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► **To cite this version:**

Frédéric Chyzak. Creative Telescoping for Parametrised Integration and Summation. École thématique. France. 2011, pp.1-37. hal-04194004

**HAL Id: hal-04194004**

**<https://inria.hal.science/hal-04194004>**

Submitted on 1 Sep 2023

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# Creative Telescoping for Parametrised Integration and Summation

Frédéric Chyzak



*Journées nationales de Calcul formel* (Nov. 14–18, 2011)

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A Nice Story: Apéry's Proof of Irrationality of  $\zeta(3)$ 

Proof, as explained in (van der Poorten, 1979)

Define:

$$b_{n,k} = \binom{n}{k}^2 \binom{n+k}{k}^2, \quad c_{n,k} = \sum_{m=1}^n \frac{1}{m^3} + \sum_{m=1}^k \frac{(-1)^{m-1}}{2m^3 \binom{n}{m} \binom{n+m}{m}},$$

$$a_{n,k} = b_{n,k} c_{n,k}, \quad a_n = \sum_{k=0}^n a_{n,k}, \quad b_n = \sum_{k=0}^n b_{n,k},$$

$$g_n = \gcd(1, \dots, n), \quad p_n = 2g_n^3 a_n, \quad q_n = 2g_n^3 b_n.$$

Then,  $(a_n)$  and  $(b_n)$  satisfy the same 2nd-order recurrence, and:

$$\zeta(3) - \frac{a_n}{b_n} = \mathcal{O}(b_n^{-2}), \quad p_n \in \mathbb{Z}, \quad q_n \in \mathbb{N}, \quad \zeta(3) - \frac{p_n}{q_n} = \mathcal{O}(q_n^{-1.08}).$$

Classical irrationality criterion for  $\alpha \in \mathbb{R}$ :

$$\left( \forall \epsilon > 0, \exists \frac{p}{q}, \left| \alpha - \frac{p}{q} \right| < \frac{\epsilon}{q} \right) \implies \alpha \notin \mathbb{Q}.$$

# Apéry's Recurrence for $(a_n)$ and $(b_n)$

Second-order recurrence (Apéry, 1979)

$$(n+1)^3 u_{n+1} - (34n^3 + 3n^2 + 27n + 5) u_n + n^3 u_{n-1} = 0$$

Cohen and Zagier's "[Creative Telescoping](#)" (van der Poorten, 1979)

"[They] cleverly construct

$$B_{n,k} = 4(2n+1)(k(2k+1) - (2n+1)^2) b_{n,k}$$

with the motive that

$$B_{n,k} - B_{n,k-1} =$$

$$(n+1)^3 b_{n+1,k} - (34n^3 + 51n^2 + 27n + 5) b_{n,k} + n^3 b_{n-1,k}."$$

After summation over  $k$  from 0 to  $n+1$ :

$$\underbrace{B_{n,n+1} - B_{n,-1}}_{0-0=0} = (n+1)^3 b_{n+1} - (34n^3 + 3n^2 + 27n + 5) b_n + n^3 b_{n-1}.$$

# Differentiating under the Integral Sign

Zeilberger's derivation (1982) of a classical integral

$$f(b) = \int_{-\infty}^{+\infty} e^{-x^2} \cos 2bx \, dx = ?$$

$$f'(b) = \int_{-\infty}^{+\infty} -2xe^{-x^2} \sin 2bx \, dx =$$

$$\left[ e^{-x^2} \sin 2bx \right]_{x=-\infty}^{x=+\infty} + \int_{-\infty}^{+\infty} -2be^{-x^2} \cos 2bx \, dx = -2bf(b)$$

Continuous analogue of creative telescoping:

$$\frac{dF}{dx}(b, x) = \frac{df}{db}(b, x) + 2bf(b, x) \quad \text{for} \quad F(b, x) = -\frac{1}{2x} \frac{df}{db}(b, x)$$

After integration over  $x$  from  $-\infty$  to  $+\infty$ :

$$\underbrace{\left[ \frac{dF}{dx}(b, x) \right]_{x=-\infty}^{x=+\infty}}_{0-0=0} = f'(b) + 2bf(b)$$

# Sums and Integrals (1/5)

## Binomial sums

$$\sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^k \binom{k}{j}^3 \quad (\text{Strehl, 1994})$$

$$\sum_{i=0}^n \sum_{j=0}^n \binom{i+j}{i}^2 \binom{4n-2i-2j}{2n-2i} = (2n+1) \binom{2n}{n}^2 \quad (\text{Blodgelt, 1990})$$

## Integrals of the theory of special functions

Four types of **Bessel functions** (Glasser & Montaldi, 1994):

$$\int_0^{+\infty} x J_1(ax) I_1(ax) Y_0(x) K_0(x) dx = -\frac{\ln(1-a^4)}{2\pi a^2}$$

No explicit form, but a 2nd-order linear ODE:

$$\int_0^{\infty} \int_0^{\infty} J_1(x) J_1(y) J_2(c\sqrt{xy}) \frac{dx dy}{e^{x+y}}$$

# Sums and Integrals (2/5)

## Extractions of coefficients

Theory of **orthogonal polynomials**, here, Hermite (Doetsch, 1930):

$$\frac{1}{2\pi i} \oint \frac{(1 + 2xy + 4y^2) \exp\left(\frac{4x^2y^2}{1+4y^2}\right)}{y^{n+1}(1 + 4y^2)^{\frac{3}{2}}} dy = \frac{H_n(x)}{[n/2]!}$$

**Scalar products** involving orthogonal/parametrised families

Chebyshev polynomials, Bessel functions, modified Bessel functions:

$$\int_{-1}^{+1} \frac{e^{-px} T_n(x)}{\sqrt{1-x^2}} dx = (-1)^n \pi I_n(p)$$

$$\int_0^{+\infty} x e^{-px^2} J_n(bx) I_n(cx) dx = \frac{1}{2p} \exp\left(\frac{c^2 - b^2}{4p}\right) J_n\left(\frac{bc}{2p}\right)$$



# Sums and Integrals (3/5)

$q$ -Sums, e.g., from the theory of combinatorial partitions

Finite forms of the Rogers–Ramanujan identities and a generalisation: setting  $(q; q)_n = (1 - q) \cdots (1 - q^n)$ ,

$$\sum_{k=0}^n \frac{q^{k^2}}{(q; q)_k (q; q)_{n-k}} = \sum_{k=-n}^n \frac{(-1)^k q^{(5k^2-k)/2}}{(q; q)_{n-k} (q; q)_{n+k}} \quad (\text{Andrews, 1974})$$

$$\sum_{j=0}^n \sum_{i=0}^{n-j} \frac{q^{(i+j)^2 + j^2}}{(q; q)_{n-i-j} (q; q)_i (q; q)_j} = \sum_{k=-n}^n \frac{(-1)^k q^{7/2k^2 + 1/2k}}{(q; q)_{n+k} (q; q)_{n-k}} \quad (\text{Paule, 1985})$$

Scalar products in algebraic combinatorics

For  $p_1 = x_1 + x_2 + \cdots$  and  $p_2 = x_1^2 + x_2^2 + \cdots$ :

$$\left\langle \exp((p_1^2 - p_2)/2 - p_2^2/4) \mid \exp(t(p_1^2 + p_2)/2) \right\rangle = \frac{e^{-\frac{1}{4}t(t+2)}}{\sqrt{1-t}}$$

# Sums and Integrals (4/5)

## Combinatorial identities

In the graph-counting sequence  $k^{k-1}$ :

$$\sum_{k=0}^n \binom{n}{k} i (k+i)^{k-1} (n-k+j)^{n-k} = (n+i+j)^n \quad (\text{Abel})$$

In Stirling numbers of the second kind (partitions) and Eulerian numbers (ascents in permutations):

$$\sum_{k=0}^n (-1)^{m-k} k! \binom{n-k}{m-k} \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\} = \left\langle \begin{matrix} n \\ m \end{matrix} \right\rangle \quad (\text{Frobenius})$$

In Bernoulli numbers (Taylor expansion of  $\tan(x)$ ):

$$\sum_{k=0}^m \binom{m}{k} B_{n+k} = (-1)^{m+n} \sum_{k=0}^n \binom{n}{k} B_{m+k} \quad (\text{Gessel, 2003})$$

# Sums and Integrals (5/5)

Identities in more special functions (related, e.g., to number theory)

In Hurwitz's zeta function and the beta function:

$$\int_0^{\infty} x^{k-1} \zeta(n, \alpha + \beta x) dx = \beta^{-k} B(k, n-k) \zeta(n-k, \alpha)$$

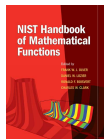
In the polylogarithm functions:

$$\int_0^{\infty} x^{\alpha-1} \text{Li}_n(-xy) dx = \frac{\pi (-\alpha)^n y^{-\alpha}}{\sin(\alpha\pi)}$$

In the (upper) incomplete Gamma function:

$$\int_0^{\infty} x^{s-1} \exp(xy) \Gamma(a, xy) dx = \frac{\pi y^{-s}}{\sin((a+s)\pi)} \frac{\Gamma(s)}{\Gamma(1-a)}$$

+ a lot more in:



or in web sites, like Victor Moll's web site on GR

# Looking Inside PBM's "Integrals and Series, Vol. 2"

$$6. \int_0^{\infty} x^{-1/2} \cos b \sqrt{x} J_0(cx) dx = -\frac{\pi b}{8c} \{Y_{1/4}(z) [Y_{1/4}(z) - J_{1/4}(z)] - J_{1/4}(z) [J_{1/4}(z) + Y_{1/4}(z)]\}_{z = b^2/(8c^2)}; \quad b, c > 0$$

$$7. \int_0^{\infty} x^{-1/2} \sin b \sqrt{x} \frac{\sin cx}{\cos cx} J_0(cx) dx = \sqrt{\frac{\pi}{2c}} \cos \frac{b^2}{16c} \pm \frac{\pi}{4} J_0 \left( \frac{b^2}{16c} \right) \quad [b, c > 0]$$

$$8. \int_0^{\infty} x^{-1/2} \cos \sqrt{x} \frac{\sin cx}{\cos cx} J_0(cx) dx = -\frac{1}{2} \sqrt{\frac{\pi}{2c}} \left[ \frac{\cos t}{\sin t} J_0 \left( \frac{b^2}{16c} \right) \mp \frac{\sin t}{\cos t} Y_0 \left( \frac{b^2}{16c} \right) \right]_{[t = b^2/(16c) - \pi/4; \quad b, c > 0]}$$

