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## On annular short-time stability conditions for generalized Persidskii systems

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### ABSTRACT

This paper studies the trajectory behavior evaluation for generalized Persidskii systems with an essentially bounded input on a finite time interval. Also, the notions of annular settling and output annular settling for general nonlinear systems are introduced. We propose conditions for annular short-time stability, short-time boundedness with a nonzero initial state, annular settling, and output annular settling for a class of Persidskii systems. These conditions are based on the verification of linear matrix inequalities. An application to recurrent neural networks illustrates the usefulness of the proposed notions and conditions.

### KEYWORDS

Annular settling; output annular settling; general nonlinear systems; annular short-time stability; generalized Persidskii systems

## 1. Introduction

The stability analysis of dynamical systems is a complicated issue, especially for nonlinear dynamics with external inputs [1, 2, 3], for which the Lyapunov theory and the asymptotic stability concepts [4] are commonly used. However, in some scenarios, analyzing the system behavior at all positive times is unnecessary; one may focus on the trajectories in a bounded time interval. A representative example is given by the short-time stability concept, introduced in the 1950s [5, 6] to address the problem that the solutions of a system stay in a given domain, starting from a bounded set of initial conditions, during a finite time interval. Various investigations of short-time stability and stabilization of linear time-varying systems were introduced [7, 8, 9], as well as the robust short-time stability analysis for linear systems [10]. Further studies in distinct directions include techniques of stochastic systems [11] or annular short-time stability [12] (called annular finite-time stability [12]), to mention several examples.

In this work, we call the mentioned stability concept short-time [6] instead of finite-time stability (also frequently used to denote the same property [5]). The reason is that there is another notion of stability with the same name, dealing with a finite time of convergence of the system trajectories to an invariant mode (*e.g.*, an equilibrium or

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a desired set) combined with Lyapunov stability for all positive times. Nevertheless, all analysis is carried out in a bounded time interval for short-time stability notion.

We study in this note several concepts. Following [13, 12], robust annular short-time stability (ASTS) and its extensions to short-time boundedness with a nonzero initial state (STBNZ) are investigated, while annular settling (AS) and output annular settling (oAS) are first time introduced. Roughly speaking, a system is said to be ASTS if the system state stays in a bounded domain during a specified time interval, given an initial bounded region separated from the origin, while in AS case, only boundedness at the end of the time interval is required. The notion of ASTS was introduced for safety problems, where the state must behave between given lower and upper bounds; for instance, the water level in a tank should not exceed the prescribed thresholds [12].

The dynamical models considered in this work belong to the class of so-called generalized Persidskii systems (extended from the dynamics in [14, 15]), which have been extensively studied in the context of neural networks [16], biological models [17], and power systems [18]. Recent advances for generalized Persidskii models include, for example, the conditions of input-to-state stability, input-to-output stability and convergence, and the synthesis of a state observer [19, 20, 21, 22]. Note that the most existing approaches to synthesizing Lyapunov functions for stability analysis in nonlinear systems involve various canonical forms of the studied differential equations, *e.g.*, Lur'e systems [23], homogeneous models [24], Persidskii systems [15], and Lipschitz dynamics. Due to the intrinsic nature of nonlinearities, the relevant stability conditions may be rather sophisticated. However, in generalized Persidskii systems, a Lyapunov function is found, where the stability conditions can be formulated in the form of linear matrix inequalities (LMIs) [19], which is a rare case for nonlinear dynamics.

Continuous-time recurrent neural networks constitute an example of generalized Persidskii systems. An important application of these neural networks consists in classifying temporal sequences defined on a finite time interval. In such a scenario, the output of a trained network has to approach desired levels (typically, a small compact set) in the considered time interval, indicating a class of the input signals. As we will show such a behavior can be quantified using the oAS notion (if the output equals the state, the concept of AS can be utilized) that is introduced in this paper. Therefore, the main contributions of this work include the introduction of these new notions: oAS and AS for general nonlinear systems, the formulation of ASTS, STBNZ, AS, and oAS conditions for a family of generalized Persidskii systems, as well as the illustration of the usefulness and efficacy of the proposed conditions in recurrent neural networks.

The organization of this paper is as follows. Section 2 introduces the preliminaries and the definitions of considered stability properties. The generalized Persidskii system is presented in Section 3. In Section 4, ASTS, STBNZ, AS, and oAS conditions for the class of considered systems are given, and in Section 5, an application to continuous-time recurrent neural networks is investigated to examine the efficiency of the proposed results.

## Notation

- $\mathbb{N}$ ,  $\mathbb{R}$  and  $\mathbb{R}_+$  represent the sets of natural, real and nonnegative real numbers, respectively.
- $\mathbb{R}^n$  and  $\mathbb{R}^{m \times n}$  denote the vector spaces of real  $n$ -vectors and  $m \times n$  real matrices, respectively. The set of  $n \times n$  diagonal matrices (with nonnegative diagonal

- elements) is denoted by  $\mathbb{D}^n$  ( $\mathbb{D}_+^n$ ).
- $I_n$  and  $O_{m \times n}$  stand for the identity  $n \times n$ -matrix and the zero  $m \times n$ -matrix, respectively. Denote the vector of dimension  $n$  with all elements equal 1 by  $\mathbf{1}_n$ .
  - The symbol  $\|\cdot\|$  refers to the Euclidean norm on  $\mathbb{R}^n$  (and the induced matrix norm  $\|A\|$  for a matrix  $A \in \mathbb{R}^{m \times n}$ ).
  - For  $p, n \in \mathbb{N}$  with  $p \leq n$ , the notation  $\overline{p, n}$  is used to represent the set  $\{p, \dots, n\}$ .
  - $A^{(s)}$  stands for the  $s$ th row of the matrix  $A \in \mathbb{R}^{m \times n}$ ,  $s \in \overline{1, m}$ . For all  $i, j \in \overline{p, n}$ , let  $(A_{i,j})_{i,j=p}^n$  denote the block matrix 
$$\begin{bmatrix} A_{p,p} & \cdots & A_{p,n} \\ \vdots & \ddots & \vdots \\ A_{n,p} & \cdots & A_{n,n} \end{bmatrix}.$$
  - $B_n(\delta) = \{y \in \mathbb{R}^n : \|y\| \leq \delta\}$  stands for the closed ball of radius  $\delta > 0$  centered at the origin;  $\text{cl}(E)$  denotes the closure of a set  $E \subset \mathbb{R}^n$ .
  - For a Lebesgue measurable function  $u: \mathbb{R} \rightarrow \mathbb{R}^m$ , define the norm  $\|u\|_{(t_1, t_2)} = \text{ess sup}_{t \in (t_1, t_2)} \|u(t)\|$  for  $(t_1, t_2) \subseteq \mathbb{R}$ . Denote by  $\mathcal{L}_{(t_1, t_2)}^m$  (or  $\mathcal{L}_\infty^m$ ) the Banach space of functions  $u$  with  $\|u\|_{(t_1, t_2)} < +\infty$  (or  $\|u\|_\infty := \|u\|_{(-\infty, +\infty)} < +\infty$ ).
  - A continuous function  $\sigma: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to class  $\mathcal{K}$  if it is strictly increasing and  $\sigma(0) = 0$ ; it belongs to class  $\mathcal{K}_\infty$  if it is also unbounded.
  - For a continuously differentiable function  $V: \mathbb{R}^n \rightarrow \mathbb{R}$ , denote by  $\nabla V(\nu)f(\nu)$  the derivative of  $V$  along the vector field  $f$  evaluated at point  $\nu \in \mathbb{R}^n$ .
  - The maximum eigenvalue of a symmetric matrix  $P$  is denoted by  $\lambda_{\max}(P)$ .

## 2. Preliminaries

Consider the differential equation

$$\begin{aligned} \dot{x}(t) &= F(x(t), u(t)), \quad t \in \Delta = [0, T] \subset \mathbb{R}, \\ y(t) &= h(x(t)), \end{aligned} \tag{1}$$

where  $x(t) \in \mathbb{R}^n$  is the state vector;  $u(t) \in \mathbb{R}^m$  is the external input,  $u \in \mathcal{L}_\infty^m$ ;  $\Delta$  is the time interval of interest with  $0 < T < +\infty$ . Moreover,  $F: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a continuous function and  $h: \mathbb{R}^n \rightarrow \mathbb{R}^p$  is a continuously differentiable function. In the rest of the paper, to lighten the notation, the time-dependency of variables might remain implicitly understood; for instance we will write  $x$  for  $x(t)$ .

For an initial state  $x(0) = x_0 \in \mathbb{R}^n$  and  $u \in \mathcal{L}_\infty^m$ , we denote the corresponding solution of system (1) by  $x(t, x_0, u)$  for the values of  $t \in \mathbb{R}_+$  the solution exists, so the corresponding output is  $y(t, x_0, u) = h(x(t, x_0, u))$ . In the sequel, we assume that such a solution of (1) is uniquely defined, for any  $x_0 \in \mathbb{R}^n$  and  $u \in \mathcal{L}_\infty^m$ , for all  $t \in \Delta$ .

