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# Design of controls for ISS and Integral ISS Stabilization of Multistable State Periodic Systems<sup>\*</sup>

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**Abstract:** This paper is devoted to the construction of continuous and periodic controllers for ISS or iISS stabilization of a class of multistable state periodic systems. For this purpose, the concepts of practical ISS-, ISS- and iISS-control Leonov functions are introduced. Then, it is proven that the existence of these functions is a sufficient condition to ensure that the system under study is ISS- or iISS stabilizable, and explicit feedback control laws can be obtained by means of a mild adaptation of Sontag's universal formula. The proposed method is illustrated by the design of an excitation controller for a synchronous generator ensuring ISS of the closed-loop system.

*Keywords:* Input-to-state stability, Control Lyapunov functions, Multistable systems, State periodic systems.

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## 1. INTRODUCTION

The concept of Input-to-State Stability (ISS) is a useful tool for the analysis of stability and robustness of nonlinear systems affected by bounded external inputs (Sontag, 2008; Dashkovskiy et al., 2011). Likewise, the concept of integral ISS (iISS) has been developed for investigation of robustness against disturbances with bounded energy (Angeli et al., 2000). Commonly, engineering applications concern systems with multiple invariant solutions (e.g., power systems, biological systems, etc.). Therefore, the extensions of the ISS and iISS properties for the multistable case, and their characterization in terms of ISS- or iISS-Lyapunov functions have been proposed by (Angeli and Efimov, 2015) and (Forni and Angeli, 2016), respectively.

A particular class of multistable systems is formed by ones with periodic dynamics in a part of the state. Some examples of such a class of systems are the forced nonlinear pendulum (Forni and Sepulchre, 2014), power systems (Galaz et al., 2003; Schiffer and Dörfler, 2016), and microgrids (Schiffer et al., 2014). Unfortunately, the aforementioned Lyapunov based approaches cannot be straightforwardly applied for the case of state periodic systems. For instance, the utilization of the results of (Angeli and Efimov, 2015) or (Forni and Angeli, 2016) requires the construction of periodic ISS- or iISS-Lyapunov functions in order to keep the positive definiteness with respect to the invariant sets.

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This fact represents a strong limitation and in most cases just local stability results can be obtained, see for example (Schiffer et al., 2014; Dorfler and Bullo, 2012).

Alternatively, the *cell structure* approach proposed by (Leonov, 1974) and (Noldus, 1977), independently, is a fitting tool for the study of global boundedness of solutions for periodic systems in a scalar state variable. Recently, (Efimov and Schiffer, 2019) has presented an extension of the cell structure approach for systems with multiple state periodic variables by introducing the concept of *Leonov functions*. These functions are sign-indefinite in the periodic states, radially unbounded in the non-periodic ones, and its derivative is sign-definite with respect to its zero level. Inspired by these ideas, (Efimov et al., 2017) and (Mendoza-Avila et al., 2022) have introduced the concepts of *ISS-* and *iISS-Leonov function*, respectively, as relaxed characterizations of the ISS and iISS properties for multistable state periodic systems.

One of the main problems in control theory is the design of feedback control laws that endow the systems with robustness properties. This task has been mainly addressed through the approach of ISS- and iISS-control Lyapunov functions (Sontag and Wang, 1995; Liberzon et al., 2002). Especially, (Liberzon et al., 2002) proposes a unified framework for ISS and iISS stabilization of nonlinear perturbed systems, where the construction of continuous feedback control laws is based on the original idea of (Artstein, 1983), further developed by (Sontag, 1989).

Recently, those results have been extended for multistable affine nonlinear systems by (Barroso et al., 2019).

Now, this paper addresses ISS- and iISS-stabilization of multistable state periodic systems. For this, we introduce the concepts of practical ISS-, ISS- and iISS-control Leonov functions. Then, we show that the existence of these functions is a sufficient condition to ensure the existence of a continuous and periodic controller rendering the closed-loop system with the corresponding property, whose construction is based on a mild adjustment of the Sontag's universal formula and a kind of small control property (SCP). Finally, the proposed method is illustrated by designing an excitation controller for ISS stabilization of a synchronous generator.

The remainder of the paper is structured as follows. Section 2 presents the considered class of systems and some preliminary results. Section 3 exposes our main results. Section 4 provides the example to illustrate the usefulness of our proposal. Finally, Section 5 contains our conclusions.

## 2. PRELIMINARIES

### 2.1 Notation

$\mathbb{N}$ ,  $\mathbb{Z}$  and  $\mathbb{R}$  stand for the sets of natural, integer and real numbers, respectively. Moreover,  $\mathbb{R}_+$  represents the set of non-negative real numbers, i.e.,  $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ .

A function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{K}$  if it is continuous, strictly increasing and  $\alpha(0) = 0$ . A function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{K}_\infty$  if  $\alpha \in \mathcal{K}$  and it is unbounded. A continuous function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  belongs to the class  $\mathcal{KL}$  if, for each fixed  $t \in \mathbb{R}_+$ ,  $\beta(\cdot, t) \in \mathcal{K}_\infty$  and, for each fixed  $s \in \mathbb{R}_+$ ,  $\beta(s, \cdot)$  is non-increasing and it tends to zero for  $t \rightarrow \infty$ .

The distance from a point  $\tilde{x} \in \mathbb{R}^n$  to the set  $S \subset \mathbb{R}^n$  is defined as  $|\tilde{x}|_S = \inf_{a \in S} |\tilde{x} - a|$ , where  $|\tilde{x}| = |\tilde{x}|_{\{0\}}$  is the usual Euclidean norm for a vector  $\tilde{x} \in \mathbb{R}^n$ . Furthermore, we introduce the vector norm  $|\tilde{x}|_\infty = \max_{1 \leq i \leq n} |\tilde{x}_i|$ , such that  $|\tilde{x}|_\infty \leq |\tilde{x}| \leq \sqrt{n}|\tilde{x}|_\infty$ . Besides, for a signal  $d : \mathbb{R} \rightarrow \mathbb{R}^m$  the essential supremum norm is defined as  $\|d\|_\infty = \text{ess sup}_{t \geq 0} |d(t)|$ , and the set of such inputs with a finite norm is denoted by  $\mathcal{L}_\infty^m$ .

