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# On observer design for a class of Persidskii systems based on steady-state estimation

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**Abstract:** This work aims to propose conditions for the existence of an observer for a particular class of nonlinear systems, presented as an interconnection of two Persidskii systems. First, we establish analytical expressions for steady-state solutions of an interconnected system. Next, a reduced-order observer for this system is designed, and the stability and boundedness of the error dynamics are proven. An academic example and an example considering Chua's circuits illustrate our results.

*Keywords:* Observer design, Output regulation, Lyapunov methods

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## 1. INTRODUCTION

The *output regulation* of nonlinear systems, introduced in Isidori and Byrnes (1990), is a concept dedicated to controlling a nonlinear dynamical plant using another system. This framework inspired an extension of *moment theory* for nonlinear systems in Astolfi (2010), Scarsiotti and Astolfi (2017a). Both approaches are based on the existence of a steady-state solution connecting two systems, provided by the *center manifold* theory (Carr (1981)).

Recently developed, the moment theory for nonlinear systems expresses more general conditions on the existence of such a solution. However, despite the theoretical approach presented in Astolfi (2010), calculating this response for nonlinear systems is not an easy task and must be solved separately for different classes of systems. Here, we borrow tools from the moment theory to look for the steady-state of a particular class of interconnected systems in Persidskii form Persidskij (1969), including Lur'e systems, with the same nonlinearity.

In Astolfi (2010), a cascade of a signal generator and plant is studied. In this work, we analyze a full feedback interconnection. By interconnection, we mean a set of two systems, where the input of the first is the output of the second and *vice versa*. Such generalization allows us to apply the presented result to a broader class of problems.

Persidskii systems are common in theoretical analysis and practical applications. Roughly speaking, it is an asymptotically stable linear system under the presence of a *weak* nonlinearity, which makes the model suitable for quasi-linear approaches and greatly simplifies the analysis. The Lur'e form of the system covers many models in mechanics, power systems, electrical circuits (Chua (1994)), bio-reactors (Bernard et al. (2001)), genetic (Li et al. (2006)), and neural networks (Forgione and Piga (2021), Kambhampati et al. (2000)). Many of these applications usually consider multi-plant scenarios (networks, multi-agent tasks, multi-circuits, *etc.*), which makes an interconnection to be justified for the analysis. Moreover, as shown in this work, an advantage of the models describing

these systems is that under mild restrictions, the global steady-state solution can be derived analytically (without applying numeric interpolation tools) and is formed by a linear hyperplane relating the solutions of both systems.

After obtaining conditions for the existence of an analytical steady-state solution, it will be used to compose the main result of the paper: *a design of a reduced-order observer for the cascade scenario* (which is one of several possible applications of the approach, presented in Section 3, for other applications see Khalin et al. (2022)), with consideration of the effects of measurement noises. One of the presented examples also considers the synchronization of two Chua circuits.

The outline of this work is as follows. Notation and definitions used in the paper and the problem statement are presented in Section 2. The main results, finding the analytical steady-state solution of interconnected Persidskii systems and the design of a reduced-order observer for the state estimation, are placed in sections 3 and 4. The theorem and proof are based on linear matrix inequalities (LMIs) and the Lyapunov function approach to analyze the error dynamics stability. In Section 5, the efficacy of the proposed observer is illustrated in two examples. Section 6 provides some conclusions and remarks.

## 2. PRELIMINARIES

### 2.1 Notation

- The set of real numbers is denoted by  $\mathbb{R}$  and we write  $\mathbb{R}_+ := \{t \in \mathbb{R} \mid t \geq 0\}$ . The vector spaces of real vectors of dimension  $n$  and real matrices of dimension  $n \times m$  are denoted by  $\mathbb{R}^n$  and  $\mathbb{R}^{n \times m}$ , respectively.
- For a Lebesgue measurable function  $u : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ , define the norm  $\|u\|_{[t_1, t_2]} = \text{ess sup}_{t \in [t_1, t_2]} \|u(t)\|$  for  $[t_1, t_2] \subset \mathbb{R}_+$ , where  $\|\cdot\|$  refers to the Euclidian norm in  $\mathbb{R}^n$ . We denote by  $\mathcal{L}_\infty^n$  the set of functions  $u : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  such that  $\|u\|_\infty := \|u\|_{[0, +\infty)} < +\infty$ .
- The transpose of matrix  $A \in \mathbb{R}^{n \times n}$  is denoted by  $A^\top$ . Let  $\lambda_{\min}(A)$  and  $\lambda_{\max}(A)$  denote the minimal

and maximal eigenvalue of a symmetric matrix  $A$ , respectively. Denote by the  $n \times n$  identity matrix  $I_n$ .