2.12.20. Integrals containing  $x^{\alpha} \frac{\sin(bx+ax)}{\cos(bx+ax)} J_{\nu}(cx)$ .

$$1. \int_0^{\infty} x^{\alpha-1} \frac{\sin(bx)}{\cos(bx)} J_{\nu}(cx) dx = 2^{\alpha-1} b^{-\alpha} c^{-\alpha} \Gamma \left[ \frac{\nu+(\alpha-1)/2}{1+(\delta+\nu-\alpha)/2} \right] \times \\ \times {}_2F_1 \left( 1 - \frac{\nu+\alpha-\delta}{2}, \frac{1}{2} + \delta, 1 + \frac{\delta+\nu-\alpha}{2}; \frac{b^2 \delta^{\alpha}}{16c^{\alpha}} \right) \mp \\ \mp b^{\alpha+\nu} \left( \frac{c}{2} \right) \frac{\sin[(\alpha+\nu)\pi/2]}{\cos[(\alpha+\nu)\pi/2]} \left\{ \left[ -\frac{\nu-\alpha}{1+\nu} \right] {}_2F_1 \left( 1 + \frac{\alpha+\nu}{2}, 1 + \nu, \frac{1+\alpha+\nu}{2}; \frac{b^2 \delta^{\alpha}}{16c^{\alpha}} \right) \right. \\ \left. [b, c > 0; -1 - \operatorname{Re} \nu < \operatorname{Re} \alpha < \delta + 3/2; \delta = \begin{cases} 0 \\ 1 \end{cases}] \right\}$$

$$2. \int_0^{\infty} \frac{1}{x} \sin bx \frac{\sin(ax)}{\cos(ax)} J_{\nu}(cx) dx = -\frac{\pi}{2} J_{\nu}(z, \sqrt{a}) \left\{ \frac{\sin(\nu\pi/2)}{\cos(\nu\pi/2)} J_{\nu}(z, \sqrt{a}) \pm \frac{\cos(\nu\pi/2)}{\sin(\nu\pi/2)} Y_{\nu}(z, \sqrt{a}) \right\} + \\ + \left\{ \frac{\cos(\nu\pi/2)}{\sin(\nu\pi/2)} \right\} J_{\nu}(z, \sqrt{a}) K_{\nu}(z, \sqrt{a}) \quad [0 < c < b; a > 0; z_{\pm} = \sqrt{b^2 \mp z} \pm \sqrt{b^2 - z}]$$

$$3. \int_0^{\infty} \frac{1}{x} \cos bx \frac{\sin(ax)}{\cos(ax)} J_{\nu}(cx) dx = \pm \frac{\pi}{2} J_{\nu}(z, \sqrt{a}) \left\{ \frac{\cos(\nu\pi/2)}{\sin(\nu\pi/2)} J_{\nu}(z, \sqrt{a}) \mp \frac{\sin(\nu\pi/2)}{\cos(\nu\pi/2)} Y_{\nu}(z, \sqrt{a}) \right\} \mp \\ \mp \left\{ \frac{\sin(\nu\pi/2)}{\cos(\nu\pi/2)} \right\} J_{\nu}(z, \sqrt{a}) K_{\nu}(z, \sqrt{a}) \quad [0 < c < b; a > 0; z_{\pm} = \sqrt{b^2 \mp z} \pm \sqrt{b^2 - z}]$$

$$4. \int_0^{\infty} \frac{1}{x} \frac{\sin(bx-a/x)}{\cos(bx-a/x)} J_{\nu}(cx) dx = 2 \left\{ \frac{\sin(\nu\pi/2)}{\cos(\nu\pi/2)} \right\} J_{\nu}(z, \sqrt{a}) K_{\nu}(z, \sqrt{a}) \\ [0 < c < b; a > 0; z_{\pm} = \sqrt{b^2 \mp z} \pm \sqrt{b^2 - z}]$$

$$5. \int_0^{\infty} \frac{1}{x} \frac{\sin(bx+a/x)}{\cos(bx+a/x)} J_{\nu}(cx) dx = -J_{\nu}(z, \sqrt{a}) \left[ \pm \frac{\cos(\nu\pi/2)}{\sin(\nu\pi/2)} J_{\nu}(z, \sqrt{a}) - \left\{ \frac{\sin(\nu\pi/2)}{\cos(\nu\pi/2)} \right\} Y_{\nu}(z, \sqrt{a}) \right] \\ [0 < c < b; a > 0; z_{\pm} = \sqrt{b^2 \mp z} \pm \sqrt{b^2 - z}]$$

2.12.21. Integrals of  $A(x) \frac{\sin(b\sqrt{a^2-x^2})}{\cos(b\sqrt{a^2-x^2})} J_{\nu}(cx)$ .

$$1. \int_0^a x^{2\nu+1} \sin(b\sqrt{a^2-x^2}) J_{\nu}(cx) dx = -\sqrt{\frac{\pi}{2}} a^{2\nu+3/2} b c^{\nu} (b^2+c^2)^{-(2\nu+1)/4} J_{\nu+3/2}(a\sqrt{b^2+c^2}) \\ [a > 0; \operatorname{Re} \nu > -1]$$

$$2. \int_0^a x \sin(b\sqrt{a^2-x^2}) J_{\nu}(cx) dx = \frac{ab}{b^2+c^2} \left[ \frac{\sin(a\sqrt{b^2+c^2})}{a\sqrt{b^2+c^2}} - \cos(a\sqrt{b^2+c^2}) \right] \\ [a > 0]$$

$$3. \int_0^a \frac{\cos(b\sqrt{a^2-x^2})}{\sqrt{a^2-x^2}} J_{\nu}(cx) dx = \frac{\pi}{2} J_{\nu+1/2} \left( \frac{a\sqrt{b^2+c^2+ab}}{2} \right) J_{\nu+1/2} \left( \frac{a\sqrt{b^2+c^2-ab}}{2} \right) \\ [a > 0; \operatorname{Re} \nu > -1]$$

$$4. \int_0^a \frac{\cos(b\sqrt{a^2-x^2})}{\sqrt{a^2-x^2}} J_1(cx) dx = \frac{1}{ac} \cos ab - \frac{1}{ac} \cos(a\sqrt{b^2+c^2}) \\ [a > 0]$$

$$5. \int_0^a x^{\nu+1} \frac{\cos(b\sqrt{a^2-x^2})}{\sqrt{a^2-x^2}} J_{\nu}(cx) dx = -\sqrt{\frac{\pi}{2}} a^{\nu+1/2} c^{\nu} (b^2+c^2)^{-(2\nu+1)/4} J_{\nu+1/2}(a\sqrt{b^2+c^2}) \\ [a > 0; \operatorname{Re} \nu > -1]$$

$$6. \int_0^a x \frac{\cos(b\sqrt{a^2-x^2})}{\sqrt{a^2-x^2}} J_0(cx) dx = \frac{\sin(a\sqrt{b^2+c^2})}{\sqrt{b^2+c^2}} \\ [a > 0]$$

2.12.22. Integrals of  $A(x) \frac{\sin(b\sqrt{x^2-a^2})}{\cos(b\sqrt{x^2-a^2})} J_{\nu}(cx)$ .

$$1. \int_a^{\infty} \sin(b\sqrt{x^2-a^2}) J_1(cx) dx = \frac{b}{c} (c^2-b^2)^{-1/2} \cos(a\sqrt{c^2-b^2}) \\ [a, b, c > 0; b \neq c]$$

$$2. \int_a^{\infty} x^{1-\nu} \sin(b\sqrt{x^2-a^2}) J_{\nu}(cx) dx = -\sqrt{\frac{\pi}{2}} a^{1/2-\nu} b c^{-\nu} (c^2-b^2)^{(2\nu-3)/4} J_{\nu-3/2}(a\sqrt{c^2-b^2}) \\ [a > 0; b \neq c; \operatorname{Re} \nu > -1/2]$$

# Creative Telescoping for Sums/Integrals

$$U_n = \sum_{k=a}^b u_{n,k} = ?$$

**Given** a relation  $a_r(n)u_{n+r,k} + \dots + a_0(n)u_{n,k} = v_{n,k+1} - v_{n,k}$ , summation leads by “telescoping” to

$$a_r(n)U_{n+r} + \dots + a_0(n)U_n = v_{n,b+1} - v_{n,a} \stackrel{\text{often}}{=} 0.$$

$$U(x) = \int_a^b u(x,y) dy = ?$$

**Given** a relation  $a_r(x)\frac{\partial^r u}{\partial x^r} + \dots + a_0(x)u = \frac{\partial}{\partial y}v(x,y)$ , integrating leads by “telescoping” to

$$a_r(x)\frac{\partial^r U}{\partial x^r} + \dots + a_0(x)U = v(x,b) - v(x,a) \stackrel{\text{often}}{=} 0.$$

Adapts easily to  $U(x) = \sum_{k=a}^b u_k(x)$  and  $U_n = \int_a^b u_n(y) dy$ .

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# History for Algorithms Based on Fasenmyer's Ansatz

- (Fasenmyer, 1945, 1947, 1949) "Some generalized hypergeometric polynomials", "A note on pure recurrence relations"
- (Rainville, 1960) "Special functions"
- (Verbaeten, 1974, 1976) "The automatic construction of pure recurrence relations", "Rekursiebetrekkingen voor lineaire hypergeometrische funkties"
- (Zeilberger, 1982) "Sister Celine's technique and its generalizations"
- (Lipshitz, 1988) "The diagonal of a D-finite power series is D-finite"
- (Zeilberger, 1990) "A holonomic systems approach to special functions identities"
- (Wilf and Zeilberger, 1992) "An algorithmic proof theory for hypergeometric (ordinary and 'q') multisum/integral identities"
- (Hornegger, 1992) "Hypergeometrische Summation und polynomiale Rekursion"
- (Wegschaider, 1997) "Computer generated proofs of binomial multi-sum identities"
- (Tefera, 2000, 2002) "Improved algorithms and implementations in the multi-WZ theory", "MultiInt, a MAPLE package for multiple integration by the WZ method"
- (Riese, 2003) "qMultiSum: a package for proving q-hypergeometric multiple summation identities"

# Hypergeometric Terms

Definition (hypergeometric terms)

An element  $h$  of a  $\mathbb{K}(n, k)$ -vector space closed under shifts is *hypergeometric* if the quotients

$$\frac{h_{n+1,k}}{h_{n,k}} \quad \text{and} \quad \frac{h_{n,k+1}}{h_{n,k}}$$

exist and are rational functions.

Basic observation: for all  $(i, j) \in \mathbb{Z}^2$ ,

$$\frac{h_{n+i,k+j}}{h_{n,k}} \in \mathbb{K}(n, k).$$

Generalises to more indices.

An **idealisation** of sequences like:  $n!$ ,  $\binom{n}{k}$ , falling factorials  $n^{\underline{k}} = n(n-1)\cdots(n-k+1)$ , etc.



# Fasenmyer's Heuristic, as Revisited by Zeilberger

Fasenmyer's ansatz: For a given hypergeometric  $h_{n,k}$ , solve

$$\sum_{i=0}^r \sum_{j=0}^s c_{i,j}(n) h_{n+i,k+j} = 0 \quad \text{where the } c\text{'s don't involve } k.$$

Motivation:

$$\sum_{i=0}^r \left( \sum_{j=0}^s c_{i,j}(n) \right) h_{n+i,k} = g_{n,k+1} - g_{n,k} \quad \text{where} \quad g_{n,k} = \sum_{i=0}^r \sum_{j=0}^{s-1} \tilde{c}_{i,j}(n) h_{n+i,k+j}.$$

Idea: Use the *rational functions*  $h_{n+i,k+j}/h_{n,k}$ , then **clear denominators**

$$\sum_{i=0}^r \sum_{j=0}^s c_{i,j}(n) p_{i,j}(n, k) = 0 \quad \text{for} \quad p_{i,j} \in \mathbb{K}(n)[k],$$

then solve a linear system over  $\mathbb{K}(n)$ . **If no solution, increase  $(r, s)$ .**

Question: possibility to enforce  $\max \deg_k p_{i,j} < (r+1)(s+1)$  ?

# Two Motivating Examples

Common denominator  $C_r$  of  $h_{n+i,k+j}/h_{n,k}$  when  $r = s$  increases?

- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ :

$$C_r = (k+1) \cdots (k+r)(n+1-k) \cdots (n+r-k)$$

has **linearly many terms**. The method **succeeds** easily (by finding Pascal's triangle rule).

- $\frac{1}{n^2 + k^2}$ :

$$C_r = \prod_{i,j \in \mathbb{N}, i+j \leq r} ((n+i)^2 + (k+j)^2)$$

has **quadratically many terms**. The method seems to **fail** (out of memory).

# Wilf and Zeilberger's Proper Hypergeometric Terms

Definition (proper hypergeometric term)

A hypergeometric term  $h$  is *proper* if it can be written, for  $\epsilon_\ell = \pm 1$ ,

$$h_{n,k} = \underbrace{P(n,k)}_{P \in \mathbb{K}[n,k]} \underbrace{\zeta^n \xi^k}_{\zeta, \xi \in \mathbb{K}} \prod_{\ell=1}^L \Gamma(\underbrace{a_\ell n + b_\ell k + c_\ell}_{a_\ell, b_\ell \in \mathbb{Z}, c_\ell \in \mathbb{K}})^{\epsilon_\ell}.$$

$$\Gamma(s+1) = s \Gamma(s) \implies \Gamma(s+u)/\Gamma(s) = \text{poly. of degree } u \text{ in } s$$

Key observation

$$h_{n+i,k+j} \rightarrow \Gamma(a_\ell n + b_\ell k + c_\ell + u) \quad \text{where} \quad |u| \leq |a_\ell|r + |b_\ell|s =: \sigma_\ell,$$

$$h_{n+i,k+j} = \zeta^n \xi^k \prod_{\ell=1}^L \Gamma(a_\ell n + b_\ell k + c_\ell - \epsilon_\ell \sigma_\ell)^{\epsilon_\ell} \times \underbrace{p_{i,j}(n,k)}_{\in \mathbb{K}(n)[k]}.$$

Examples:  $\binom{n}{k}$ ,  $\frac{1}{n-k}$ ,  $n^k$ .