### 2.2 Multistable state periodic system

Let the map  $f(\tilde{x}, d) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  be of class  $\mathcal{C}^1$  (continuously differentiable with respect to its arguments). Consider the system

$$\dot{\tilde{x}} = f(\tilde{x}, d), \quad (1)$$

where  $\tilde{x} \in \mathbb{R}^n$  is the state and  $d \in \mathcal{L}_\infty^m$  is the input signal. For any  $\tilde{x}_0 \in \mathbb{R}^n$  and  $d \in \mathcal{L}_\infty^m$ , we denote by  $\tilde{X}(t, \tilde{x}_0; d)$  the uniquely defined solution of (1) at time  $t$  fulfilling  $\tilde{X}(0, \tilde{x}_0; d) = \tilde{x}_0$ .

A set  $S \subset \mathbb{R}^n$  is invariant for the system (1), with  $d \equiv 0$ , if  $\tilde{X}(t, \tilde{x}; 0) \in S$  for all  $t \in \mathbb{R}$  and for all  $\tilde{x} \in S$ ; for  $\tilde{x} \in \mathbb{R}^n$  the point  $y \in \mathbb{R}^n$  belongs to its  $\omega$ -limit ( $\alpha$ -limit) set if there is a sequence of time instants  $t_i$ ,  $\lim_{i \rightarrow +\infty} t_i = +\infty$ , such that  $\lim_{i \rightarrow +\infty} \tilde{X}(t_i, \tilde{x}; 0) = y$  ( $\lim_{i \rightarrow +\infty} \tilde{X}(-t_i, \tilde{x}; 0) = y$ ); for any  $\tilde{x} \in \mathbb{R}^n$  its  $\alpha$ - and  $\omega$ -limit sets are invariant (Guckenheimer and Holmes, 1983).

*Assumption 1.* Let  $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$ , where  $\tilde{z} \in \mathbb{R}^k$  and  $\tilde{\theta} \in \mathbb{R}^q$  are two subsets of the state vector,  $n = k + q$ ,  $k > 0$  and  $q > 0$ . For  $d \equiv 0$ , the vector field  $f$  in (1) is  $2\pi$ -periodic with respect to  $\tilde{\theta}$ , i.e. for all  $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$

$$f(\tilde{x}, 0) = f(\tilde{x} + \xi_j, 0), \quad (2)$$

where  $\xi_j = [0_k^\top \ 2\pi j^\top]^\top \in \mathbb{R}^n$ , with  $0_k$  being a zero vector of dimension  $k$  and  $j \in \mathbb{Z}^q$ .

In other words, we suppose that the system (1), with  $d \equiv 0$ , can be embedded in a manifold  $M = \mathbb{R}^k \times \mathbb{S}^q$ , where  $\mathbb{S}$  is the unit sphere, by a simple projection of the variables  $\tilde{\theta}$  on the sphere  $\mathbb{S}^q$ . Denote this projection by  $\mathcal{P} : \mathbb{R}^q \rightarrow \mathbb{S}^q$ , that is just a modulus of the  $\theta$  coordinates over  $2\pi$ , such that  $x = (z, \theta) \in M$  with  $z = \tilde{z} \in \mathbb{R}^k$  and  $\theta = \mathcal{P}(\tilde{\theta}) \in \mathbb{S}^q$ . Hence, for any  $\tilde{x} \in \mathbb{R}^n$  the solution  $\tilde{X}(t, \tilde{x}; d) \in \mathbb{R}^n$  of the system (1) can be projected to the solution  $X(t, x; d) \in M$  with  $x \in M$ , such that both solutions are defined on the same time interval. Similarly, the set  $\tilde{\mathcal{W}} \subset \mathbb{R}^n$ , containing all  $\alpha$ - and  $\omega$ -limit sets of the unperturbed system (1) (with  $d \equiv 0$ ), can be projected to  $\mathcal{W} \subset M$  by using the periodicity of the last  $q$  variables. Moreover,  $|x|_{\mathcal{W}} = \inf_{y \in \mathcal{W}} |x - y|$  is a distance to the set  $\mathcal{W}$  for  $x \in M$  (then  $|\cdot|$  represents the distance on  $M$ ). Note that the elements of set  $\tilde{\mathcal{W}}$  are uniformly distributed in the whole  $\mathbb{R}^n$  due to the periodicity of (1). Therefore, we can say that  $|x|_{\mathcal{W}} = |\tilde{x}|_{\tilde{\mathcal{W}}}$ . Moreover, even if  $\mathcal{W}$  is compact, the set  $\tilde{\mathcal{W}}$  becomes unbounded in  $\mathbb{R}^n$ , in a common case.

### 2.3 Decomposable sets

Let  $\Lambda \subset M$  be a compact invariant set for the unperturbed system (1) (with  $d \equiv 0$ ).

*Definition 2.* (Nitecki and Shub, 1975). A decomposition of  $\Lambda$  is a finite and disjoint family of compact invariant sets  $\Lambda_1, \dots, \Lambda_k$  such that  $\Lambda = \cup_{i=1}^k \Lambda_i$ , where  $\Lambda_i$  is called an *atom* of the decomposition.

For an invariant set  $\Lambda$ , its attracting and repulsing subsets are defined as follows:

$$\begin{aligned} \mathfrak{A}(\Lambda) &= \{x \in M : |X(t, x; 0)|_\Lambda \rightarrow 0 \text{ as } t \rightarrow +\infty\}, \\ \mathfrak{R}(\Lambda) &= \{x \in M : |X(t, x; 0)|_\Lambda \rightarrow 0 \text{ as } t \rightarrow -\infty\}. \end{aligned}$$

Then, define a relation on  $\mathcal{B} \subset M$  and  $\mathcal{D} \subset M$  by  $\mathcal{B} \prec \mathcal{D}$  if  $\mathfrak{A}(\mathcal{B}) \cap \mathfrak{R}(\mathcal{D}) \neq \emptyset$ .

