One-Bend Drawing of K_n in 3D, revisited

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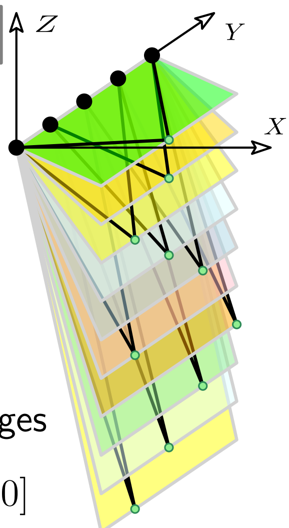
2.1 General idea of [2,3] constructions

- Make k groups of $\frac{n}{k}$ vertices
- Each group is on a line parallel to the Y-axis
- The complete graph of each group is below its line
- The complete bi-partite graph between each pair of groups is above one of the lines
- [2,3]: use distinct k and distinct placements

2.2 Construction from [2,3]

Collinear one-bend drawing of complete graph K_k

- Vertices on Y-axis
- Bends at $X = 1$, $Z < 0$
- Distinct edges: have bends at distinct Z $\Rightarrow$ they lie in distinct pages
- Box $= [0,1] \times [0,k] \times [\frac{k^2}{2}, 0]$



3.2 Our ideas: Intersecting wedges

- Make $\sqrt{n}$ groups of $\sqrt{n}$ vertices at $Z = 0$
- Complete graph of each group below $Z = 0$
- Complete bi-partite graph of each pair of groups

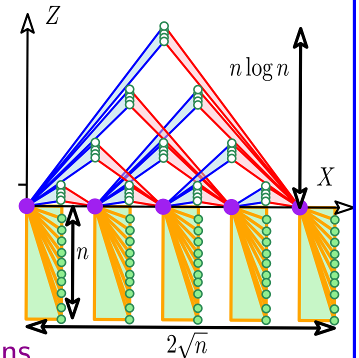
Disjoint upgoing wedges

Disjoint dowgoing wedges

Up/Down intersection

Iterative top-down drawing

Shift new bend in Y to avoid edge intersections



Drawing in $[0, 2\sqrt{n}] \times [0, Y_{\max}] \times [-\frac{n}{2}, n \log n]$

Volume $= O(Y_{\max} n^{\frac{3}{2}} \log n)$

1. The problem:

- Vertices with integer coordinates
- Edges = two line segments (bend at int)
- Minimize volume of enclosing box

Known:

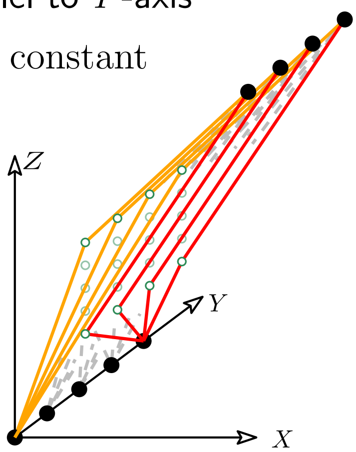
$\Omega(n^2)$	[1] Bose et al. 2004	doi:10.7155/jgaa.00079
$O(n^3/\log^2 n)$	[2] Morin & Wood 2004	doi:10.7155/jgaa.00095
$O(n^{2.5})$	[3] Devillers et al. 2006	doi:10.7155/jgaa.00128

Our contribution: $O(n^2 \log^2 n)$ Conjecture
Experimental validation

2.3 Construction from [2,3]

Drawing of complete bi-partite graph $K_{k,k}$

- Vertices on lines parallel to Y-axis
- Grid of bends at $X = \text{constant}$
- Can be done in a box $[0,2] \times [0,k] \times [0,k]$



3.1 Our ideas:

Shift bend points

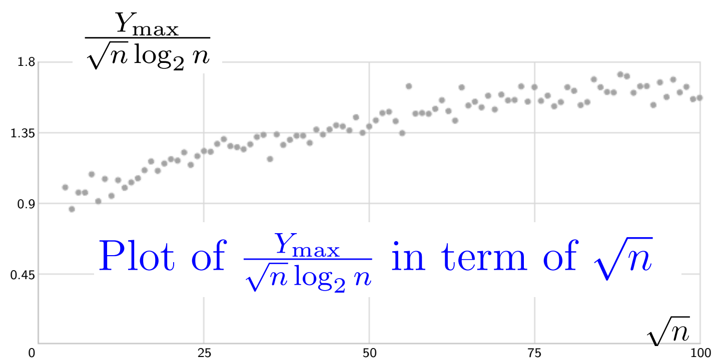
Given X and Z
Consecutive Y unnecessary
Use monotonic Y