Counter-example:  $\frac{1}{n^2+k^2}$ .

# Wilf and Zeilberger's Algorithm

## Theorem

Fasenmyer's ansatz can be solved for the setting:

$$r = B \quad \text{and} \quad s = (A - 1)B + \deg_k(P) + 1,$$

where

$$A = \sum_{\ell} |a_{\ell}| \quad \text{and} \quad B = \sum_{\ell} |b_{\ell}|.$$

Proof: Observe  $\deg_k p_{i,j} \leq \deg_k P + Ar + Bs$ , set  $r = B$ , and enforce

$$\deg_k P + Ar + Bs + 1 < (r + 1)(s + 1).$$

## Generalisations:

- to more than 2 indices, with non-explicit bounds;
- to proper  $q$ -hypergeometric terms, with similar bounds:

$$h_{n,k} = P(q^n, q^k) \zeta^n \bar{\zeta}^k q^{\alpha n^2 + \beta nk + \gamma k^2 + \lambda \binom{n}{2} + \mu \binom{k}{2}} \prod_{\ell=1}^L ((q; c_{\ell})_{a_{\ell} n + b_{\ell} k})^{\epsilon_{\ell}}.$$

# Really an Algorithm?

What if Wilf and Zeilberger's output is *zero*?

$$\underbrace{\sum_{i=0}^r \underbrace{\left( \sum_{j=0}^s c_{i,j}(n) \right)}_{=0} h_{n+i,k}}_{=0} = g_{n,k+1} - g_{n,k} ?$$

→ Summation over  $k$  will deliver nothing, so?

# Change of Notation: Recurrence Operators

Sequences:

$$u : (n, k) \mapsto u_{n,k}.$$

Shift operators:

$$S_n u : (n, k) \mapsto u_{n+1,k} \quad \text{and} \quad S_k u : (n, k) \mapsto u_{n,k+1}.$$

Multiplication operators:

$$n u : (n, k) \mapsto n u_{n,k} \quad \text{and} \quad k u : (n, k) \mapsto k u_{n,k}.$$

Operator algebras, e.g.,  $\mathbb{K}(n)[k]\langle S_n, S_k \rangle$ , in which:

$$S_n n = (n + 1) S_n \quad \text{and} \quad S_k k = (k + 1) S_k.$$

# Wegschaider's Fix

Given  $L = \sum_{i=0}^r \sum_{j=0}^s c_{i,j}(n) S_n^i S_k^j \in \mathbb{K}(n)\langle S_n, S_k \rangle$  such that  $Lh = 0$ , find a nonzero  $P \in \mathbb{K}(n)\langle S_n \rangle$  such that  $Ph = (S_k - 1)(\dots)$ , in the bad case  $L = (S_k - 1)^m \tilde{L}$ .

Observe the commutation:

$$(k-a)^\ell (S_k - 1) = (S_k - 1)(k-a-1)^\ell - \ell(k-a-1)^{\ell-1},$$

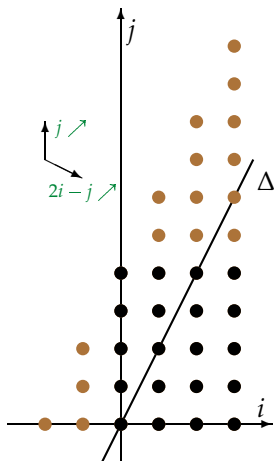
so that, after iterating,

$$k^m (S_k - 1)^m = (-1)^m m! + (S_k - 1)(\dots).$$

As  $(-1)^m m!^{-1} k^m Lh = 0$ , write  $\tilde{L} = P + (S_k - 1)Q$  to get:

$$0 = \underbrace{P(n, S_n)}_{\neq 0} h + (S_k - 1) \underbrace{\hat{Q}(n, k, S_n, S_k)}_{\substack{\text{dependency in } k \\ \text{still sums right!}}} h.$$

## Verbaeten's Completion (1/2)



$$\text{Ex: } \sum_{k=0}^{2n} \underbrace{(-1)^k \binom{2n}{k}}_{:=h_{n,k}} = (-1)^n \frac{(3n)!}{n!^3} \quad (\text{Dixon, 1891}).$$

$$r_{i,j} := \frac{h_{n-i,k-j}}{h_{n,k}} = (\dots) \left( \frac{k!}{(k-j)!} \frac{(2n-k)!}{(2n-k-(2i-j))!} \right)^3$$

Denom. of  $r_{i,j}$  constant on a parallel to  $\Delta$ .

Common denom. of  $r_{i,j}$  over the  $\bullet$ 's =

$$(k - 2n - 4)^3 \cdots (k - 2n - 1)^3 =: C.$$

$$\deg_k Cr_{i,j} = 3(2i + 4) \leq 30 \text{ over } \bullet\text{'s and } \bullet\text{'s.}$$

# of unknowns = # of  $\bullet$ 's and  $\bullet$ 's = 36

# equations = 1 + maximal degree in  $k$  = 31

$\implies$  success after completion!



# Verbaeten's Completion (2/2)

- Verbaeten could already solve Fasenmyer's ansatz for a class of special proper hypergeometric terms;
- Elegant geometric interpretation by Hornegger (volumes of polytopes);
- Nicely put to practice, also in the multisum case, by Wegschaider for sums, resp. Riese for  $q$ -sums.

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- 2 Fasenmyer's (a.k.a. "*k*-Free") Ansatz
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# Differentiably Finite Functions and Series

Definition (D-finite term; hyperexponential term)

An element  $f$  of a finite-dimensional  $\mathbb{K}(x, y)$ -vector space closed under derivations is called *differentiably finite*, in short *D-finite*.

$$\dim \operatorname{span} \sum_{i, j \geq 0} \mathbb{K}(x, y) D_x^i D_y^j f < \infty.$$

The 1-dimensional case is called *hyperexponential*.

Vast closure properties, under: derivation, addition, product, and ...

# Lipshitz's Closure under Diagonals

$$\text{Series } f = \sum_{n,m \in \mathbb{N}} c_{n,m} x^n y^m \in \mathbb{Q}[[x, y]] \rightarrow \text{diagonal } \Delta f = \sum_{n \in \mathbb{N}} c_{n,n} x^n \in \mathbb{Q}[[x]].$$

Theorem (Lipshitz, 1989)

The diagonal of a D-finite series is D-finite.

To prepare for the proof, introduce

$$g = \frac{1}{s} f \left( s, \frac{x}{s} \right) \in \bigcup_{m \in \mathbb{Z}} \left\{ \sum_{(p,q) \in \mathbb{Z}^2, p+q \geq m} \phi_{p,q} s^p x^q \right\},$$

so that  $\Delta f = \text{res}_s g$ , where

$$\text{res}_s \sum_{(p,q) \in \mathbb{Z}^2, p+q \geq m} \phi_{p,q} s^p x^q = \sum_{q=m+1}^{\infty} \phi_{-1,q} x^q.$$

# Differential Operators

Functions:

$$u : (x, y) \mapsto u(x, y).$$

Derivation operators:

$$D_x u : (x, y) \mapsto \frac{\partial u}{\partial x}(x, y) \quad \text{and} \quad D_y u : (x, y) \mapsto \frac{\partial u}{\partial y}(x, y).$$

Multiplication operators:

$$xu : (x, y) \mapsto xu(x, y) \quad \text{and} \quad yu : (x, y) \mapsto yu(x, y).$$

Operator algebras, e.g.,  $\mathbb{K}(x)[y]\langle D_x, D_y \rangle$ , in which:

$$D_x x = x D_x + 1 \quad \text{and} \quad D_y y = y D_y + 1.$$

Remark: if  $p \in \mathbb{K}[x, y]$ ,

$$D_x^a D_y^b p = p D_x^a D_y^b + (\text{terms of total order less than } a + b).$$

# Lipshitz's Elementary Proof

By D-finiteness,  $g$  annihilated by:

$$\begin{aligned} A(s, x, D_s) &= \lambda(s, x) D_s^m + (\text{lower order terms in } D_s \text{ only}), \\ B(s, x, D_x) &= \lambda(s, x) D_x^{m'} + (\text{lower order terms in } D_x \text{ only}), \\ &\text{for } \lambda \in \mathbb{K}[s, x] \quad \text{and} \quad \text{all degrees} \leq h. \end{aligned}$$

By induction using  $\lambda D_s^a D_x^b g = \sum_{0 \leq i+j \leq a+b-1} (\text{deg} \leq h) D_s^i D_x^j g$ ,

$$\lambda^{a+b} x^c D_s^a D_x^b g = \sum_{0 \leq i < m, 0 \leq j < m'} (\text{deg} \leq (a+b)h + c) D_s^i D_x^j g.$$

Dimension analysis: for  $a + b + c \leq N$ ,

$$\dim. \binom{N+3}{3} \simeq \Theta(N^3) \longrightarrow \dim. \binom{(h+1)N+2}{2} \simeq \Theta(N^2).$$

$g$  killed by  $L(x, D_x, D_s) = P(x, D_x) D_s^v + (\text{higher-order deriv. w.r.t. } s)$ ,  
so that  $P (\neq 0!)$  kills  $\text{res}_s g = \Delta f$ .

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# History for Zeilberger's Fast Algorithm(s)

- (Gosper, 1978) "Decision procedure for indefinite hypergeometric summation"
- (Almkvist and Zeilberger, 1990) "The method of differentiating under the integral sign"
- (Zeilberger, 1991) "The method of creative telescoping"
- (Chyzak, 2000) "An extension of Zeilberger's fast algorithm to general holonomic functions"
- (Koutschan, 2010) "A fast approach to creative telescoping"



# Gosper's Algorithm (1/2)

## Specifications

INPUT: a hypergeometric term  $f_k$ .

OUTPUT: a rational function  $R(k)$  such that  $Rf$  is an **indefinite sum** w.r.t.  $k$  of  $f$ , or  $\nexists$  ("a proof that no such  $R$  exists").

## Simplified variant (basing on Abramov's algorithm)

- 1 Rewrite the equation

$$f_k = R(k+1)f_{k+1} - R(k)f_k$$

in a rational function  $R$  into

$$1 = R(k+1)\rho(k) - R(k).$$

- 2 Solve by **Abramov's decision algorithm**: finding no  $R$  is a **proof** that none exists.

# Gosper's Algorithm (2/2)

## Original variant

- ① Write the ratio  $\rho(k) = f_{k+1}/f_k$  in the (unique) form

$$\rho(k) = \frac{p(k+1)}{p(k)} \frac{q(k)}{r(k+1)}$$

so that  $\gcd(p, q) = \gcd(p, r) = \gcd(q(k), r(k+h)) = 1$  for all integer  $h > 0$ .

- ② The change of variables  $R = \frac{rS}{p}$  yields an equation in a polynomial  $S$ :

$$p(k) = q(k) S(k+1) - r(k) S(k).$$

- ③ Solve (using **explicit bounds**). If an  $S$  is found, return  $R = rS/p$ , else, this is a proof that no  $R$  exists.

Both variants reduce to linear-system solving.

# Example of Use of Gosper's Algorithm

$$\sum_{k=0}^{n-1} \boxed{\frac{(-1)^k (4k+1) \binom{2k+1}{k}}{4^k (4k^2-1)}} = -\frac{2(n+1)}{4n+1} \boxed{\frac{(-1)^n (4n+1) \binom{2n+1}{n}}{4^n (4n^2-1)}} - 2.$$

The hypergeometric summand is given by:

$$\rho = -\frac{1}{2} \frac{(4k+5)(2k-1)}{(4k+1)(k+2)},$$

$$p(k) = 4k+1, \quad q(k) = \frac{1}{2} - k, \quad r(k) = k+2.$$

# Parametrised Gosper Algorithm

## Specifications

INPUT: a hypergeometric  $f_k$  and rational functions  $s_0(k), \dots, s_m(k)$ .