*Definition 3.* (Nitecki and Shub, 1975). Let  $\Lambda_1, \dots, \Lambda_k$  be a decomposition of  $\Lambda$ , then

- (1) An  $r$ -cycle ( $r \geq 2$ ) is an ordered  $r$ -tuple of distinct indices  $i_1, \dots, i_r$  such that  $\Lambda_{i_1} \prec \dots \prec \Lambda_{i_r} \prec \Lambda_{i_1}$ .
- (2) A 1-cycle is an index  $i$  such that  $\mathfrak{A}(\Lambda_i) \cap \mathfrak{R}(\Lambda_i) \setminus \Lambda_i \neq \emptyset$ .
- (3) A filtration ordering is a numbering of the  $\Lambda_i$  so that  $\Lambda_i \prec \Lambda_j \implies i \leq j$ .

*Definition 4.* The set  $\Lambda$  is called decomposable if it admits a finite decomposition without cycles,  $\Lambda = \cup_{i=1}^k \Lambda_i$ , for some non-empty disjoint compact sets  $\Lambda_i$ , which form a filtration ordering of  $\Lambda$ , as detailed in Definitions 2 and 3.

*Assumption 5.* Let a set  $\tilde{\mathcal{W}} \subset \mathbb{R}^n$  contain all  $\alpha$ - and  $\omega$ -limit sets of (1) with  $d \equiv 0$ , and  $\mathcal{W}$ , its projection on  $M = \mathbb{R}^k \times \mathbb{S}^q$ , be compact and decomposable (as in Definition 4).

The definitions of pAG, pGS, limit, UBEBS, and 0-GATT properties used below are not provided due to space limitation but they can be revised in (Angeli and Efimov, 2015) and (Forni and Angeli, 2016).

*Definition 6.* (Angeli and Efimov, 2015). Suppose that the system (1) satisfies Assumption 5 and possesses the practical asymptotic gain (pAG), or the practical global stability (pGS) and the limit properties. Then it is said to be ISS with respect to the set  $\mathcal{W}$  and the input  $d \in \mathcal{L}_\infty^m$ .

*Definition 7.* (Forni and Angeli, 2016). The system (1) is said to be iISS with respect to the set  $\mathcal{W}$  and the input  $d \in \mathcal{L}_\infty^m$  if it satisfies Assumption 5, and possesses the uniform bounded energy bounded state (UBEBS) and zero global attractiveness (0-GATT) properties.

*Definition 8.* A  $\mathcal{C}^1$  function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be a practical ISS-Leonov function for the system (1) if there exist functions  $\vartheta_1, \vartheta_2, \sigma_1, \gamma, \lambda \in \mathcal{K}_\infty$ , and scalars  $g_1, \ell_1, r \geq 0$ , such that for all  $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$ :

$$\vartheta_1(|\tilde{x}|_{\tilde{\mathcal{W}}}) - \sigma_1(|\tilde{\theta}|) - g_1 \leq V(\tilde{x}) \leq \vartheta_2(|\tilde{x}|_{\tilde{\mathcal{W}}} + \ell_1), \quad (3)$$

and the following dissipation inequality

$$\frac{\partial V(\tilde{x})}{\partial \tilde{x}} f(\tilde{x}, d) \leq -\lambda(V(\tilde{x})) + \gamma(|d|) + r, \quad (4)$$

holds for all  $\tilde{x} \in \{\tilde{x} \in \mathbb{R}^n : V(\tilde{x}) \geq 0\}$  and all  $d \in \mathbb{R}^m$ . If the latter inequality holds for  $r = 0$ , then  $V$  is said to be an ISS-Leonov function. Moreover, if inequality (4) holds with  $r = 0$  and a non-negative function  $\lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , then  $V$  is said to be an iISS-Leonov function.

Practical ISS- and iISS-Leonov functions are commonly sign-indefinite and non-periodic permitting to address global analysis of robustness for the considered class of systems.

*Lemma 9.* Let Assumptions 1 and 5 be satisfied. Then, the existence of a practical ISS-Leonov function as in Definition 8 for the system (1) implies the pGS and pAG properties with respect to  $\mathcal{W}$ .

The concept of practical ISS-Leonov functions and the result of Lemma 9 were originally established in (Efimov et al., 2017). However, the original definition of a practical ISS-Leonov function is more restrictive than the provided here, since the dissipation inequality (4) must be satisfied for all  $\tilde{x} \in \mathbb{R}^n$ , with  $\lambda \in \mathcal{K}_\infty$  only for nonnegative arguments. Despite of this, the proof of (Efimov et al., 2017, Lemma1) requires such a dissipation inequality only for the set  $\{\tilde{x} \in \mathbb{R}^n : V(\tilde{x}) \geq 0\}$ . Thus, the claim of Lemma 9 can be conclude by using Definition 8. A detailed proof is omitted due to the space limitation.

*Lemma 10.* (Mendoza-Avila et al., 2022). Let Assumptions 1 and 5 be satisfied. Then, the existence of an iISS-Leonov function as in Definition 8 for the system (1) implies 0-GATT and UBEBS properties with respect to  $\mathcal{W}$ .

The concept iISS-Leonov functions appears for the first time in (Mendoza-Avila et al., 2022) just as it is presented in Definition 8, and the proof of Lemma 10 is also given there.

Consider a system of the form

$$\dot{\tilde{\theta}} = f_1(\tilde{x}, d), \quad \dot{\tilde{z}} = f_2(\tilde{x}, d) + g(\tilde{x})u, \quad (5)$$

where  $\tilde{x} \in \mathbb{R}^n$  is the state vector,  $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$  with  $\tilde{z} \in \mathbb{R}^k$  and  $\tilde{\theta} \in \mathbb{R}^q$ ,  $n = k + q$  for  $k, q > 0$ ,  $u \in \mathcal{L}_\infty^p$  is a control input, and  $d \in \mathcal{L}_\infty^m$  is a vector of perturbation signals. Moreover,  $f_1 : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^q$ ,  $f_2 : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^k$  and  $g : \mathbb{R}^n \rightarrow \mathbb{R}^{k \times p}$  are supposed to be locally Lipschitz continuous vector functions.

*Assumption 11.* The vector fields  $f_1, f_2$  and  $g$  in (5) are  $2\pi$ -periodic with respect to  $\tilde{\theta}$ .