## 2.2 The nonlinear moment theory

Consider a nonlinear continuous-time system:

$$\begin{aligned}\dot{x}(t) &= F(x(t), u(t)), \quad t \in \mathbb{R}_+, \\ y(t) &= h(x(t)),\end{aligned}\tag{1}$$

where  $x : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  is the state,  $u : \mathbb{R}_+ \rightarrow \mathbb{R}^m$  the input and  $y : \mathbb{R}_+ \rightarrow \mathbb{R}^p$  the output signal,  $F : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ ,  $h : \mathbb{R}^n \rightarrow \mathbb{R}^p$  are analytic functions with  $F(0, 0) = 0$  and  $h(0) = 0$ . Consider the signal generator system given by:

$$\begin{aligned}\dot{\omega}(t) &= s(\omega(t)), \\ u(t) &= \ell(\omega(t)),\end{aligned}\tag{2}$$

where  $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}^q$  is the state variable,  $s : \mathbb{R}^q \rightarrow \mathbb{R}^q$ ,  $\ell : \mathbb{R}^q \rightarrow \mathbb{R}^m$  are analytic functions with  $s(0) = 0$  and  $\ell(0) = 0$ . The *interconnected system* is given by (1) and (2)<sup>1</sup>:

$$\dot{x} = F(x, \ell(\omega)), \quad \dot{\omega} = s(\omega), \quad y = h(x).\tag{3}$$

The following conditions should be imposed:

*Assumption 1.* (Scarciotti and Astolfi (2017b)) The system (1) is minimal, and the zero equilibrium is locally exponentially stable. The signal generator system (2) is observable and neutrally stable.

The minimality condition implies the *observability* and *accessibility* of the system (as defined in Scarciotti and Astolfi (2017b), definitions 2.10., 2.11). Under this hypothesis, the steady-state behavior of (3) can be characterized as:

*Lemma 2.* (Lemma 2.1 in Scarciotti and Astolfi (2017b)) Consider the system (1) and the signal generator system (2). Suppose Assumption 1 holds. Then there is a mapping  $\pi : \mathbb{R}^q \rightarrow \mathbb{R}^n$ , locally defined in a neighborhood  $\mathcal{W} \subset \mathbb{R}^q$  of 0, with  $\pi(0) = 0$ , which solves the partial differential equation:

$$\frac{\partial \pi}{\partial w}(w)s(w) = F(\pi(w), \ell(w)),\tag{4}$$

for all  $w \in \mathcal{W}$ . In addition, the steady-state response of the system (3) is  $x^{ss}(t) = \pi(\omega(t))$  for any  $x(0)$  and  $\omega(0)$  sufficiently small.

Note that other results and definitions accompanying the moment theory can be found in Scarciotti and Astolfi (2017b) (see Def. 2.1-2.9). We will focus here on the practical applicability of the solution of (4).

## 2.3 Convergent systems

The solution of (4) defines a trajectory for (3):  $x(t) = \pi(\omega(t))$ , but if, in addition, we ask that such a solution be attractive for the surrounding trajectories, then we come to the concept of convergence:

<sup>1</sup> Throughout the paper, we simplify the notation by writing, for instance,  $x$  in place of  $x(t)$  when the time-dependency is clear.

*Definition 3.* (Pavlov et al. (2005)) For a given input  $u \in \mathcal{L}^\infty_m$  the system (1) is said to be *convergent* if there is a solution  $\bar{x}(t)$ , defined and bounded for all  $t \in \mathbb{R}_+$ , which is globally asymptotically stable. Such a solution  $\bar{x}$  is called the *limit solution*.

