximum state perturbation $\delta > 0$ and maximum measurement error $\omega > 0$ such that

$$\|d(k)\|_2 \leq \delta, \quad \|w(k)\|_2 \leq \omega, \quad \forall k \geq 0.$$

The system is equipped with a smart sensor (a sensor admitting some computational capacities, which allows it to perform additional computations on the measured data) and it is connected to a remote location via a data-rate constrained communication channel, which can only send messages that are of finite size. Our goal is to provide estimates $\hat{x}(k)$ of $x(k)$ at the remote location by sending messages over this communication channel. Note that the measurements of input $u(k)$ are available at the plant side and also have to be communicated to the remote location. The sensor and the remote location are aware of an initial estimate $\hat{x}(0)$ which verifies

$$\|x(0) - \hat{x}(0)\|_2 \leq \varepsilon_0, \quad (2)$$

where ε_0 is a selectable parameter corresponding to the error of initial conditions.

The communication is operating using the following steps. In order to generate the estimates, messages $m(k_j)$ are sent,

where k_j are the transmission times and $j = \{0, 1, \dots\}$ is the index of communication. Four ingredients interact with these messages: a sampler \mathcal{S} , a coder \mathcal{C} , an alphabet size function \mathcal{A} , and a decoder \mathcal{D} . The four devices together form a communication protocol. The following constants/parameters are known by all devices: the system matrices A , B , H , the mapping φ , the constants L and $\bar{\varphi}$, the maximum state perturbation δ , the maximum measurement error ω , and the initial estimate $\hat{x}(0)$ with its accuracy ε_0 . At the system side, the sampler \mathcal{S} generates the instants of transmission in the following way

$$k_{j+1} = \mathcal{S}(\{u(k)\}_{k < k_{j+1}}, k_j, \{y(k)\}_{k < k_{j+1}}, \{m(k_i)\}_{i \leq j}), \quad (3)$$

$k_0 = 0$. The coder then generates the messages in the following way

$$m(k_j) = \mathcal{C}(\{u(k)\}_{k \leq k_j}, k_j, \hat{x}(0), \{y(k)\}_{k \leq k_j}, \{m(k_i)\}_{i \leq j-1}), \quad (4)$$

$\forall k_j : j > 0$. At each communication instant, the different possible messages are encoded into a finite-sized alphabet (the finite-sizedness being due to the data-rate constraints). The alphabet size function \mathcal{A} determines the number of different messages l_j in the following way

$$l_j = \mathcal{A}(\hat{x}(0), \{m(k_i)\}_{i \leq j-1}), \quad \forall j > 0. \quad (5)$$

The restriction on the choice of messages is then

$$m(k_j) \in \{1, \dots, l_j\}, \quad \forall j > 0.$$

When a message is sent, it is encoded by using bits. The number of bits b_j required to encode a message at communication instant k_j is defined as

$$b_j := \lceil \log_2 l_j \rceil \quad j > 0. \quad (6)$$

At the remote location, the decoder \mathcal{D} receives the messages and interprets them to generate an estimate of the state $\hat{x}(k)$. For simplicity, we assume that there is no transmission delay, i.e., that the messages are received at the same time as they are sent. The decoder functions in the following way

$$\hat{x}(k) = \mathcal{D}(\hat{x}(0), m(k_1), \dots, m(k_j)), \quad \forall k \in \{k_j, \dots, k_{j+1} - 1\},$$

$\forall j \geq 0$. Because of the perturbation, measurement error and finite data-rate, it is impossible to provide exact estimates at the remote location. Instead, the design of the communication protocol should ensure that the estimation error $\|x(k) - \hat{x}(k)\|$ is bounded by a quantity which we call the **maximum observation error** ξ and is defined as

$$\xi := \sup_{k \geq 0} \|x(k) - \hat{x}(k)\|_2. \quad (7)$$

In order to properly define the goal of the paper, we need a quantity that evaluates the rate at which bits are sent through the communication channel. This quantity depends on both b_j , which can vary from one communication instant to another, and on $k_{j+1} - k_j$, which is fluctuating as well. We thus define the **maximum communication rate** R as

$$R := \limsup_{\bar{j} \rightarrow \infty} \frac{\sum_{j=0}^{\bar{j}} b_j}{\bar{j} + 1}, \quad (8)$$

which can be commented as follows: first we define the average number of bits sent per unit of time during a window of \bar{j} consecutive communications, next we start counting at time instant j such that this quantity is the largest possible, and finally, we take \bar{j} such that the quantity is the smallest possible. This quantity is called maximum communication rate because we will provide results that guarantee that this rate is not exceeded.

The first objective of the paper is to design a data-rate constrained observation scheme, to investigate what maximum observation error ξ it guarantees, and to determine the maximum communication rate R required to implement it. **The second objective** is to ensure that this data-rate constrained observation scheme requires on average a lower communication rate than R when the perturbations occur in a favourable way. This will be achieved by using an event-triggered operation.

Remark 2: The problem statement allows for an observation scheme that uses all previous inputs and outputs in order to generate estimates. Therefore, it is necessary to store these input-output data locally in memory, which can be costly. The solution that is presented in this work does not rely on storing all previous inputs and outputs. However, we decided to leave this possibility, to keep the problem statement as general as possible.

IV. THE OBSERVATION SCHEME

In this section, we describe the different components that form the suggested estimator: sampler, coder, alphabet, decoder, and related observers. The observation scheme needs to provide accurate estimates of $x(k)$ for all k . The naive solution is to simply send an estimate of $x(k)$ at every time instant. This solution is extremely inefficient in terms of data-rate, and a way to decrease the transmission rate is to only occasionally send estimates of the states and to utilize the system's dynamics on the decoder side in order to complement the estimates between the communications. This is the solution that will be used in this paper.

The problem is that the decoder has no information about the driving signal $u(k)$, so it cannot reconstruct estimates based on the system's dynamics. The solution we are going to explore in this paper, is to communicate an **estimate of the driving signal** $\hat{u}(k)$ via the messages at every time instant (i.e. $k_j = k$) and to sometimes send an estimate of the state $\hat{x}(k)$ in addition to $\hat{u}(k)$. The reasoning behind this idea is that the state and the driving signal have different dimensions (respectively n and m) and hence, their communication produces different loads on the channel. Typically, $m \leq n$ and it is less "expensive" to transmit an estimate of the driving signal than the state of the system.

Therefore, the observation scheme is going to send two different types of messages: messages which only contain the estimate of driving signal and messages which contain both estimates of the driving signal and of the state. We denote j_x is the index of the last communication instant when an estimate of the state was transmitted. The following dynamics are used to generate new estimates at the remote site in between

messages containing \hat{x} :

$$\hat{x}(k) = A\hat{x}(k-1) + B\hat{u}(k-1) + \varphi(H\hat{x}(k-1), \hat{u}(k-1)), \quad (9)$$

$\forall k : k \neq k_{j_x}$, that is, for all time instants when the current message does not contain an estimate of the state.

Since these estimates are based entirely on the messages, the sampler and coder also maintain local copies of these estimates $\hat{x}_c(k) = \hat{x}(k)$ and $\hat{u}_c(k) = \hat{u}(k)$. These copies are used by the smart sensor to determine when to trigger communications, as will be explained further in this paper. Figure 1 displays how the two agents, one containing sampler with coder and another with decoder, interact.

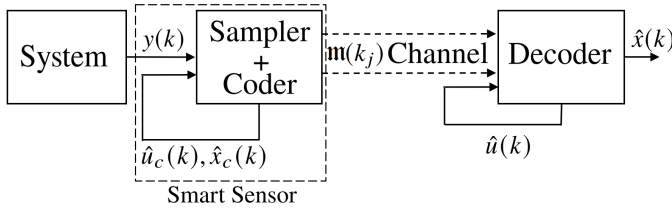


Fig. 1. Structure of setup.

The event-triggered mechanism determines the instants of communication. Since the sampler knows $\hat{x}_c(k)$ and $\hat{u}_c(k)$, it will communicate new estimates $\hat{x}(k)$ only when the distance between $x(k)$ and $\hat{x}(k)$ becomes too large (including a margin of error to account for the measurement noise). Also, in order to reduce the amount of communications, the sampler will space out subsequent communications of estimates of $x(k)$ by at least \bar{k} time instants. The quantity \bar{k} is a tunable integer constant larger or equal to one, which allows one to tune the error and the maximum communication rate, as will be proven later in the paper. We assume that, as a part of the observation scheme, this quantity is known by all agents.

In order to properly define the observation scheme, we will need several additional notions.

First of all, we define the sets of points \mathcal{V} , \mathcal{W} and \mathcal{W}_x , where $\mathcal{V} = \{v_l \in \mathbb{R}^m, l = 1, \dots, |\mathcal{V}|\}$ and $\mathcal{W} = \{w_l \in \mathbb{R}^n, l = 1, \dots, |\mathcal{W}|\}$, while the set \mathcal{W}_x is equal to the set \mathcal{W} , shifted by the vector x : $\{w_l^x = x + w_l, l = 1, \dots, |\mathcal{W}|\}$ (since the set is a shift of \mathcal{W} , it has the same number of elements). The points v_l and w_l represent the centers of balls that cover the sets of admissible values for the input $\hat{u}(k)$ and the state $\hat{x}(k)$, respectively, and the closest elements in the sets \mathcal{V} and $\mathcal{W}_{\hat{x}_c(k)}$ to the current data will be transmitted by the channel.