OUTPUT: a rational function  $R(k)$  and constants  $\eta_0, \dots, \eta_m$  such that  $Rf$  is an **indefinite sum** w.r.t.  $k$  of

$$(\eta_0 s_0(k) + \dots + \eta_m s_m(k)) f_k,$$

or  $\nexists$  (“a proof that no such  $R$  exists for any family  $\{\eta_i\}$ ”).

## Sketch of algorithm

$\eta_i$  involved only **linearly** and **in inhomogeneous side** of equation  $\rightarrow$  simply linear solving for  $S(k)$  with additional unknowns  $\eta_0, \dots, \eta_m$ .

# Zeilberger's ("Fast") Algorithm

## Specifications

INPUT: hypergeometric term  $f_{n,k}$ .

OUTPUT: rational functions  $\eta_0(n), \dots, \eta_r(n), \phi(n, k)$  for minimal  $r \in \mathbb{N}$  such that

$$\eta_r(n)f_{n+r,k} + \dots + \eta_0(n)f_{n,k} = \phi(n, k+1)f_{n,k+1} - \phi(n, k)f_{n,k}.$$

*Termination not guaranteed in general, but it is in "holonomic" case.  
Explicit criterion due to Abramov.*

## Sketch

For increasing values  $r = 0, 1, \dots$ :

- Compute the rational functions  $s_i(n, k) = f_{n+i,k}/f_{n,k}$ .
- Appeal to the parametrised Gosper algorithm.
- If  $(\phi, \{\eta_i\})$  is found, return it (else loop).

A  $q$ -analogue exists for  $q$ -hypergeometric terms.

# Example of Use of Zeilberger's Algorithm (1/5)

Jacobi's **orthogonal polynomials**  $P_n^{(\alpha, \beta)}(x)$  can be expressed in terms of Gauss's **hypergeometric function**  ${}_2F_1\left(\begin{smallmatrix} a, b \\ c \end{smallmatrix} \middle| z\right)$ :

$$P_n^{(\alpha, \beta)}(x) = \frac{(\alpha + 1)_n}{n!} {}_2F_1\left(\begin{smallmatrix} -n, n + \alpha + \beta + 1 \\ \alpha + 1 \end{smallmatrix} \middle| \frac{1-x}{2}\right),$$

for

$${}_2F_1\left(\begin{smallmatrix} a, b \\ c \end{smallmatrix} \middle| z\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k.$$

Zeilberger's algorithm provides **recurrences** in  $n$ ,  $\alpha$ , or  $\beta$ , like:

$$\begin{aligned} 0 &= 2(n+2)(n+\alpha+\beta+2)(2n+\alpha+\beta+2)P_{n+2}^{(\alpha, \beta)}(x) \\ &\quad - \left((2n+\alpha+\beta+2)_3x + (2n+\alpha+\beta+3)(\alpha-\beta)(\alpha+\beta)\right)P_{n+1}^{(\alpha, \beta)}(x) \\ &\quad + 2(n+\alpha+1)(n+\beta+1)(2n+\alpha+\beta+4)P_n^{(\alpha, \beta)}(x). \end{aligned}$$

Slight extension: mixed recurrences, contiguity relations.

# Example of Use of Zeilberger's Algorithm (2/5)

The summand is

$$f_k = \frac{(\alpha + 1)_n (-n)_k (n + \alpha + \beta + 1)_k (1 - x)^k}{n! (\alpha + 1)_k k! 2^k}.$$

The ratio  $f_{k+1}/f_k$  is

$$\rho = \frac{1}{2} \frac{(n - k) (n + k + \alpha + \beta + 1) (x - 1)}{(k + 1) (k + \alpha + 1)}.$$

# Example of Use of Zeilberger's Algorithm (3/5)

The parametrised recurrence to solve for rational solutions is

$$\begin{aligned}
 & (n+2-k)(n+1-k)(n-k)(n+\alpha+\beta+2) \\
 & \quad (n+\alpha+\beta+1)(n+k+\alpha+\beta+1)(x-1)R(k+1) \\
 & - 2(k+1)(k+\alpha+1)(n+\alpha+\beta+1) \\
 & \quad (n+2-k)(n+1-k)(n+\alpha+\beta+2)R(k) \\
 & \quad = 2\eta_0(k+1)(k+\alpha+1)(n+\alpha+\beta+1) \\
 & \quad \quad (n+1-k)(n+2-k)(n+\alpha+\beta+2) \\
 & + 2\eta_1(n+k+\alpha+\beta+1)(n+\alpha+1)(k+1) \\
 & \quad \quad (k+\alpha+1)(n+2-k)(n+\alpha+\beta+2) \\
 & + 2\eta_2(n+\alpha+2)(n+\alpha+1)(n+\alpha+\beta+2+k) \\
 & \quad \quad (n+k+\alpha+\beta+1)(k+1)(k+\alpha+1).
 \end{aligned}$$



# Example of Use of Zeilberger's Algorithm (4/5)

A multiple of the denominators of all solutions is

$$(n + 2 - k)(n + 1 - k).$$

The recurrence rewrites

$$\begin{aligned} & (-(n + \alpha + \beta + 2)(n + \alpha + \beta + 1)(x - 1)k^2 + \dots) S(k + 1) \\ & + (-2(n + \alpha + \beta + 2)(n + \alpha + \beta + 1)k^2 + \dots) S(k) \\ & = (c_0k^4 + \dots) \eta_0 + (c_1k^4 + \dots) \eta_1 + (c_2k^4 + \dots) \eta_2. \end{aligned}$$

Therefore, any solution  $S(k)$  has degree at most 2 in  $k$ .

# Example of Use of Zeilberger's Algorithm (5/5)

Solving yields  $P(n, k) = Q(n, k + 1) - Q(n, k)$  where

$$\begin{aligned}
 P(n, k) = & 2(n + \beta + 1)(n + \alpha + 1)(2n + \alpha + \beta + 4)u(n, k) \\
 & - (2n + \alpha + \beta + 3)(\alpha^2 + \alpha^2x + 4\alpha nx + 2\alpha\beta x + 6\alpha x + 12nx \\
 & \quad + \beta^2x + 4nx\beta + 4n^2x + 6\beta x + 8x - \beta^2)u(n + 1, k) \\
 & \quad + 2(n + 2)(n + \alpha + \beta + 2)(2n + \alpha + \beta + 2)u(n + 2, k),
 \end{aligned}$$

$$Q(n, k) = \frac{N(n, k)}{D(n, k)}u(n, k) \quad \text{for}$$

$$\begin{aligned}
 N(n, k) = & -2(2n + \alpha + \beta + 4)(2n + \alpha + \beta + 3)(2n + \alpha + \beta + 2) \\
 & \quad \times (n + \alpha + 1)(\alpha + k)k
 \end{aligned}$$

$$D(n, k) = (n + \alpha + \beta + 1)(n + 1 - k)(n + 2 - k).$$

The announced recurrence is obtained after summation.

# Almkvist and Zeilberger's Algorithm

Strict analogue for bivariate hyperexponential functions:

$$\frac{D_x f(x, y)}{f(x, y)} = R(x, y) \in \mathbb{K}(x, y) \quad \text{and} \quad \frac{D_y f(x, y)}{f(x, y)} = S(x, y) \in \mathbb{K}(x, y).$$

Example:

$$F(x) = \int_{-\infty}^{+\infty} f(x, y) dy \quad \text{for} \quad f(x, y) = \exp\left(-\frac{x^2}{y^2} - y^2\right).$$

$$\left(xD_x^3 + 3D_x^2 - 4xD_x - 12\right)f = D_y \left(\frac{2(3y^2 - 2x^2)}{y^3}f\right),$$

$$\left(xD_x^3 + 3D_x^2 - 4xD_x - 12\right)F = 0,$$

$$F(x) = \sqrt{\pi} \exp(-2x).$$

Termination always guaranteed by “holonomy”!

# Functions versus Equations versus Vector Space

**Special function** or **combinatorial sequence**  $f$ :

$$f(n, z) = J_n(z) \quad (\text{Bessel function}).$$

[Also: algebraic, trigonometric, elementary, transcendental, hypergeometric functions; binomial coefficients, harmonic numbers, hypergeometric sequences; orthogonal polynomials;  $q$ -analogues.]

**Linear** functional system (+ initial conditions):

$$\begin{aligned} z^2 J_n''(z) + z J_n'(z) + (z^2 - n^2) J_n(z) &= 0, & z J_n'(z) + z J_{n+1}(z) - n J_n(z) &= 0, \\ z J_{n+2}(z) - 2(n+1) J_{n+1}(z) + z J_n(z) &= 0. \end{aligned}$$

**Vector space** closed under  $D_z$  and  $S_n$ , here **finite-dimensional**:

$$V = \mathbb{C}(n, z) J_n(z) \oplus \mathbb{C}(n, z) J_{n+1}(z) = \mathbb{C}(n, z) J_n(z) \oplus \mathbb{C}(n, z) J_n'(z).$$

# $\partial$ -Finite Terms

$$\partial_i = D_i \text{ or } S_i$$

$t$  is  $\partial$ -finite w.r.t. the operator algebra

$$\mathbb{K}(x_1, \dots, x_m) \langle \partial_1, \dots, \partial_m \rangle$$



the  $\partial_1^{\alpha_1} \dots \partial_m^{\alpha_m} t$ 's span a  
finite-dimensional  $\mathbb{K}(x_1, \dots, x_m)$ -vector space:

$$\dim_{\mathbb{K}(x_1, \dots, x_m)} \left( \mathbb{K}(x_1, \dots, x_m) \langle \partial_1, \dots, \partial_m \rangle t \right) < +\infty$$

$t$  is described by  
higher-order linear functional equations.

Algorithmic closures under  $+$ ,  $\times$ , the  $\partial_i$ 's, integration, summation  
 $\implies$  simplification and zero test of  $\partial$ -finite expressions.

# Extended Gosper Decision Algorithm

ALGORITHM:  $T = \mathbf{Indefinite}(t, (b_i)_{i=1,\dots,d}, A)$ .

INPUT:  $\left\{ \begin{array}{l} \text{a } \partial\text{-finite term } t \in V = \bigoplus_{i=1}^d \mathbb{K}(x) b_i, \\ \text{an operator algebra } A = \mathbb{K}(x)\langle \partial \rangle, \\ \text{the action of } A \text{ on } V. \end{array} \right.$

OUTPUT:  $T \in V$  such that  $\partial T = t$ ; or  $\nexists$ .

- ① let  $T = \phi_1 b_1 + \dots + \phi_d b_d$  for **undetermined coefficients**  $\phi_i \in \mathbb{K}(x)$ ;
- ② extract the coefficients of  $\partial T - t$  in the  $b_i$ 's to obtain a **first-order functional system** in the  $\phi_i$ 's;
- ③ solve for **rational solutions** (uncoupling + **Abramov's decision algorithms**);
- ④ if solvable return  $T$ ; otherwise return  $\nexists$ .