In our case, the limit solution  $\bar{x}$  is supposed to be the steady-state solution  $x^{ss}$  under the input  $u(t) \in \mathbb{R}^m$ , found from (4) (using Lemma 2).

## 2.4 Input excitation

*Definition 4.* A function  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  is *persistently exciting* (or  $\phi$  is *PE*), if there exist  $T, \mu > 0$  such that for all  $t \in \mathbb{R}_+$ ,

$$\int_t^{t+T} \phi(s)\phi(s)^\top ds > \mu I_n,$$

where  $\phi(s)^\top$  denotes the function  $\mathbb{R}_+ \rightarrow \mathbb{R}^{1 \times n}$ ,  $t \mapsto \phi(t)^\top$ .

*Definition 5.* Two PE functions  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ ,  $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}^m$  are *linearly independent*, if there exists  $T > 0$  such that for all  $t \in \mathbb{R}_+$ , the following matrix functions:

$$\begin{aligned}&\left( \int_t^{t+T} \phi(s)\phi(s)^\top ds \right)^{-1} \int_t^{t+T} \phi(s)\psi(s)^\top ds \text{ and} \\ &\left( \int_t^{t+T} \psi(s)\psi(s)^\top ds \right)^{-1} \int_t^{t+T} \psi(s)\phi(s)^\top ds\end{aligned}$$

are not constant.

The PE property in the context of the moment theory has been studied in Padoan et al. (2017).

## 2.5 Problem statement

The class of systems of the interest is composed by Persidskii systems (Persidskij (1969), Efimov and Aleksandrov (2019), Efimov and Alexandrov (2021)). This system is also often addressed in literature as Lur'e system (Yalçin et al. (2001)). Therefore, consider an interconnection of two such nonlinear systems:

$$\begin{cases} \dot{x}(t) = A_0 x(t) + A_1 f(Hx(t)) + Bu(t), \quad t \in \mathbb{R}_+, \\ y(t) = C_0 x(t) + C_1 f(Hx(t)), \end{cases}\tag{5}$$

$$\begin{cases} \dot{\omega}(t) = S_0 \omega(t) + S_1 f(J\omega(t)) + Gy(t), \\ u(t) = L_0 \omega(t) + L_1 f(J\omega(t)), \end{cases}\tag{6}$$

where  $x : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  is the state function and  $y : \mathbb{R}_+ \rightarrow \mathbb{R}^p$  is the output function of (5);  $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}^q$  is the state function and  $u : \mathbb{R}_+ \rightarrow \mathbb{R}^m$  is the output function of (6);  $f : \mathbb{R}^r \rightarrow \mathbb{R}^k$  is a nonlinear continuous function,  $A_0 \in \mathbb{R}^{n \times n}$ ,  $A_1 \in \mathbb{R}^{n \times k}$ ,  $H \in \mathbb{R}^{r \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $C_0 \in \mathbb{R}^{p \times n}$ ,  $C_1 \in \mathbb{R}^{p \times k}$ ,  $S_0 \in \mathbb{R}^{q \times q}$ ,  $S_1 \in \mathbb{R}^{q \times k}$ ,  $J \in \mathbb{R}^{r \times q}$ ,  $G \in \mathbb{R}^{q \times p}$ ,  $L_0 \in \mathbb{R}^{m \times q}$ ,  $L_1 \in \mathbb{R}^{m \times k}$  are known constant matrices. We assume that the function  $f$  allows the forward existence and uniqueness of solutions of the systems (5), (6).

*Remark 6.* Note that for  $G = 0$ , the dynamics of (6) is an example of (2), and it becomes a signal generator for (5). Hence, in such a case, we have a nonlinear system as in (1) and (2).

This paper aims to find a global invariant/steady-state solution of the interconnection of systems (5), (6) analytically. In the case of noisy measurements, using the found steady-state responses, a reduced-order observer is designed, where stability conditions of the estimation error are formulated using LMIs.

