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Stability analysis and stabilization of systems with hyperexponential rates [★]

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Abstract: Hyperexponential stability is investigated for dynamical systems with the use of both, explicit and implicit, Lyapunov function methods. A nonlinear hyperexponential control is designed for stabilizing linear systems. The tuning procedure is formalized in LMI form. Through numeric experiments, it is observed that the proposed hyperexponential control is less sensitive with respect to noises and discretization errors than its finite-time analog. It also demonstrates better performance in the presence of delays as well. Theoretical results are supported by numerical simulations.

1. INTRODUCTION

A nonlinear system is said to be fast (or hyperexponential) if its transients are faster than any linear one (Caraballo, 2001; Palamarchuk, 2018; Polyakov, 2018; Inoue et al., 2011; Wang and Song, 2019; Moulay and Perruquetti, 2006). Such control/observer design problems have often been the object of research, as they provide rapid error convergence (i.e., faster than any exponential) and useful robust properties (see, e.g., Bhat and Bernstein, 2000; Orlov, 2004; Polyakov, 2018; Polyakov et al., 2015b and references therein). Most of the research is devoted to ensuring finite/fixed-time stability, which guarantees the termination of all transients in a finite time.

Interest in a weaker type of stability, a hyperexponential one, has increased in recent years. This interest is mostly related to time-delay and discrete-time systems and based on the fact that finite/fixed-time stability (stabilization) is not natural for these classes of models, and it is only possible in some rare special cases (see, e.g., Efimov et al., 2014; Nekhoroshikh et al., 2020), in contrast to hyperexponential stability (see, for example, Polyakov et al., 2015a, 2018).

This paper investigates hyperexponential stability for systems without delays. Hyperexponential stability conditions are proposed for both, the rated and unrated cases. It is shown, that based on Implicit Lyapunov Function (ILF) method finite- and fixed-time controls (as in Polyakov et al., 2015b, 2016; Zimenko et al., 2020) under sampled-time realization provide hyperexponential stability only. In this regard, the question arises whether hyperexponential, but not finite-time, control can have better robust/discretization properties? The answer is yes, and it is investigated below.

An ILF-based hyperexponential but not finite-time control is proposed for linear systems. Through numeric experiments, it is shown that proposed hyperexponential control

is less sensitive with respect to noises than its finite-time analog. Moreover, under sampled-time realization it preserves the hyperexponential stability. In that case, further development and robustness investigation of hyperexponential controls is promising, since in some cases it will allow to obtain in practise better transients and robust properties in comparison with finite-time ones.

The outline of this paper is as follows. Preliminary results on ILF method are given in Section 2. Hyperexponential and non-asymptotic stability conditions are presented in Section 3. Section 4 presents the main results on hyperexponential stability conditions and hyperexponential control design for linear systems. Finally, concluding remarks are discussed in Section 5.

Notation: \mathbb{R} is the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$; \mathbb{N} is the set of natural numbers; the symbol $\overline{1, m}$ is used to denote a sequence of integers $1, \dots, m$; the order relation $P > 0$ (< 0 ; ≥ 0 ; ≤ 0) for $P \in \mathbb{R}^{n \times n}$ means that P is symmetric and positive (negative) definite (semidefinite); the minimal and maximal eigenvalues of a symmetric matrix P are denoted by $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$, respectively; $\text{diag}\{\lambda_i\}_{i=1}^n$ is a diagonal matrix with elements λ_i .

2. PRELIMINARIES

The Implicit Lyapunov Function method allows to determine the stability of a system without presenting the Lyapunov function in an explicit form (e.g., the Lyapunov function can be introduced as a solution of some algebraic equation, and it is not necessary to solve this equation for stability analysis).

Consider the system of the form

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \geq 0, \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a nonlinear vector field, $f(0) = 0$, which can be discontinuous with respect to the state variable. In this case the solutions $\Phi(t, x_0)$ of the system (1) are understood in the sense of Filippov (Filippov, 1988).

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The next theorem presents the Implicit Lyapunov function method (Korobov, 1979; Adamy and Flemming, 2004).