# Example of Indefinite $\partial$ -Finite Summation

$$\text{Input: } \begin{cases} \sum_{j=1}^k t_j \text{ for } t_k = \binom{k}{p} H_k, & H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}, \\ (\cdots) t_{k+2} + (\cdots) t_{k+1} + (\cdots) t_k = 0. \end{cases}$$

$$\text{Algorithmic reformulation: } A = \mathbf{Q}(p, k) \langle S_k \rangle, \quad T = \phi_0 t_k + \phi_1 t_{k+1}.$$

$$\left\{ \begin{array}{l} (k+2-p)\phi_0(k+1) + (2k+3)\phi_1(k+1) - (k+2-p)\phi_1(k) = 0, \\ (k+2-p)(k+1-p)\phi_0(k) + (k+1)^2\phi_1(k+1) = \\ \quad - (k+2-p)(k+1-p). \end{array} \right.$$

$$\longrightarrow (\cdots)\phi_1(k+2) + (\cdots)\phi_1(k+1) + (\cdots)\phi_1(k) = (\cdots).$$

$$\text{Output: } \begin{cases} \phi_0(k) = \frac{(k-p)(k+p+2)}{(p+1)^2}, \quad \phi_1(k) = -\frac{(k-p)(k-p+1)}{(p+1)^2}, \\ T = \sum t_k \delta k = \binom{k}{p} \frac{k-p}{(p+1)^2} ((p+1)H_k - 1). \end{cases}$$

# Example of Indefinite $\partial$ -Finite Integration

$$\text{Input: } \begin{cases} \int \text{Ci}(z) dz \text{ for } t(z) = \text{Ci}(z) = \int_0^z \frac{\cos(t) - 1}{t} dt, \\ (\dots) t'''(z) + (\dots) t''(z) + (\dots) t'(z) = 0. \end{cases}$$

$$\text{Algorithmic reformulation: } A = \mathbb{Q}(z)\langle D_z \rangle, \quad T = \phi_0 t + \phi_1 t' + \phi_2 t''.$$

$$\left\{ \begin{array}{l} z\phi_1 + z\phi_2' - 2\phi_2 = 0, \\ z^2\phi_0 + z^2\phi_1' - z\phi_1 - z\phi_2' + (2 - z^2)\phi_2 = 0, \\ \phi_0' = 0. \end{array} \right.$$

$$\longrightarrow z^3\phi_2''' - 2z^2\phi_2'' + (z^3 + 4z)\phi_2' - 4\phi_2 = 0.$$

$$\text{Output: } \begin{cases} \phi_0(z) = z, & \phi_1(z) = 1, & \phi_2(z) = z, \\ T = \int \text{Ci}(z) dz = z \text{Ci}(z) - \sin(z). \end{cases}$$



# Extended Zeilberger (a.k.a. Chyzak's) Algorithm

ALGORITHM:  $(P, Q) = \mathbf{Definite}(u, (b_i)_{i=1, \dots, d}, A)$ .

INPUT:  $\begin{cases} \text{a } \partial\text{-finite term } u \text{ w.r.t. } A = \mathbb{K}(x, y)\langle \partial_x, \partial_y \rangle, \\ \text{a finite } \mathbb{K}(x, y)\text{-basis } (b_i)_{i=1, \dots, d} \text{ of } Au. \end{cases}$

OUTPUT:  $\begin{cases} P \in \mathbb{K}(x, y)\langle \partial_x \rangle, \\ Q \in A \text{ such that } Pu = \partial_y Qu, \text{ resp. } Pu = (\partial_y - 1)Qu. \end{cases}$

For increasing values of  $r$ :

- ① let  $P = \sum_{i=0}^r \eta_i \partial_x^i$  and  $t = Pu$  for **undetermined coefficients**  $\eta_i(x)$ ;
- ② let  $(T, (\eta_i)) = \mathbf{ParamIndefinite}(t, (b_i), A, \{\eta_0, \dots, \eta_r\})$ .
- ③ if  $T \neq \ddagger$  return  $(P, Q)$  for  $Qu = T$ .

# A $\partial$ -Finite Integral: $\frac{2}{\pi} \int_0^1 \frac{\cos(z t)}{\sqrt{1-t^2}} dt = J_0(z)$

$$A = \mathbb{Q}(z, t) \langle D_z, D_t \rangle, \quad f = \frac{\cos(zt)}{\sqrt{1-t^2}} \leftrightarrow \text{basis} = \left( f, D_z f = -t \frac{\sin(zt)}{\sqrt{1-t^2}} \right)$$

$$\left\{ \begin{array}{l} D_z^2 + t^2, \\ t(1-t^2)D_t - (1-t^2)zD_z - t^2. \end{array} \right.$$

$$P = zD_z^2 + D_z + z, \quad Q = \frac{1-t^2}{t}D_z, \quad P \int_0^1 \frac{\cos(zt)}{\sqrt{1-t^2}} + \left[ Q \frac{\cos(zt)}{\sqrt{1-t^2}} \right]_0^1 = 0:$$

$$(zD_z^2 + D_z + z) \frac{2}{\pi} \int_0^1 \frac{\cos(zt)}{\sqrt{1-t^2}} dt = 0 \quad + \quad 2 \text{ initial conditions.}$$

# A Neumann Addition Theorem $J_0(z)^2 + 2 \sum_{k=1}^{\infty} J_k(z)^2 = 1$

①  $A = \mathbb{Q}(z, k) \langle D_z, S_k \rangle, \quad J_k(z) \leftrightarrow \text{basis} = (J_k(z), D_z J_k(z)):$

$$\left\{ \begin{array}{l} z^2 D_z^2 + z D_z + z^2 - k^2, \quad z S_k + z D_z - k. \end{array} \right.$$

② By closure,  $J_k(z)^2 \leftrightarrow \text{basis} = (J_k(z)^2, S_k J_k(z)^2, D_z J_k(z)^2):$

$$\left\{ \begin{array}{l} z D_z^2 + (-2k + 1) D_z - 2z S_k + 2z, \\ z D_z S_k + z D_z + (2k + 2) S_k - 2k, \\ z^2 S_k^2 - 4(k + 1)^2 S_k - 2z(k + 1) D_z + 4k(k + 1) - z^2. \end{array} \right.$$

③  $P = D_z, Q = \frac{k}{z} + \frac{1}{2} D_z, \quad P \sum_{k=0}^{\infty} J_k(z)^2 + \left[ Q J_k(z)^2 \right]_{k=0}^{\infty} = 0:$

$$D_z \left( 2 \sum_{k=1}^{\infty} J_k(z)^2 + J_0(z)^2 - 1 \right) = 0 \quad + \text{initial condition at 0.}$$

# Koutschan's Heuristics

Chyzak's fast algorithm can be **slow**:

- ① **uncouple** the system for the  $\phi$ 's (non-commutative Gauss elim.)
- ② **algorithmically** bound denominators (Abramov's bound)
- ③ **algorithmically** bound numerator degrees (indicial equation)

Handles multiple sums/integrals iteratively.

**Always** finds a solution (at least theoretically).

Koutschan's **fast** heuristics

- ① **don't uncouple!**
- ② **heuristically** set denominator exponents (in accordance to  $Pu$ )
- ③ **heuristically** set numerator degrees (prop. to the denom. degs)

Simultaneous multiple sums/integrals possible, at times faster.

**May fail** to find a solution.

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- 1 Introduction: Early Days and Examples
- 2 Fasenmyer's (a.k.a. "*k*-Free") Ansatz
- 3 Lipshitz's Diagonals
- 4 Zeilberger's Ansatz
- 5 Gröbner Bases: Closures and Elimination**
- 6 Beyond Holonomy
- 7 Termination, Bounds, Complexity
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# History of Gröbner Bases for Creative Telescoping

- (Galligo, 1985) “Some algorithmic questions on ideals of differential operators”
- (Takayama, 1989) “Gröbner basis and the problem of contiguous relations”
- (Kandri-Rody and Weispfenning, 1990) “Noncommutative Gröbner bases in algebras of solvable type”
- (Takayama, 1990) “Gröbner basis, integration and transcendental functions”, “An algorithm of constructing the integral of a module: an infinite dimensional analog of Gröbner basis”
- (Kredel, 1993) “Solvable polynomial rings”
- (Chyzak and Salvy, 1998) “Non-commutative elimination in Ore algebras proves multivariate identities”
- (Oaku and Takayama, 1998) “Algorithms for  $D$ -modules: restriction, tensor product, localization, and local cohomology groups”
- (Saito, Sturmfels, and Takayama, 2000) “Gröbner deformations of hypergeometric differential equations”

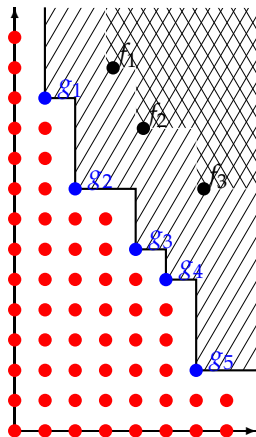
# Crash Course on Gröbner Bases

Generalize Euclidean division and the Euclidean (gcd) algorithm to ideals in polynomial rings  $\mathbb{K}[x_1, \dots, x_m]$ .

- ① **Monomial order(ing)**: total order  $<$  on the monomials, compatible with the product, with minimal element 1
- ② **Gröbner basis** of a ideal  $I$  w.r.t.  $<$ : generators **matching the stairs** of  $I$

$$\text{GB}(f_1, f_2, f_3) = (g_1, \dots, g_5)$$

- ③ **Quotient**: vector basis **below the stairs**; a ring in the commutative case
- ④ **Reduction** of  $P$  modulo  $I$ : unique remainder written on this basis
- ⑤ **Elimination**: procedure for computing  $I \cap \mathbb{K}[x_1, \dots, x_s]$ ,  $s < m$



# Annihilating Ideals

For an operator algebra  $A = \mathbb{K}\langle x_1, \dots, n_1, \dots \rangle \langle D_1, \dots, S_1, \dots \rangle$ :

- **left ideal:**  $\text{ann} f = \{P \in A \mid Pf = 0\} \leftrightarrow$  “all equations on  $f$ ”
- **left module:**  $A / \text{ann} f \simeq Af \leftrightarrow$  “space of shifted derivatives of  $f$ ”

Calculations modulo  $\text{ann} f$ .

Example of Bessel functions,  $J_n(z)$ :

$$\text{ann} J = A (z^2 D_z^2 + z D_z + (z^2 - n^2), z D_z + z S_n - n, z S_n^2 - 2(n+1)S_n + z).$$

$$V = AJ \simeq A / \text{ann} J$$

$$(J_n(z), J_{n+1}(z), J'_n(z)) \leftrightarrow (1 + \text{ann} J, S_n + \text{ann} J, D_z + \text{ann} J)$$

Also for  $\mathbb{K}\langle x_1, \dots, n_1, \dots \rangle \langle D_1, \dots, S_1, \dots \rangle$  and intermediate cases.



# Skew Gröbner Bases

**Buchberger's algorithm** extends to left ideals.

Gröbner bases are still **finite** (noetherianity).

(Galligo, 1985; Takayama, 1989; Kandry-Rody and Weispfenning, 2000; Kredel, 1993)

Direct extensions to **Gröbner bases for left modules**.

Technical restriction on commutations:

$$\partial_i x_j = (a_{i,j} x_j + b_{i,j}) \partial_i + c_{i,j}(x_1, \dots, x_m) \quad \text{with } a_{i,j} \neq 0.$$

Restriction on orders allowed:  $c_{i,j}(x_1, \dots, x_m) < x_j \partial_i$ .

Both OK for  $\partial = D$  and  $\partial = S$ , as  $Dx = xD + 1$  and  $Sx = xS + S$ .

Closures of  $\partial$ -Finite Functions under  $+$ ,  $\times$ ,  $\partial$ 

Finite dimensions make algorithms terminate.

Example: the product  $h = T_n(x) \times \frac{e^{-ux}}{\sqrt{1-x^2}}$ .

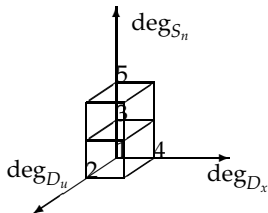
$A = \mathbb{Q}(x, n, u) \langle D_x, S_n, D_u \rangle$ , Total degree in  $D_x > S_n > D_u$ .