3.3 Conjecture

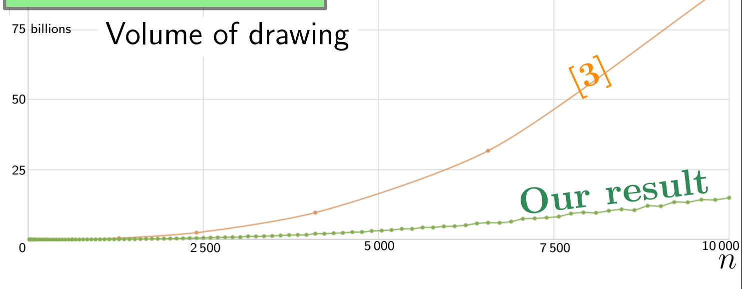
$Y_{\max} = O(\sqrt{n} \log n)$

Volume $= O(n^2 \log^2 n)$

3.4 Experimental validation



Comparison with [3]



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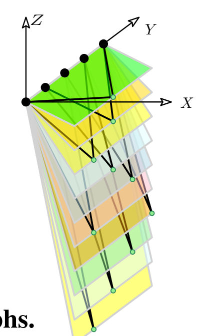
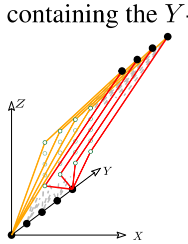
Although graphs are usually drawn in 2D, some authors have considered 3D-drawings with vertices placed on an integer grid and try to minimize the volume of the enclosing box. If the edges are constrained to be straight line segments, Cohen et al. [2] proposed a crossing free drawing of the complete graph K_n within an optimal bounding box of volume $\Theta(n^3)$. Some authors consider the possibility of allowing some bends per edges. Bose, et al. [1] showed a $\Omega(n^2)$ lower bound on the volume of a k-bend drawing of K_n (for any k). Dyck et al. [4] achieved a $O(n^2)$ construction with two bends per edge. If we restrict the number of bends to at most one per edge, Morin and Wood [5] get a volume of $O(\frac{n^3}{\log^2 n})$, later improved by Devillers et al. to $O(n^{2.5})$ [3]. Both constructions split the set of vertices in k groups of $\frac{n}{k}$ vertices and use two basic tools: the collinear drawing of the complete graph of each group (all vertices of a group are collinear) and the bi-collinear drawing of the complete bi-partite graph for each pair of groups (vertices are on two parallel lines). The trick is to arrange these subgraphs so that their hulls are disjoints.

In this paper, we allow the hulls of some bi-partite graphs to intersect provided that the bunches of edges do not intersect. Edges are drawn iteratively, checking the possible conflicts with already drawn edges. We draw the complete graph in a box of size $[0, \sqrt{n}] \times [0, Y_{\max}] \times [-n, n \log n]$, where $Y_{\max}$ is computed for each n. We conjecture and check experimentally that $Y_n = O(\sqrt{n} \log n)$ reaching a volume of $O(n^2 \log^2 n)$ close to the quadratic lower bound.



Collinear drawing of complete graphs.
Morin and Wood proposed to draw the complete graph K_k with all vertices on the Y-axis and $O(k^3)$ volume (with various aspect ratio for the enclosing box). We will use it with a box $[0, 1] \times [0, k] \times [-\frac{k^2}{2}, 0]$ by placing the vertices at $(0, i, 0)$ for $i \in [0, k)$ and bend at $(1, y, j)$ where y is any integer in $[0, k]$ and j is used to number the $\frac{k(k-1)}{2}$ edges of K_k. Each edge is drawn in its own page (plane containing the Y-axis), and thus there are no crossings.