Next, the notation $A \hat{\sim} B$ refers to the concatenation of the text A and B (i.e. "12" $\hat{\sim}$ "34" = "1234"), $K \in \mathbb{R}^{m \times n}$ is a tunable constant gain matrix, α is a tunable constant. Since we communicate at every time instant, we have $k = k_j = j$. For the sake of correctness and to highlight their different meaning, we keep referring to each of these quantities separately, despite the fact that they are always equal. In any message $\mathbf{m}(k_j)$, $\mathbf{m}_u(k_j)$ refers to the part of the message that encodes information necessary to reconstruct $\hat{u}(k)$, and $\mathbf{m}_x(k_j)$ to the part that encodes information necessary to reconstruct $\hat{x}(k)$. Finally, we define the provisional estimate $\hat{x}^-(k) := A\hat{x}(k-1) + B\hat{u}(k-1) + \varphi(H\hat{x}(k-1), \hat{u}(k-1))$ and its local copy $\hat{x}_c^-(k) = \hat{x}^-(k)$ (which are maintained by the sampler and coder). Note that

$\hat{x}^-(k) = \hat{x}(k)$ only if $k \neq k_{j_x}$. It is a provisional estimate and will coincide with the actual estimate $\hat{x}(k)$ only if no message is sent. If a message containing information to reconstruct $\hat{x}(k)$ is sent, $\hat{x}^-(k)$ is discarded and that message is used to generate the estimate $\hat{x}(k)$.

Procedure 1:

Sampler \mathcal{S} :

- 1: **if** $k = 0$ **then**
- 2: $j \leftarrow 0$
- 3: $j_x \leftarrow 0$
- 4: $k_j \leftarrow 0$
- 5: **else**
- 6: $j \leftarrow k_j \leftarrow k$
- 7: **end if**

Alphabet Size Function \mathcal{A} :

- 1: $l_j \leftarrow |\mathcal{V}|$
- 2: **if** $k_j \geq k_{j_x} + \bar{k}$ **and** $\|y(k) - \hat{x}_c^-(k)\|_2 \geq \alpha$ **then**
- 3: $l_j \leftarrow l_j |\mathcal{W}|$
- 4: **end if**

Coder \mathcal{C} :

- 1: $\hat{x}_c^-(k) = A\hat{x}_c(k-1) + B\hat{u}_c(k-1) + \varphi(H\hat{x}_c(k-1), \hat{u}_c(k-1))$
- 2: $\mathbf{m}(k_j) \leftarrow \arg \min_{l \in \{1, \dots, |\mathcal{V}|\}} \|u(k) + K(y(k) - \hat{x}_c(k)) - v_l\|_2$
- 3: $\hat{u}_c(k) \leftarrow v_{\mathbf{m}(k_j)}$
- 4: **if** $k_j \geq k_{j_x} + \bar{k}$ **and** $\|y(k) - \hat{x}_c^-(k)\|_2 \geq \alpha$ **then**
- 5: $\mathbf{m}_x(k_j) \leftarrow \arg \min_{l \in \{1, \dots, |\mathcal{W}|\}} \|y(k) - w_l^{\hat{x}_c^-(k)}\|_2$
- 6: $\mathbf{m}(k_j) \leftarrow \mathbf{m}(k_j) \hat{\sim} \mathbf{m}_x(k_j)$
- 7: $\hat{x}_c(k) \leftarrow w_{\mathbf{m}_x(k_j)}^{\hat{x}_c^-(k)}$
- 8: $j_x \leftarrow k_j$
- 9: **else**
- 10: $\hat{x}_c(k) \leftarrow \hat{x}_c^-(k)$
- 11: **end if**

Decoder \mathcal{D} :

- 1: $\hat{x}^-(k) = A\hat{x}(k-1) + B\hat{u}(k-1) + \varphi(H\hat{x}(k-1), \hat{u}(k-1))$
- 2: $\hat{u}(k) \leftarrow v_{\mathbf{m}_u(k_j)}$
- 3: **if** $\mathbf{m}_x(k_j) \neq \emptyset$ **then**
- 4: $\hat{x}(k) \leftarrow w_{\mathbf{m}_x(k_j)}^{\hat{x}_c^-(k)}$
- 5: **else**
- 6: $\hat{x}(k) \leftarrow \hat{x}^-(k)$
- 7: **end if**

The Sampler \mathcal{S} is relatively simple: there is an initialization step (lines 2-4) and an incremental step (lines 6), where j and k_j are simply updated to the current time instant (since we communicate \hat{u} at every time instant).

The Alphabet Size Function \mathcal{A} starts (line 1) by being equal to the cardinality of \mathcal{V} (since \mathcal{V} is used to transmit \hat{u} and we communicate \hat{u} at every time instant). Next (line 2), it checks whether two conditions are verified: firstly that enough time has elapsed since the last communication of \hat{x} ($k_j \geq k_{j_x} + \bar{k}$) and secondly that the triggering condition is verified ($\|y(k) - \hat{x}_c^-(k)\|_2 \geq \alpha$). If both condition are true, then an estimate \hat{x} will be communicated in addition to \hat{u} so we increase the length of the alphabet by multiplying the previous value (which was set to the cardinality of \mathcal{V} at line 1) with the cardinality of \mathcal{W} (line 3).

The Coder \mathcal{C} starts by computing the local copy of the

provisional estimate \hat{x}_c^- (line 1). It then computes the point in \mathcal{V} closest to $u(k) + K(y(k) - \hat{x}(k))$, uses its index as the message (line 2), and updates the local copy of \hat{u}_c (line 3). Next, it verifies whether the triggering condition is verified (line 4). If it is, then it computes the point in $\mathcal{W}_{\hat{x}_c^-(k)}$ closest to $y(k)$, concatenates its index to the previous part of the message (line 5), updates the local copy of \hat{x} accordingly (line 6), and updates j_x (line 7). If the triggering condition is not verified then the provisional estimate is used for $\hat{x}_c(k)$ (line 9).

The Decoder \mathcal{D} starts by computing the provisional estimate \hat{x}^- (line 1) and uses the index it received via the message to set the current estimate of driving signal to the point in \mathcal{V} to which this index corresponds (line 2). Next, if it received a message containing information on \hat{x} (line 3), then it uses the point in $W_{\hat{x}_c^-(k)}$ corresponding to the received index (line 4), otherwise, it updates \hat{x} to be equal to \hat{x}^- (line 6). Since $|\mathcal{V}|$ and $|\mathcal{W}|$ are constant, the length of the alphabet indicates whether a message contains information about $\hat{u}(k)$, or both $\hat{u}(k)$ and $\hat{x}(k)$. By looking at the number of bits it received, the decoder can thus clearly distinguish between the different types of messages. The decoder can also easily distinguish which bits encode information about the driving signal, and which bits encode information about the state, when estimates of both are sent at the same time (the first bits encode for the driving signal, the last bits for the state).

As it follows from this procedure, the estimate of driving signal $\hat{u}(k)$ includes a state correction term:

$$\hat{u}(k) = u(k) + K(y(k) - \hat{x}(k)).$$

We conclude this section with several remarks regarding the observation scheme.

Remark 3:

- The observation scheme relies on choices of the quantities \bar{k} , ε_u , \mathcal{V} , ε_x , \mathcal{W} , K , α . At this stage, there is no guarantee that, no matter the choices for these quantities, the observation scheme will always be implementable. Moreover, the choices of these quantities greatly impact the resulting maximum observation error and maximum communication rate. This will be discussed extensively in the next section.
- The sets \mathcal{V} and \mathcal{W} play an essential role in the observation scheme. They will be used to make a covering of the sets containing $\hat{u}(k)$ and $\hat{x}(k_{j_x})$. The observation scheme relies on them having the following properties:
 - For any possible $\hat{u}(k)$, there should exist $v_l \in \mathcal{V}$ such that $\|\hat{u}(k) - v_l\| \leq \varepsilon_u$ (*precision of the covering*).
 - For any possible $\hat{x}(k_{j_x})$, there should exist $w_l^{\hat{x}^-(k_{j_x})} \in \mathcal{W}_{\hat{x}^-(k_{j_x})}$ such that $\|\hat{x}(k_{j_x}) - w_l^{\hat{x}^-(k_{j_x})}\| \leq \varepsilon_x$ (*precision of the covering*).
 - The cardinalities $|\mathcal{V}|$ and $|\mathcal{W}|$ are finite (*size of the covering*).

V. CHOICES, ERROR AND RATES

With the observation scheme and its agents fully introduced, this section aims to answer the following questions: "What is the resulting maximum error ξ ?" and "What is the resulting

maximum communication rate R ?". The answers come in the form of two theorems: one for each quantity.

Before the theorems can be presented, there is a need to provide proper choices for the constant α , gain matrix K , and sets \mathcal{V} and \mathcal{W} .