A Gröbner basis for  $\text{ann} f$ ,  $f = T_n(x)$ , provides **normal forms** in  $Af$ :

$$D_u, \quad (x^2 - 1) D_x - nS_n + nx, \quad S_n^2 - 2xS_n + 1.$$

A Gröbner basis for  $\text{ann} f$ ,  $g = \frac{e^{-ux}}{\sqrt{1-x^2}}$ , provides **normal forms** in  $Ag$ :

$$D_u + x, \quad S_n - 1, \quad (x^2 - 1) D_x + ux^2 + x - u.$$



$$1 < \boxed{D_u} < S_n < \boxed{D_x} < D_u^2 < D_u S_n < D_u D_x \\ < \boxed{S_n^2} < S_n D_x < D_x^2 < \dots$$

**Normal forms** in  $Af \otimes_{\mathbb{Q}(x,n,u)} Ag$ :

$$\begin{aligned} 1(f \otimes g) &= f \otimes g, \\ D_u(f \otimes g) &= D_u f \otimes g + f \otimes D_u g = -x(f \otimes g), \\ S_n(f \otimes g) &= S_n f \otimes S_n g = S_n f \otimes g, \\ D_x(f \otimes g) &= D_x f \otimes g + f \otimes D_x g \\ &= (x^2 - 1)^{-1} \left( n S_n f \otimes g - (ux^2 + (n+1)x - u)(f \otimes g) \right), \\ S_n^2(f \otimes g) &= S_n^2 f \otimes S_n^2 g = 2x S_n f \otimes g - f \otimes g. \end{aligned}$$

Gröbner basis **computed** for  $f \otimes g$ :

$$\boxed{D_u} + x, \quad (x^2 - 1)\boxed{D_x} - nS_n + ux^2 + (n+1)x - u, \quad \boxed{S_n^2} - 2xS_n + 1.$$

An Integral: 
$$\int_{-1}^{+1} T_n(x) \frac{e^{-ux}}{\sqrt{1-x^2}} dx = \pi (-1)^n I_n(u)$$

Now, in  $\mathbb{Q}[n, u, x] \langle D_x, S_n, D_u \rangle$ , **heuristically eliminating  $x$**  by the order

$$x \gg D_x > S_n > D_u > n > u$$

yields another Gröbner basis:

$$\underbrace{S_n^2 + 2S_n D_u + 1, \quad D_x(D_u^2 - 1) - nS_n + uD_u^2 - (n-1)D_u - u, \quad x + D_u.}_{\text{non-trivial by chance!}}$$

The integral  $H = \int_{-1}^{+1} h dx$  satisfies

$$(S_n^2 + 2S_n D_u + 1)H = 0,$$

$$(nS_n - uD_u^2 + (n-1)D_u + u)H = \left[ (D_u^2 - 1)h \right]_{x=-1}^{x=+1} = 0.$$

# Takayama's Approach

(Chyzak and Salvy's variant)

Optimisation dedicated to integrals/sums over natural boundaries

Compute the  $P$ 's such that, for some  $Q$ ,

$$P(n, u, D_x, S_n, D_u) - D_x Q(x, n, u, D_x, S_n, D_u) \in \text{ann } h$$

without computing the  $Q$ 's.

Idea: View ideal  $\text{ann } h$  as a module

$$\sum_{i=1}^{\ell} \mathbb{Q}[n, u, x] \langle D_x, S_n, D_u \rangle g_i = \lim_{N \rightarrow \infty} \sum_{i=1}^{\ell} \sum_{j=0}^N \mathbb{Q}[n, u] \langle D_x, S_n, D_u \rangle x^j g_i.$$

Adapts to mixed simultaneous integrations/summations.

# Example: Same Integral

- For  $N = 3$ :

$$\begin{aligned}
 & 1D_u + x, \quad xD_u + x^2, \quad x^2D_u + x^3, \\
 & 1(-D_x - nS_n - u) + x(n+1) + x^2(u + D_x), \quad x(-D_x - nS_n - u) + x^2(n+1) + x^3(u + D_x), \\
 & 1(S_n^2 + 1) - x2S_n, \quad x(S_n^2 + 1) - x^22S_n, \quad x^2(S_n^2 + 1) - x^32S_n.
 \end{aligned}$$

- Moding out  $D_x$  on the left, by  $p(x) D_x = D_x p(x) - p'(x)$ , yields

$$\begin{aligned}
 & D_u 1 + x, \quad D_u x + x^2, \quad D_u x^2 + x^3, \\
 & (-nS_n - u)1 + (n-1)x + ux^2, \quad 1 + (-nS_n - u)x + (n-2)x^2 + ux^3, \\
 & (S_n^2 + 1)1 - 2S_n x, \quad (S_n^2 + 1)x - 2S_n x^2, \quad (S_n^2 + 1)x^2 - 2S_n x^3.
 \end{aligned}$$

- Gröbner basis for  $x \gg S_n > D_u$ :

$$\begin{aligned}
 & \boxed{(-u - nS_n - (n-1)D_u + uD_u^2)1}, \quad \boxed{(S_n^2 + 1 + 2D_u S_n)1}, \\
 & D_u 1 + x, \quad (-u - nS_n - (n-1)D_u)1 + ux^2, \\
 & (unS_n D_u + n(n-2)S_n - u + (u^2 + n^2 - 3n + 2)D_u + nu)1 + u^2 x^3.
 \end{aligned}$$

# Dimension of Ideals

$M_k(I) := \langle m : m \text{ is below the stairs and of total degree } \leq s \rangle$

## Theorem (Hilbert)

For any  $I$ , there is an integer  $\delta(I)$  such that  $\#M_k(I) = \Theta(s^{\delta(I)})$ .

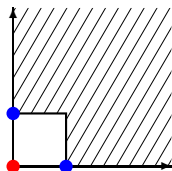
- $\delta(I)$  is the (Hilbert) dimension of  $I$ .
- Finite measure of infinite-dimensional vector-spaces.
- Also measures capacity for elimination.
- Can be obtained by a Gröbner-basis calculation.
- $I \subset J \Rightarrow \delta(I) \geq \delta(J)$

Caution!

dimension in  $\mathbb{K}(\underline{x})\langle \underline{\partial} \rangle \neq$  dimension in  $\mathbb{K}[\underline{x}]\langle \underline{\partial} \rangle$

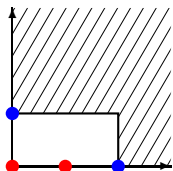
Examples of Dimensions in  $\mathbb{K}\langle x \rangle \langle \partial \rangle$ 

Binomial coeffs  $\binom{n}{k}$  w.r.t.  $S_n, S_k$ ;  
Hypergeometric sequences:



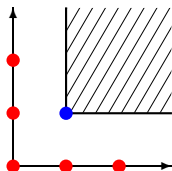
$$\delta(I) = 0, \dim S/I = 1$$

Bessel  $J_\nu(x)$  w.r.t.  $S_\nu, D_x$ ;  
Orthogonal polys w.r.t.  $S_n, D_x$ :



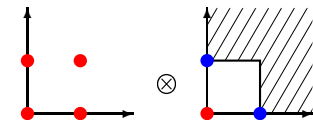
$$\delta(I) = 0, \dim S/I = 2$$

Stirling nbs w.r.t.  $S_n, S_k$ :



$$\delta(I) = 1, \dim S/I = \infty$$

Abel-type w.r.t.  $S_m, S_k, S_r, S_s$ ,  
 $h(m, k) (k+r)^k (m-k+s)^{m-k} \frac{r}{k+r}$ :



$$\delta(I) = 2 \text{ in space of dimension } 4$$



Sample Dimensions in  $\mathbb{K}(\underline{x})\langle\underline{\partial}\rangle$  vs in  $\mathbb{K}[\underline{x}]\langle\underline{\partial}\rangle$ 

Experimentation “shows”:

## Differential case

algebra \ function	1	$\exp(x+y)$	$\Gamma(x+y)$	$\Gamma(x)\Gamma(y)$
$\mathbb{C}(x,y)\langle D_x, D_y \rangle$	0	0	1	2
$\mathbb{C}[x,y]\langle D_x, D_y \rangle$	2	2	3	4

## Recurrence case

algebra \ function	1	$\frac{1}{nk}$	$\frac{1}{1+nk}$	$\binom{n}{k}$	$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$
$\mathbb{C}(n,k)\langle S_n, S_k \rangle$	0	0	0	0	1
$\mathbb{C}[n,k]\langle S_n, S_k \rangle$	2	2	3	2	3

# Elimination for Holonomic Functions

In the **differential setting**:  $D_i x_j = x_j D_i + \delta_{i,j}$ .

Holonomic function

a function of  $x_1, \dots, x_m$  is **holonomic**



the  $x_1^{\alpha_1} \dots x_m^{\alpha_m} D_1^{\beta_1} \dots D_m^{\beta_m} f$  for  $\alpha_1 + \dots + \alpha_m + \beta_1 + \dots + \beta_m \leq N$   
span a  $\mathbb{K}$ -vector space of dimension  $\mathcal{O}(N^m)$ .

Elimination lemma

For a holonomic function  $f$  and **any choice**

$$\{u_1, \dots, u_{m+1}\} \subset \{x_1, \dots, x_m, D_1, \dots, D_m\},$$

there is a **non-zero**  $P(u_1, \dots, u_{m+1})$  such that  $Pf = 0$ .

Corollary 1: A holonomic function is D-finite.

Corollary 2: Creative telescoping succeeds for holonomic functions.

# D-Finite Functions are Holonomic

**Finite-dimensional** confinement of derivatives:

$$f \in V = \bigoplus_{j=1}^d \mathbb{K}(x_1, \dots, x_m) b_j, \quad D_s b_i = \sum_{j=1}^d \frac{a_{s,i,j}}{w} b_j, \quad a_{s,i,j}, w \in \mathbb{K}[x_1, \dots, x_m]$$

**Action of  $x$ 's and  $D$ 's on degrees:** for  $\mu = 1 + \max \{ \deg w, \deg a_{s,i,j} \}$ ,

$$D_s \left( \frac{p}{w^N} b_i \right) = \frac{D_s p}{w^N} b_i - N \frac{p D_s w}{w^{N+1}} b_i + \sum_{j=1}^d \frac{a_{s,i,j} p}{w^{N+1}} b_j, \quad x_s \left( \frac{p}{w^N} b_i \right) = \frac{x_s w p}{w^{N+1}}.$$

$$\frac{(\deg \leq \alpha)}{w^N} \rightarrow \frac{(\deg \leq \alpha + \mu)}{w^{N+1}}$$

**Dimension analysis:** for  $\alpha_1 + \dots + \alpha_m + \beta_1 + \dots + \beta_m \leq N$ ,

$$x_1^{\alpha_1} \dots x_m^{\alpha_m} D_1^{\beta_1} \dots D_m^{\beta_m} f = \frac{(\deg \leq \mu N)}{w^N} \rightarrow \dim. \binom{\mu N + m}{m} \simeq \Theta(N^m).$$

# Other Techniques by Homogenisation

For  $F(x) = \int_a^b f(x, y) dy$ . Common themes:

- reduce by weights  $-1$  for  $y$ ,  $+1$  for  $\partial_y$ ;
- “homogenise” to recover termination.

## Saito, Sturmfels, Takayama

- Homogenise the algebra by  $\partial_y y = y \partial_y + h^2$ .
- Compute Gröbner basis for homogenised generators of  $\text{ann} f$ , then dehomogenise the output.

## Oaku’s “homogenisation” for the V-filtration

- Add a variable  $v$  to the algebra.
- Mark monomials in generators by  $v$  raised to the V-degree.
- Compute Gröbner basis that eliminates  $v$ , then set  $v = 1$ .

# Weyl Closure

A problem:  $f = p^{-1}$  for  $p = y^2 - y + x$  (Chyzak and Salvy, 1998)

$$g_1 = pD_x + 1 \quad \text{and} \quad g_2 = pD_y + 2y - 1$$

In  $A = \mathbb{Q}(x, y)\langle D_x, D_y \rangle$ :  $\text{ann}_A f = Ag_1 + Ag_2$ .

In  $W = \mathbb{Q}[x, y]\langle D_x, D_y \rangle$ :  $Wg_1 + Wg_2$  has Hilbert dimension 3 (!)  
 $\rightarrow$  elimination of  $y$  from  $\{g_1, g_2\}$  fails

$$g_3 = pD_x D_y + 2D_y \in W \cap \text{ann}_A f$$

$Wg_1 + Wg_2 + Wg_3$  has Hilbert dimension 2

$\rightarrow$  elimination of  $y$  from  $\{g_1, g_2, g_3\}$  finds  $(1 - 4x)D_x^2 - 6D_x - D_y^2$

Problem: GB, Takayama, etc., require  $\text{ann}_W f$  as inputs, not  $\text{ann}_A f$ .