### 3. STEADY-STATE SOLUTIONS OF THE INTERCONNECTION

Using tools from moment theory, we can formulate the conditions for the existence of a steady-state/invariant solution for (5), (6) in the following Proposition. The proof is omitted for brevity.

*Proposition 7.* Assume that there exists  $\Pi \in \mathbb{R}^{n \times q}$  such that  $J = H\Pi$  and

$$\begin{aligned} (A_0 - \Pi G C_0)\Pi - \Pi S_0 + B L_0 &= 0, \\ A_1 + B L_1 - \Pi(S_1 + G C_1) &= 0, \end{aligned} \quad (7)$$

then

$$x(t) = \Pi\omega(t), \quad \forall t \in \mathbb{R}_+$$

is a solution of the system (5), (6), for any  $\omega(0) \in \mathbb{R}^q$  and  $x(0) = \Pi\omega(0)$ .

In addition, if for some  $\omega(0) \in \mathbb{R}^q$  and  $x(0) \in \mathbb{R}^n$  the functions  $\omega$  and  $g = f(J\omega)$  are linearly independent, condition  $J = H\Pi$  is satisfied, and  $x = \Pi\omega$  is a solution of the system (5), (6), then the equalities (7) are necessarily verified.

Since both systems, (5) and (6), have the same shape of nonlinearity, under suitable relations between the matrices given in Proposition 7, the obtained steady-state mapping is linear, further simplifying the use of this property and significantly reducing the complexity of analysis for this interconnection.

In Proposition 7, only the existence of an invariant solution  $x(t) = \Pi\omega(t)$  is established for all  $t \in \mathbb{R}_+$ . If the solution  $x(t) = \Pi\omega(t)$  is also (locally) attractive for (5), (6) (see Khalil (2002) or Definition 3), then  $\pi(\omega) = \Pi\omega$  represents the steady-state response of the system (5) with the input generator (6) (for  $G = 0$ ). The advantage of considering a particular form of a nonlinear system and its input generator, as in (5) and (6), is that our solution  $\pi$  is global in  $\omega \in \mathbb{R}^q$  and analytically obtained.

*Remark 8.* Assume that there exists  $\Pi \in \mathbb{R}^{n \times q}$  such that

$$\begin{aligned} (A_0 - \Pi G C_0)\Pi - \Pi S_0 + B L_0 &= 0, \\ A_1 - \Pi G C_1 &= 0, \quad B L_1 - \Pi S_1 = 0, \end{aligned}$$

then  $x(t) = \Pi\omega(t)$ ,  $\forall t \in \mathbb{R}_+$ , is a solution of the system (5), (6). Therefore, we can skip the requirement  $J = H\Pi$  for cases where it is not satisfied.

### 4. STATE ESTIMATION OF THE CASCADE CONNECTION

We assume that the output of the system (5) is available for measurement with a noise signal  $v \in \mathcal{L}_\infty^p$ :

$$Y(t) = y(t) + v(t), \quad \forall t \in \mathbb{R}_+, \quad (8)$$

and that the input signal  $u$  of (5) (the output of (6)) is not measured. For brevity, let  $G = 0$ . Our goal is to estimate the state of the cascade (5), (6).

We need the following hypotheses:

*Assumption 9.* The equalities (7) and  $J = H\Pi$  are satisfied for (5), (6) with  $G = 0$ .

According to Proposition 7, it implies that  $x = \Pi\omega$  is a solution of the system, but this steady-state solution may not be unique, and to streamline our presentation, we exclude such a situation by supposing that:

*Assumption 10.* The system (5) is globally convergent.

In the last hypothesis, the convergence property is assumed to be global (*i.e.*, for any initial conditions  $x(0) \in \mathbb{R}^n$ , the corresponding trajectories of (5) converge to a trajectory entrained by the autonomous (since  $G = 0$ ) signal generator (6) with a fixed  $\omega(0) \in \mathbb{R}^q$ ).