To this end, we first introduce the following Bilinear Matrix Inequality program, which follows the proof of the main results given later in Appendix I.

$$\begin{aligned} (\mu_i^*, N^*, Q_i^*, S^*) &:= \arg \min_{\mu_i, N, Q_i, S} (\delta^2 \mu_4 + \varepsilon_u^2 \mu_5 + \omega^2 \mu_6), \\ \text{subject to: (11) and} \\ \mu_i I_n &\succeq Q_i, \quad \forall i \in \{2, \dots, 4\}, \\ \mu_i I_m &\succeq Q_i, \quad \forall i \in \{5, 6\}, \\ \mu_1 &\geq 0, \\ N &\succ 0, \\ Q_i &\succeq 0, \quad \forall i \in \{2, \dots, 6\}, \end{aligned} \quad (10)$$

Based on the solution of this program $(\mu_i^*, N^*, Q_i^*, S^*)$ (due to the structure of this particular BMI, such a solution always exists), we define

$$P^* := (N^*)^{-1}, \quad (12)$$

which will be used to introduce a quadratic Lyapunov function characterizing the properties of the estimation error dynamics, and the matrix T which is a matrix verifying $T^T T = P^*$. Note that since P^* is symmetric and positive definite, this decomposition always exists. The gain K used in the observation scheme is defined as follows:

$$K^* := P^* S^*. \quad (13)$$

As was previously mentioned, an important part of the observation scheme is the sets \mathcal{V} and \mathcal{W} . We will use the concept of covering of a set which is defined as follows:

Definition 1: A set S_1 with elements s_l **generates a covering** of radius ε_s of the set S_2 if the following holds

$$S_2 \subseteq \bigcup_{l=1}^{|S_1|} \mathfrak{B}_{\varepsilon_s}(s_l).$$

The set \mathcal{V} is then determined as follows: first we compute η_u such that the distance between $\hat{u}(k)$ and $\frac{u+\hat{u}}{2}$ is at most η_u (that is $\hat{u}(k) \in \mathfrak{B}_{\eta_u}(\frac{u+\hat{u}}{2})$), next we define \mathcal{V} such that it generates a covering of radius ε_u of $\mathfrak{B}_{\eta_u}(\frac{u+\hat{u}}{2})$. This necessarily implies that $\min_{l \in \{1, \dots, |\mathcal{V}|\}} \|\hat{u}(k_j) - v_l\|_2 \leq \varepsilon_u$.

For the set \mathcal{W} , we proceed similarly by computing η_x such that $\hat{x}(k_{j_x}) \in \mathfrak{B}_{\eta_x}(\hat{x}^-(k_{j_x}))$, then we define \mathcal{W} such that it is a covering of radius ε_x of $\mathfrak{B}_{\eta_x}(0)$. The sets $\mathcal{W}_{\hat{x}^-(k_{j_x})}$ are then constructed by simply shifting the set \mathcal{W} by a vector $\hat{x}^-(k_{j_x})$.

$$\begin{bmatrix} -\mu_1 N & * & * & * & \mu_3 \bar{\varphi}^2 S^\top & \mu_3 \bar{\varphi}^2 S^\top & NA^\top - S^\top B^\top & \mu_3 \bar{\varphi}^2 S^\top \\ * & -Q_2 & * & * & * & * & I_n & * \\ * & * & -Q_3 & * & * & * & I_n & * \\ * & * & * & -Q_4 & * & * & I_n & * \\ \mu_3 \bar{\varphi}^2 S & * & * & * & -Q_5 + \mu_3 \bar{\varphi}^2 I_m & \mu_3 \bar{\varphi}^2 I_m & B^\top & * \\ \mu_3 \bar{\varphi}^2 S & * & * & * & \mu_3 \bar{\varphi}^2 I_m & -Q_6 + \mu_3 \bar{\varphi}^2 I_m & B^\top & * \\ AN - BS & I_n & I_n & I_n & B & B & -N + L^2 \mu_2 H^\top H & * \\ \mu_3 \bar{\varphi}^2 S & * & * & * & * & * & * & -I_m \end{bmatrix} \leq 0 \quad (11)$$

For the sequel we will need the following quantities:

$$\begin{aligned} \eta_u &:= \left\| \frac{u - \bar{u}}{2} \right\|_2 + \sigma_1(K^*)\omega + \varepsilon_u \\ &+ \sigma_1(K^*) \max \left\{ \varepsilon_x + \omega, \frac{(\mu_1^*)^{\frac{\bar{k}-1}{2}} \sigma_1(T)}{\sigma_n(T)} (\varepsilon_x + \omega) \right. \\ &\left. + \frac{1}{\sigma_n(T)} (\mu_4^* \delta^2 + \mu_5^* \varepsilon_u^2 + \mu_6^* \sigma_1(K^*)^2 \omega^2)^{\frac{1}{2}} \sum_{i=0}^{\bar{k}-2} (\mu_1^*)^{\frac{i}{2}} \right\}, \end{aligned} \quad (14)$$

$$\begin{aligned} \eta_x &:= \frac{(\mu_1^*)^{\frac{\bar{k}}{2}} \sigma_1(T)}{\sigma_n(T)} (\varepsilon_x + \omega) + \omega \\ &+ \frac{1}{\sigma_n(T)} (\mu_4^* \delta^2 + \mu_5^* \varepsilon_u^2 + \mu_6^* \sigma_1(T)^2 \omega^2)^{\frac{1}{2}} \sum_{i=0}^{\bar{k}-1} (\mu_1^*)^{\frac{i}{2}}. \end{aligned} \quad (15)$$

In order to implement Procedure 1, it is also necessary to define α , which is used in the triggering condition of the observation scheme and it is defined as $\alpha :=$

$$\begin{aligned} \min &\left\{ \frac{\sigma_n(T) [\eta_x - \omega - (\mu_4^* \delta^2 + \mu_5^* \varepsilon_u^2 + \mu_6^* \sigma_1(K^*)^2 \omega^2)^{\frac{1}{2}}]}{(\mu_1^*)^{\frac{1}{2}} \sigma_1(T)} - \omega, \right. \\ \max &\left\{ \varepsilon_x + \omega, \frac{(\mu_1^*)^{\frac{\bar{k}-1}{2}} \sigma_1(T)}{\sigma_n(T)} (\varepsilon_x + \omega) \right. \\ &\left. \left. + \frac{1}{\sigma_n(T)} (\mu_4^* \delta^2 + \mu_5^* \varepsilon_u^2 + \mu_6^* \sigma_1(K^*)^2 \omega^2)^{\frac{1}{2}} \sum_{i=0}^{\bar{k}-2} (\mu_1^*)^{\frac{i}{2}} \right\} \right\}. \end{aligned} \quad (16)$$

We now present the two main results of the paper: first a theorem that provides a bound on the maximum observation error and then a theorem that gives a bound on the maximum communication rate.

In the following theorem, \bar{k} , ε_x and ε_u are tunable constants.

Theorem 1: Let Assumptions 1 to 3 hold for the system (1). Then for any $\bar{k} \geq 1$, $\varepsilon_x > 0$, $\varepsilon_0 \leq \varepsilon_x + \omega$ and $\varepsilon_u > 0$, Procedure 1 guarantees the following bound on the **maximum observation error**:

$$\begin{aligned} \xi &\leq \max \left\{ \varepsilon_x + \omega, \frac{(\mu_1^*)^{\frac{\bar{k}-1}{2}} \sigma_1(T)}{\sigma_n(T)} (\varepsilon_x + \omega) \right. \\ &\left. + \sum_{i=0}^{\bar{k}-2} (\mu_1^*)^{\frac{i}{2}} \frac{1}{\sigma_n(T)} (\mu_4^* \delta^2 + \mu_5^* \varepsilon_u^2 + \mu_6^* \sigma_1(K^*)^2 \omega^2)^{\frac{1}{2}} \right\} \end{aligned} \quad (17)$$

The proof of this theorem is provided in Appendix I-A.

In the following theorem, the notation $\lceil \cdot \rceil$ refers to the ceiling function (i.e. the function that rounds up to the nearest integer).

Theorem 2: Let Assumptions 1 to 3 hold for the system (1). For any $\bar{k} \geq 1$, $\varepsilon_x > 0$, $\varepsilon_0 \leq \varepsilon_x + \omega$ and $\varepsilon_u > 0$, Procedure 1 results in the following bound on the **maximum communication rate**:

$$R \leq \left\lceil m \log_2 \frac{2\eta_u \sqrt{m}}{\varepsilon_u} \right\rceil + \frac{\left\lceil n \log_2 \frac{2\eta_x \sqrt{n}}{\varepsilon_x} \right\rceil}{\bar{k}} \quad (18)$$

The proof of this theorem is presented in Appendix I-B.

VI. SIMULATIONS

In this section, various systems are simulated under the observation scheme. The objectives are:

- To compare the bound on R with the actual rate observed in simulations;
- To show how the Lipschitz-nonlinear term of the system's dynamics affects R , the actual rate, and ξ ;
- To show how R , the actual rate, and ξ are influenced by the choices of \bar{k} , ε_x , and ε_u ;
- To show how R , the actual rate, and ξ are influenced by the perturbations $d(k)$ and $w(k)$.

We will consider two examples of Lipschitz-nonlinear systems. First, a simple two-dimensional system with a trigonometric nonlinearity, which is a structure typical of mechanical systems. Secondly, a model for a flexible joint robot, which has already been studied in [29].