*Assumption 12.* For  $d = 0$  and  $u = 0$ , let the set  $\tilde{\mathcal{W}} \subset \mathbb{R}^n$  contain all  $\alpha$ - and  $\omega$ -limit sets of (5) and  $\mathcal{W}$ , its projection on  $M = \mathbb{R}^k \times \mathbb{S}^q$ , be compact and decomposable (in the sense of Definition 4).

Hence, the system (5) is said to be ISS or iISS stabilizable if there exists a continuous controller  $u : \mathbb{R}^n \rightarrow \mathbb{R}^p$ , periodic in  $\tilde{\theta}$ , which endows the system with the respective property. Therefore, the problem consists in determine conditions ensuring the existence of such a controller. The solution proposed in this work is based on the concepts of (practical) ISS and iISS control Leonov functions (CLeFs).

*Definition 13.* A  $\mathcal{C}^1$  function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is said to be a practical ISS-CLeF for (5) if  $\frac{\partial V(\tilde{x})}{\partial \tilde{z}}$  is  $2\pi$ -periodic in  $\tilde{\theta}$ , and there exist functions  $\underline{\vartheta}, \bar{\vartheta}, \underline{\sigma}, \bar{\sigma}, \varpi, v \in \mathcal{K}_\infty$  and scalars  $\underline{g}, \bar{g}, \zeta \geq 0$ , such that

$$\underline{\vartheta}(|\tilde{x}|_{\tilde{\mathcal{W}}}) - \underline{\sigma}(|\tilde{\theta}|) - \underline{g} \leq V(\tilde{x}) \leq \bar{\vartheta}(|\tilde{x}|_{\tilde{\mathcal{W}}} + \bar{g}) - \bar{\sigma}(|\tilde{\theta}|), \quad (6)$$

for all  $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$ , and

$$\inf_{u \in \mathbb{R}^p} \{a(\tilde{x}, d) + b(\tilde{x})u\} \leq -\varpi(V(\tilde{x})) + v(|d|) + \zeta, \quad (7)$$

for all  $\tilde{x} \in \mathbb{R}^n \setminus \Omega_1$  and all  $d \in \mathbb{R}^m$  where  $a(\tilde{x}, d) = \frac{\partial V(\tilde{x})}{\partial \tilde{\theta}} f_1(\tilde{x}, d) + \frac{\partial V(\tilde{x})}{\partial \tilde{z}} f_2(\tilde{x}, d)$  and  $b(\tilde{x}) = \frac{\partial V(\tilde{x})}{\partial \tilde{z}} g(\tilde{x})$ , and  $\mathcal{W}'$  is formed by those atoms  $\mathcal{W}_i$  of the decomposition of  $\mathcal{W}$ , such that  $(\mathcal{W}_i + \{\xi_j\}) \subset \Omega_1 \cup \Omega_2$  for some  $j \in \mathbb{Z}^q$ , where  $\Omega_1 = \{\tilde{x} \in \mathbb{R}^n : V(\tilde{x}) < 0\}$ ,  $\Omega_2 = \{\tilde{x} \in \mathbb{R}^n : b(\tilde{x}) = 0\}$ . Moreover, if (7) holds for  $\zeta = 0$ , then  $V$  is said to be an ISS-CLeF. Besides, if the inequality (7) holds for  $\zeta = 0$  and a non-negative function  $\varpi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , then  $V$  is said to be an iISS-CLeF.

In general, (practical) ISS-CLeFs and iISS-CLeFs are non-periodic, hence the partial derivative of  $V$  with respect to  $\tilde{\theta}$  may contain non-periodic terms of  $\tilde{\theta}$ . Therefore, the term  $a(\tilde{x}, d)$  is usually non-periodic. On the other hand, since both  $\frac{\partial V(\tilde{x})}{\partial \tilde{z}}$  and  $g(\tilde{x})$  are imposed to be  $2\pi$ -periodic in  $\tilde{\theta}$ , the term  $b(\tilde{x})$  inherits this property.

We can easily see that the condition (6) for practical ISS-CLeFs and iISS-CLeFs is even more restrictive than the condition (3) for practical ISS- and iISS-Leonov functions. Moreover, under a proper design of the control input  $u$ , the dissipation inequality (7) is equivalent to (3). Thus, we can say that practical ISS-CLeFs (or an iISS-CLeFs) are indeed practical ISS-Leonov functions (iISS-Leonov functions, respectively) for the closed-loop system.

Let us introduce the function

$$\omega(\tilde{x}) = \sup_{d \in \mathbb{R}^m} \{a(\tilde{x}, d) - \bar{v}(|d|) - \bar{\zeta}\}, \quad (8)$$

which is well-defined for an appropriate  $\mathcal{K}_\infty$ -function  $\bar{v}(s) \geq v(s)$  for all  $s \geq 0$ , and  $\bar{\zeta} \geq \zeta$ . Note that  $\omega(\tilde{x})$  is non-periodic since the term  $a(\tilde{x}, d)$  usually is like that. However, from (6) we have that the restriction  $V(\tilde{x}) \geq 0$  implies  $\bar{\sigma}(|\tilde{\theta}|) \leq \bar{\vartheta}(|\tilde{x}|_{\tilde{\mathcal{W}}'} + \bar{g})$ . Basically, the radially unbounded terms of  $\tilde{z}$  dominates the non-periodic terms of  $\tilde{\theta}$  wherever  $V(\tilde{x}) \geq 0$ . Then, we can define a function

$$\phi(|\tilde{x}|_{\tilde{\mathcal{W}}'}, \tilde{z}) = \sup_{\tilde{x} \in \mathbb{R}^n \setminus \Omega_1} \{\omega(\tilde{x}) + \frac{1}{2}\varpi(V(\tilde{x}))\}, \quad (9)$$

which is  $2\pi$ -periodic in  $\tilde{\theta}$  since  $|\tilde{x}|_{\tilde{\mathcal{W}}'} = |\tilde{x} - \xi_j|_{\tilde{\mathcal{W}}'}$ .

Then, we say that the function  $V(\tilde{x})$  satisfies a SCP in the framework of Leonov functions (SCP-LeF) if for each constant  $\varepsilon > 0$  there is a  $\delta > 0$ , such that

$$\sup_{0 < |\tilde{x}|_\Gamma \leq \delta} \frac{\phi(|\tilde{x}|_{\tilde{\mathcal{W}}'}, \tilde{z})}{|b(\tilde{x})|^2} < \varepsilon. \quad (10)$$

where  $\Gamma = \{\tilde{x} \in \mathbb{R}^n : V(\tilde{x}) = 0; b(\tilde{x}) \equiv 0\}$ .