*Remark 11.* Note that under Assumption 10, the system (5) admits an observer:

$$\begin{aligned} \dot{\hat{x}}(t) &= A_0\hat{x}(t) + A_1f(H\hat{x}(t)) + Bu(t) + L(Y(t) - \hat{y}(t)), \\ \hat{y}(t) &= C_0\hat{x}(t) + C_1f(H\hat{x}(t)), \quad \forall t \in \mathbb{R}_+ \end{aligned}$$

where  $L \in \mathbb{R}^{n \times p}$  is a gain matrix to be tuned. For  $L = 0$ , we have that  $\hat{x}(t) \rightarrow x(t)$  when  $t \rightarrow +\infty$  due to the convergence property, and the gain  $L$  can be used to accelerate the convergence taking into account the attenuation of the noise influence  $v$ . However, such an observer cannot be realized since the input  $u$  is not measured, and we have to design an observer in any case, for the input generator (6).

If we wish to design an observer for (5), (6) under Assumption 10, that might be impossible due to a lack of observability. Then Assumption 9 helps us: the mapping  $x = \Pi\omega$  corresponds to the system (5) behavior in response to the particular input produced by the generator (6), then its utilization simplifies the observer design while reducing its dimension.

Indeed, assumptions 9 and 10 have a consequence:

$$Y(t) = \psi(t) + \tilde{v}(t), \quad \psi(t) = C_0\Pi\omega(t) + C_1f(J\omega(t)),$$

where  $\tilde{v}(t) = v(t) + \epsilon(t)$  is a new measurement noise for the output system (6) and

$$\epsilon(t) = C_0(x(t) - \Pi\omega(t)) + C_1(f(Hx(t)) - f(H\Pi\omega(t)))$$

is a converging asymptotically to zero residual signal (recall that  $f$  is a continuous function). Therefore, the following observer can be designed for (6):

$$\begin{aligned} \dot{\hat{\omega}}(t) &= S_0\hat{\omega}(t) + S_1f(J\hat{\omega}(t)) + M(Y(t) - \hat{\psi}(t)), \\ \hat{\psi}(t) &= C_0\Pi\hat{\omega}(t) + C_1f(J\hat{\omega}(t)), \end{aligned} \quad (9)$$

where  $\hat{\omega} \in \mathbb{R}^q$  is the estimate of  $\omega$ ,  $M \in \mathbb{R}^{q \times p}$  is the gain to be selected. The dynamics of the estimation error  $e := \omega - \hat{\omega}$  can be written as follows:

$$\begin{aligned} \dot{e}(t) &= (S_0 - M C_0 \Pi) e(t) - M \tilde{v}(t) \\ &\quad + (S_1 - M C_1) (f(Je(t) + J\hat{\omega}(t)) - f(J\hat{\omega}(t))), \end{aligned} \quad (10)$$

where  $M\tilde{v}$  represents an accumulated measurement perturbation.

*Remark 12.* It is worth noting that assumptions 9 and 10 do not imply observability of (5), (6). Therefore, we must introduce additional restrictions leading to the convergence of the estimation error  $e$  in (10). For example, if  $S_1 = 0$  and  $C_1 = 0$ , then the pair  $(S_0, C_0\Pi)$  should be detectable.

Various approaches can be used to prove robust stability or boundedness of the estimation error  $e$ , which depend on the form and the role of the nonlinearities. For example, the techniques for Lur'e (Arcak and Kokotovic (2001)) or Persidskii (Mei et al. (2020)) systems can be applied. In this note, we use the most straightforward approach assuming a type of Lipschitzness of  $f$  in Theorem 13. For conciseness, we skip the proof.

*Theorem 13.* Let assumptions 9 and 10 be satisfied, assume there exist matrices  $F \in \mathbb{R}^{q \times k}$  and  $W = W^\top \in \mathbb{R}^{q \times q}$  such that

$$e^\top F (f(Je + J\hat{\omega}) - f(J\hat{\omega})) \leq e^\top W e$$

for all  $e, \hat{\omega} \in \mathbb{R}^q$ , and assume there exist  $P = P^\top \in \mathbb{R}^{q \times q}$ ,  $Q = Q^\top \in \mathbb{R}^{q \times q}$ ,  $\Gamma = \Gamma^\top \in \mathbb{R}^{p \times p}$  and  $U \in \mathbb{R}^{q \times p}$  such that the linear matrix inequalities