A. Example 1

We consider the discretization, by using an Euler method of the following continuous-time system (inspired by Example 2 of [32]):

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} \gamma \sin(x_2(t)) \\ 0 \end{bmatrix} + d(t), \\ y(t) &= x(t) + w(t), \end{aligned}$$

where $\gamma > 0$ is a parameter, with a discretization step of 10 ms. This yields the following discrete-time system:

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 1 & 0.01 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0.01 \\ 0 \end{bmatrix} u(k) + \begin{bmatrix} 0.01 \gamma \sin(x_2(k)) \\ 0 \end{bmatrix} \\ &\quad + d(k), \\ y(k) &= x(k) + w(k), \end{aligned} \quad (19)$$

For this system, Assumption 2 holds with $L = 0.01\gamma$ and $\bar{\varphi} = 0$. We first consider $\gamma = 2$, Assumption 1 with $\underline{u}_1 = -1$ and $\bar{u}_1 = 1$, Assumption 3 with $\delta = 0.01$ and $\omega = 0.01$, and the following choices for the observer constants

$$\varepsilon_u = 0.01, \varepsilon_x = 0.01, \bar{k} = 2.$$

Solving (10) and using Theorems 1 and 2, we obtain

$$\begin{aligned} \xi &\leq 0.0519 \\ R &\leq 17. \end{aligned}$$

We used Monte-Carlo methods to simulate the observation scheme for 10,000 iterations from $k = 0$ to $k = 1000$ each. The simulated communication rate R^* is $R^* = 12.0068$. The number of bits N_u used to transmit \hat{u} is $N_u = 12$ and the number of bits N_x required to transmit \hat{x} is $N_x = 10$. Since we communicate \hat{u} at every instant, we sent 12 bits at each time instant. Since the communications of \hat{x} are spaced out by at least \bar{k} instants, we at most send 10 bits every \bar{k} instants. When decomposing the rate between the rate that is used to send estimates of u (R_u^*) and the rate that is used to send estimates of x (R_x^*), we observe $R_u^* = 12$ and $R_x^* = 0.0068$. This means that the coder extremely rarely sends estimates of the state and instead relies on the estimate of driving signal to keep the error low. We compared these quantities for varying γ and the same values for the other system parameters and observer constants. The results are displayed in Table I. We observe that:

- The nonlinearity influences ξ : the bigger γ , the bigger the maximum error. Since the Lipschitz nonlinearity is modelled as a perturbation, it is natural that a larger nonlinearity implies a larger ξ ;
- The communication rate is mostly due to communications of \hat{u} ;
- In terms of communication rate, we observe that a significant decrease happens, when γ is increased. This is due to the fact that α grows with L and hence with γ , which implies that the triggering condition is verified less often. For $\gamma = 10$, there are simply no communications of the state at all whilst ξ remains small, which implies that our observer makes good usage of the estimate of driving signal.

γ	1	2	5	10
ξ	0.0479	0.0519	0.0607	0.0706
N_u	12	12	12	12
N_x	10	10	11	11
R^*	12.0279	12.0068	12.0003	12
R_u^*	8	8	8	8
R_x^*	0.0279	0.0068	0.0003	0

TABLE I

EXAMPLE 1: RESULTS FOR VARIOUS VALUES γ .

Next, we simulated the observation scheme, using the same process, for $\gamma = 2$ and various choices of \bar{k} . The results are displayed in Table II. We observe that ξ is greatly influenced

by \bar{k} . The error is multiplied by 5 while \bar{k} increases from 2 to 5. The same effect as for the nonlinearity happens here: the increase in \bar{k} also increases α which means that the triggering condition is verified less and less often. The event-triggering mechanism is very useful in this case as it completely removes the need to communicate estimates of the state.

\bar{k}	1	2	3	5
ξ	0.0200	0.0519	0.0958	0.2172
N_u	11	12	12	13
N_x	8	10	11	13
R^*	18.992	12.0068	12	13
R_u^*	11	12	12	13
R_x^*	7.992	0.0068	0	0

TABLE II

EXAMPLE 1: RESULTS FOR VARIOUS CHOICES \bar{k} .

B. Example 2

Now that the effects of L and \bar{k} have been illustrated on a simple system, we turn to a higher order system to illustrate the effects of changing \bar{k} , ε_u , ε_x , δ , and ω . We consider the discretization of the following continuous-time system, which was first introduced in [29] (*Flexible joint robot*):

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 1 & 0 & 0 \\ 19.5 & 0 & -19.5 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 21.6u(t) \\ 0 \\ -3.33 \sin(x_3(t)) \end{bmatrix}$$

We consider a discretization step of 10 ms. The system matrices and mapping are then

$$A = \begin{bmatrix} 1 & 0.01 & 0 & 0 \\ -0.4860 & 0.9875 & 0.486 & 0 \\ 0 & 0 & 1 & 0.01 \\ 0.195 & 0 & -0.195 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0.216 \\ 0 \\ 0 \end{bmatrix},$$

$$H = [0 \ 0 \ 1 \ 0], \varphi: \zeta \rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.033 \sin(\zeta) \end{bmatrix}$$

For this system, Assumption 2 holds with $L = 0.033$ and $\bar{\varphi} = 0$. For the simulations, the default values for the different parameters and constants are as follow: we assume Assumption 1 with $\underline{u}_1 = -10$ and $\bar{u}_1 = 10$, Assumption 3 with $\delta = 0.5$ and $\omega = 0.2$, and the following choices for the observer constants

$$\varepsilon_u = 0.1, \varepsilon_x = 0.05, \bar{k} = 2.$$

For all constants that are not mentioned in the upcoming description of the simulations, the reader should assume that the constant takes the aforementioned default value. The first simulations investigated the impact of changes in \bar{k} . The results are displayed in Table III. We observe the following effects:

- The error is greatly influenced by the choice of \bar{k} ;
- In terms of transmitted number of bits, both N_u and N_x increase with \bar{k} ;

- For the case $\bar{k} = 1$, we can see that the maximum error is extremely close to the sum of the measurement error and the discretization error of \hat{x} , which is why the event-triggering condition is triggered almost at every communication instant and hence why the theoretical rate of 37 is almost equal to the actual rate. For larger \bar{k} , the error increases drastically and hence the need for communications of \hat{x} decreases also.

\bar{k}	1	2	3	5
ξ	0.2678	1.2164	2.7549	7.6805
N_u	9	10	10	11
N_x	28	32	35	40
R^*	36.5411	12.6822	11.0757	11.468
R_u^*	9	10	10	11
R_x^*	27.5411	2.6822	1.0757	0.468

TABLE III

EXAMPLE 2: RESULTS FOR VARIOUS CHOICES \bar{k} .

We then analysed the impact of the choices of ε_u and ε_x . The results of the simulations are displayed in Tables IV and V. Several observations are to be made:

- ε_u has a smaller impact on the error than ε_x proportionally;
- Increasing both, ε_u and ε_x , enlarges the discretization error, leads to a larger error, and a smaller communication rate;
- The impact in terms of rate is greater for ε_u than for ε_x ;
- The conclusion of these simulations is to send less precise estimates of the driving signal and more precise estimates of the state rather than the other way around.

ε_u	0.01	0.1	1	ε_x	0.025	0.05	0.1
ξ	1.2087	1.2164	1.3382	ξ	1.1641	1.2164	1.3195
N_u	13	10	6	N_u	10	10	10
N_x	32	32	32	N_x	36	32	28
R^*	15.7028	12.6822	8.3312	R^*	13.2434	12.6822	12.0589
R_u^*	13	10	6	R_u^*	10	10	10
R_x^*	2.7028	2.6822	2.3312	R_x^*	3.2434	2.6822	2.0589

TABLE IV

EXAMPLE 2: RESULTS FOR VARIOUS CHOICES ε_u .

TABLE V

EXAMPLE 2: RESULTS FOR VARIOUS CHOICES ε_x .

Finally, we analysed the impact of the size of the state perturbations and measurement error, through the bounds on their maximum norm δ and ω . The results for various values of δ and ω are displayed in Tables VI and VII. We make the following observations about these results:

- The impact of the state perturbation on the error and on the communication rate is greater than the impact of the measurement error;
- Even in situation with high perturbations, the number of communications stays well below the theoretical maximum (for $\delta = 1$, the theoretical maximum is $10 + 35/2 = 27.5$);

- The measurement noise only affects the rate up to a certain point. When it becomes too large, α becomes large as well and hence the triggering condition is more rarely verified, which explains the reduction in communication rate.

δ	0.25	0.5	1
ξ	0.9137	1.2164	1.8088
N_u	9	10	10
N_x	30	32	35
R^*	10.5982	12.6822	14.4613
R_u^*	9	10	10
R_x^*	1.5982	2.6822	4.4613

TABLE VI

EXAMPLE 2: RESULTS FOR VARIOUS VALUES OF δ .

ω	0.01	0.1	1
ξ	0.9823	1.2164	1.6688
N_u	9	10	10
N_x	31	32	33
R^*	12.0663	12.6822	12.0865
R_u^*	9	10	10
R_x^*	2.0663	2.6822	2.0865

TABLE VII

EXAMPLE 2: RESULTS FOR VARIOUS CHOICES ω .

Based on these simulations, we make the following concluding observations:

- The minimum duration between subsequent communications \bar{k} is the constant that has the biggest impact on ξ ;
- It is better to choose ε_u large rather than ε_x , both in terms of error and in terms of communication rate;
- The event-triggering mechanism greatly reduces the actual communication rate compared to the theoretical maximum;
- Most of the communication rate is due to transmitting the estimate of driving signal.

VII. CONCLUSION

In this paper, an approach was proposed for the remote observation of a dynamical system with a Lipschitz nonlinearity through a data-rate constrained communication channel. A solution, named observation scheme, in the form of several interacting agents (sampler/coder and decoder) was proposed and evaluated. The main features of the observation scheme are:

- The maximum observation error is upper-bounded by a quantity that can be computed from the system's features as well as selectable parameters;
- The maximum communication rate is also upper bounded by a quantity that can be computed from the system's features as well as selectable parameters;
- The scheme uses an event-triggering mechanism to reduce the overall required communication rate, without reducing the performance in terms of maximum observation error.

As was demonstrated through simulations, the actual communication rate is much lower than the theoretically evaluated maximum one, which is due to two factors. First of all, the error bounds are conservative, which is due to the usage of a basic quadratic Lyapunov function. The second factor is the event-triggered communication protocol, which greatly helps in reducing the resulting communication rate.

The continuation of this work includes:

- Improving the error bounds through finding a better Lyapunov function;

- Extending the observer to deal with a larger class of nonlinear systems;
- Using this observation scheme for consensus problems in networks of perturbed dynamical systems with data-rate constraints.

APPENDIX I PROOFS OF SECTION V

We first provide several auxiliary lemmata, which will be used next in the proof of the main theorems. We define $e(k) := x(k) - \hat{x}(k)$ and $V(e(k)) := e(k)^\top P^* e(k)$, where P^* is defined in (12).