**Definition 1** ([13, 12]). *System (1) is said to be:*

- (1) short-time bounded *with respect to*  $(\Delta, \gamma, \delta)$ , *if for given*  $\gamma \geq 0$  *and*  $\delta > 0$ ,

$$x_0 = 0, \|u\|_\Delta \leq \gamma \quad \Rightarrow \quad \|x(t, x_0, u)\| \leq \delta, \quad \forall t \in \Delta.$$

- (2) short-time bounded with nonzero initial state (STBNZ) *with respect to*  $(\Delta, \epsilon, \gamma, \delta)$ , *if for given*  $\epsilon > 0$ ,  $\gamma \geq 0$  *and*  $\delta > 0$ ,

$$\|x_0\| \leq \epsilon, \|u\|_\Delta \leq \gamma \quad \Rightarrow \quad \|x(t, x_0, u)\| \leq \delta, \quad \forall t \in \Delta.$$

- (3) annular short-time stable (ASTS) *with respect to*  $(\Delta, \epsilon_1, \epsilon_2, \gamma, \delta_1, \delta_2)$ , *if for given*

$(\epsilon_1, \epsilon_2) \subseteq (\delta_1, \delta_2) \subset \mathbb{R}_+$  and  $\gamma \geq 0$ ,

$$\begin{aligned} & x_0 \in \text{cl}(B_n(\epsilon_2) \setminus B_n(\epsilon_1)), \|u\|_\Delta \leq \gamma \\ \Rightarrow & x(t, x_0, u) \in \text{cl}(B_n(\delta_2) \setminus B_n(\delta_1)), \quad \forall t \in \Delta. \end{aligned}$$

In the classification of temporal sequences by neural networks, the resulting output level at the time instant  $T$  is considered, whose value characterizes the class to which the given input signal belongs (a complete analysis over the whole time interval  $\Delta$  is nevertheless unnecessary in this scenario, as it is considered in the case of ASTS). For investigation of this behavior, we further introduce two useful notions:

**Definition 2.** *System (1) is said to be:*

- (1) annular settled (AS) with respect to  $(T, \epsilon_1, \epsilon_2, \gamma, \delta_1, \delta_2)$ , if for given  $(\epsilon_1, \epsilon_2) \subset \mathbb{R}_+$ ,  $(\delta_1, \delta_2) \subset \mathbb{R}_+$  and  $\gamma \geq 0$ ,

$$\begin{aligned} & x_0 \in \text{cl}(B_n(\epsilon_2) \setminus B_n(\epsilon_1)), \|u\|_\Delta \leq \gamma \\ \Rightarrow & x(T, x_0, u) \in \text{cl}(B_n(\delta_2) \setminus B_n(\delta_1)). \end{aligned}$$

- (2) output annular settled (oAS) with respect to  $(T, x_0, \gamma, \delta_1, \delta_2)$ , if for given  $x_0 \in \mathbb{R}^n$ ,  $(\delta_1, \delta_2) \subset \mathbb{R}_+$  and  $\gamma \geq 0$ ,

$$\|u\|_\Delta \leq \gamma \Rightarrow y(T, x_0, u) \in \text{cl}(B_p(\delta_2) \setminus B_p(\delta_1)).$$

Therefore, all considered properties are not related to the attractiveness of a set, but with visiting this set by trajectories during the interval of time  $\Delta$  for ASTS or just at the end of the interval for AS and oAS. Next, the trajectories may leave the sets of interest.

### 3. Problem Statement

Consider the following system in a generalized Persidskii form [15]:

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{j=1}^M A_j F_j(H_j x(t)) + u(t), \quad t \in [0, T], \\ y(t) &= C x(t), \end{aligned} \tag{2}$$

where  $x = [x_1 \ \dots \ x_n]^\top \in \mathbb{R}^n$  is the state,  $x(0) = x_0$ ;  $y(t) \in \mathbb{R}^p$  is the output signal;  $[0, T]$  is the interval of interest for some  $0 < T < +\infty$ ;  $u \in \mathcal{L}_\infty^n$  is the exogenous input;  $A_0 \in \mathbb{R}^{n \times n}$ ,  $A_j \in \mathbb{R}^{n \times k_j}$ ,  $H_j \in \mathbb{R}^{k_j \times n}$  ( $j \in \overline{1, M}$ ),  $C \in \mathbb{R}^{p \times n}$  are constant matrices (to shorten further writing, we define  $k_0 = n$  and  $H_0 = I_n$ );  $F_j : \mathbb{R}^{k_j} \rightarrow \mathbb{R}^{k_j}$  are continuous functions,

$$F_j(\ell^j) = [f_j^1(\ell_1) \ \dots \ f_j^{k_j}(\ell_{k_j})]^\top \in \mathbb{R}^{k_j}, \quad \forall j \in \overline{1, M}$$

with  $\ell^j = [\ell_1 \ \dots \ \ell_{k_j}]^\top \in \mathbb{R}^{k_j}$ . Thus, the nonlinearity  $F_j$  has a special structure: each component  $f_j^i : \mathbb{R} \rightarrow \mathbb{R}$  of  $F_j$  depends only on  $H_j^{(i)} x$ , for  $i \in \overline{1, k_j}$ .

Sector restrictions on  $F_j$  ( $j \in \overline{1, M}$ ) are imposed in the following assumption:

**Assumption 1.** Assume that for any  $j \in \overline{1, M}$  and  $i \in \overline{1, k_j}$

$$\nu f_j^i(\nu) > 0, \quad \forall \nu \in \mathbb{R} \setminus \{0\}.$$

Under Assumption 1, with a reordering of nonlinearities and their decomposition, there exists an index  $c \in \overline{0, M}$  such that for all  $a \in \overline{1, c}$ ,  $i \in \overline{1, k_a}$

$$\lim_{\nu \rightarrow \pm\infty} f_a^i(\nu) = \pm\infty,$$

and that there exists  $\mu \in \overline{c, M}$  such that for all  $b \in \overline{1, \mu}$ ,  $i \in \overline{1, k_b}$

$$\lim_{\nu \rightarrow \pm\infty} \int_0^\nu f_b^i(\tau) d\tau = +\infty.$$

In this case,  $c = 0$  implies that all nonlinearities are bounded (at least for positive or negative argument). Clearly,  $\mu \geq c$ .

In this study, to perform the stability analysis, we also need a mild assumption of upper and lower bounds on the integrals of the nonlinearities:

**Assumption 2.** Assume that for any  $j, j' \in \overline{1, M}$ ,  $z \in \overline{j'+1, M}$ ,  $i \in \overline{1, k_j}$ ,  $i' \in \overline{1, n}$  and  $\Lambda_j = \text{diag}(\Lambda_j^1, \dots, \Lambda_j^{k_j}) \in \mathbb{D}_+^{k_j}$ , there exist  $\kappa_{0,j}^i, \kappa_{1,jj'}^i, \kappa_{2,jj'}^i, \kappa_{3,jj'z}^i, \eta_{0,j}^i, \eta_{1,jj'}^i, \eta_{2,jj'}^i, \eta_{3,jj'z}^i \geq 0$ , such that

$$\begin{aligned} & \mathbf{x}^\top H_j^\top \kappa_{0,j} H_j \mathbf{x} + \sum_{j'=1}^M f_{j'}(H_{j'} \mathbf{x})^\top \left( \kappa_{1,jj'} f_{j'}(H_{j'} \mathbf{x}) + 2\kappa_{2,jj'} H_{j'} \mathbf{x} \right) \\ & + 2 \sum_{j'=1}^M \sum_{z=j'+1}^M f_{j'}(H_{j'} \mathbf{x})^\top H_{j'} \cdot \kappa_{3,jj'z} \cdot H_z^\top f_z(H_z \mathbf{x}) \\ & \leq 2\mathbf{1}_{k_j}^\top \Lambda_j \begin{bmatrix} \int_0^{H_j^{(1)} \mathbf{x}} f_j^1(s) ds \\ \vdots \\ \int_0^{H_j^{(k_j)} \mathbf{x}} f_j^{k_j}(s) ds \end{bmatrix} \leq \\ & \mathbf{x}^\top H_j^\top \eta_{0,j} H_j \mathbf{x} + \sum_{j'=1}^M f_{j'}(H_{j'} \mathbf{x})^\top \left( \eta_{1,jj'} f_{j'}(H_{j'} \mathbf{x}) + 2\eta_{2,jj'} H_{j'} \mathbf{x} \right) \\ & + 2 \sum_{j'=1}^M \sum_{z=j'+1}^M f_{j'}(H_{j'} \mathbf{x})^\top H_{j'} \cdot \eta_{3,jj'z} \cdot H_z^\top f_z(H_z \mathbf{x}) \end{aligned}$$

for all  $x \in \mathbb{R}^n$ , where

$$\begin{aligned}\kappa_{0,j} &= \text{diag}(\kappa_{0,j}^1, \dots, \kappa_{0,j}^{k_j}), \quad \kappa_{1,jj'} = \text{diag}(\kappa_{1,jj'}^1, \dots, \kappa_{1,jj'}^{k_j}), \\ \kappa_{2,jj'} &= \text{diag}(\kappa_{2,jj'}^1, \dots, \kappa_{2,jj'}^{k_j}), \quad \kappa_{3,jj'z} = \text{diag}(\kappa_{3,jj'z}^1, \dots, \kappa_{3,jj'z}^n), \\ \eta_{0,j} &= \text{diag}(\eta_{0,j}^1, \dots, \eta_{0,j}^{k_j}), \quad \eta_{1,jj'} = \text{diag}(\eta_{1,jj'}^1, \dots, \eta_{1,jj'}^{k_j}), \\ \eta_{2,jj'} &= \text{diag}(\eta_{2,jj'}^1, \dots, \eta_{2,jj'}^{k_j}), \quad \eta_{3,jj'z} = \text{diag}(\eta_{3,jj'z}^1, \dots, \eta_{3,jj'z}^n).\end{aligned}$$

This hypothesis is satisfied by many nonlinear functions: sigmoid functions in neural networks; for polynomials, for example, it is sufficient to select  $\kappa_{2,jj'} \neq 0$  and  $\eta_{2,jj'} \neq 0$ .

In this work, if the upper bound is smaller than the lower one for an index, then the corresponding term (in sum or a sequence) must be omitted.

#### 4. Stability conditions

In this section, ASTS, STBNZ, AS and oAS sufficient conditions for the generalized Persidskii system (2) in the presence of an essentially bounded input are formulated.

Following [25, 20, 21, 22], the stability analysis of (2) can be performed using a Lyapunov function

$$V(x) = x^\top P x + 2 \sum_{j \in \overline{1, M}} \sum_{i \in \overline{1, k_j}} \Lambda_j^i \int_0^{H_j^{(i)} x} f_j^i(\nu) d\nu,$$

where  $0 \leq P = P^\top \in \mathbb{R}^{n \times n}$  and  $\Lambda_j = \text{diag}(\Lambda_j^1, \dots, \Lambda_j^{k_j}) \in \mathbb{D}_+^{k_j}$  are tuning matrices. If they are selected in a way ensuring positive definiteness of  $V$  under Assumption 1, then there exist  $\alpha_1^{P, \Lambda_1, \dots, \Lambda_M}, \alpha_2^{P, \Lambda_1, \dots, \Lambda_M} \in \mathcal{K}_\infty$  such that

$$\alpha_1^{P, \Lambda_1, \dots, \Lambda_M}(\|x\|) \leq V(x) \leq \alpha_2^{P, \Lambda_1, \dots, \Lambda_M}(\|x\|)$$

for all  $x \in \mathbb{R}^n$ . For example, the upper bound can be always taken as

$$\alpha_2^{P, \Lambda_1, \dots, \Lambda_M}(\tau) = \lambda_{\max}(P)\tau^2 + 2 \left( \sum_{j=1}^M k_j \right) \cdot \max_{j \in \overline{1, M}, i \in \overline{1, k_j}} \left\{ \Lambda_j^i \int_0^{\|H_j^{(i)}\| \tau} f_j^i(\nu) d\nu \right\}.$$

These functions  $\alpha_1^{P, \Lambda_1, \dots, \Lambda_M}, \alpha_2^{P, \Lambda_1, \dots, \Lambda_M}$  will be used later in the proofs.