Theorem 1 *Let there exists a continuous function*

$$Q: \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R} \\ (V, x) \rightarrow Q(V, x)$$

satisfying the conditions

C1) Q is continuously differentiable outside the origin of $\mathbb{R}_+ \times \mathbb{R}^n \setminus \{0\}$;

C2) for any $x \in \mathbb{R}^n \setminus \{0\}$ there exists $V \in \mathbb{R}_+$ such that $Q(V, x) = 0$;

C3) for $\Omega = \{(V, x) \in \mathbb{R}_+ \times \mathbb{R}^n : Q(V, x) = 0\}$

$$\lim_{\substack{x \rightarrow 0 \\ (V, x) \in \Omega}} V = 0^+, \quad \lim_{\substack{V \rightarrow 0^+ \\ (V, x) \in \Omega}} \|x\| = 0, \quad \lim_{\substack{\|x\| \rightarrow \infty \\ (V, x) \in \Omega}} V = +\infty;$$

C4) the inequality $\frac{\partial Q(V, x)}{\partial V} < 0$ holds for all $V \in \mathbb{R}_+$ and $\forall x \in \mathbb{R}^n \setminus \{0\}$.

If $\frac{\partial Q(V, x)}{\partial x} f(x) < 0$ for all $(V, x) \in \Omega$, then the origin of (1) is globally asymptotically stable.

Remark 1 Theorem 1 represents the well-known stability result on asymptotic stability, where the conditions *C1)*, *C2)* and *C4)* guarantee existence and uniqueness of a positive definite function $V \in \mathbb{R}_+ : Q(V, x) = 0$ for all $x \in \mathbb{R}^n$; *C3)* implies that $V(x) \rightarrow 0$ as $x \rightarrow 0$ ($V(x) \rightarrow +\infty$ as $x \rightarrow \infty$); and the condition $\frac{\partial Q(V, x)}{\partial x} f(x) < 0$ implies $\dot{V}(x) < 0$.

3. HYPEREXPONENTIAL AND NON-ASYMPTOTIC STABILITY DEFINITIONS

Over the past several decades, increased interest has been paid to stability notions with non-asymptotic convergence, e.g., finite-time and fixed-time stability (see, Bhat and Bernstein, 2000; Orlov, 2004; Polyakov et al., 2015b; Polyakov, 2012; Zimenko et al., 2020 and references therein).

Let D be an open neighborhood of the origin in \mathbb{R}^n .

Definition 1 (Bhat and Bernstein, 2000; Orlov, 2004) *The origin of (1) is said to be finite-time stable if it is asymptotically stable and for any $x_0 \in D$ any solution $\Phi(t, x_0)$ of the system (1) reaches the origin at some finite time moment, i.e. $\Phi(t, x_0) = 0 \forall t \geq T(x_0)$ and $\Phi(t, x_0) \neq 0 \forall t \in [0, T(x_0))$, $x_0 \neq 0$, where $T: \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\}$, $T(0) = 0$ is the settling-time function.*

Definition 2 (Polyakov, 2012) *The origin of (1) is said to be fixed-time stable if it is finite-time stable and the settling-time function $T(x_0)$ is bounded, i.e., $\exists T_{\max} > 0: T(x_0) \leq T_{\max}, \forall x_0 \in D$.*

If $D = \mathbb{R}^n$, then the corresponding stability properties are called global.

Definition 3 (Polyakov, 2018) *A set $\mathcal{M} \subset \mathbb{R}^n$ is said to be globally fixed-time attractive if any solution $\Phi(t, x_0)$ of (1) reaches \mathcal{M} in some finite time moment $t = T(x_0) < T_{\max}$, $T_{\max} \in \mathbb{R}_+$ and remains there for all $t \geq T(x_0)$, where $T: \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\}$. The origin is said to be nearly fixed-time stable if it is globally Lyapunov stable and any neighborhood of the origin is fixed-time attractive.*

The interest in finite/fixed-time stability is due to existence of many fast control applications with time con-

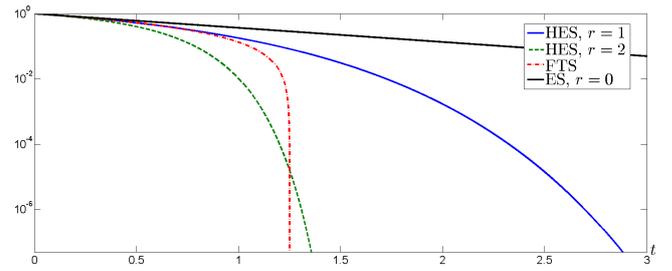


Fig. 1. Rates of convergence: exponential (ES); hyperexponential (HES); finite-time (FTS)

straints, where a high precision is needed in the presence of significant perturbations (e.g., rejection of non-Lipschitz disturbances).