Now, following (Sontag, 1989), consider the so-called ‘‘universal formula’’ given by

$$K(\phi, |b|) = \begin{cases} \frac{\phi + \sqrt{\phi^2 + |b|^4}}{|b|^2} & \text{if } b \neq 0, \\ 1 & \text{if } b = 0, \end{cases} \quad (11)$$

where the arguments of the functions  $b(\tilde{x})$  and  $\phi(|\tilde{x}|_{\tilde{\mathcal{W}}'}, \tilde{z})$  have been omitted for compactness. Let us check the continuity of (11) at the points of the set  $\tilde{x} \in \mathbb{R}^n \setminus \Omega_1$ . First, recall the property  $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$  for any  $a, b \geq 0$ . Hence,

$$K(\phi, |b|) \leq \frac{\phi + |\phi|}{|b|^2} + 1. \quad (12)$$

From (7), we can see that  $b(\tilde{x}) \equiv 0$  implies  $a(\tilde{x}, d) \leq -\varpi(V(\tilde{x})) + v(|d|) + \zeta$ , wherever  $V(\tilde{x}) \geq 0$ , and for all  $d \in \mathbb{R}^m$ . Then, from (8) we obtain that  $b(\tilde{x}) \equiv 0$  leads to  $\omega(\tilde{x}) \leq -\varpi(V(\tilde{x}))$  in the same domain. Consequently, from (9) we obtain that  $b(\tilde{x}) \equiv 0$  also implies

$$\phi(|\tilde{x}|_{\tilde{\mathcal{W}}'}, \tilde{z}) \leq \sup_{\tilde{x} \in \mathbb{R}^n \setminus \Omega_1} \{-\frac{1}{2}\varpi(V(\tilde{x}))\}, \quad (13)$$

for all  $\tilde{x} \in \mathbb{R}^n \setminus \Omega_1$ . Clearly, by continuity of the functions  $b(\tilde{x})$  and  $\phi(|\tilde{x}|_{\tilde{\mathcal{W}}'}, \tilde{z})$  we can readily see from (12) that the function  $K(\phi(|\tilde{x}|_{\tilde{\mathcal{W}}'}, \tilde{z}), |b(\tilde{x})|)$  tends to 1 as  $b(\tilde{x})$  tends to 0, wherever  $V(\tilde{x})$  is strictly positive. Moreover, if the SCP-LeF (10) is verified, then the formula (11) satisfies

$$K(\phi(|\tilde{x}|_{\tilde{\mathcal{W}}'}, \tilde{z}), |b(\tilde{x})|) \leq \varepsilon + 1,$$

for all  $0 < |\tilde{x}|_\Gamma \leq \delta$ . Since we can make  $\varepsilon$  arbitrarily small then we conclude that (11) is continuous in the surface  $V(\tilde{x}) = 0$  as well. Accordingly, the function (11) is continuous in the set  $\tilde{x} \in \mathbb{R}^n \setminus \Omega_1$  as it is needed.

Thus, there exists a function  $\bar{K}(|\tilde{x}|_{\tilde{\mathcal{W}}'}, \tilde{z})$  being continuous in  $\mathbb{R}^n$  and  $2\pi$ -periodic in  $\tilde{\theta}$ , such that

$$K(\phi(|\tilde{x}|_{\tilde{\mathcal{W}}'}, \tilde{z}), |b(\tilde{x})|) \leq \bar{K}(|\tilde{x}|_{\tilde{\mathcal{W}}'}, \tilde{z})$$

for all  $\tilde{x} \in \mathbb{R}^n \setminus \Omega_1$ .

Therefore, we propose a feedback control law given by

$$u(\tilde{x}) = -\min \left\{ 1, \frac{|\tilde{x}|_{\tilde{\mathcal{W}}^-}}{\varepsilon} \right\} \bar{K}(|\tilde{x}|_{\tilde{\mathcal{W}}'}, \tilde{z}) b^\top(\tilde{x}), \quad (14)$$

where  $\mathcal{W}^- = \cup_{i=1}^l \mathcal{W}_i$  such that  $(\mathcal{W}_i + \{\xi_j\}) \subset \Omega_1 \setminus \Omega_2$  for some  $j \in \mathbb{Z}^q$ ,  $\mathcal{W}_i$  are atoms of the decomposition of  $\mathcal{W}$  (hence,  $\mathcal{W}^- \subset \mathcal{W}$ ), and  $0 < \varepsilon \leq \varepsilon^*$ , where  $\varepsilon^* > 0$  is the distance between the sets  $\{\tilde{x} \in \mathbb{R}^n : V(\tilde{x}) = 0\}$  and  $\tilde{\mathcal{W}}^-$ .

*Remark 14.* Note that if  $b(\tilde{x}) \equiv 0$  implies  $\phi(|\tilde{x}|_{\tilde{\mathcal{W}}'}, \tilde{z}) \leq 0$  for all  $\tilde{x} \in \mathbb{R}^n$ , and the SCP-LeF (10) is satisfied for all the points where  $\phi(|\tilde{x}|_{\tilde{\mathcal{W}}'}, \tilde{z})$  and  $|b(\tilde{x})|$  are zero, simultaneously, then the function (11) is continuous in  $\mathbb{R}^n$ . Hence, it can be directly used in the design of the controller (14), just like in the control Lyapunov function framework with the Sontag’s formula.

By construction, the controller (14) is continuous everywhere and  $2\pi$ -periodic in  $\tilde{\theta}$ . Hence, we can easily check that the closed-loop system (5),(14) inherits these properties. Moreover, let us define the set  $\tilde{\mathcal{W}}_{cl}$  containing all invariant solutions of (5),(14). Since the controller (14) vanishes at every atom of the decomposition of  $\tilde{\mathcal{W}}'$ , and they are invariant solutions of the (5), we can be sure that  $\tilde{\mathcal{W}}'$  is contained in  $\tilde{\mathcal{W}}_{cl}$ . Hence,  $\tilde{\mathcal{W}}_{cl} \cap \tilde{\mathcal{W}} \neq \emptyset$ .