The system (2) is highly nonlinear with multiple nonlinearities, which makes the analysis of ASTS complicated as those nonlinearities may override the linear part readily and significantly influence the behavior of  $x(t)$ . The following theorem presents ASTS conditions for system (2), and it is the principal theoretical contribution of the paper.

**Theorem 1.** *Let assumptions 1 and 2 be satisfied and  $T > 0$ ;  $\gamma_0 \geq 0$ ;  $(\epsilon_1, \epsilon_2) \subseteq (\delta_1, \delta_2) \subset \mathbb{R}_+$  be given. If there exist  $0 \leq \bar{P} = \bar{P}^\top$ ,  $\underline{P} = \underline{P}^\top \in \mathbb{R}^{n \times n}$ ;  $\{\bar{\Lambda}_j = \text{diag}(\bar{\Lambda}_j^1, \dots, \bar{\Lambda}_j^{k_j}), \underline{\Lambda}_j = \text{diag}(\underline{\Lambda}_j^1, \dots, \underline{\Lambda}_j^{k_j})\}_{j=1}^M \subset \mathbb{D}_+^{k_j}$ ;  $\{\bar{\Upsilon}_{0,j}, \underline{\Upsilon}_{0,j}\}_{j=1}^M \subset$*

$\mathbb{D}^{k_j}; \{\bar{\Upsilon}_{s,z}, \underline{\Upsilon}_{s,z}\}_{1 \leq s < z \leq M} \subset \mathbb{D}^n; \{\bar{\Omega}^j, \underline{\Omega}^j\}_{j=1}^M \subset \mathbb{R}^{n \times k_j};$  symmetric matrices  
 $\bar{\Gamma}, \underline{\Gamma}, \bar{\Psi}, \underline{\Psi} \in \mathbb{R}^{n \times n}, \{\bar{\Xi}^s, \underline{\Xi}^s\}_{s=0}^M \subset \mathbb{R}^{k_s \times k_s}; \bar{\gamma}, \underline{\gamma} > 0$  and  $\beta_1, \beta_2, \bar{\rho}, \underline{\rho} \in \mathbb{R}$  such that

$$\underline{P} + \underline{\rho} \sum_{j=1}^{\mu} H_j^\top \underline{\Lambda}_j H_j > 0, \quad (3)$$

$$\underline{Q} = \underline{Q}^\top = \left( \underline{Q}_{a,b} \right)_{a,b=1}^{M+3} \geq 0, \quad (4)$$

$$\begin{cases} \underline{\gamma} \leq \frac{e^{\beta_1 T} \underline{\alpha}_1(\epsilon_1) - \underline{\alpha}_2(\delta_1)}{\frac{\gamma_0}{2} [e^{\beta_1 T} - 1]}, & \text{if } \beta_1 < 0 \text{ or } \beta_1 > 0, \underline{\gamma} > \frac{\beta_1}{\gamma_0} \underline{\alpha}_1(\epsilon_1), \\ \underline{\gamma} \leq \frac{\underline{\alpha}_1(\epsilon_1) - \underline{\alpha}_2(\delta_1)}{T \gamma_0^2}, & \text{if } \beta_1 = 0, \\ \underline{\alpha}_2(\delta_1) \leq \underline{\alpha}_1(\epsilon_1), & \text{if } \beta_1 > 0, \underline{\gamma} \leq \frac{\beta_1}{\gamma_0^2} \underline{\alpha}_1(\epsilon_1). \end{cases} \quad (5)$$

$$\begin{cases} \underline{\Lambda}_j \leq \Lambda_j, \\ -\underline{\Xi}^0 \geq \beta_1 \left( \underline{P} + \sum_{j=1}^M H_j^\top \eta_{0,j} H_j \right), \\ -\underline{\Xi}^j \geq \beta_1 \sum_{j'=1}^M \eta_{1,jj'}, & \text{if } \beta_1 \geq 0, \\ -\underline{\Upsilon}_{0,j} \geq \beta_1 \sum_{j'=1}^M \eta_{2,jj'}, \\ -\underline{\Upsilon}_{j,z} \geq \beta_1 \sum_{j'=1}^M \eta_{3,jj'z}, \end{cases} \quad (6)$$

$$\begin{cases} \underline{\Lambda}_j \geq \Lambda_j, \\ -\beta_1 \left( \underline{P} + \sum_{j=1}^M H_j^\top \kappa_{0,j} H_j \right) \geq \underline{\Xi}^0, \\ -\beta_1 \sum_{j'=1}^M \kappa_{1,jj'} \geq \underline{\Xi}^j, & \text{if } \beta_1 < 0, \\ -\beta_1 \sum_{j'=1}^M \kappa_{2,jj'} \geq \underline{\Upsilon}_{0,j}, \\ -\beta_1 \sum_{j'=1}^M \kappa_{3,jj'z} \geq \underline{\Upsilon}_{j,z}. \end{cases}$$

$$\bar{P} + \bar{\rho} \sum_{j=1}^{\mu} H_j^\top \bar{\Lambda}_j H_j > 0, \quad (7)$$

$$\bar{Q} = \bar{Q}^\top = \left( \bar{Q}_{a,b} \right)_{a,b=1}^{M+3} \leq 0, \quad (8)$$

$$\begin{cases} \bar{\gamma} \leq \frac{\bar{\alpha}_1(\delta_2) - e^{\beta_2 T} \bar{\alpha}_2(\epsilon_2)}{\frac{\gamma_0}{2} [e^{\beta_2 T} - 1]}, & \text{if } \beta_2 > 0 \text{ or } \beta_2 < 0, \bar{\gamma} > -\frac{\beta_2}{\gamma_0}, \\ \bar{\gamma} \leq \frac{\bar{\alpha}_1(\delta_2) - \bar{\alpha}_2(\epsilon_2)}{T \gamma_0^2}, & \text{if } \beta_2 = 0, \\ \bar{\alpha}_2(\epsilon_2) \leq \bar{\alpha}_1(\delta_2), & \text{if } \beta_2 < 0, \bar{\gamma} \leq -\frac{\beta_2}{\gamma_0^2} \bar{\alpha}_2(\epsilon_2). \end{cases} \quad (9)$$



$$\begin{cases}
\bar{\Lambda}_j \leq \Lambda_j, \\
\bar{\Xi}^0 \geq -\beta_2 \left( \bar{P} + \sum_{j=1}^M H_j^\top \eta_{0,j} H_j \right), \\
\bar{\Xi}^j \geq -\beta_2 \sum_{j'=1}^M \eta_{1,jj'}, & \text{if } \beta_2 < 0, \\
\bar{\Upsilon}_{0,j} \geq -\beta_2 \sum_{j'=1}^M \eta_{2,jj'}, \\
\bar{\Upsilon}_{j,z} \geq -\beta_2 \sum_{j'=1}^M \eta_{3,jj'z}, \\
\bar{\Lambda}_j \geq \Lambda_j, \\
\beta_2 \left( \bar{P} + \sum_{j=1}^M H_j^\top \kappa_{0,j} H_j \right) \geq -\bar{\Xi}^0, \\
\beta_2 \sum_{j'=1}^M \kappa_{1,jj'} \geq -\bar{\Xi}^j, & \text{if } \beta_2 \geq 0, \\
\beta_2 \sum_{j'=1}^M \kappa_{2,jj'} \geq -\bar{\Upsilon}_{0,j}, \\
\beta_2 \sum_{j'=1}^M \kappa_{3,jj'z} \geq -\bar{\Upsilon}_{j,z}.
\end{cases} \tag{10}$$

where

$$\begin{aligned}
\alpha_1(s) &= \alpha_1^{\bar{P}, \bar{\Lambda}_1, \dots, \bar{\Lambda}_M}(s), & \alpha_2(s) &= \alpha_2^{\bar{P}, \bar{\Lambda}_1, \dots, \bar{\Lambda}_M}(s), \\
\bar{\alpha}_1(s) &= \alpha_1^{\bar{P}, \bar{\Lambda}_1, \dots, \bar{\Lambda}_M}(s), & \bar{\alpha}_2(s) &= \alpha_2^{\bar{P}, \bar{\Lambda}_1, \dots, \bar{\Lambda}_M}(s).
\end{aligned}$$

$$\begin{aligned}
\bar{Q}_{1,1} &= -\bar{\Psi}^\top - \bar{\Psi}; & \bar{Q}_{1,2} &= \bar{\Psi}^\top A_0 + \bar{P} - \bar{\Gamma}; \\
\bar{Q}_{1,j+2} &= \bar{\Omega}_j + H_j^\top \bar{\Lambda}_j + \bar{\Psi}^\top A_j; & \bar{Q}_{1,M+3} &= \bar{\Psi}^\top; \\
\bar{Q}_{2,2} &= \bar{\Gamma}^\top A_0 + A_0^\top \bar{\Gamma} + \bar{\Xi}_0; & \bar{Q}_{2,j+2} &= \bar{\Gamma}^\top A_j + H_j^\top \bar{\Upsilon}_{0,j} - A_0^\top \bar{\Omega}_j, \\
\bar{Q}_{2,M+3} &= \bar{\Gamma}^\top; & \bar{Q}_{j+2,j+2} &= -\bar{\Omega}_j^\top A_j - A_j^\top \bar{\Omega}_j + \bar{\Xi}^j, \\
\bar{Q}_{s+2,z'+2} &= -\bar{\Omega}_s^\top A_{z'} - A_s^\top \bar{\Omega}_{z'} + H_s \bar{\Upsilon}_{s,z'} H_{z'}^\top, \\
\bar{Q}_{j+2,M+3} &= -\bar{\Omega}_j^\top; & \bar{Q}_{M+3,M+3} &= -\bar{\gamma} I_n.
\end{aligned}$$