$$P > 0, Q > 0, \Gamma > 0, PS_1 - UC_1 = F$$

$$\begin{pmatrix} Y & -U \\ -U^\top & -\Gamma \end{pmatrix} \leq 0,$$

$$Y = S_0^\top P + PS_0 - \Pi^\top C_0^\top U^\top - UC_0 \Pi + 2W + Q$$

have a solution. Then for  $M = P^{-1}U$  in (5), (6), (8), (9):

$$\|e(t)\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} e^{-0.5 \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} t} \|e(0)\| + \sqrt{\frac{\lambda_{\max}(P)\lambda_{\max}(\Gamma)}{\lambda_{\min}(P)\lambda_{\min}(Q)}} \|\tilde{v}\|_\infty$$

for all  $t \geq 0$ .

*Remark 14.* The LMIs presented in Theorem 13 are conventionally used for controlling or estimating a Lipschitz continuous system. These LMIs are feasible if the linear part given by the matrix  $S_0$  is detectable for the output matrix  $C_0 \Pi$  and the Lipschitz constant of  $f$  reflected by the matrix  $W$  is sufficiently small.

According to Theorem 13 (assuming that the respective assumptions are satisfied), the variable  $\hat{\omega}$  asymptotically approaches  $\omega$  in the noise-free case (or stay in a vicinity of the ideal value with an error proportional to the amplitude of the perturbations  $v$ ), and we can select

$$\hat{x}(t) = \Pi \hat{\omega}(t) \quad (11)$$

as an asymptotic estimate of the state of the system (5).

Note that (9) with (11) is a reduced-order observer for the system (5), (6).

## 5. EXAMPLES

Two examples illustrate the studied problematics. One of the given examples is academic, and the second one is more practical, considering a cascade of two Chua's circuits.

*Example 15.* Consider systems (5), (6) with  $G = 0$ ,  $q = m = 2$ ,  $k = r = p = 1$  and  $n = 3$ :

$$A_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$f(\xi) = \arctan(\xi), \quad \xi \in \mathbb{R}$$

$$L_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad C_0 = (1 \ 0 \ 0), \quad H = (0 \ 0 \ 1), \quad C_1 = 0,$$

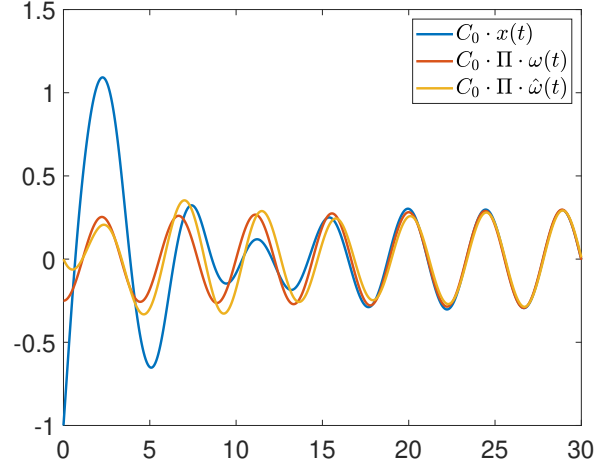


Fig. 1. Simulation results for example 1. Initial conditions:  $\omega(0) = (\frac{1}{8} \ 0)^\top$ ,  $x(0) = (\frac{1}{2} \ -\frac{1}{2} \ \frac{1}{2})^\top$ ,  $\hat{\omega}(0) = (\frac{1}{4} \ 0)^\top$ .