*Theorem 15.* Let the system (5) satisfy Assumptions 11 and 12, and the set  $\mathcal{W}_{cl}$  be decomposable in the sense of Definition 4. If there exists a practical ISS-CLeF (iISS-CLeF) for the system (5) satisfying the SCP-LeF (10), then the feedback control law (14) is continuous on  $\mathbb{R}^n$  and provides the ISS (iISS) property to the system (5) with respect to  $\mathcal{W}_{cl}$ .

## 4. APPLICATION TO EXCITATION CONTROL DESIGN FOR A SYNCHRONOUS GENERATOR

### 4.1 Control design with ISS-CLeFs

Consider the following model of a single generator connected to an infinite bus through a transmission line (under standard assumptions, including that the mechanical input power is constant) (Galaz et al., 2003):

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -b_1 x_3 \sin(x_1) - b_2 x_2 + P, \\ \dot{x}_3 &= b_3 \cos(x_1) - b_4 x_3 + E + u + d, \end{aligned} \quad (15)$$

where  $x = [x_1, x_2, x_3]^\top \in \mathbb{R}^3$  is the state,  $b_i > 0$ ,  $i = 1, \dots, 4$ , are parameters,  $P, E \in \mathbb{R}$  are constants,  $d \in \mathcal{L}_\infty$  is a perturbation, and  $u \in \mathcal{L}_\infty$  is the control input. For  $E > b_1^{-1} b_4 P - b_3$ , the unperturbed system ( $u = 0$  and  $d = 0$ ) has equilibrium points at  $x_0 = [x_{10}, 0, x_{30}]^\top$ , and  $x'_0 = [x'_{10}, 0, x'_{30}]^\top$ , where the pairs  $(x_{10}, x_{30})$  and  $(x'_{10}, x'_{30})$  are solutions of the equations

$$b_1 x_3 \sin(x_1) = P \quad E = b_4 x_3 - b_3 \cos(x_1) \quad (16)$$

in the manifold  $\{x \in \mathbb{R}^3 : -\pi \leq x_1 \leq \pi\}$ . Accordingly, we can define the set  $\tilde{\mathcal{W}}_o = \{[x_{10} + 2j\pi, 0, x_{30}]^\top\} \cup \{[x'_{10} + 2j\pi, 0, x'_{30}]^\top\}$ , where  $j \in \mathbb{Z}$ , see (Pai, 1989) for more details about the properties of the system (15).

Consider the coordinates  $\zeta = [\zeta_1, \zeta_2, \zeta_3]^\top \in \mathbb{R}^3$ , where  $\zeta_1 = x_1 - x_{10}$  and  $\zeta_2 = x_2$ , and  $\zeta_3 = x_3 - x_{30}$ . Then, the system (15) can be rewritten in the coordinates  $\zeta$  as

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2, \\ \dot{\zeta}_2 &= -b_1 \zeta_3 \sin(\zeta_1 + x_{10}) - b_1 x_{30} \varphi_2(\zeta_1) - b_2 \zeta_2, \\ \dot{\zeta}_3 &= b_3 \varphi_1(\zeta_1) - b_4 \zeta_3 + u + d, \end{aligned} \quad (17)$$

where  $\varphi_1(\zeta_1) = \cos(\zeta_1 + x_{10}) - \cos(x_{10})$  and  $\varphi_2(\zeta_1) = \sin(\zeta_1 + x_{10}) - \sin(x_{10})$ . So, for  $u = 0$  and  $d = 0$ , all equilibria of the system (17) are contained in the set  $\tilde{\mathcal{W}} = \{[2j\pi, 0, 0]^\top\} \cup \{[x'_{10} - x_{10} + 2j\pi, 0, x'_{30} - x_{30}]^\top\}$ , for any  $j \in \mathbb{Z}$ .

An ISS-CLeF candidate for the system (17) is given by

$$V(\zeta) = \frac{1}{2}\zeta_2^2 + \frac{1}{2}\zeta_3^2 - \frac{1}{2}\Delta\zeta_1^2 - b_1(\zeta_3 + x_{30})\varphi_1(\zeta_1) - P\zeta_1, \quad (18)$$

where  $\Delta$  is a constant parameter to be determined.

Let us define  $\varrho(\zeta_1) = -b_1x_{30}\varphi_1(\zeta_1) - P\zeta_1 - \frac{1}{2}\kappa b_1x_{30}\zeta_1^2$ , for some  $\kappa \geq 1$ , such that,

$$\begin{aligned} \varrho'(\zeta_1) &= b_1x_{30} \sin(\zeta_1 + x_{10}) - P - \kappa b_1x_{30}\zeta_1, \\ \varrho''(\zeta_1) &= b_1x_{30} \cos(\zeta_1 + x_{10}) - \kappa b_1x_{30}. \end{aligned}$$

Then, for any  $\kappa > 1$  the equation  $\varrho'(\zeta_1) = 0$  has a unique solution at  $\zeta_1 = 0$  and the inequality  $\varrho''(0) < 0$  holds. Moreover, the equation  $\varrho''(\zeta_1) = 0$  has no solution for  $\kappa > 1$ . Hence,  $\varrho'(\zeta_1)$  is strictly decreasing in such a case. Thus, for  $\kappa > 1$  we have that  $\varrho(\zeta_1)$  is negative definite, which implies that  $-b_1x_{30}\varphi_1(\zeta_1) - P\zeta_1 \leq \frac{1}{2}\kappa b_1x_{30}\zeta_1^2$  under such a condition. By using similar arguments, we can deduce that  $-b_1x_{30}\varphi_1(\zeta_1) - P\zeta_1 \geq -\frac{1}{2}\kappa b_1x_{30}\zeta_1^2$ , for  $\kappa > 1$  as well. Besides, by using the Young's inequality we obtain  $-\frac{1}{4}\zeta_3^2 - b_1^2(\cos(x_{10}) + 1)^2 \leq b_1\zeta_3\varphi_1(\zeta_1) \leq \frac{1}{4}\zeta_3^2 + b_1^2(|\cos(x_{10})| + 1)^2$ .