$$\begin{aligned}
\underline{Q}_{1,1} &= -\underline{\Psi}^\top - \underline{\Psi}; & \underline{Q}_{1,2} &= \underline{\Psi}^\top A_0 + \underline{P} - \underline{\Gamma}; \\
\underline{Q}_{1,j+2} &= \underline{\Omega}_j + H_j^\top \underline{\Gamma} + \underline{\Psi}^\top A_j; & \underline{Q}_{1,M+3} &= \underline{\Psi}^\top; \\
\underline{Q}_{2,2} &= \underline{\Gamma}^\top A_0 + A_0^\top \underline{\Gamma} + \underline{\Xi}_0; & \underline{Q}_{2,j+2} &= \underline{\Gamma}^\top A_j + H_j^\top \underline{\Upsilon}_{0,j} - A_0^\top \underline{\Omega}_j; \\
\underline{Q}_{2,M+3} &= \underline{\Gamma}^\top; & \underline{Q}_{j+2,j+2} &= -\underline{\Omega}_j^\top A_j - A_j^\top \underline{\Omega}_j + \underline{\Xi}^j, \\
\underline{Q}_{s+2,z'+2} &= -\underline{\Omega}_s^\top A_{z'} - A_s^\top \underline{\Omega}_{z'} + H_s \underline{\Upsilon}_{s,z'} H_{z'}^\top, \\
\underline{Q}_{j+2,M+3} &= -\underline{\Omega}_j^\top; & \underline{Q}_{M+3,M+3} &= \underline{\gamma} I_n.
\end{aligned}$$

$$j, j' \in \overline{1, M}; \quad z \in \overline{j+1, M}; \quad s \in \overline{1, M-1}; \quad z' \in \overline{s+1, M},$$

then the system (2) is ASTS with respect to  $([0, T], \epsilon_1, \epsilon_2, \gamma_0, \delta_1, \delta_2)$ .

The idea of the proof of this theorem is to consider two Lyapunov functions,  $\bar{V}$  and  $\underline{V}$ , whose upper and lower estimates, respectively, provide the corresponding bounds for the behavior of the state  $\|x(t)\|$  (this explains why there are two sets of conditions).

Considering the computation of expressions for  $\dot{\bar{V}}$  and  $\dot{\underline{V}}$  in the proof of Theorem 1, the term  $\dot{x}$  in  $(\dot{x}^\top \bar{P}x + x^\top \bar{P}\dot{x})$  can also be expanded using (2). As a result, the terms  $A_0^\top \underline{P} + \underline{P}A_0$  and  $A_0^\top \bar{P} + \bar{P}A_0$  will appear in the elements  $\underline{Q}_{2,2}$  and  $\bar{Q}_{2,2}$ , respectively (together with other corresponding modifications). Depending on the properties of  $A_0$ , such a substitution may provide more possibilities for the LMIs solution in Theorem 1.

One shall note that the conditions (3)–(6) and the ones (7)–(10) are independent of each other, which means that the computational complexity of the LMIs is not high comparing with their variations successfully solved in the literature as in [26], for example. Also, in the given conditions, the selections of the parameters  $\epsilon_1, \epsilon_2, \delta_1, \delta_2$  are per the practical demands on the behavior of the solution of the system (2),  $\beta_1$  and  $\beta_2$  can take positive or negative values, and  $\gamma_0$  should be not smaller than  $\|u\|_{[0,T]}$  (for practical verification, one can select them based on error and trial method). Taking into account the general shape of the functions  $\underline{V}$  and  $\bar{V}$  used for establishing stable or unstable bounding compartments, one shall see that the corresponding conditions are not restrictive and under proper selections of the tuning parameters, the given LMIs can be satisfied.

For the formulation of the STBNZ conditions for system (2), we set  $\epsilon_1 = \delta_1 = 0$  in Theorem 1 and obtain the following corollary:

**Corollary 1.** *Assume that all conditions of Theorem 1 are satisfied under the substitutions  $\epsilon_1 \rightarrow 0, \delta_1 \rightarrow 0$ , and the eliminations of (3)–(6). Then system (2) is STBNZ with respect to  $([0, T], \epsilon_2, \gamma_0, \delta_2)$ .*

**Proof.** For STBNZ property, the conditions (3)–(6) under the restrictions of Corollary 1 can be omitted due to  $\|x\| \geq 0$  for  $x \in \mathbb{R}^n$ .  $\square$

As it follows from definitions 1 and 2, the requirements of AS are much weaker than those of ASTS (the constraints on the state norm are imposed only for  $t = T$ ). At the same time, the former notion is sufficient for investigating the classification problem in neural networks. The following corollary formulates the sufficient conditions for the AS property of (2), which will be used in the next section for an application to recurrent neural networks.

**Corollary 2.** *Let assumptions 1 and 2 be satisfied and  $T > 0; \gamma_0 \geq 0; (\epsilon_1, \epsilon_2), (\delta_1, \delta_2) \subset \mathbb{R}_+$  be given. If there exist  $0 \leq \bar{P} = \bar{P}^\top, \underline{P} = \underline{P}^\top \in \mathbb{R}^{n \times n}; \{\bar{\Lambda}_j = \text{diag}(\bar{\Lambda}_j^1, \dots, \bar{\Lambda}_j^{k_j}), \underline{\Lambda}_j = \text{diag}(\underline{\Lambda}_j^1, \dots, \underline{\Lambda}_j^{k_j})\}_{j=1}^M \subset \mathbb{D}_+^{k_j}; \{\bar{\Upsilon}_{0,j}, \underline{\Upsilon}_{0,j}\}_{j=1}^M \subset \mathbb{D}^{k_j}; \{\bar{\Upsilon}_{s,z}, \underline{\Upsilon}_{s,z}\}_{1 \leq s < z \leq M} \subset \mathbb{D}^n; \{\bar{\Omega}^j, \underline{\Omega}^j\}_{j=1}^M \subset \mathbb{R}^{n \times k_j};$  symmetric matrices  $\bar{\Gamma}, \underline{\Gamma}, \bar{\Psi}, \underline{\Psi} \in \mathbb{R}^{n \times n}, \{\bar{\Xi}^s, \underline{\Xi}^s\}_{s=0}^M \subset \mathbb{R}^{k_s \times k_s}; \bar{\gamma}, \underline{\gamma} > 0$  and  $\beta_1, \beta_2, \bar{\rho}, \underline{\rho} \in \mathbb{R}$  such*

that

$$\begin{aligned}
& \underline{P} + \underline{\rho} \sum_{j=1}^{\mu} H_j^\top \underline{\Lambda}_j H_j > 0, \\
& \underline{Q} = \underline{Q}^\top = \left( \underline{Q}_{a,b} \right)_{a,b=1}^{M+3} \geq 0, \\
& \begin{cases} \gamma \leq \frac{e^{\beta_1 T} \underline{\alpha}_1(\epsilon_1) - \underline{\alpha}_2(\delta_1)}{\frac{\gamma_0^2}{\beta_1} [e^{\beta_1 T} - 1]}, & \text{if } \beta_1 \neq 0, \\ \gamma \leq \frac{\underline{\alpha}_1(\epsilon_1) - \underline{\alpha}_2(\delta_1)}{T \gamma_0^2}, & \text{if } \beta_1 = 0, \end{cases} \\
& \begin{cases} \underline{\Lambda}_j \leq \Lambda_j, \\ -\underline{\Xi}^0 \geq \beta_1 \left( \underline{P} + \sum_{j=1}^M H_j^\top \eta_{0,j} H_j \right), \\ -\underline{\Xi}^j \geq \beta_1 \sum_{j'=1}^M \eta_{1,jj'}, \\ -\underline{\Upsilon}_{0,j} \geq \beta_1 \sum_{j'=1}^M \eta_{2,jj'}, \\ -\underline{\Upsilon}_{j,z} \geq \beta_1 \sum_{j'=1}^M \eta_{3,jj'z}, \end{cases} & \text{if } \beta_1 \geq 0, \\
& \begin{cases} \underline{\Lambda}_j \geq \Lambda_j, \\ -\beta_1 \left( \underline{P} + \sum_{j=1}^M H_j^\top \kappa_{0,j} H_j \right) \geq \underline{\Xi}^0, \\ -\beta_1 \sum_{j'=1}^M \kappa_{1,jj'} \geq \underline{\Xi}^j, \\ -\beta_1 \sum_{j'=1}^M \kappa_{2,jj'} \geq \underline{\Upsilon}_{0,j}, \\ -\beta_1 \sum_{j'=1}^M \kappa_{3,jj'z} \geq \underline{\Upsilon}_{j,z}. \end{cases} & \text{if } \beta_1 < 0,
\end{aligned}$$

$$\begin{aligned}
& \overline{P} + \overline{\rho} \sum_{j=1}^{\mu} H_j^\top \overline{\Lambda}_j H_j > 0, \\
& \overline{Q} = \overline{Q}^\top = \left( \overline{Q}_{a,b} \right)_{a,b=1}^{M+3} \leq 0, \\
& \begin{cases} \overline{\gamma} \leq \frac{\overline{\alpha}_1(\delta_2) - e^{\beta_2 T} \overline{\alpha}_2(\epsilon_2)}{\frac{\gamma_0^2}{\beta_2} [e^{\beta_2 T} - 1]}, & \text{if } \beta_2 \neq 0, \\ \overline{\gamma} \leq \frac{\overline{\alpha}_1(\delta_2) - \overline{\alpha}_2(\epsilon_2)}{T \gamma_0^2}, & \text{if } \beta_2 = 0, \end{cases} \\
& \begin{cases} \overline{\Lambda}_j \leq \Lambda_j, \\ \overline{\Xi}^0 \geq -\beta_2 \left( \overline{P} + \sum_{j=1}^M H_j^\top \eta_{0,j} H_j \right), \\ \overline{\Xi}^j \geq -\beta_2 \sum_{j'=1}^M \eta_{1,jj'}, \\ \overline{\Upsilon}_{0,j} \geq -\beta_2 \sum_{j'=1}^M \eta_{2,jj'}, \\ \overline{\Upsilon}_{j,z} \geq -\beta_2 \sum_{j'=1}^M \eta_{3,jj'z}, \end{cases} & \text{if } \beta_2 < 0, \\
& \begin{cases} \overline{\Lambda}_j \geq \Lambda_j, \\ \beta_2 \left( \overline{P} + \sum_{j=1}^M H_j^\top \kappa_{0,j} H_j \right) \geq -\overline{\Xi}^0, \\ \beta_2 \sum_{j'=1}^M \kappa_{1,jj'} \geq -\overline{\Xi}^j, \\ \beta_2 \sum_{j'=1}^M \kappa_{2,jj'} \geq -\overline{\Upsilon}_{0,j}, \\ \beta_2 \sum_{j'=1}^M \kappa_{3,jj'z} \geq -\overline{\Upsilon}_{j,z}. \end{cases} & \text{if } \beta_2 \geq 0,
\end{aligned}$$