$$S_0 = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad J = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}, \quad L_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For this choice of matrices the equalities (7) and  $J = H\Pi$  are satisfied for

$$\Pi = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}.$$

Thus, Assumption 9 is verified, and it is possible to show that Assumption 10 is also valid in a vicinity of the origin (the system (5) can be linearized having a canonical form with a Hurwitz matrix). One of the solutions of the LMIs given in Theorem 13 is

$$P = \begin{pmatrix} 0.677 & -0.001 \\ -0.001 & 0.3267 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.077 & -0.011 \\ -0.011 & 0.001 \end{pmatrix},$$

$$\Gamma = 0.7377, \quad U = \begin{pmatrix} -0.317 \\ -0.007 \end{pmatrix}.$$

Then the observer (9), (11) can be applied with gain matrix

$$M = P^{-1}U = \begin{pmatrix} -0.468 \\ -0.035 \end{pmatrix}.$$

In Fig. 1, we can see that the observer output  $\hat{\psi} = C_0 \Pi \hat{\omega} + C_1 f(J\hat{\omega})$  is converging to the virtual output  $\psi(t) = C_0 \Pi \omega + C_1 f(J\omega)$ , as well as to the measured output  $Y = C_0 x + C_1 f(Hx) + v$ . This means that  $\hat{\omega}$  converges to the state  $\omega$  and  $x = \Pi \omega$  is indeed the steady-state solution of the system (5), which corresponds to the results of Theorem 13.

*Example 16.* One of the possible applications of our approach can be the synchronization and observer design for a pair of Chua's circuits with a cubic nonlinearity in the resistor, introduced in Zhong (1994). Consider one of them to be autonomous and oscillating due to the choice of coefficients ( $k_1 = 10$ ,  $k_2 = \frac{100}{7}$ ):

$$\begin{aligned}\dot{\omega}_1(t) &= k_1 \left( \omega_2(t) - \frac{1}{7}(2\omega_1^3(t) - \omega_1(t)) \right), \\ \dot{\omega}_2(t) &= \omega_1(t) - \omega_2(t) + \omega_3(t), \\ \dot{\omega}_3(t) &= -k_2\omega_2(t).\end{aligned}$$

Such a circuit has been a popular research subject for many years since its introduction by Leon Chua (Chua (1994)). It had many applications in chaos theory, dynamical systems (it appeared as an example for observer design (Howell and Hedrick (2002))), and other fields.

Consider that the second system is affected by a control input  $U \in \mathbb{R}^3$  that has to synchronize the two circuits. For a non-trivial case, let us select two non-identical circuits, due to, for example, a different resistance in a circuit or a different capacitance with  $k_3 = 10, k_4 = 16$  and  $k_5 = -2.857$ :

$$\begin{aligned}\dot{x}_1(t) &= k_3 (x_2(t) - x_1^3(t) + k_5 x_1(t)) + U_1(t), \\ \dot{x}_2(t) &= x_1(t) - x_2(t) + x_3(t) + U_2(t), \\ \dot{x}_3(t) &= -k_4 x_2(t) + U_3(t).\end{aligned}$$

A similar setup with two or more Chua's circuits can be frequently found in synchronization problems (Yassen (2003)), with the aim of implementation in data encryption.

*Problem 2a).* Even though the main problem presented in this paper is not synchronization, it can still be investigated using the same approach (Nijmeijer and Mareels (1997)). We can view our setup as a typical master-slave system, where the signal  $x$  has to follow dynamics of  $\omega$ , which in our case is oscillating.

It is easy to notice that both systems are in the required form (5),(6) with  $G = 0$ :

$$\begin{aligned}A_0 &= \begin{pmatrix} -28.57 & 10 & 0 \\ 1 & -1 & 1 \\ 0 & -16 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -10 \\ 0 \\ 0 \end{pmatrix}, \quad f(x) = x^3, \\ C_0 &= (0 \ 0 \ 1), \quad H = (1 \ 0 \ 0), \quad C_1 = 1, \\ S_0 &= \begin{pmatrix} \frac{10}{7} & 10 & 0 \\ 1 & -1 & 1 \\ 0 & -\frac{100}{7} & 0 \end{pmatrix}, \quad S_1 = \begin{pmatrix} -\frac{20}{7} \\ 0 \\ 0 \end{pmatrix}, \quad J = (1 \ 0 \ 0).\end{aligned}$$