The following lemma provides a bound on the one-step evolution of $V(e(k))$, provided that $\hat{u}(k) = u(k) - K^*(y(k) - \hat{x}(k)) + e_u(k)$, where $\|e_u(k)\|_2 \leq \varepsilon_u$.

Lemma 1: Let Assumptions 1 to 3 hold for the system (1), and there is a solution to (10). For any $k \geq 0$, any $e_u(k)$ such that $\|e_u(k)\|_2 \leq \varepsilon_u$, any $x(k), \hat{x}(k) \in \mathbb{R}^n$, any $w(k), d(k) \in \mathbb{R}^n$ satisfying Assumption 3, any $u(k)$ satisfying Assumption 1, and for $y(k), \hat{u}(k), e(k+1), x(k+1)$, and $\hat{x}(k+1)$ such that $y(k) = x(k) + w(k)$, $\hat{u}(k) = u(k) + K^*(y(k) - \hat{x}(k)) + e_u(k)$, $e(k+1) = x(k+1) - \hat{x}(k+1)$, $x(k+1) = Ax(k) + Bu(k) + \varphi(Hx(k), u(k)) + d(k)$, and $\hat{x}(k+1) = A\hat{x}(k) + B\hat{u}(k) + \varphi(H\hat{x}(k), \hat{u}(k))$ the following holds

$$V(e(k+1)) \leq \mu_1^* V(e(k)) + \mu_4^* \delta^2 + \mu_5^* \varepsilon_u^2 + \mu_6^* \sigma_1 (K^*)^2 \omega^2. \quad (20)$$

Proof: Our goal is to demonstrate that under the restrictions of the lemma, the estimate (20) can be derived, while the respective conditions can be formulated in terms of bilinear matrix inequalities given in (10). From

$$x(k+1) = Ax(k) + Bu(k) + \varphi(Hx(k), u(k)) + d(k),$$

$$\hat{u}(k) = u(k) + K^*(y(k) - \hat{x}(k)) + e_u(k), \text{ and}$$

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + B\hat{u}(k) + \varphi(H\hat{x}(k), \hat{u}(k)) \\ &= A\hat{x}(k) + Bu(k) + BK^*(y(k) - \hat{x}(k)) + Be_u(k) \\ &\quad + \varphi(H\hat{x}(k), u(k) + K^*(y(k) - \hat{x}(k)) + e_u(k)) \\ &= A\hat{x}(k) + Bu(k) + BK^*(x(k) - \hat{x}(k)) + Be_u(k) \\ &\quad + \varphi(H\hat{x}(k), u(k) + K^*(y(k) - \hat{x}(k)) + e_u(k)) \\ &\quad + BK^*w(k), \end{aligned}$$

we have

$$\begin{aligned} e(k+1) &= Ax(k) + Bu(k) + \varphi(Hx(k), u(k)) + d(k) - A\hat{x}(k) \\ &\quad - Bu(k) - BK^*(x(k) - \hat{x}(k)) - Be_u(k) - BK^*w(k) \\ &\quad - \varphi(H\hat{x}(k), u(k) + K^*(y(k) - \hat{x}(k)) + e_u(k)) \\ &= (A - BK^*)e(k) + \varphi(Hx(k), u(k)) + d(k) - Be_u(k) - BK^*w(k) \\ &\quad - \varphi(H\hat{x}(k), u(k) + K^*(y(k) - \hat{x}(k)) + e_u(k)). \end{aligned}$$

We add and subtract $\varphi(H\hat{x}(k), u(k))$ from the previous equation to obtain

$$\begin{aligned} e(k+1) &= (A - BK^*)e(k) + \varphi(Hx(k), u(k)) - \varphi(H\hat{x}(k), u(k)) \\ &\quad + \varphi(H\hat{x}(k), u(k)) - \varphi(H\hat{x}(k), u(k) + K^*(y(k) - \hat{x}(k)) + e_u(k)) \\ &\quad + d(k) - Be_u(k) - BK^*w(k). \end{aligned}$$

We then have

$$\begin{aligned} V(e(k+1)) &= e(k+1)^\top P^* e(k+1) \\ &= [(A - BK^*)e(k) + \varphi(Hx(k), u(k)) - \varphi(H\hat{x}(k), u(k)) \\ &\quad + \varphi(H\hat{x}(k), u(k)) - \varphi(H\hat{x}(k), u(k) + K^*(y(k) - \hat{x}(k)) + e_u(k)) \\ &\quad + d(k) - Be_u(k) - BK^*w(k)]^\top P^* \\ &= [(A - BK^*)e(k) + \varphi(Hx(k), u(k)) - \varphi(H\hat{x}(k), u(k)) \\ &\quad + \varphi(H\hat{x}(k), u(k)) - \varphi(H\hat{x}(k), u(k) + K^*(y(k) - \hat{x}(k)) + e_u(k)) \\ &\quad + d(k) - Be_u(k) - BK^*w(k)]. \end{aligned}$$

By adding and subtracting the following terms from the previous equation

- $\mu_1 e(k)^\top P^* e(k)$;
- $L^2 \mu_2^* e(k)^\top H^\top H e(k)$;
- $[\varphi(Hx(k), u(k)) - \varphi(H\hat{x}(k), u(k))]^\top Q_2^* [\varphi(Hx(k), u(k)) - \varphi(H\hat{x}(k), u(k))]$;
- $[\varphi(H\hat{x}(k), u(k)) - \varphi(H\hat{x}(k), u(k) + K^*(y(k) - \hat{x}(k)) + e_u(k))]^\top Q_3^* [\varphi(H\hat{x}(k), u(k)) - \varphi(H\hat{x}(k), u(k) + K^*(y(k) - \hat{x}(k)) + e_u(k))]$;
- $d(k)^\top Q_4^* d(k)$;
- $e_u(k)^\top Q_5^* e_u(k)$;
- $w(k)^\top (K^*)^\top Q_6^* K^* w(k)$;

it can be rewritten as

$$\begin{aligned} V(e(k+1)) &= z(k)^\top M z(k) + \mu_1^* e(k)^\top P^* e(k) \\ &\quad - L^2 \mu_2^* e(k)^\top H^\top H e(k) + [\varphi(Hx(k), u(k)) - \varphi(H\hat{x}(k), u(k))]^\top Q_2^* \\ &\quad [\varphi(Hx(k), u(k)) - \varphi(H\hat{x}(k), u(k))] + [\varphi(H\hat{x}(k), u(k)) \\ &\quad - \varphi(H\hat{x}(k), u(k) + K^*(y(k) - \hat{x}(k)) + e_u(k))]^\top Q_3^* [\varphi(H\hat{x}(k), u(k)) \\ &\quad - \varphi(H\hat{x}(k), u(k) + K^*(y(k) - \hat{x}(k)) + e_u(k))] + d(k)^\top Q_4^* d(k) \\ &\quad + e_u(k)^\top Q_5^* e_u(k) + w(k)^\top (K^*)^\top Q_6^* K^* w(k), \end{aligned}$$

where $z(k) =$

$$\begin{bmatrix} e(k) \\ \varphi(Hx(k), u(k)) - \varphi(H\hat{x}(k), u(k)) \\ \varphi(Hx(k), u(k)) - \varphi(Hx(k), u(k) + K^*(y(k) - \hat{x}(k)) + e_u(k)) \\ d(k) \\ e_u(k) \\ K^* w(k) \end{bmatrix}$$

and M as in (21). From (10), we have $\mu_2^* I_n \succeq Q_2^*$, $\mu_3^* I_n \succeq Q_3^*$, $\mu_4^* I_n \succeq Q_4^*$, $\mu_5^* I_m \succeq Q_5^*$, $\mu_6^* I_m \succeq Q_6^*$ and hence

$$\begin{aligned} V(e(k+1)) &= z(k)^\top M z(k) + \mu_1^* e(k)^\top P^* e(k) \\ &\quad - L^2 \mu_2^* e(k)^\top H^\top H e(k) + \mu_2^* [\varphi(Hx(k), u(k)) - \varphi(H\hat{x}(k), u(k))]^\top \\ &\quad [\varphi(Hx(k), u(k)) - \varphi(H\hat{x}(k), u(k))] + \mu_3^* [\varphi(H\hat{x}(k), u(k)) \\ &\quad - \varphi(H\hat{x}(k), u(k) + K^*(y(k) - \hat{x}(k)) + e_u(k))]^\top [\varphi(H\hat{x}(k), u(k)) \\ &\quad - \varphi(H\hat{x}(k), u(k) + K^*(y(k) - \hat{x}(k)) + e_u(k))] + \mu_4^* d(k)^\top d(k) \\ &\quad + \mu_5^* e_u(k)^\top e_u(k) + \mu_6^* w(k)^\top (K^*)^\top K^* w(k). \end{aligned}$$