where

$$\begin{aligned}\alpha_1(s) &= \alpha_1^{\overline{P}, \overline{\Lambda}_1, \dots, \overline{\Lambda}_M}(s), \quad \underline{\alpha}_2(s) = \alpha_2^{\overline{P}, \overline{\Lambda}_1, \dots, \overline{\Lambda}_M}(s), \\ \overline{\alpha}_1(s) &= \alpha_1^{\overline{P}, \overline{\Lambda}_1, \dots, \overline{\Lambda}_M}(s), \quad \overline{\alpha}_2(s) = \alpha_2^{\overline{P}, \overline{\Lambda}_1, \dots, \overline{\Lambda}_M}(s).\end{aligned}$$

$$\begin{aligned}\overline{Q}_{1,1} &= -\overline{\Psi}^\top - \overline{\Psi}; \quad \overline{Q}_{1,2} = \overline{\Psi}^\top A_0 + \overline{P} - \overline{\Gamma}; \\ \overline{Q}_{1,j+2} &= \overline{\Omega}_j + H_j^\top \overline{\Lambda}_j + \overline{\Psi}^\top A_j; \quad \overline{Q}_{1,M+3} = \overline{\Psi}^\top; \\ \overline{Q}_{2,2} &= \overline{\Gamma}^\top A_0 + A_0^\top \overline{\Gamma} + \overline{\Xi}_0; \quad \overline{Q}_{2,j+2} = \overline{\Gamma}^\top A_j + H_j^\top \overline{\Upsilon}_{0,j} - A_0^\top \overline{\Omega}_j, \\ \overline{Q}_{2,M+3} &= \overline{\Gamma}^\top; \quad \overline{Q}_{j+2,j+2} = -\overline{\Omega}_j^\top A_j - A_j^\top \overline{\Omega}_j + \overline{\Xi}^j, \\ \overline{Q}_{s+2,z'+2} &= -\overline{\Omega}_s^\top A_{z'} - A_s^\top \overline{\Omega}_{z'} + H_s \overline{\Upsilon}_{s,z'} H_{z'}^\top, \\ \overline{Q}_{j+2,M+3} &= -\overline{\Omega}_j^\top; \quad \overline{Q}_{M+3,M+3} = -\overline{\gamma} I_n.\end{aligned}$$

$$\begin{aligned}Q_{1,1} &= -\underline{\Psi}^\top - \underline{\Psi}; \quad Q_{1,2} = \underline{\Psi}^\top A_0 + \underline{P} - \underline{\Gamma}; \\ Q_{1,j+2} &= \underline{\Omega}_j + H_j^\top \underline{\Lambda}_j + \underline{\Psi}^\top A_j; \quad Q_{1,M+3} = \underline{\Psi}^\top; \\ Q_{2,2} &= \underline{\Gamma}^\top A_0 + A_0^\top \underline{\Gamma} + \underline{\Xi}_0; \quad Q_{2,j+2} = \underline{\Gamma}^\top A_j + H_j^\top \underline{\Upsilon}_{0,j} - A_0^\top \underline{\Omega}_j; \\ Q_{2,M+3} &= \underline{\Gamma}^\top; \quad Q_{j+2,j+2} = -\underline{\Omega}_j^\top A_j - A_j^\top \underline{\Omega}_j + \underline{\Xi}^j, \\ Q_{s+2,z'+2} &= -\underline{\Omega}_s^\top A_{z'} - A_s^\top \underline{\Omega}_{z'} + H_s \underline{\Upsilon}_{s,z'} H_{z'}^\top, \\ Q_{j+2,M+3} &= -\underline{\Omega}_j^\top; \quad Q_{M+3,M+3} = \underline{\gamma} I_n.\end{aligned}$$

$$j, j' \in \overline{1, M}; \quad z \in \overline{j+1, M}; \quad s \in \overline{1, M-1}; \quad z' \in \overline{s+1, M},$$

then system (2) is AS with respect to  $(T, \epsilon_1, \epsilon_2, \gamma_0, \delta_1, \delta_2)$ .

**Proof.** In such a case, our goal is to ensure the fulfillment of the relaxed conditions  $\overline{V}(x(T)) \leq \overline{\alpha}_1(\delta_2)$ ,  $\underline{V}(x(T)) \geq \underline{\alpha}_2(\delta_1)$ , which can be directly deduced under the applied substitutions, then the conditions of this corollary imply that all necessary counterparts from Theorem 1 are verified, and the conclusion follows.  $\square$

Note that in this corollary, there is no restriction on relations between  $(\epsilon_1, \epsilon_2)$  and  $(\delta_1, \delta_2)$ . Hence, these intervals may be inside one another or even not intersecting. In applications of neural networks, there is usually an output that has to approach desired levels to classify different temporal sequences. We further consider oAS conditions for system (2) to address this problem.

**Theorem 2.** Let assumptions 1 and 2 be satisfied and  $T > 0$ ,  $\gamma_0 \geq 0$ ,  $(\delta_1, \delta_2) \in \mathbb{R}_+$ , and an initial condition  $x_0 \in \mathbb{R}^n$  with  $Cx_0 \neq 0$  be given. If there exist  $0 \leq \overline{P}_1, \underline{P}_1 \in \mathbb{R}^{n \times n}$ ,  $0 < \overline{P}_2, \underline{P}_2 \in \mathbb{R}^{p \times p}$ ;  $\{\overline{\Lambda}_j = \text{diag}(\overline{\Lambda}_j^1, \dots, \overline{\Lambda}_j^{k_j}), \underline{\Lambda}_j = \text{diag}(\underline{\Lambda}_j^1, \dots, \underline{\Lambda}_j^{k_j})\}_{j=1}^M \subset \mathbb{D}_+^{k_j}$ ;  $\{\overline{\Upsilon}_{0,j}, \underline{\Upsilon}_{0,j}\}_{j=1}^M \subset \mathbb{D}^{k_j}$ ;  $\{\overline{\Upsilon}_{s,z}, \underline{\Upsilon}_{s,z}\}_{1 \leq s < z \leq M} \subset \mathbb{D}^n$ ;  $\{\overline{\Omega}^j, \underline{\Omega}^j\}_{j=1}^M \subset \mathbb{R}^{n \times k_j}$ ; symmetric matrices  $\overline{\Gamma}, \underline{\Gamma}, \overline{\Psi}, \underline{\Psi} \in \mathbb{R}^{n \times n}$ ,  $\{\overline{\Xi}^s, \underline{\Xi}^s\}_{s=0}^M \subset \mathbb{R}^{k_s \times k_s}$ ;  $\overline{\gamma}, \underline{\gamma} > 0$  and  $\beta_1, \beta_2, \underline{\rho}, \underline{\ell} \in \mathbb{R}$

such that

$$\begin{aligned}
\underline{P} + \underline{\rho} \sum_{j=1}^{\mu} H_j^\top \underline{\Lambda}_j H_j &\leq \underline{\ell} C^\top C, \quad \underline{P} := \underline{P}_1 + C^\top \underline{P}_2 C, \\
\underline{Q} = \underline{Q}^\top &= \left( \underline{Q}_{a,b} \right)_{a,b=1}^{M+3} \geq 0, \\
\begin{cases} \underline{\gamma} \leq \frac{e^{\beta_1 T} \underline{\alpha}_1 (\|C x_0\|) - \underline{\alpha}_2 (\delta_1) \frac{\gamma_0^2}{\beta_1}}{\gamma_0^2} [e^{\beta_1 T} - 1], & \text{if } \beta_1 \neq 0, \\ \underline{\gamma} \leq \frac{\underline{\alpha}_1 (\|C x_0\|)}{T \gamma_0^2} - \underline{\alpha}_2 (\delta_1) T \gamma_0^2, & \text{if } \beta_1 = 0, \end{cases}
\end{aligned}$$

$$\begin{cases} \underline{\Lambda}_j \leq \Lambda_j, \\ -\underline{\Xi}^0 \geq \beta_1 \left( \underline{P} + \sum_{j=1}^M H_j^\top \eta_{0,j} H_j \right), \\ -\underline{\Xi}^j \geq \beta_1 \sum_{j'=1}^M \eta_{1,jj'}, \\ -\underline{\Upsilon}_{0,j} \geq \beta_1 \sum_{j'=1}^M \eta_{2,jj'}, \\ -\underline{\Upsilon}_{j,z} \geq \beta_1 \sum_{j'=1}^M \eta_{3,jj'z}, \end{cases} \quad \text{if } \beta_1 \geq 0,$$

$$\begin{cases} \underline{\Lambda}_j \geq \Lambda_j, \\ -\beta_1 \left( \underline{P} + \sum_{j=1}^M H_j^\top \kappa_{0,j} H_j \right) \geq \underline{\Xi}^0, \\ -\beta_1 \sum_{j'=1}^M \kappa_{1,jj'} \geq \underline{\Xi}^j, \\ -\beta_1 \sum_{j'=1}^M \kappa_{2,jj'} \geq \underline{\Upsilon}_{0,j}, \\ -\beta_1 \sum_{j'=1}^M \kappa_{3,jj'z} \geq \underline{\Upsilon}_{j,z}. \end{cases} \quad \text{if } \beta_1 < 0,$$