Completion of  $\text{ann}_A f$  to  $\text{ann}_W f$  by Tsai's Weyl-closure algorithms (2000, 2002)

(Differential case only.)

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# Examples of Non- $\partial$ -Finite Terms

- Stirling numbers of the second kind,  $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$  = numbers of partitions of a set of  $n$  objects into  $k$  non-empty subsets:

$$\left\{ \begin{smallmatrix} n+1 \\ k \end{smallmatrix} \right\} = k \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} + \left\{ \begin{smallmatrix} n \\ k-1 \end{smallmatrix} \right\} \quad (k > 0)$$

with initial conditions  $\left\{ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right\} = 1$  and  $\left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} 0 \\ n \end{smallmatrix} \right\} = 0$ , ( $n > 0$ ).

- Hurwitz' zeta function,  $\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^s}$ :

$$\frac{\partial \zeta}{\partial x} = -s \zeta(s+1, x)$$

with initial conditions  $\zeta(0, x) = 1/2 - x$  and  $\zeta(s, 1) = \zeta(s)$   
(Riemann's zeta function).

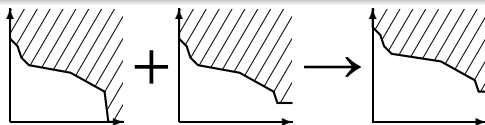
$$\text{ann} \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = A_{n,k} (S_n S_k - (k+1)S_k - 1), \quad \text{ann} \zeta = A_{x,s} (D_x + sS_s).$$

# History for Algorithms in Positive Dimension

- (Majewicz, 1996) “WZ-style certification and Sister Celine’s technique for Abel-type sums”
- (Kauers, 2007) “Summation algorithms for Stirling number identities”
- (Chen, Sun, 2009) “Extended Zeilberger’s algorithm for identities on Bernoulli and Euler polynomials”
- (Chyzak, Kauers, Salvy, 2009) “A non-holonomic systems approach to special function identities”



## Algebraic Closure Properties in Positive Dimension



## Proposition

$\delta$  of sum  $\leq \max$ . of  $\delta$ 's,  $\delta$  of product  $\leq \text{sum}$  of  $\delta$ 's,  $\delta$  of der.  $\leq \delta$ .

Algorithm (for a product  $fg$ ): for a *graded* ordering,

For  $s = 0, 1, 2, \dots$ , **until**  $\delta(I) \leq \text{bound}$ :

for each  $|\underline{\alpha}| \leq s$ , **reduce**  $\partial^{\underline{\alpha}}(fg)$  to a sum  $\sum u_{\alpha, \beta, \gamma}(\underline{x}) \partial^{\beta} f \partial^{\gamma} g$

over  $\beta \in M_s(\text{ann } f)$ ,  $\gamma \in M_s(\text{ann } g)$  ( $M_s$ : "monomials under the stairs")

search for  $\mathbb{Q}(x)$ -linear relations, set  $I$  to the ideal they generate

return  $I$ , a subideal of  $\text{ann } fg$

Example (Stirling numbers of the second kind):  $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ ,  $\delta(I) = 1$ ,

1st-order rec.  $\xrightarrow{s=3}$   $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} m \\ k \end{smallmatrix} \right\}$ ,  $\delta(I) = 2$ , 2nd-order rec.

# A Slightly Different View of Fasenmyer's Technique

## Rediscovering Pascal's Triangle Rule

Reduce all monomials of degree  $\leq s = 2$ :

$$1 \rightarrow \mathbf{1}, \quad S_n \rightarrow \frac{n+1}{n+1-k} \mathbf{1}, \quad S_k \rightarrow \frac{n-k}{k+1} \mathbf{1}$$

$$S_n^2 \rightarrow \frac{(n+2)(n+1)}{(n+2-k)(n+1-k)} \mathbf{1}, \quad S_k^2 \rightarrow \frac{(n-k-1)(n-k)}{(k+2)(k+1)} \mathbf{1}, \quad S_n S_k \rightarrow \frac{n+1}{k+1} \mathbf{1}.$$



Common denominator:  $C_2 = (k+1)(k+2)(n+1-k)(n+2-k)$ .

$C_2, C_2 S_n, C_2 S_k, C_2 S_n^2, C_2 S_k^2, C_2 S_n S_k$  **confined** in

$\langle \mathbf{1}, k \mathbf{1}, k^2 \mathbf{1}, k^3 \mathbf{1}, k^4 \mathbf{1} \rangle$  over  $\mathbb{K}(n)$

$$\rightarrow C_2 (S_n S_k - S_k - 1) \in \text{ann} \binom{n}{k}.$$

$\deg C_s = \mathcal{O}(s) \Rightarrow$  this **had to happen** for some  $s$ .

# More Examples from this Perspective

- Proper hypergeometric terms: essentially the same situation.

- $\frac{1}{n^2 + k^2}$ : confinement in a space of dimension  $\mathcal{O}(s^2)$ ,  
no elimination of  $k$  succeeds.

- $f = \frac{a(x, y_1, \dots, y_r)}{b(x, y_1, \dots, y_r)}$ :  $C_s = b^s$ ,  
confinement in a space of dimension  $\mathcal{O}(s^r)$  over  $\mathbb{K}(x)$ ,  
elimination of  $y_1, \dots, y_r$  has to succeed.  
Base case of proof that D-finite functions are holonomic.

# Adding a Left Part and a Right Part

## Specification of a Creative-Telescoping Algorithm

Input: generators of (a subideal of)  $\text{ann} f$

Output: all  $(A, B)$  such that:

- $A + \partial_y B \in \text{ann} f$ , resp.  $A + (\partial_y - 1)B \in \text{ann} f$ ,
- $A$  is free of  $y$  and  $\partial_y$ .

## Definition (telescoping ideal of $I$ w.r.t. $y$ )

$T_y(I) := (I + \partial_y S_{x,y}) \cap S_x$  where

$$S_{x,y} := \mathbb{K}(x, y)\langle \partial_x, \partial_y \rangle \text{ and } S_x := \mathbb{K}(x)\langle \partial_x \rangle.$$

When  $I = S_{x,y}G_1 + \cdots + S_{x,y}G_\ell$ , this involves

$$(S_{x,y}G_1 + \cdots + S_{x,y}G_\ell) + \partial_y S_{x,y}.$$

# Polynomial Growth and Creative Telescoping when $\delta > 0$

Definition (polynomial growth  $p$ )

An ideal has polynomial growth  $p$  if there exists a sequence of polynomials  $P_s(x, y)$ , s.t. if  $|a| + b \leq s$ ,

$P_s \partial_{x_1}^{a_1} \dots \partial_{x_k}^{a_k} \partial_y^b$  reduces to polys of degree  $\mathcal{O}(s^p)$  in  $y$ .

Theorem (Chyzak, Kauers & Salvy, 2009)

$$\delta(T_y(I)) \leq \max(\delta(I) + p - 1, 0).$$

Proof:

$$P_s \partial_x^a \partial_y^b - \sum_{\beta \in M_s(I)} (\text{deg. in } y \leq \mathcal{O}(s^p)) \partial^\beta \in I$$

$\Rightarrow$  any choice of  $(\delta(I) + p - 1) + 1$  generators among  $\{\partial_{x_1}, \dots, \partial_{x_k}, \partial_y\}$  is algebraically dependent modulo  $I$ .

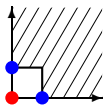
Corollary (sufficient condition for creative telescoping)

$\delta(I) + p - 1 < k \Rightarrow$  identities exist for the sum/int. w.r.t.  $y$ .

# Examples with Polynomial Growth $p = 1$

- Proper hypergeometric (Wilf & Zeilberger, 1992):

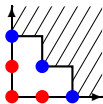
$$Q(n, k) \zeta^k \frac{\prod_{i=1}^u (a_i n + b_i k + c_i)!}{\prod_{i=1}^v (u_i n + v_i k + w_i)!}$$



$Q$  polynomial,  $a_i, b_i, u_i, v_i$  **integers**.

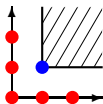
- Differential D-finite (special case of holonomy).
- Stirling:  $\delta = 1 \rightarrow$  for  $\geq 3$  vars, e.g., Frobenius' id.:

$$\sum_{k=0}^n (-1)^{m-k} k! \binom{n-k}{m-k} \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\} = \left\langle \begin{matrix} n \\ m \end{matrix} \right\rangle.$$



- Abel type:  $\delta = 2 \rightarrow$  for  $\geq 4$  vars, e.g., Abel's id.:

$$\sum_{k=0}^n \binom{n}{k} i(k+i)^{k-1} (n-k+j)^{n-k} = (n+i+j)^n.$$



# Algorithm I: Fasenmyer's Style

Polynomial growth + linear algebra  $\rightarrow J := I \cap S_x[\partial_y]$ .

## Algorithm

For increasing values of  $s$ , until  $\delta(J) \leq \text{bound}$ :

- ① Reduce all  $\partial_x^a \partial_y^b$  with  $a + b \leq s$ ;
- ② Normalize to a common denominator;
- ③ Set up a linear system to cancel the positive powers of  $y$ ;
- ④ If a non-zero solution is found, it has the form  $A(x, \partial_x) + \partial_y B(y, \partial_x, \partial_y)$ ; return it.

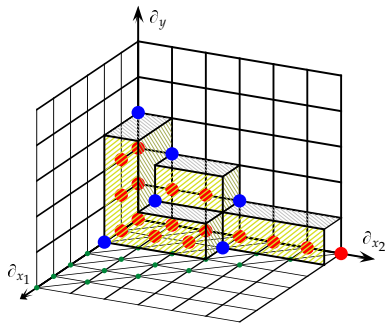
This computes  $A$  in

$$(J + \partial_y S_x[\partial_y]) \cap S_x, \quad \text{not in } T_t(I) := (I + \partial_y S_{x,y}) \cap S_x.$$

$$S_x = \mathbb{K}(x)\langle \partial_x \rangle \subset S_x[\partial_y] \subset S_{x,y} := \mathbb{K}(x, y)\langle \partial_x, \partial_y \rangle.$$

Algorithm II. Zeilberger's Style Extended to  $\text{Dim} > 0$ 

Compute  $A$  in  $T_t(I) := (I + \partial_y S_{x,y}) \cap S_x \leftrightarrow$  faster, more precise.