Therefore, for all  $\zeta \in \mathbb{R}^3$ :

$$\frac{1}{2}\zeta_2^2 + \frac{1}{4}\zeta_3^2 - \sigma\zeta_1^2 - g \leq V(\zeta) \leq \frac{1}{2}\zeta_2^2 + \zeta_3^2 - \bar{\sigma}\zeta_1^2 + g,$$

where  $g = b_1^2(\cos(x_{10}) + 1)^2$ ,  $\sigma = \frac{1}{2}(\Delta + \kappa b_1x_{30})$  and  $\bar{\sigma} = \frac{1}{2}(\Delta - \kappa b_1x_{30})$ . Hence, we can say that the function (18) satisfies the condition (6) if  $\Delta > \kappa b_1x_{30}$ , with  $\tilde{\mathcal{W}}' = \{[2j\pi, 0, 0]^\top\}$ , for all  $j \in \mathbb{Z}$ .

Now, the derivative of the function  $V$  along the trajectories of the system (17) is given by

$$\dot{V}(\zeta) = a(\zeta, d) + b(\zeta)u,$$

where  $b(\zeta) = \zeta_3 - b_1\varphi_1(\zeta_1)$ , and

$$\begin{aligned} a(\zeta, d) &= -b_2\zeta_2^2 - b_4\zeta_3^2 - \Delta\zeta_1\zeta_2 - b_1b_3\varphi_1^2(\zeta_1) \\ &\quad + (b_3 + b_1b_4)\zeta_3\varphi_1(\zeta_1) + (\zeta_3 - b_1\varphi_1(\zeta_1))d. \end{aligned}$$

Let us define the vectors  $\Phi_1 = [\zeta_1, \zeta_2]^\top$ ,  $\Phi_2 = [\zeta_3, d]^\top$  and the matrices

$$\Xi_1 = \begin{bmatrix} 2b_4\bar{\sigma} & \Delta \\ \Delta & 2b_2 - b_4 \end{bmatrix}, \quad \Xi_2 = \begin{bmatrix} b_4 & 1 \\ 1 & 2\nu \end{bmatrix},$$

where  $\nu > 0$  is a constant parameter. Hence, we obtain

$$\begin{aligned} a(\zeta, d) + b_4V(\zeta) &\leq -\frac{1}{2}\Phi_1\Xi_1\Phi_1 - \frac{1}{2}\Phi_2\Xi_2\Phi_2 + v(|d|) \\ &\quad - b_1b_3\varphi_1^2(\zeta_1) + b_3\zeta_3\varphi_1(\zeta_1), \quad (19) \end{aligned}$$

where  $v(|d|) = \nu d^2 + b_1(1 + |\cos(x_{10})|)|d|$ . The matrices  $\Xi_1$  and  $\Xi_2$  are positive semi-definite if the parameters  $\kappa$ ,  $\Delta$  and  $\nu$  are chosen, such that

$$1 \leq \kappa \leq \frac{b_2^2}{4b_1x_{30}}, \quad \nu \geq \frac{1}{b_4}, \quad (20)$$

$$\max \left\{ b_{24} - \sqrt{D}, 2\kappa b_1x_{30} \right\} \leq 2\Delta \leq b_{24} + \sqrt{D}$$

where  $b_{24} = b_4(2b_2 - b_4)$ , and  $D = b_{24}(b_{24} - 4\kappa b_1x_{30})$ , provided that  $b_2^2 > 4b_1x_{30}$ , and

$$b_2 - \sqrt{b_2^2 - 4\kappa b_1x_{30}} \leq b_4 \leq b_2 + \sqrt{b_2^2 - 4\kappa b_1x_{30}}.$$

So, from (19) we can readily check that

$$b(\zeta) = 0 \implies a(\zeta, d) \leq -b_4V(\zeta) + v(|d|),$$

which is equivalent to the condition (7). Therefore, we can say that (18) is an ISS-CLeF for the system (17).

Now, consider a function  $\bar{v}(s) = v(s)$ , such that

$$\begin{aligned} a(\zeta, d) - \bar{v}(|d|) &= -b_2\zeta_2^2 - \left(b_4 - \frac{1}{4\nu}\right)\zeta_3^2 \\ &\quad - \Delta\zeta_1\zeta_2 + (b_3 + b_1b_4)\zeta_3\varphi_1(\zeta_1) \\ &\quad - b_1b_3\varphi_1^2(\zeta_1) - \left(\frac{1}{2\sqrt{\nu}}\zeta_3 - \sqrt{\nu}d\right)^2. \end{aligned}$$

So, following (8) we have

$$\begin{aligned} \omega(\zeta) &= -b_2\zeta_2^2 - \left(b_4 - \frac{1}{4\nu}\right)\zeta_3^2 - \Delta\zeta_1\zeta_2 \\ &\quad - b_1b_3\varphi_1^2(\zeta_1) + (b_3 + b_1b_4)\zeta_3\varphi_1(\zeta_1). \end{aligned}$$

Then, from (9) we obtain

$$\begin{aligned} \phi(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2, \zeta_3) &\leq \sup_{\zeta \in \mathbb{R}^3 \setminus \Omega_1} \left\{ -\left(b_2 - \frac{1}{4}b_4 - \frac{\Delta^2}{2b_4\bar{\sigma}}\right)\zeta_2^2 \right. \\ &\quad - \left(\frac{3}{4}b_4 - \frac{1}{4\nu}\right)\zeta_3^2 + (b_3 + \frac{1}{2}b_1b_4)\zeta_3\varphi_1(\zeta_1) \\ &\quad \left. - b_1b_3\varphi_1^2(\zeta_1) - \left(\sqrt{\frac{1}{2}b_4\bar{\sigma}}\zeta_1 + \frac{\Delta}{\sqrt{2b_4\bar{\sigma}}}\zeta_2\right)^2 \right\} \\ &\leq -\left(b_2 - \frac{1}{4}b_4 - \frac{\Delta^2}{2b_4\bar{\sigma}}\right)\zeta_2^2 - \left(\frac{3}{4}b_4 - \frac{1}{4\nu}\right)\zeta_3^2 \\ &\quad - b_1b_3\varphi_1^2(\zeta_1) + (b_3 + \frac{1}{2}b_1b_4)\zeta_3\varphi_1(\zeta_1) \\ &= \bar{\phi}(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2, \zeta_3). \end{aligned}$$