$$\begin{aligned}
\overline{P} &:= \overline{P}_1 + C^\top \overline{P}_2 C, \\
\overline{Q} = \overline{Q}^\top &= \left( \overline{Q}_{a,b} \right)_{a,b=1}^{M+3} \leq 0, \\
\begin{cases} \overline{\gamma} \leq \frac{\overline{\alpha}_1 (\delta_2) - e^{\beta_2 T} \overline{\alpha}_2 (\|x_0\|)}{\frac{\gamma_0^2}{\beta_2} [e^{\beta_2 T} - 1]}, & \text{if } \beta_2 \neq 0, \\ \overline{\gamma} \leq \frac{\overline{\alpha}_1 (\delta_2) - \overline{\alpha}_2 (\|x_0\|)}{T \gamma_0^2}, & \text{if } \beta_2 = 0, \end{cases} \tag{11}
\end{aligned}$$

$$\begin{cases} \overline{\Lambda}_j \leq \Lambda_j, \\ \overline{\Xi}^0 \geq -\beta_2 \left( \overline{P} + \sum_{j=1}^M H_j^\top \eta_{0,j} H_j \right), \\ \overline{\Xi}^j \geq -\beta_2 \sum_{j'=1}^M \eta_{1,jj'}, \\ \overline{\Upsilon}_{0,j} \geq -\beta_2 \sum_{j'=1}^M \eta_{2,jj'}, \\ \overline{\Upsilon}_{j,z} \geq -\beta_2 \sum_{j'=1}^M \eta_{3,jj'z}, \end{cases} \quad \text{if } \beta_2 < 0,$$

$$\begin{cases} \overline{\Lambda}_j \geq \Lambda_j, \\ \beta_2 \left( \overline{P} + \sum_{j=1}^M H_j^\top \kappa_{0,j} H_j \right) \geq -\overline{\Xi}^0, \\ \beta_2 \sum_{j'=1}^M \kappa_{1,jj'} \geq -\overline{\Xi}^j, \\ \beta_2 \sum_{j'=1}^M \kappa_{2,jj'} \geq -\overline{\Upsilon}_{0,j}, \\ \beta_2 \sum_{j'=1}^M \kappa_{3,jj'z} \geq -\overline{\Upsilon}_{j,z}. \end{cases} \quad \text{if } \beta_2 \geq 0,$$

where

$$\begin{aligned}\alpha_1(s) &= \lambda_{\min}(P_2)s^2, \quad \alpha_2(s) = \alpha_2^{P, \Lambda_1, \dots, \Lambda_M}(s), \\ \bar{\alpha}_1(s) &= \lambda_{\min}(\bar{P}_2)s^2, \quad \bar{\alpha}_2(s) = \alpha_2^{\bar{P}, \bar{\Lambda}_1, \dots, \bar{\Lambda}_M}(s).\end{aligned}$$

$$\begin{aligned}\bar{Q}_{1,1} &= -\bar{\Psi}^\top - \bar{\Psi}; \quad \bar{Q}_{1,2} = \bar{\Psi}^\top A_0 + \bar{P} - \bar{\Gamma}; \\ \bar{Q}_{1,j+2} &= \bar{\Omega}_j + H_j^\top \bar{\Lambda}_j + \bar{\Psi}^\top A_j; \quad \bar{Q}_{1,M+3} = \bar{\Psi}^\top; \\ \bar{Q}_{2,2} &= \bar{\Gamma}^\top A_0 + A_0^\top \bar{\Gamma} + \bar{\Xi}_0; \quad \bar{Q}_{2,j+2} = \bar{\Gamma}^\top A_j + H_j^\top \bar{\Upsilon}_{0,j} - A_0^\top \bar{\Omega}_j, \\ \bar{Q}_{2,M+3} &= \bar{\Gamma}^\top; \quad \bar{Q}_{j+2,j+2} = -\bar{\Omega}_j^\top A_j - A_j^\top \bar{\Omega}_j + \bar{\Xi}^j, \\ \bar{Q}_{s+2,z'+2} &= -\bar{\Omega}_s^\top A_{z'} - A_s^\top \bar{\Omega}_{z'} + H_s \bar{\Upsilon}_{s,z'} H_{z'}^\top, \\ \bar{Q}_{j+2,M+3} &= -\bar{\Omega}_j^\top; \quad \bar{Q}_{M+3,M+3} = -\bar{\gamma} I_n.\end{aligned}$$

$$\begin{aligned}Q_{1,1} &= -\Psi^\top - \Psi; \quad Q_{1,2} = \Psi^\top A_0 + P - \Gamma; \\ Q_{1,j+2} &= \Omega_j + H_j^\top \Lambda_j + \Psi^\top A_j; \quad Q_{1,M+3} = \Psi^\top; \\ Q_{2,2} &= \Gamma^\top A_0 + A_0^\top \Gamma + \Xi_0; \quad Q_{2,j+2} = \Gamma^\top A_j + H_j^\top \Upsilon_{0,j} - A_0^\top \Omega_j; \\ Q_{2,M+3} &= \Gamma^\top; \quad Q_{j+2,j+2} = -\Omega_j^\top A_j - A_j^\top \Omega_j + \Xi^j, \\ Q_{s+2,z'+2} &= -\Omega_s^\top A_{z'} - A_s^\top \Omega_{z'} + H_s \Upsilon_{s,z'} H_{z'}^\top, \\ Q_{j+2,M+3} &= -\Omega_j^\top; \quad Q_{M+3,M+3} = \gamma I_n.\end{aligned}$$

$$j, j' \in \overline{1, M}; \quad z \in \overline{j+1, M}; \quad s \in \overline{1, M-1}; \quad z' \in \overline{s+1, M},$$

then system (2) is oAS with respect to  $(T, x_0, \gamma_0, \delta_1, \delta_2)$ .

**Proof.** This proof follows the arguments of Theorem 1 and Corollary 2, except that the Lyapunov functions

$$\begin{aligned}\bar{V}(x) &= x^\top \left( \bar{P}_1 + C^\top \bar{P}_2 C \right) x + 2 \sum_{j=1}^M \sum_{i=1}^{k_j} \bar{\Lambda}_j^i \int_0^{H_j^{(i)} x} f_j^i(\tau) d\tau, \\ \underline{V}(x) &= x^\top \left( \underline{P}_1 + C^\top \underline{P}_2 C \right) x + 2 \sum_{j=1}^M \sum_{i=1}^{k_j} \underline{\Lambda}_j^i \int_0^{H_j^{(i)} x} f_j^i(\tau) d\tau\end{aligned}$$

and the conditions

$$\begin{aligned}\bar{\alpha}_1(\|Cx\|) &\leq \bar{V}(x) \leq \bar{\alpha}_2(\|x\|), \\ \alpha_1(\|Cx\|) &\leq \underline{V}(x) \leq \alpha_2(\|Cx\|), \quad \forall x \in \mathbb{R}^n\end{aligned}$$

are considered. The descriptor method is also utilized.  $\square$

Note also that the conditions in Theorem 2 are much milder than the ones in

Theorem 1, which allows the corresponding LMIs in Theorem 2 to be solved more easily. The condition  $Cx_0 \neq 0$  can be avoided by allowing  $\underline{\gamma}$  to change its sign.

## 5. Application to Recurrent Neural Networks

In this section, we illustrate the usefulness of the proposed conditions for continuous-time recurrent neural networks (CTRNNs).

The considered CTRNN is [27]:

$$\begin{aligned}\dot{\chi}(t) &= A\chi(t) + W_0g(W_1\chi(t)) + u(t), \quad t \in \mathbb{R}_+, \\ y(t) &= \tilde{C}\chi(t),\end{aligned}\tag{12}$$

where  $\chi(t) \in \mathbb{R}^n$  is the state vector,  $\chi(0) = \chi_0$ ;  $A \in \mathbb{R}^{n \times n}$ ,  $W_0 \in \mathbb{R}^{n \times N}$ ,  $W_1 \in \mathbb{R}^{N \times n}$  are the weight matrices;  $g: \mathbb{R}^N \rightarrow \mathbb{R}^N$  is the activation function;  $u \in \mathcal{L}_\infty^n$  is the input;  $y(t) \in \mathbb{R}^p$  is the output and  $\tilde{C} \in \mathbb{R}^{p \times n}$ ,  $N \geq n$  is the number of neurons in the hidden layer. It is clear that the CTRNN (12) is in the form of (2), and the Hopfield neural network [28] is a special example of (12).

Here we consider typical cases of non-polynomial activation functions [29], *e.g.*, rectified linear unit, sigmoid, and hyperbolic tangent functions, satisfying the sector condition of Assumption 1.

During the training process, it is of primary importance to analyze AS and oAS of a trained CTRNN (12) at each step. The following corollary is for studying AS of (12).

**Corollary 3.** *If the conditions of Corollary 2 are satisfied under the substitutions  $A_0 \rightarrow A$ ,  $M \rightarrow 1$ ,  $A_1 \rightarrow W_0$ ,  $F_1 \rightarrow g$ ,  $H_1 \rightarrow W_1$ , then the CTRNN (12) is AS with respect to  $(T, \epsilon_1, \epsilon_2, \gamma_0, \delta_1, \delta_2)$ .*

**Proof.** This result is a direct consequence, considering the expressions of (2), (12) and Corollary 2.  $\square$

We then propose the following main theorem of oAS for the CTRNN (12).