Using Assumption 2, we conclude that

$$\begin{aligned} V(e(k+1)) &= z(k)^\top M z(k) + \mu_1^* e(k)^\top P^* e(k) \\ &\quad - L^2 \mu_2^* e(k)^\top H^\top H e(k) + L^2 \mu_2^* [Hx(k) - H\hat{x}(k)]^\top [Hx(k) - H\hat{x}(k)] \\ &\quad + \mu_3^* \bar{\varphi}^2 [u(k) - u(k) - K^*(y(k) - \hat{x}(k)) - e_u(k)]^\top [u(k) - u(k) \\ &\quad - K^*(y(k) - \hat{x}(k)) - e_u(k)] + \mu_4^* d(k)^\top d(k) + \mu_5^* e_u(k)^\top e_u(k) \\ &\quad + \mu_6^* w(k)^\top (K^*)^\top K^* w(k). \end{aligned}$$

$$\hat{M} = \begin{bmatrix} \hat{M}_1 & (A-BK^*)\mathcal{P}^* & (A-BK^*)^\top \mathcal{P}^* & (A-BK^*)^\top \mathcal{P}^* & \hat{M}_2^\top & \hat{M}_2^\top \\ P^*(A-BK^*) & P^* - Q_2^* & P^* & P^* & P^*B & P^*B \\ P^*(A-BK^*) & P^* & P^* - Q_3^* & P^* & P^*B & P^*B \\ P^*(A-BK^*) & P^* & P^* & P^* - Q_4^* & P^*B & P^*B \\ \hat{M}_2 & B^\top P^* & B^\top P^* & B^\top P^* & B^\top P^*B - Q_5^* + \mu_3^* \bar{\varphi}^2 I_m & B^\top P^*B + \mu_3^* \bar{\varphi}^2 I_m \\ \hat{M}_2 & B^\top P^* & B^\top P^* & B^\top P^* & B^\top P^*B + \mu_3^* \bar{\varphi}^2 I_m & B^\top P^*B - Q_6^* + \mu_3^* \bar{\varphi}^2 I_m \end{bmatrix} \quad (22)$$

$$\hat{M}_1 = (A-K^*B)^\top P^*(A-BK^*) - \mu_1^* P^* + L^2 \mu_2^* H^\top H + (\mu_3^*)^2 \bar{\varphi}^2 (K^*)^\top K^*$$

$$\hat{M}_2 = B^\top P^*(A-BK^*) + \mu_3^* \bar{\varphi}^2 K^*$$

$$\begin{bmatrix} -\mu_1^* P^* + L^2 \mu_2^* H^\top H + (\mu_3^*)^2 \bar{\varphi}^4 (K^*)^\top K^* & * & * & * & \mu_3^* \bar{\varphi}^2 (K^*)^\top & \mu_3^* \bar{\varphi}^2 (K^*)^\top & (A-BK^*)^\top \\ * & -Q_2^* & * & * & * & * & I_n \\ * & * & -Q_3^* & * & * & * & I_n \\ * & * & * & -Q_4^* & * & * & I_n \\ \mu_3^* \bar{\varphi}^2 K^* & * & * & * & -Q_5^* + \mu_3^* \bar{\varphi}^2 I_m & \mu_3^* \bar{\varphi}^2 I_m & B^\top \\ \mu_3^* \bar{\varphi}^2 K^* & * & * & * & \mu_3^* \bar{\varphi}^2 I_m & -Q_6^* + \mu_3^* \bar{\varphi}^2 I_m & B^\top \\ (A-BK^*) & I_n & I_n & I_n & B & B & -(P^*)^{-1} \end{bmatrix} \preceq 0 \quad (23)$$

$$\begin{bmatrix} -\mu_1^* P^* + (\mu_3^*)^2 \bar{\varphi}^4 (K^*)^\top K^* & * & * & * & \mu_3^* \bar{\varphi}^2 (K^*)^\top & \mu_3^* \bar{\varphi}^2 (K^*)^\top & A^\top - (K^*)^\top B^\top \\ * & -Q_2^* & * & * & * & * & I_n \\ * & * & -Q_3^* & * & * & * & I_n \\ * & * & * & -Q_4^* & * & * & I_n \\ \mu_3^* \bar{\varphi}^2 K^* & * & * & * & -Q_5^* + \mu_3^* \bar{\varphi}^2 I_m & \mu_3^* \bar{\varphi}^2 I_m & B^\top \\ \mu_3^* \bar{\varphi}^2 K^* & * & * & * & \mu_3^* \bar{\varphi}^2 I_m & -Q_6^* + \mu_3^* \bar{\varphi}^2 I_m & B^\top \\ A-BK^* & I_n & I_n & I_n & B & B & -(P^*)^{-1} + L^2 \mu_2^* H^\top H \end{bmatrix} \preceq 0 \quad (24)$$

$$\begin{bmatrix} -\mu_1^* P^* & * & * & * & \mu_3^* \bar{\varphi}^2 (K^*)^\top & \mu_3^* \bar{\varphi}^2 (K^*)^\top & A^\top - (K^*)^\top B^\top & \mu_3^* \bar{\varphi}^2 (K^*)^\top \\ * & -Q_2^* & * & * & * & * & I_n & * \\ * & * & -Q_3^* & * & * & * & I_n & * \\ * & * & * & -Q_4^* & * & * & I_n & * \\ \mu_3^* \bar{\varphi}^2 K^* & * & * & * & -Q_5^* + \mu_3^* \bar{\varphi}^2 I_m & \mu_3^* \bar{\varphi}^2 I_m & B^\top & * \\ \mu_3^* \bar{\varphi}^2 K^* & * & * & * & \mu_3^* \bar{\varphi}^2 I_m & -Q_6^* + \mu_3^* \bar{\varphi}^2 I_m & B^\top & * \\ A-BK^* & I_n & I_n & I_n & B & B & -(P^*)^{-1} + L^2 \mu_2^* H^\top H & * \\ \mu_3^* \bar{\varphi}^2 K^* & * & * & * & * & * & * & -I_m \end{bmatrix} \preceq 0 \quad (25)$$

Consequently,

$$\alpha_i \leq \alpha_k, \quad \forall i \in \{0, \dots, k\}.$$

If $\alpha_0 > \frac{\gamma}{1-\beta}$, then we have

$$\alpha_i = \beta^i \alpha_0 + \gamma \frac{1-\beta^i}{1-\beta}.$$

Since $\alpha_0 > \frac{\gamma}{1-\beta}$, we have

$$\alpha_i \leq \beta^i \alpha_0 + \alpha_0(1-\beta^i) = \alpha_0,$$

which completes the proof. \blacksquare

The following lemma provides a condition on the choice of η_u such that $\hat{u}(k) \in \mathfrak{B}_{\eta_u}(\frac{u+\bar{u}}{2})$.

Lemma 4: Let Assumptions 1 to 3 hold for the system (1). For any $k \geq 0$, any $e_u(k)$ such that $\|e_u(k)\|_2 \leq \epsilon_u$, any $x(k), \hat{x}(k) \in \mathbb{R}^n$, any $w(k), d(k) \in \mathbb{R}^n$ satisfying Assumption 3, any $u(k)$ satisfying Assumption 1, and for $y(k)$ and $\hat{u}(k)$, such

that $y(k) = x(k) + w(k)$ and $\hat{u}(k) = u(k) - K^*(y(k) - \hat{x}(k)) + e_u(k)$, choosing η_u as

$$\eta_u \geq \left\| \frac{u-\bar{u}}{2} \right\|_2 + \sigma_1(K^*) \|x(k) - \hat{x}(k)\|_2 + \sigma_1(K^*) \omega + \epsilon_u \quad (29)$$

implies $\hat{u}(k) \in \mathfrak{B}_{\eta_u}(\frac{u+\bar{u}}{2})$.

Proof: We have

$$\begin{aligned} \left\| \hat{u}(k) - \frac{u+\bar{u}}{2} \right\|_2 &\leq \left\| u(k) - K^*(y(k) - \hat{x}(k)) + e_u(k) - \frac{u+\bar{u}}{2} \right\|_2 \\ &\leq \left\| u(k) - \frac{u+\bar{u}}{2} \right\|_2 + \|K^*(y(k) - \hat{x}(k))\|_2 + \|e_u(k)\|_2 \\ &\leq \left\| u(k) - \frac{u+\bar{u}}{2} \right\|_2 + \|K^*(x(k) - \hat{x}(k))\|_2 + \|K^*w(k)\|_2 + \|e_u(k)\|_2 \end{aligned}$$

$$\begin{bmatrix} -\mu_1^*(P^*)^{-1} & \star & \star & \star & \mu_3^*\bar{\varphi}^2(P^*)^{-1}(K^*)^\top & \mu_3^*\bar{\varphi}^2(P^*)^{-1}(K^*)^\top & \hat{M}_3 & \mu_3^*\bar{\varphi}^2(P^*)^{-1}(K^*)^\top \\ \star & -Q_2^* & \star & \star & \star & \star & I_n & \star \\ \star & \star & -Q_3^* & \star & \star & \star & I_n & \star \\ \star & \star & \star & -Q_4^* & \star & \star & I_n & \star \\ \mu_3^*\bar{\varphi}^2 K^*(P^*)^{-1} & \star & \star & \star & -Q_5^* + \mu_3^*\bar{\varphi}^2 I_m & \mu_3^*\bar{\varphi}^2 I_m & B^\top & \star \\ \mu_3^*\bar{\varphi}^2 K^*(P^*)^{-1} & \star & \star & \star & \mu_3^*\bar{\varphi}^2 I_m & -Q_6^* + \mu_3^*\bar{\varphi}^2 I_m & B^\top & \star \\ \hat{M}_3 & I_n & I_n & I_n & B & B & \hat{M}_4 & \star \\ \mu_3^*\bar{\varphi}^2 K^*(P^*)^{-1} & \star & \star & \star & \star & \star & \star & -I_m \end{bmatrix} \succeq 0, \quad (26)$$

$$\hat{M}_3 = A(P^*)^{-1} - BK^*(P^*)^{-1},$$

$$\hat{M}_4 = -(P^*)^{-1} + L^2 \mu_2^* H^\top H.$$

$$\begin{aligned} &\leq \left\| u(k) - \frac{u + \bar{u}}{2} \right\|_2 + \sigma_1(K^*) \|x(k) - \hat{x}(k)\|_2 \\ &+ \sigma_1(K^*) \|w(k)\|_2 + \|e_u(k)\|_2 \\ &\leq \left\| u(k) - \frac{u + \bar{u}}{2} \right\|_2 + \sigma_1(K^*) \|x(k) - \hat{x}(k)\|_2 + \sigma_1(K^*) \omega + \varepsilon_u. \end{aligned} \quad (30)$$

From Assumption 1, we have

$$\left\| u(k) - \frac{u + \bar{u}}{2} \right\|_2 \leq \left\| \frac{u - \bar{u}}{2} \right\|_2$$

which, when combined with (30), concludes the proof. \blacksquare

The next lemma provides a condition on the choices of η_x and α such that $y(k_{j_x})$ will be contained in $\mathfrak{B}_{\eta_x}(\hat{x}^-(k_{j_x}))$, provided that an estimate of precision ε_x was provided at the last communication instant contained the state guess. To this end, we define $\bar{l}(j_x)$: the communication instant before j_x where an estimate of the state was provided.