Note that the restriction  $b(\zeta) = \zeta_3 - b_1\varphi_1(\zeta_1) = 0$  implies

$$\bar{\phi}(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2, \zeta_3) \leq -\left(b_2 - \frac{1}{4}b_4 - \frac{\Delta^2}{2b_4\bar{\sigma}}\right)\zeta_2^2 - \left(\frac{3}{4}b_4 - \frac{1}{4\nu}\right)\zeta_3^2 \leq 0,$$

for all  $\zeta \in \mathbb{R}^3$ , for a correct selection of the parameters  $\Delta, \kappa > 0$  satisfying the conditions in (20). Additionally,  $b(\zeta)$  and  $\bar{\phi}(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2, \zeta_3)$  are zero, simultaneously, at the points  $[2\pi j, 0, 0]$  for any  $j \in \mathbb{R}^n$ . Nevertheless, since the functions  $|b(\zeta)|^2$  and  $\bar{\phi}(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2, \zeta_3)$  are quadratic with respect to  $\zeta_3$  and  $\varphi_1(\zeta_1)$ , then we can easily verified that the SCP-LeF (10) is satisfied for all those points. Thus, the formula (11), with  $b(\zeta)$  and  $\bar{\phi}(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2, \zeta_3)$ , is continuous everywhere. Hence, following Remark 14 we can set  $K(\bar{\phi}(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2, \zeta_3), b(\zeta)) = \bar{K}(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2, \zeta_3)$ .

Besides, note that the equations in (16) are satisfied wherever  $|\zeta|_{\tilde{\mathcal{W}}} = 0$ . Therefore, we can define

$$|\zeta|_{\tilde{\mathcal{W}}} = \sqrt{x_2^2 + (b_1x_3 \sin(x_1) - P)^2 + (b_4x_3 - b_3 \cos(x_1) - E)^2}.$$

Also, for the system (17) and the function (18) we have  $\tilde{\mathcal{W}}^- = \{[x'_{10} - x_{10} + 2j\pi, 0, x'_{30} - x_{30}]^\top\}$ , for any  $j \in \mathbb{Z}$ , then

$$|\zeta|_{\tilde{\mathcal{W}}^-} = |\zeta|_{\tilde{\mathcal{W}}} + \sqrt{\sin^2(x_1 - x'_{10}) + (x_3 - x'_{30})^2}.$$

Hence, the constant  $\epsilon^* = \min_{\zeta \in \{\zeta \in \mathbb{R}^3, V(\zeta) = 0\}} |\zeta|_{\tilde{\mathcal{W}}^-}$  is the shorter distance between the set  $\tilde{\mathcal{W}}^-$  and the surface  $V(\zeta) = 0$ .

Therefore, similarly to (14), we can define the controller

$$u(\zeta) = -\min \left\{ 1, \frac{|\zeta|_{\tilde{\mathcal{W}}^-}}{\epsilon} \right\} \bar{K}(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2, \zeta_3)b(\zeta), \quad (21)$$

where  $\epsilon \leq \epsilon^*$ . We can see that the controller (21) preserves the properties of periodicity and continuity in the closed-loop with the system (17). Moreover, the controller (21) vanishes at every element of the set  $\tilde{\mathcal{W}}$ , since its first function vanishes in  $\tilde{\mathcal{W}}^-$ , and all elements of  $\tilde{\mathcal{W}}'$  lay in the surface  $b(\zeta) = 0$ . Hence,  $\tilde{\mathcal{W}} \subseteq \tilde{\mathcal{W}}_{cl}$ , and let us assume that the latter is decomposable in the sense of Definition 4. Finally, by invoking Theorem 15 we conclude that the closed-loop system (17)-(21) has the ISS property with respect to the set  $\mathcal{W}_{cl}$ .

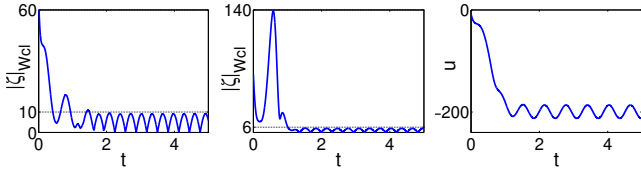


Fig. 1. Distance of the trajectories of (15),(21) to  $\mathcal{W}_{cl}$ , with  $d = 10 \sin(10t)$ . Left: for  $u = 0$  Center: for  $u$  as in (21). Right: control signal.

#### 4.2 Numerical example

Select the parameters of the system (15) as  $b_1 = 36$ ,  $b_2 = 18$ ,  $b_3 = 2$ ,  $b_4 = 20$ ,  $P = 3$ , and  $E = 15$ . Then, the set  $\mathcal{W}_o = \{[0.09825 + 2j\pi, 0, 0.8498]^T\} \cup \{[3.013 + 2j\pi, 0, 0.6509]^T\}$ , for all  $j \in \mathbb{Z}$ , contains the equilibria of (15). Moreover, the parameters of the function  $\phi(|\zeta|_{\mathcal{W}_o}, \zeta_2, \zeta_3)$  are chosen as  $\kappa = 1.9$ ,  $\Delta = 130$ , and  $\nu = 0.5$ . Besides, the constant  $\epsilon = 0.0001$  is just selected sufficiently small. Figure 1 show the distance of the trajectories of the closed-loop system (15)-(21), with  $\zeta_0 = [1, 5, 2]^T$  and  $d = 2 \sin(10t)$ , to the set  $\mathcal{W}_{cl}$ . We can see that the proposed controller increases the robustness of the system (15), such that, the system's trajectories remain in a closer vicinity of  $\mathcal{W}_{cl}$ .

### 5. CONCLUSIONS

The concepts of practical ISS-, ISS- and iISS-CLeFs were introduced for a class of multistable systems, whose dynamics are periodic with respect to a part of the state. It was proven that the existence of these functions, satisfying a SCP-LeF, is a sufficient condition to ensure that the system under study is ISS- or iISS stabilizable. Then, by using a mild adaptation of the Sontag's universal formula we can design a continuous and periodic controller, which endows the closed-loop system with the respective property. The proposed methodology was illustrated by the design of an excitation controller for a synchronous generator increasing the robustness the closed-loop system.

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