For  $s = 0, 1, 2, \dots$ , until  $\delta(J) \leq \text{bound}$ :

set  $A := \sum_{|\alpha| \leq s} \eta_\alpha \partial^\alpha$

for undetermined coeffs  $\eta_\alpha(y) \in \mathbb{K}(x)$

set  $B := \sum_{\beta \in M_s(I)} \phi_\beta(y) \partial^\beta$

for undetermined coeffs  $\phi_\beta(y) \in \mathbb{K}(x, y)$

reduce  $A - \partial_y B$

onto the basis  $M_{s+1}(I)$

extract coeffs to form a linear system  
of first order w.r.t.  $\partial_y$

solve and set  $J$  to the ideal of the  $A$ 's  
return the pairs  $(A, B)$



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- 6 Beyond Holonomy
- 7 Termination, Bounds, Complexity**
- 8 Conclusions

# History for Termination, Bounds, and Complexity

- (Wilf and Zeilberger, 1992) "An algorithmic proof theory for hypergeometric (ordinary and 'q') multisum/integral identities"
- (Yen, 1996, 1997) "A two-line algorithm for proving terminating hypergeometric identities", "A two-line algorithm for proving q-hypergeometric identities"
- (Abramov, 2002, 2003) "Applicability of Zeilberger's algorithm to hypergeometric terms", "When does Zeilberger's algorithm succeed?"
- (Zhang, 2003) "A new elementary algorithm for proving q-hypergeometric identities"
- (Apagodu and Zeilberger, 2005, 2006) "Sharp upper bounds for the orders of the recurrences output by the Zeilberger and q-Zeilberger algorithms", "Multi-variable Zeilberger and Almkvist-Zeilberger algorithms and the sharpening of Wilf-Zeilberger theory"
- (Bostan, Cluzeau, and Salvy, 2005) "Fast algorithms for polynomial solutions of linear differential equations"
- (Bostan, Chyzak, Salvy, and Cluzeau, 2006) "Low complexity algorithms for linear recurrences"
- (Guo, Hou, Sun, 2008) "Proving hypergeometric identities by numerical verifications"
- (Bostan, Chen, Chyzak, and Li, 2010) "Complexity of creative telescoping for bivariate rational functions"
- (Chen, 2011) "Some applications of differential-difference algebra to creative telescoping"
- (Chen, Chyzak, Feng, and Li) "On the existence of telescopers for hyperexponential-hypergeometric sequences" (in preparation)

# Order Bounds

For proper hypergeometric terms  $P(n, k) \zeta^n \xi^k \prod_{\ell=1}^L \Gamma(a_\ell n + b_\ell k + c_\ell)^{\epsilon_\ell}$ :

$$r \leq \sum_{\ell=1}^L |b_\ell| \quad (\text{Wilf and Zeilberger, 1992})$$

$$r \leq \sum_{\ell=1, \epsilon=+1}^L b_\ell^+ + \sum_{\ell=1, \epsilon=-1}^L (-b_\ell)^+ + \left( - \sum_{\ell=1, \epsilon=\pm 1}^L \epsilon b_\ell \right)^+ \quad (\text{Yen, 1993})$$

For hyperexponential terms  $p(x, y) \exp\left(\frac{a(x, y)}{b(x, y)}\right) \prod_{s \in S} s(x, y)^{\alpha_s}$ :

(Apagodu and Zeilberger, 2006), if  $a \neq 0$ ,  $\alpha_s \notin \bar{\mathbb{Q}}$ :

$$r \leq \deg_y(b) + \max(\deg_y(a), \deg_y(b)) + \left( \sum_{s \in S} \deg_y(s) \right) - 1.$$

# Proving Summation Identities by Evaluations

<b>Linear-algebraic</b> system to encode output recurrence	→ <b>Bounds on orders, degrees, and heights</b> (Cramer's rules)	→ <b>Bound on maximal integer singularity of recurrence</b>
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(Yen, 1996)

To prove  $\sum_{k=0}^{\infty} P(n, k) \zeta^n \zeta^k \prod_{\ell=1}^L \Gamma(a_{\ell}n + b_{\ell}k + c_{\ell})^{\epsilon_{\ell}} = 1$  for all  $n$ , it is sufficient to verify it at  $n = 0, \dots, N$ , where  $N = ((3\sigma)^{3\beta} \delta^5 h)^{\beta} + 1$ .

$$\lambda_+ = \#\{\ell \mid \epsilon_{\ell} = +1\}, \quad \lambda_- = \#\{\ell \mid \epsilon_{\ell} = -1\},$$

$$\sigma = \max(\lambda_+, \lambda_-) \max\{|a_{\ell}|, |b_{\ell}|, |c_{\ell}|\},$$

$$\delta = 1 + \max(\deg_n P, \deg_k P), \quad \beta = (\delta + 1)(2\sigma)^3,$$

$$h = \text{height}(P) = \text{maximal absolute value of coeffs.}$$

Proof: For the linear-algebra formulation, bound Fasenmyer's ansatz by Wilf and Zeilberger, and translate it to Zeilberger's ansatz.

# Proving Summation Identities by Fewer Evaluations

(Guo, Hou, and Sun, 2008)

To reduce  $N$ :

- directly use Zeilberger's ansatz with tighter order bound and explicit denominator bound;
- perform explicit majorisations on the concrete matrix, rather than use a generic bound;
- simplify the matrix by algorithmic manipulations.

→ **new  $N$  given by a procedure**, not a formula.

Examples:

	$\sum_k \binom{n}{k} = 2^n$	$\sum_k \binom{n}{k}^2 = \binom{2n}{n}$
Yen	$10^{11}$	$10^{115}$
Guo, Hou, Sun	4	12091

# Definite Integration of Bivariate Rational Functions

First step toward complexity analysis of creative telescoping

$$f \in \mathbb{K}(x, y) \text{ with } \deg_x = \deg_y = \delta \longrightarrow L(x, D_x)f = D_y g \text{ for } g \in \mathbb{K}(x, y)$$

(Bostan, Chen, Chyzak, Li, 2010): **Complexity is polynomial!**

**New algorithm based on Hermite reduction + Analyses:**

	Method	$\deg_{D_x} L$	$\deg_x L$	$\deg_x g$	$\deg_y g$	Complexity
Minimal	<b>Hermite</b>	$\leq \delta$	$\mathcal{O}(\delta^3)$	$\mathcal{O}(\delta^3)$	$\mathcal{O}(\delta^2)$	$\tilde{\mathcal{O}}(\delta^{\omega+4})$ L.V.
Telescopier	A. & Z.	$\leq \delta$	$\mathcal{O}(\delta^3)$	$\mathcal{O}(\delta^3)$	$\mathcal{O}(\delta^2)$	$\tilde{\mathcal{O}}(\delta^{2\omega+3})$ L.V.
Nonminimal	Lipshitz	$\leq 6(\delta + 1)^2$	$\mathcal{O}(\delta^2)$	$\mathcal{O}(\delta^3)$	$\mathcal{O}(\delta^3)$	$\mathcal{O}(\delta^{6\omega})$ det.
Telescopier	Cubic	$\leq 6\delta$	$\mathcal{O}(\delta^2)$	$\mathcal{O}(\delta^2)$	$\mathcal{O}(\delta^2)$	$\mathcal{O}(\delta^{4\omega})$ det.

Hermite-based: matrix size =  $\delta$ , entries of degree  $\delta^3$ .

A. & Z.: matrix size =  $\delta^2$ , entries of degree  $\delta^2$ .

Lipshitz: total-degree filtration at degree =  $\delta \rightarrow$  matrix size =  $\delta^6$ .

Cubic: bi-filtration at (degree, order) =  $(\delta^2, \delta) \rightarrow$  matrix size =  $\delta^4$ .

# Hermite Reduction for Indefinite Integration in $\mathbb{K}(y)$

One step, given a Bézout relation  $1 = sq' + tq$  for irreducible  $q$ :

$$\int \frac{p}{q^j} = \int \frac{-sp}{j-1} \cdot \frac{-(j-1)q'}{q^j} + \int \frac{tp}{q^{j-1}} = \frac{-sp}{(j-1)q^{j-1}} + \int \frac{tp + (sp)'/(j-1)}{q^{j-1}}.$$

By iterating:

$$\int \sum_{j=1}^v \frac{p_j}{q^j} = \sum_{j=1}^{v-1} \frac{a_j}{q^j} + \int \frac{u}{q} \quad \text{where} \quad \deg_y u < \deg_y q.$$

Viewed differently:

$$\sum_{j=1}^v \frac{p_j}{q^j} = \left( \sum_{j=1}^{v-1} \frac{a_j}{q^j} \right)' + \frac{u}{q} \quad \text{where} \quad \deg_y u < \deg_y q.$$

Complexity for general  $\frac{p}{q}$ :  $\tilde{O}(\max(\deg p, \deg q))$ .

# An Algorithm for Definite Integration in $\mathbb{K}(x, y)$

Given  $f = \frac{p}{q} \in \mathbb{K}(x)(y)$ , to compute  $\int_a^b f(x, y) dy$ :

$$f(x, y) = D_y \left( \begin{array}{c} \cdots \\ \cdots \end{array} \right) + \frac{u_0}{q^*}, \quad \dots, \quad D_x^r f(x, y) = D_y \left( \begin{array}{c} \cdots \\ \cdots \end{array} \right) + \frac{u_r}{q^*},$$

where:

- ①  $q^* = q_1 \cdots q_m$  is the squarefree part of  $q = q_1^1 q_2^2 \cdots q_m^m$ ;
- ②  $\deg_y u_i < \deg_y q^*$  for  $0 \leq i \leq r$ .

Linear algebra over  $\mathbb{K}(x)$ : for some  $r \leq \deg_y q^*$ ,

$$\sum_{i=0}^r \eta_i(x) D_x^i f(x, y) = D_y \left( \begin{array}{c} \cdots \\ \cdots \end{array} \right) + \frac{0}{q^*}.$$

Matrix size ruled by  $\deg_y u$ , not by  $\deg_y g$  !

Complexity:

- bivariate Hermite by evaluation/interpolation;
- linear algebra over  $\mathbb{K}[x]$  by Storjohann–Villard.



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# Non-Minimality of Computed Operators

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{pk}{n} = (-p)^n \quad (\text{order } p-1 \text{ instead of } 1),$$

$$\sum_{k=0}^n \frac{1}{pk+1} \binom{pk+1}{k} \binom{p(n-k)}{n-k} = \binom{pn+1}{n}$$

(order 2 instead of 1 for  $p = 3$ ).

This non-minimality is not understood well.

$$\eta_r(n) u_{n+r,k} + \cdots + \eta_0(n) u_{n,k} = v_{n,k+1} - v_{n,k}$$

The existence of  $v$  constrains  $d$ !

This relation does not take the summation range into account!

# An Integral Not Amenable to Creative Telescoping?

Borrowed from (Borwein, Straub, Wan, Zudilin, 2011).

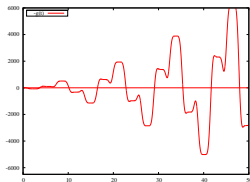
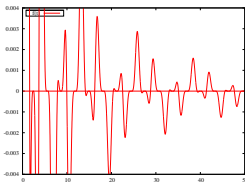
$$F(x) = \int_0^{\infty} \underbrace{xt J_0(xt) J_0(t)^4}_{=f(x,t)} dt \quad \rightarrow \quad Pf = D_t g \quad \text{for} \quad g = Qf.$$

$$P = (x-4)(x-2)(x+2)(x+4)x^3 D_x^3 + 6(x^2-10)x^4 D_x^2 + (7x^4-32x^2+64)x D_x + (x^2-8)(x^2+8)$$

$$Q = 5x^3 t^2 D_x D_t^3 - x^2 t^3 D_t^4 - t^{-1} x^2 (5x^4 t^4 - 60x^2 t^4 + 64t^4 - 28t^2 - 4) - 7x^2 t^2 D_t^3$$

$$+ x^2 t (10x^2 t^2 - 20t^2 - 1) D_t^2 - 5x^3 (2x^2 t^2 - 12t^2 - 1) D_x D_t$$

$$+ 4x^2 (5x^2 t^2 - 15t^2 - 1) D_t + 5t^{-1} x^3 (2x^2 t^2 - 12t^2 - 1) D_x$$

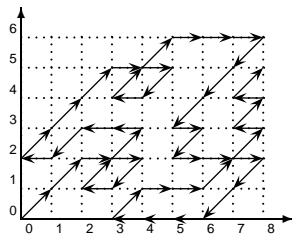
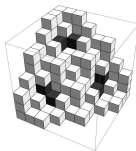
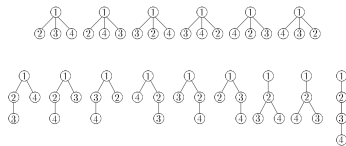


$$D_x^i f|_{x=3/2} = \Theta(\cos(3\pi/2 + \pi/4) \sin(t + \pi/4) t^{i-3/2}) ! \quad \text{No limit of } g \text{ at } \infty !$$

$$\text{For } i > 0, \text{ no limit of } \int_0^T D_x^i f|_{x=3/2} dt \text{ at } \infty !$$

# Successful Applications of Creative Telescoping

- *Descendants in heap-ordered trees* (Prodinger, 1996): Zeilberger's algorithm ( $2 \times$ )
- *Totally symmetric plane partition* (Kauers, Koutschan & Zeilberger, 2011):  $q$ -Koutschan ( $n \times$ )
- Lattice walks confined in a quadrant, in particular *Gessel's conjecture* for  $\leftarrow, \rightarrow, \swarrow, \nearrow$  (Kauers, Koutschan & Zeilberger, 2009): variant of Takayama for specialisation ( $n \leftrightarrow S_n$ )



All were first proofs!