**Theorem 3.** *If the conditions of Theorem 2 are satisfied under the substitutions  $A_0 \rightarrow A$ ,  $M \rightarrow 1$ ,  $A_1 \rightarrow W_0$ ,  $F_1 \rightarrow g$ ,  $H_1 \rightarrow W_1$ ,  $C \rightarrow \tilde{C}$ , then the CTRNN (12) is oAS with respect to  $(T, x_0, \gamma_0, \delta_1, \delta_2)$ .*

**Proof.** This proof is again straightforward under the consideration of the forms of (2), (12) and Theorem 2.  $\square$

**Remark 1.** *Note that if there exist non-zero biases in the activation function, then it is possible to rewrite the system (12) as follows:*

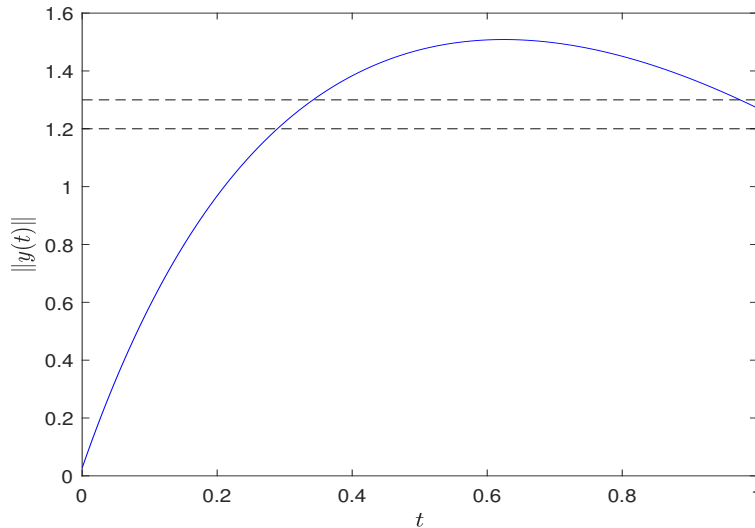
$$\begin{aligned}\dot{\xi}(t) &= \tilde{A}\xi(t) + \tilde{W}_0\tilde{g}([W_1 \quad b_0]\xi(t)) + \tilde{u}(t), \\ y(t) &= \hat{C}\xi(t)\end{aligned}$$

which is again in the form of (2), where

$$\begin{aligned}\xi(t) &= \begin{bmatrix} \chi(t) \\ \eta \end{bmatrix} \in \mathbb{R}^{n+N}, \quad \tilde{A} = \text{diag}(A, O_{N \times N}), \\ \tilde{g}(\tau) &= \begin{bmatrix} g(\tau) \\ g(\tau) \end{bmatrix}, \quad \tilde{C} = [\tilde{C} \quad O_{p \times N}], \\ \tilde{W}_0 &= \text{diag}(W_0, O_{N \times N}), \quad \tilde{u}(t) = \begin{bmatrix} u(t) \\ O_{N \times 1} \end{bmatrix}.\end{aligned}$$

Then a similar analysis can be performed under the guarantee that  $\chi(T)$  or  $y(T)$  will be in  $\text{cl}(B_{n+N}(\delta_2) \setminus B_{n+N}(\delta_1))$  or  $\text{cl}(B_p(\delta_2) \setminus B_p(\delta_1))$ , respectively.

**Remark 2.** The proposed conditions can also be applied to numerous variations of the CTRNNs, e.g., a variable activation function (VAF) sub-network scheme was introduced in [29], which can be embedded into a full connected CTRNN, resulting in a CTRNN taking the form of (2) with a sufficiently large  $M$ .



**Figure 1.** The trajectory of  $\|y(t)\|$

### 5.1. Numerical example

In this subsection, we consider Hopfield neural networks [28] with one hidden layer, which has  $n$  recurrent nodes of bipolar sigmoids and  $N$  neurons in the hidden layer.

**Example 1.** Let  $n = 2$ ,  $N = 50$ , and the activation function  $g^\ell = \tanh$  for all  $\ell \in \overline{1, N}$ . The weight matrix  $A$  and  $\tilde{C}$  are

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -5 \end{bmatrix}, \quad \tilde{C} = [-2.2 \quad 1.3].$$

We also obtained the values of  $W_0, W_1$  in the Hopfield neural network randomly and analyzed the oAS property. Assumption 1 is fulfilled due to the form of  $g$ . Also, As-



sumption 2 is satisfied for

$$\begin{aligned}\kappa_{0,1}^1 &= \dots = \kappa_{0,1}^{50} = 0, \quad \kappa_{1,11}^1 = \dots = \kappa_{1,11}^{50} = 0, \\ \kappa_{2,11}^1 &= \dots = \kappa_{2,11}^{50} = 0; \\ \eta_{0,1} &= \Lambda_1, \quad \eta_{1,11}^1 = \dots = \eta_{1,11}^{50} = 0, \\ \eta_{2,11}^1 &= \dots = \eta_{2,11}^{50} = 0\end{aligned}$$

since

$$\begin{aligned}2\mathbf{1}_N^\top \Lambda_1 \begin{bmatrix} \int_0^{W_1^{(1)}\chi} g^1(\tau) d\tau \\ \vdots \\ \int_0^{W_1^{(N)}\chi} g^N(\tau) d\tau \end{bmatrix} &= 2\mathbf{1}_N^\top \Lambda_1 \begin{bmatrix} \ln(\cosh(W_1^{(1)}\chi)) \\ \vdots \\ \ln(\cosh(W_1^{(N)}\chi)) \end{bmatrix} \\ &\leq \mathbf{1}_N^\top \Lambda_1 \begin{bmatrix} (W_1^{(1)}\chi)^2 \\ \vdots \\ (W_1^{(N)}\chi)^2 \end{bmatrix} \leq \chi^\top W_1^\top \eta_{0,1} W_1 \chi.\end{aligned}$$

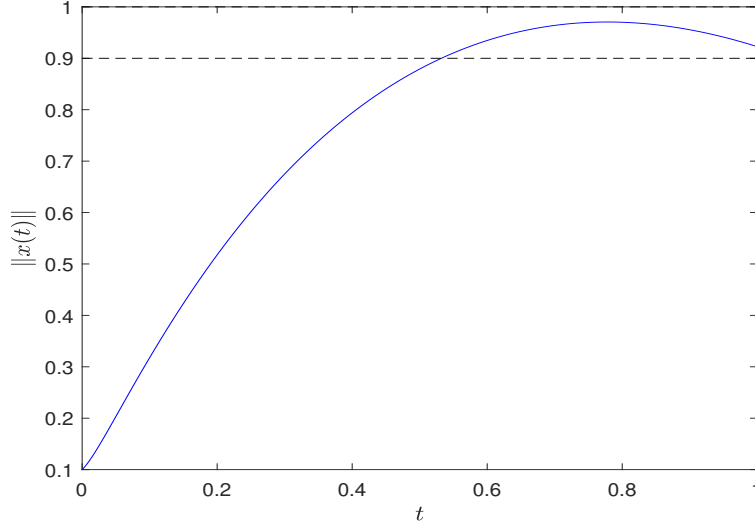
The oAS conditions in Theorem 3 with  $\delta_1 = 1.2$ ,  $\delta_2 = 1.3$ ,  $u(t) = \begin{bmatrix} 3 \cos(t) \\ \sin(t) \end{bmatrix}$ ,  $T = 1$ , and  $\gamma_0 = 16$  are verified. The trajectory of the norm of the output with a nonzero initial state on  $t \in [0, 1]$  is shown in Fig. 1. From this plot, we see that the given estimates are rather tight, considering the trajectory closely approaches the border at  $t = 1$ , which means oAS of the considered system.

Moreover, we illustrated the AS property of the considered CTRNN. We omitted the output  $y$  and let the other system parameters remain unchanged for brevity. The AS conditions are verified by

$$\begin{aligned}\epsilon_1 &= 0.09, \quad \epsilon_2 = 0.1, \quad \delta_1 = 0.9, \quad \delta_2 = 1, \quad u(t) = \begin{bmatrix} 3 \cos(t) \\ \sin(t) \end{bmatrix}, \quad T = 1, \quad \gamma_0 = 16, \\ \beta_1 &= 100, \quad \Lambda = \underline{\Lambda}_1 = 0, \quad \underline{P} > 0, \quad -\underline{\Xi}^0 \geq \beta_1 \underline{P}, \quad \underline{\Xi}^1 = O_{50 \times 50}, \quad -\underline{\Upsilon}_{0,1} \geq 0, \\ \underline{\gamma} &\leq \frac{e^{100} \underline{\alpha}_1(0.09) - \underline{\alpha}_2(0.9)}{2.56 \times [e^{100} - 1]}, \\ \underline{Q} = \underline{Q}^\top &= \begin{bmatrix} -\underline{\Psi}^\top - \underline{\Psi} & \underline{\Psi}^\top A + \underline{P} - \underline{\Gamma} & \underline{\Omega}_1 + \underline{\Psi}^\top W_0 & \underline{\Psi}^\top \\ * & \underline{\Gamma}^\top A + A^\top \underline{\Gamma} + \underline{\Xi}_0 & \underline{\Gamma}^\top W_0 + W_1^\top \underline{\Upsilon}_{0,1} - A^\top \underline{\Omega}_1 & \underline{\Gamma}^\top \\ * & * & -\underline{\Omega}_1^\top W_0 - W_0^\top \underline{\Omega}_1 & -\underline{\Omega}_1^\top \\ * & * & * & \underline{\gamma} I_n \end{bmatrix} \geq 0, \\ \beta_2 &= 0.01, \quad \Lambda = \bar{\Lambda}_1 = 0, \quad \bar{P} > 0, \quad \bar{\Xi}^0 = O_{2 \times 2}, \quad \bar{\Xi}^1 = O_{50 \times 50}, \quad \bar{\Upsilon}_{0,1} \geq 0, \\ \bar{\gamma} &\leq \frac{\bar{\alpha}_1(1) - e^{0.01} \bar{\alpha}_2(0.1)}{25600 \times [e^{0.01} - 1]}, \\ \bar{Q} = \bar{Q}^\top &= \begin{bmatrix} -\bar{\Psi}^\top - \bar{\Psi} & \bar{\Psi}^\top A + \bar{P} - \bar{\Gamma} & \bar{\Omega}_1 + \bar{\Psi}^\top W_0 & \bar{\Psi}^\top \\ * & \bar{\Gamma}^\top A + A^\top \bar{\Gamma} & \bar{\Gamma}^\top W_0 + W_1^\top \bar{\Upsilon}_{0,1} - A^\top \bar{\Omega}_1 & \bar{\Gamma}^\top \\ * & * & -\bar{\Omega}_1^\top W_0 - W_0^\top \bar{\Omega}_1 & -\bar{\Omega}_1^\top \\ * & * & * & -\bar{\gamma} I_n \end{bmatrix} \leq 0.\end{aligned}$$

Also, the trajectory of  $\|x(t)\|$  with a nonzero initial state over the time interval  $[0, 1]$

is presented in Fig. 2, from which one can easily see that the ASTS is not satisfied since  $0.9 \leq \|x(t)\| \leq 1$  does not hold for all  $t \in [0, 1]$ , while the AS property was verified (only the time instants  $t = 0$  and  $t = 1$  are taken into account). This demonstrates the main difference between ASTS and AS, and the usefulness of the latter notion (in the presence of output, oAS can be applied, for instance, in this example).