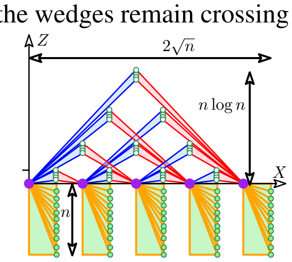
Bi-collinear drawing of complete bi-partite graphs.
The construction proposed by Morin and Wood [5, §3] uses, as building blocks, drawings of complete bi-partite graphs with exactly one bend point per edge. These drawings are called one-bend bi-collinear drawings because the two sets of vertices of $K_{n,m}$ are furthermore constrained to lie on two different lines parallel to the Y-axis.

Vertices are placed at $(0, i, 0)$ for $0 \leqslant i < n$ and $(x, y + j, z)$ for $0 \leqslant j < m$ with $(x, y, z) \in \mathbb{Z}^3$. The bend point is placed at $(x', j, z' + i)$ $0 < x' < x$ and $z' \in \mathbb{Z}$ Choosing $x' = 1$, $z' = 0$, $(x, y, z) = (2, 0, 0)$, the volume is $3nm$. Notice that this construction is contained in two convex bodies, called *wedges*, which are the convex hulls of, respectively, the bends and the n vertices, and the bends and the m vertices. Disjointness of edges is granted by considering the different planes parallel to the Y-axis they belong to.

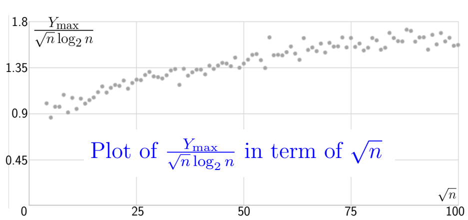
Notice that this construction still works if we place the bends of the edges from $(0, i, 0)$ to $(x, y + j, z)$ at $(x', y_{ij}, z' + i)$ (instead of $(x', j, z' + i)$) where $(y_{i,j})_j$ are sequences of integers that are strictly increasing with j (sequences may depend on i).

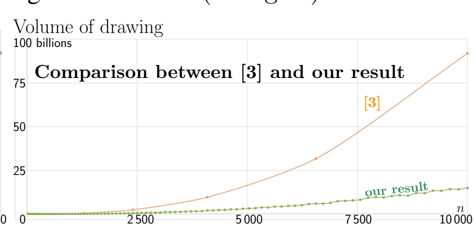
Our construction.

We follow the idea of previous constructions to (i) split the set of n vertices in k groups of size $\frac{n}{k}$, (ii) place the groups on some lines parallel to the Y-axis in the XY-plane, (iii) draw for each group its complete graph in a box below the XY-plane, and (iv) draw for each pair of groups its complete bi-partite graph in two wedges above the XY-plane. However, unlike previous constructions, we allow the possibility for some wedges to intersect, although the different line segments in the wedges remain crossing free. We thus propose the following strategy:



— We split the vertices in $k = \sqrt{n}$ groups of k vertices.
— Vertex $v_{i,j}$, $0 \leqslant i, j < k$ is at coordinates $(2i, j, 0)$.
— The complete graph of group $v_{i,\star}$ is drawn in the box $[2i, 2i + 1] \times [0, k] \times [-k^2, 0]$.
— The bend for the edge from $v_{i,j}$ to $v_{i',j'}$ with $i < i'$ is placed at $e_{i,j,i',j'} = (i' + i, y_{i,j,i',j'}, z_{i'-i}k - j)$ where $z_1 = 1$, $z_i = \left\lceil \frac{i}{i-1} z_{i-1} \right\rceil + 1$, and $y_{i,j,i',j'}$ is determined algorithmically. Such a placement guarantees disjoint up-going (resp.down-going) wedges and a maximal height of $O(n \log n)$. To avoid intersections between up-going and down-going segments the bends are placed iteratively at the smallest y that preserves the monotonicity and does not create crossing with already drawn edges. We achieve a drawing within a box $[0, 2\sqrt{n}] \times [0, Y_{\max}] \times [-n, n \log n]$, where $Y_{\max} = O(n^2)$ is determined by the algorithm. Based on the the computation of $Y_{\max}$ for many values of n (see below), we conjecture that $Y_{\max} = O(\sqrt{n} \log n)$, yielding a volume of $O(n^2 \log^2 n)$.





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