Lemma 5: Let Assumptions 1 to 3 hold for the system (1). For any $\bar{k} \geq 1$, any $\varepsilon_x > 0$, any $\varepsilon_u > 0$ such that (10) has a solution $(\mu_i^*, N^*, Q_i^*, S^*)$, for any $x(k_{\bar{l}(j_x)})$, any $w(k_{\bar{l}(j_x)})$, for $y(k_{\bar{l}(j_x)})$ such that $y(k_{\bar{l}(j_x)}) = x(k_{\bar{l}(j_x)}) + w(k_{\bar{l}(j_x)})$, for $\hat{x}(k_{\bar{l}(j_x)})$ such that $\|y(k_{\bar{l}(j_x)}) - \hat{x}(k_{\bar{l}(j_x)})\|_2 \leq \varepsilon_x$, for any $u(k_{\bar{l}(j_x)} + 1), \dots, u(k_{j_x} - 1)$, for $e_u(k)$, $k \in \{k_{\bar{l}(j_x)}, \dots, k_{j_x} - 1\}$ such that $\|e_u(k)\|_2 \leq \varepsilon_u$, for $\hat{u}(k)$, $k \in \{k_{\bar{l}(j_x)}, \dots, k_{j_x} - 1\}$, such that $\hat{u}(k) = u(k) + K^*(y(k) - \hat{x}(k)) + e_u(k)$, $\forall k \in \{k_{\bar{l}(j_x)}, \dots, k_{j_x} - 1\}$, choosing α and η_x as

$$\begin{aligned} \eta_x &\geq \frac{(\mu_1^*)^{\frac{\bar{k}}{2}} \sigma_1(T)}{\sigma_n(T)} (\varepsilon_x + \omega) + \omega \\ &+ \frac{1}{\sigma_n(T)} (\mu_4^* \delta^2 + \mu_5^* \varepsilon_u^2 + \mu_6^* \sigma_1(T)^2 \omega^2)^{\frac{1}{2}} \sum_{i=0}^{\bar{k}-1} (\mu_1^*)^{\frac{i}{2}} \end{aligned} \quad (31)$$

and

$\alpha \geq$

$$\begin{aligned} &\min \left\{ \frac{\sigma_n(T) [\eta_x - \omega - (\mu_4^* \delta^2 + \mu_5^* \varepsilon_u^2 + \mu_6^* \sigma_1(K^*)^2 \omega^2)^{\frac{1}{2}}]}{\sqrt{\mu_1^*} \sigma_1(T)} - \omega, \right. \\ &\max \left\{ \varepsilon_x + \omega, \frac{(\mu_1^*)^{\frac{\bar{k}-1}{2}} \sigma_1(T)}{\sigma_n(T)} (\varepsilon_x + \omega) \right. \\ &\left. \left. + \frac{1}{\sigma_n(T)} (\mu_4^* \delta^2 + \mu_5^* \varepsilon_u^2 + \mu_6^* \sigma_1(K^*)^2 \omega^2)^{\frac{1}{2}} \sum_{i=0}^{\bar{k}-2} (\mu_1^*)^{\frac{i}{2}} \right\} \right\}. \end{aligned} \quad (32)$$

implies $y(k_{j_x}) \in \mathfrak{B}_{\eta_x}(\hat{x}^-(k_{j_x}))$.

Proof: We distinguish two different cases: the first one is the situation where $j_x = \bar{l}(j_x) + \bar{k}$, the second is the situation where $j_x > \bar{l}(j_x) + \bar{k}$.

Situation 1: Let there be k_{j_x} such that $j_x = \bar{l}(j_x) + \bar{k}$. By the theorem statement, at $k_{\bar{l}(j_x)}$, we have

$$\begin{aligned} \|x(k_{\bar{l}(j_x)}) - \hat{x}^-(k_{\bar{l}(j_x)})\|_2 &= \|x(k_{\bar{l}(j_x)}) - \hat{x}(k_{\bar{l}(j_x)})\|_2 \\ &\leq \varepsilon_x + \omega. \end{aligned}$$

Applying Lemma 1 \bar{k} times, we obtain

$$\begin{aligned} &V(x(k_{\bar{l}(j_x)} + \bar{k}) - \hat{x}(k_{\bar{l}(j_x)} + \bar{k})) \\ &\leq (\mu_1^*)^{\bar{k}} V(x(k_{\bar{l}(j_x)}) - \hat{x}(k_{\bar{l}(j_x)})) \\ &+ (\mu_4^* \delta^2 + \mu_5^* \varepsilon_u^2 + \mu_6^* \sigma_1(K^*)^2 \omega^2) \sum_{i=0}^{\bar{k}} (\mu_1^*)^i. \end{aligned}$$

Applying Lemma 2, this implies

$$\begin{aligned} \|x(k_{\bar{l}(j_x)} + \bar{k}) - \hat{x}(k_{\bar{l}(j_x)} + \bar{k})\|_2 &\leq \frac{(\mu_1^*)^{\frac{\bar{k}}{2}} \sigma_1(T)}{\sigma_n(T)} (\varepsilon_x + \omega) \\ &+ \frac{1}{\sigma_n(T)} (\mu_4^* \delta^2 + \mu_5^* \varepsilon_u^2 + \mu_6^* \sigma_1(K^*)^2 \omega^2)^{\frac{1}{2}} \sum_{i=0}^{\bar{k}-1} (\mu_1^*)^{\frac{i}{2}}. \end{aligned}$$

And thus

$$\begin{aligned} \|y(k_{j_x}) - \hat{x}^-(k_{j_x})\|_2 &\leq \|y(k_{j_x}) - x(k_{j_x})\|_2 \\ &+ \|x(k_{j_x}) - \hat{x}^-(k_{j_x})\|_2 \leq \omega + \frac{(\mu_1^*)^{\frac{\bar{k}}{2}} \sigma_1(T)}{\sigma_n(T)} (\varepsilon_x + \omega) \\ &+ \frac{1}{\sigma_n(T)} (\mu_4^* \delta^2 + \mu_5^* \varepsilon_u^2 + \mu_6^* \sigma_1(K^*)^2 \omega^2)^{\frac{1}{2}} \sum_{i=0}^{\bar{k}-1} (\mu_1^*)^{\frac{i}{2}}, \end{aligned}$$

which implies $y(k_{j_x}) \in \mathfrak{B}_{\eta_x}(\hat{x}^-(k_{j_x}))$.

Situation 2: Since $j_x > \bar{l}(j_x) + \bar{k}$, the triggering condition $\|y(k_{j_x} - 1) - \hat{x}(k_{j_x} - 1)\|_2 \leq \alpha$ necessarily holds (otherwise, a communication would have been triggered at the previous time instant). We have

$$\begin{aligned} & \|x(k_{j_x} - 1) - \hat{x}^-(k_{j_x} - 1)\|_2 \\ & \leq \|y(k_{j_x} - 1) - x(k_{j_x} - 1)\|_2 + \|y(k_{j_x} - 1) - \hat{x}^-(k_{j_x} - 1)\|_2 \\ & \leq \omega + \|y(k_{j_x} - 1) - \hat{x}^-(k_{j_x} - 1)\|_2 \leq \omega + \alpha. \end{aligned}$$

Applying Lemma 1 and Lemma 2 together we have

$$\begin{aligned} & \|x(k_{j_x}) - \hat{x}^-(k_{j_x})\|_2 \leq \frac{\sigma_1(T)\sqrt{\mu_1^*}}{\sigma_n(T)}(\omega + \alpha) \\ & + \frac{1}{\sigma_n(T)}(\mu_4^*\delta^2 + \mu_5^*\varepsilon_u^2 + \mu_6^*\sigma_1(K^*)^2\omega^2)^{\frac{1}{2}} \end{aligned}$$

which, after replacing α with the right-hand side of (32) and some computations, yields

$$\|x(k_{j_x}) - \hat{x}^-(k_{j_x})\|_2 \leq \eta_x - \omega,$$

which implies $y(k_{j_x}) \in \mathfrak{B}_{\eta_x}(\hat{x}^-(k_{j_x}))$.