**Figure 2.** The trajectory of  $\|x(t)\|$

**Example 2.** In order to further illustrate the validity of the proposed conditions for different neural networks in practice, we then deal with a larger Hopfield neural network with  $n = 10$  recurrent nodes. Under the settings of  $N = 50$  and  $g^\ell = \tanh$  for  $\ell \in \overline{1, N}$ , we chose the tuple of matrices  $(A, W_0, W_1, C)$  in (12) arbitrarily, then the LMIs of Theorem 3 are again verified for  $\epsilon_1 = 0.05$ ,  $\epsilon_2 = 0.1$ ,  $\delta_1 = 2$ ,  $\delta_2 = 2.3$ ,  $T = 1$ , and  $\gamma_0 = 16$ . It is worth mentioning that the verification of the corresponding LMIs is feasible due to the high flexibility of the tuning parameters.

## 6. Conclusion

In this paper, notions of annular settling (AS) and output annular settling (oAS) were proposed, and sufficient conditions for annular short-time stability (ASTS), short-time boundedness with nonzero initial state (STBNZ), AS, and oAS in generalized Persidskii systems were given. The formulated conditions were obtained in the form of linear algebraic and matrix inequalities. Therefore, they can be constructively verified. An application to a continuous-time recurrent neural network was presented to validate the proposed results.

## Appendix. Proof of Theorem 1

*Proof.* Consider a candidate Lyapunov function

$$\bar{V}(x) = x^\top \bar{P}x + 2 \sum_{j=1}^M \sum_{i=1}^{k_j} \bar{\Lambda}_j^i \int_0^{H_j^{(i)}x} f_j^i(\tau) d\tau.$$

Under condition (7), there exist some functions  $\bar{\alpha}_1, \bar{\alpha}_2$  from class  $\mathcal{K}_\infty$  such that

$$\bar{\alpha}_1(\|x\|) \leq \bar{V}(x) \leq \bar{\alpha}_2(\|x\|), \quad \forall x \in \mathbb{R}^n$$

due to Finsler's Lemma and Assumption 1. Then consider the time derivative of  $\bar{V}$  along trajectories of (2), for which the following upper estimate holds true under the restriction (8):

$$\begin{aligned} \dot{\bar{V}} &= \nabla \bar{V}(x) \dot{x} \\ &= \dot{x}^\top \bar{P}x + x^\top \bar{P} \dot{x} + 2 \sum_{j=1}^M \dot{x}^\top H_j^\top \bar{\Lambda}_j F_j(H_j x) \\ &\quad + 2 \left( \sum_{j=1}^M F_j(H_j x)^\top \bar{\Omega}_j^\top - \dot{x}^\top \bar{\Psi}^\top - x^\top \bar{\Gamma}^\top \right) \left( \dot{x} - A_0 x \right. \\ &\quad \left. - \sum_{j=1}^M A_j F_j(H_j x) - u \right) \\ &= \begin{bmatrix} \dot{x} \\ x \\ F_1(H_1 x) \\ \vdots \\ F_M(H_M x) \\ u \end{bmatrix}^\top \bar{Q} \begin{bmatrix} \dot{x} \\ x \\ F_1(H_1 x) \\ \vdots \\ F_M(H_M x) \\ u \end{bmatrix} - x^\top \bar{\Xi}^0 x \\ &\quad - \sum_{j=1}^M F_j(H_j x)^\top \bar{\Xi}^j F_j(H_j x) - 2 \sum_{j=1}^M x^\top H_j^\top \bar{\Upsilon}_{0,j} F_j(H_j x) \\ &\quad - 2 \sum_{s=1}^{M-1} \sum_{z=s+1}^M F_s(H_s x)^\top H_s \bar{\Upsilon}_{s,z} H_z^\top F_z(H_z x) \\ &\quad + \bar{\gamma} u^\top u \\ &\leq -x^\top \bar{\Xi}^0 x - 2 \sum_{s=1}^{M-1} \sum_{z=s+1}^M F_s(H_s x)^\top H_s \bar{\Upsilon}_{s,z} H_z^\top F_z(H_z x) \\ &\quad - \sum_{j=1}^M F_j(H_j x)^\top \bar{\Xi}^j F_j(H_j x) - 2 \sum_{j=1}^M x^\top H_j^\top \bar{\Upsilon}_{0,j} F_j(H_j x) \\ &\quad + \bar{\gamma} u^\top u. \end{aligned}$$

Here the descriptor method [30] was applied on the second step.

Now we have to show that there exists  $\beta_2 \in \mathbb{R}$  such that

$$\begin{aligned} \beta_2 \bar{V}(x) &\geq -x^\top \bar{\Xi}^0 x - \sum_{j=1}^M F_j(H_j x)^\top \bar{\Xi}^j F_j(H_j x) \\ &- 2 \sum_{j=1}^M x^\top H_j^\top \bar{\Upsilon}_{0,j} F_j(H_j x) - 2 \sum_{s=1}^{M-1} \sum_{z=s+1}^M F_s(H_s x)^\top H_s \bar{\Upsilon}_{s,z} H_z^\top F_z(H_z x), \end{aligned}$$

which is true under Assumption 2 and the conditions (7), (10). Therefore, we have

$$\dot{\bar{V}} \leq \beta_2 \bar{V} + \bar{\gamma} u^\top u,$$

so that

$$\begin{aligned} \bar{V}(x(t)) &\leq e^{\beta_2 t} \bar{V}(x_0) + \int_0^t e^{\beta_2(t-s)} \bar{\gamma} u(s)^\top u(s) ds \\ &\leq e^{\beta_2 t} \bar{\alpha}_2(\|x_0\|) + \frac{\bar{\gamma} \gamma_0^2}{\beta_2} (e^{\beta_2 t} - 1) \\ &\leq e^{\beta_2 t} \bar{\alpha}_2(\epsilon_2) + \frac{\bar{\gamma} \gamma_0^2}{\beta_2} (e^{\beta_2 t} - 1). \end{aligned}$$

It further holds that

$$\begin{aligned} \sup_{t \in [0, T]} \bar{V}(x(t)) &\leq \sup_{t \in [0, T]} \left\{ a \eta(t) + b(\eta(t) - 1) \right\} \\ &= \begin{cases} a \eta(T) + b(\eta(T) - 1), & \text{if } \beta_2 > 0, \\ a + \bar{\gamma} \gamma_0^2 T, & \text{if } \beta_2 = 0, \\ a \eta(0) + b(\eta(0) - 1), & \text{if } \beta_2 < 0, a + b \geq 0, \\ a \eta(T) + b(\eta(T) - 1), & \text{if } \beta_2 < 0, a + b < 0 \end{cases} \end{aligned}$$

with the settings of

$$\eta(t) = e^{\beta_2 t}, \quad b = \frac{\bar{\gamma} \gamma_0^2}{\beta_2}, \quad a = \bar{\alpha}_2(\epsilon_2).$$

Therefore, under conditions (9), we obtain

$$\bar{V}(x(t)) \leq e^{\beta_2 t} \bar{\alpha}_2(\epsilon_2) + \frac{\bar{\gamma} \gamma_0^2}{\beta_2} [e^{\beta_2 t} - 1] \leq \bar{\alpha}_1(\delta_2), \quad \forall t \in [0, T],$$

by which we see that

$$\bar{\alpha}_1(\|x(t)\|) \leq \bar{V}(x(t)) \leq \bar{\alpha}_1(\delta_2) \quad \Rightarrow \quad \|x(t)\| \leq \delta_2, \quad \forall t \in [0, T].$$

Now let us evaluate the infimum of  $\|x(t)\|$  on  $t \in [0, T]$ , for which another candidate

Lyapunov function

$$\underline{V}(x) = x^\top \underline{P}x + 2 \sum_{j=1}^M \sum_{i=1}^{k_j} \underline{\Lambda}_j^i \int_0^{H_j^{(i)}x} f_j^i(\tau) d\tau$$

is considered. In a similar way as in the previous development, it follows that there exist some functions  $\underline{\alpha}_1, \underline{\alpha}_2 \in \mathcal{K}_\infty$  such that

$$\underline{\alpha}_1(\|x\|) \leq \underline{V}(x) \leq \underline{\alpha}_2(\|x\|), \quad \forall x \in \mathbb{R}^n$$

due to the condition (3), Finsler's Lemma and Assumption 1. Calculating the time derivative of  $\underline{V}$  on trajectories of (2), due to (4) and using the descriptor method, we obtain:

$$\begin{aligned} \dot{\underline{V}} &\geq -x^\top \underline{\Xi}^0 x - 2 \sum_{s=1}^{M-1} \sum_{z=s+1}^M F_s(H_s x)^\top H_s \underline{\Upsilon}_{s,z} H_z^\top F_z(H_z x) \\ &\quad - \sum_{j=1}^M F_j(H_j x)^\top \underline{\Xi}^j F_j(H_j x) - 2 \sum_{j=1}^M x^\top H_j^\top \underline{\Upsilon}_{0,j} F_j(H_j x) \\ &\quad - \underline{\gamma} u^\top u. \end{aligned}$$

Applying similar arguments, there exists  $\beta_1 \in \mathbb{R}$  such that

$$\begin{aligned} \beta_1 \underline{V}(x) &\leq -x^\top \underline{\Xi}^0 x - \sum_{j=1}^M F_j(H_j x)^\top \underline{\Xi}^j F_j(H_j x) \\ &\quad - 2 \sum_{j=1}^M x^\top H_j^\top \underline{\Upsilon}_{0,j} F_j(H_j x) - 2 \sum_{s=1}^{M-1} \sum_{z=s+1}^M F_s(H_s x)^\top H_s \underline{\Upsilon}_{s,z} H_z^\top F_z(H_z x) \end{aligned}$$

due to the condition (6) and Assumption 2. Then, it can be shown that under the conditions (5), the property

$$e^{\beta_1 t} \underline{\alpha}_1(\epsilon_1) - \frac{\gamma \gamma_0^2}{\beta_1} \left[ e^{\beta_1 t} - 1 \right] \geq \underline{\alpha}_2(\delta_1), \quad \forall t \in [0, T]$$

holds, which deduces

$$\underline{\alpha}_2(\delta_1) \leq \underline{V}(x(t)) \leq \underline{\alpha}_2(\|x(t)\|) \quad \Rightarrow \quad \|x(t)\| \geq \delta_1, \quad \forall t \in [0, T].$$

This completes the proof. □

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