Let us consider  $\Pi$  as the identity matrix (the needed relation for synchronization of these two systems). Then we can design  $BL_0$  and  $BL_1$  such that the signal  $U = u - L_0 x - L_1 f(Hx)$ , where  $u = L_0 \omega - L_1 f(J\omega)$  as before, will synchronize both dynamics using Proposition 7. In our case, the matrices will be, for example:

$$\begin{aligned}BL_0 &= S_0 - A_0 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 31.43 & 0 & 0 \\ -1.43 & 0 & 0 \\ 0 & 1.7143 & 0 \end{pmatrix}, \\ BL_1 &= S_1 - A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2.857 \\ 10 \\ 0 \end{pmatrix}.\end{aligned}$$

After this regularization we can state that the dynamics of  $x(t)$  become convergent and Assumption 10 is satisfied at

least locally. Therefore, systems (5) and (6) will converge to a common steady-state (see Fig. 2).

*Problem 2b).* As it was assumed in Section 4, we measure only  $Y = C_0 x + C_1 f(Hx) + v$  and the signal  $u$  is not measured as well as the states  $\omega$  and  $x$ . Task 2b then would be to build an observer to estimate the state of both systems, using Proposition 7 and Theorem 13.

Proposition 7 is satisfied by the choice of matrices for the input  $(B, L_0, L_1)$  for  $\Pi = I_q$  from problem 2a. Since  $\Pi$  is the identity matrix, Assumption 9 is also verified.

Since the first system is oscillating and the second one is asymptotically stable, Assumption 1 is satisfied. It is also possible to show that Assumption 10 is verified in a vicinity of the origin. By choosing corresponding matrices  $F, W$ , we can check LMIs of Theorem 13 to determine an observer gain. Consider matrices

$$F = \begin{pmatrix} 0.5 \\ -5 \\ 0.2 \end{pmatrix}, \quad W = \begin{pmatrix} 0.25 & -2.5 & 0.1 \\ -2.5 & 0 & 0 \\ 0.1 & 0 & 0 \end{pmatrix}.$$

A solution of LMIs is omitted for brevity. The resulting matrix of gains is:

$$M = \begin{pmatrix} 0.73 \\ 1.29 \\ 3.16 \end{pmatrix}.$$

Fig. 2 shows the convergence of all three signals  $(x, \omega, \hat{\omega})$ , which illustrates well one of possible applications of presented approach.

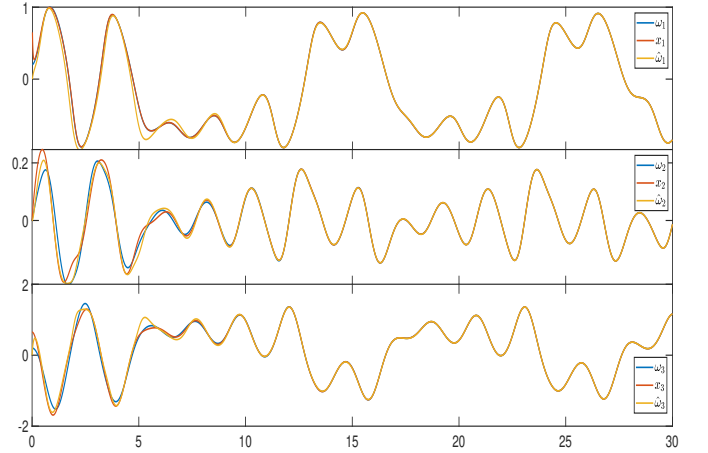


Fig. 2. Simulation results for example 2 (2a, 2b). Initial conditions:  $\omega(0) = (0.2 \ 0 \ 0.2)^\top$ ,  $x(0) = (0.65 \ 0 \ \frac{1}{2})^\top$ ,  $\hat{\omega}(0) = (0 \ 0 \ 0)^\top$ .

## 6. CONCLUSION

A particular class of nonlinear systems was investigated. One of the advantages of the considered form of systems deals with the existence of the analytical expression for steady-state solutions in the interconnection, which was found using tools from moment theory for nonlinear systems. A reduced-order observer was designed using the derived expressions of invariant solutions and imposing conditions on their global or local attractiveness. The stability of the error dynamics in the ISS framework (concerning the measurement noise) was proven using appropriate LMIs. Several examples illustrate our results.

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