A. Proof of Theorem 1

Proof: At first, from (2), we have

$$\|x(0) - \hat{x}(0)\|_2 \leq \varepsilon_0,$$

and thus from the theorem statement, it immediately follows that

$$\|x(0) - \hat{x}(0)\|_2 \leq \varepsilon_x + \omega.$$

Lemma 1 implies

$$\begin{aligned} V(e(1)) & \leq \frac{\sqrt{\mu_1^*}\sigma_1(T)}{\sigma_n(T)}V(e(0)) \\ & + \frac{1}{\sigma_2(T)}(\mu_4^*\delta^2 + \mu_5^*\varepsilon_u^2 + \mu_6^*\sigma_1(K^*)^2\omega^2)^{\frac{1}{2}}. \end{aligned}$$

and recalling Lemma 2 thus implies that

$$\begin{aligned} \|x(1) - \hat{x}(1)\|_2 & \leq \frac{\sqrt{\mu_1^*}\sigma_1(T)}{\sigma_n(T)}(\varepsilon_x + \omega) \\ & + \frac{1}{\sigma_2(T)}(\mu_4^*\delta^2 + \mu_5^*\varepsilon_u^2 + \mu_6^*\sigma_1(K^*)^2\omega^2)^{\frac{1}{2}}. \end{aligned}$$

From (14) and Lemma 4, we have that $\hat{u}(1) \in \mathfrak{B}_{\eta_u}(\frac{u+\bar{u}}{2})$ and hence we can apply Lemma 1, again. From there on, proceeding sequentially for $j \in \{2, \dots, \bar{k}\}$. Every time, we apply Lemma 1, Lemma 2 and Lemma 3 to obtain that

$$\begin{aligned} \|x(l) - \hat{x}(l)\|_2 & \leq \max \left\{ \varepsilon_x + \omega, \frac{(\sqrt{\mu_1^*})^j \sigma_1(T)}{\sigma_n(T)}(\varepsilon_x + \omega) \right. \\ & \left. + \frac{1}{\sigma_n(T)}(\mu_4^*\delta^2 + \mu_5^*\varepsilon_u^2 + \mu_6^*\sigma_1(K^*)^2\omega^2)^{\frac{1}{2}} \sum_{i=0}^{j-1} (\mu_1^*)^{\frac{i}{2}} \right\}, \end{aligned}$$

$\forall l \in \{1, \dots, j\}$. The parameters chosen in (14) and Lemma 4 then imply that $\hat{u}(j) \in \mathfrak{B}_{\eta_u}(\frac{u+\bar{u}}{2})$, which ensures that the same can be repeated until $j = \bar{k} - 1$. At this point, we have

$$\begin{aligned} \|x(j) - \hat{x}(j)\|_2 & \leq \max \left\{ \varepsilon_x + \omega, \frac{(\sqrt{\mu_1^*})^{\bar{k}-1} \sigma_1(T)}{\sigma_n(T)}(\varepsilon_x + \omega) \right. \\ & \left. + \frac{1}{\sigma_n(T)}(\mu_4^*\delta^2 + \mu_5^*\varepsilon_u^2 + \mu_6^*\sigma_1(K^*)^2\omega^2)^{\frac{1}{2}} \sum_{i=0}^{\bar{k}-2} (\mu_1^*)^{\frac{i}{2}} \right\}. \end{aligned}$$

$\forall j \in \{0, \dots, \bar{k} - 1\}$. At $j = \bar{k}$, and at all subsequent time instants, if no communication occurs at $k = \bar{k}$, $\|x(j) - \hat{x}(j)\|_2 \leq \alpha$ is verified, otherwise a communication would occur. From (16), we thus have

$$\begin{aligned} \|x(j) - \hat{x}(j)\|_2 & \leq \max \left\{ \varepsilon_x + \omega, \frac{(\sqrt{\mu_1^*})^{\bar{k}-1} \sigma_1(T)}{\sigma_n(T)}(\varepsilon_x + \omega) \right. \\ & \left. + \frac{1}{\sigma_n(T)}(\mu_4^*\delta^2 + \mu_5^*\varepsilon_u^2 + \mu_6^*\sigma_1(K^*)^2\omega^2)^{\frac{1}{2}} \sum_{i=0}^{\bar{k}-2} (\mu_1^*)^{\frac{i}{2}} \right\}. \end{aligned}$$

Finally, at $k = k_{j_x}$, we use Lemma 5, so that $y(k_{j_x}) \in \mathfrak{B}_{\eta_x}(\hat{x}^-(k_{j_x}))$ and thus an estimate of precision ε_x is transmitted, which resets the error to $\varepsilon_x + \omega$. From there on, the observation scheme simply repeats the same steps in between two communications of an estimate of the state and hence

$$\begin{aligned} \|x(k) - \hat{x}(k)\|_2 & \leq \max \left\{ \varepsilon_x + \omega, \frac{(\sqrt{\mu_1^*})^{\bar{k}-1} \sigma_1(T)}{\sigma_n(T)}(\varepsilon_x + \omega) \right. \\ & \left. + \frac{1}{\sigma_n(T)}(\mu_4^*\delta^2 + \mu_5^*\varepsilon_u^2 + \mu_6^*\sigma_1(K^*)^2\omega^2)^{\frac{1}{2}} \sum_{i=0}^{\bar{k}-2} (\mu_1^*)^{\frac{i}{2}} \right\}. \end{aligned}$$

$\forall k \geq 0$. ■

B. Proof of Theorem 2

Proof: Although the messages of the communication protocol can possibly vary at every instant, the length of their associated alphabets can only take two values: $|\mathcal{V}|$ or $|\mathcal{V}||\mathcal{W}|$. The former corresponds to communication of the driving signal only, while the latter happens when both, an estimate of driving signal and an estimate of the state, are communicated. Let us call the first situation (a) and the second (b). If (a) occurs for a particular k_j , then we have $b_j = \lceil \log_2 |\mathcal{V}| \rceil$. If (b) occurs for a particular k_j , then we have $b_j = \lceil \log_2 |\mathcal{V}||\mathcal{W}| \rceil$. At each time instant, either (a) or (b) can be realized. Let $\#_b(\bar{j})$ be the number of times that the situation (b) has occurred up to the communication instant $k_{\bar{j}}$. We have

$$\begin{aligned} \frac{\sum_{j=0}^{\bar{j}} b_j}{\bar{j} + 1} & = \#_b(\bar{j}) \frac{\lceil \log_2 |\mathcal{V}||\mathcal{W}| \rceil}{\bar{j} + 1} + (\bar{j} + 1 - \#_b(\bar{j})) \frac{\lceil \log_2 |\mathcal{V}| \rceil}{\bar{j} + 1} \\ & \leq \#_b(\bar{j}) \frac{\lceil \log_2 |\mathcal{V}| \rceil}{\bar{j} + 1} + \#_b(\bar{j}) \frac{\lceil \log_2 |\mathcal{W}| \rceil}{\bar{j} + 1} \\ & \quad + (\bar{j} + 1 - \#_b(\bar{j})) \frac{\lceil \log_2 |\mathcal{V}| \rceil}{\bar{j} + 1} \\ & = \lceil \log_2 |\mathcal{V}| \rceil + \#_b(\bar{j}) \frac{\lceil \log_2 |\mathcal{W}| \rceil}{\bar{j} + 1}. \end{aligned}$$

Since at least \bar{k} time instants elapse between two successive communications of \hat{x} (or events (b)), we have $\#_b(\bar{j}) \leq \frac{\bar{j}+1}{\bar{k}} + 1$, $\forall \bar{j} \geq 0$. This implies that

$$\begin{aligned} \frac{\sum_{j=0}^{\bar{j}} b_j}{\bar{j}+1} &\leq \lceil \log_2 |\mathcal{V}| \rceil + \left(\frac{\bar{j}+1}{\bar{k}} + 1 \right) \frac{\lceil \log_2 |\mathcal{W}| \rceil}{\bar{j}+1} \\ &= \lceil \log_2 |\mathcal{V}| \rceil + \frac{\lceil \log_2 |\mathcal{W}| \rceil}{\bar{k}} + \frac{\lceil \log_2 |\mathcal{W}| \rceil}{\bar{j}+1}. \end{aligned}$$

Starting from the definition of R , we thus have

$$\begin{aligned} R &:= \limsup_{\bar{j} \rightarrow \infty} \frac{\sum_{j=0}^{\bar{j}} b_j}{\bar{j}+1} \\ &\leq \limsup_{\bar{j} \rightarrow \infty} \left[\lceil \log_2 |\mathcal{V}| \rceil + \frac{\lceil \log_2 |\mathcal{W}| \rceil}{\bar{k}} + \frac{\lceil \log_2 |\mathcal{W}| \rceil}{\bar{j}+1} \right] \\ &= \lceil \log_2 |\mathcal{V}| \rceil + \frac{\lceil \log_2 |\mathcal{W}| \rceil}{\bar{k}} + \limsup_{\bar{j} \rightarrow \infty} \frac{\lceil \log_2 |\mathcal{W}| \rceil}{\bar{j}+1} \\ &= \lceil \log_2 |\mathcal{V}| \rceil + \frac{\lceil \log_2 |\mathcal{W}| \rceil}{\bar{k}}. \end{aligned}$$

The set \mathcal{V} is a covering of the ball of radius η_u by balls of radius ε_u . One possible solution to obtain such a covering is as follows. In any ball of radius ε_u , there is a cube of side ε_u/\sqrt{m} inscribed. Any ball of radius η_u is inscribed in a hypercube of side $2\eta_u$. Consequently, it is possible to cover $\mathfrak{B}_{\eta_u}(\frac{u+\bar{u}}{2})_n$ with $\left(\frac{2\eta_u}{\varepsilon_u/\sqrt{m}}\right)^m$ balls. By an identical reasoning, at most $\left(\frac{2\eta_x}{\varepsilon_x/\sqrt{n}}\right)^n$ balls are required to cover $\mathfrak{B}_{\eta_x}(0)$. Therefore,

$$R \leq \left\lceil m \log_2 \frac{2\eta_u \sqrt{m}}{\varepsilon_u} \right\rceil + \frac{\left\lceil n \log_2 \frac{2\eta_x \sqrt{n}}{\varepsilon_x} \right\rceil}{\bar{k}}.$$

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