An extension of Theorem 1 for finite/fixed-time stability analysis is proposed in (Polyakov et al., 2015b).

Another type of stability, which also applies to fast systems, is hyperexponential stability (see, e.g., the papers Caraballo, 2001; Palamarchuk, 2018; Polyakov, 2018; Inoue et al., 2011).

Definition 4 (Polyakov, 2018) *The system (1) is said to be hyperexponentially stable at the origin in D if the system is Lyapunov stable and*

$$\forall r \in \mathbb{R}_+ \exists t' \in \mathbb{R}_+, \kappa > 0, C > 0 : \\ \|\Phi(t, x_0)\| \leq C e^{-rt}, \quad \forall t > t', \forall x_0 \in \{x \in D : \|x\| \leq \kappa\}. \quad (2)$$

In other words, the hyperexponential stability assumes increasing of convergence rates as $\Phi(t, x_0) \rightarrow 0$.

In order to provide quantitative index to characterize hyperexponential convergence rates, the rated hyperexponential stability is presented in (Polyakov et al., 2015a).

For $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_r)^\top \in \mathbb{R}_+^{r+1}$ and $r \in \mathbb{N}$ let us introduce the function of nested exponentials $\rho_{r, \alpha}: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by the following recursive formula

$$\rho_{0, \alpha}(z) = \alpha_0 z, \\ \rho_{i, \alpha}(z) = \alpha_i \left(e^{\rho_{i-1, \alpha}(z)} - e^{\rho_{i-1, \alpha}(0)} \right), \quad i = \overline{1, r}. \quad (3)$$

Definition 5 (Polyakov et al., 2015a) *The origin of the system (1) is said to be hyperexponentially stable of degree $r \in \mathbb{N}$ in D , if it is Lyapunov stable and for each $\kappa > 0 \exists C \in \mathbb{R}_+$ and $\alpha \in \mathbb{R}_+^{r+1}$ such that*

$$\|\Phi(t, x_0)\| \leq C e^{-\rho_{r, \alpha}(t)} \quad \forall x_0 \in \{x \in D : \|x\| \leq \kappa\}. \quad (4)$$

Notice that

- for the case $r = 0$ the expression (4) corresponds to exponential stability definition;
- all finite-time and fixed-time stable systems are also hyperexponentially stable due to finite/fixed-time stability is obtained through an "infinite eigenvalue assignment" for the system at the origin.

Fig. 1 depicts in a logarithmic scale $e^{-\rho_{i, \alpha}(t)}$ for $t \geq 0$, $\alpha_i = 1$, $i = \overline{0, 2}$ and $\begin{cases} (1 - 0.8t)^{1.25} & \text{for } t \in [0, 1.25], \\ 0 & \text{for } t > 1.25 \end{cases}$, to represent the finite-time stability case in order to show the decay rates.

Hyperexponential stability (stabilization) is often the subject of research in the analysis of systems with delays (see, e.g., Polyakov et al., 2015a, 2018). Also, the problems of hyperexponential stability analysis, control and observer design via time-varying feedback are considered in (Inoue et al., 2011; Wang and Song, 2019; Efimov et al., 2022). In this paper, the hyperexponential stability is investigated with the use of both, explicit and implicit, Lyapunov function methods, and a time-invariant hyperexponential control is proposed for linear systems.

4. HYPEREXPONENTIAL STABILITY ANALYSIS USING LYAPUNOV FUNCTION METHOD

The Lyapunov function method (Lyapunov, 1992) is the main tool for stability analysis of nonlinear systems. The following theorem presents a sufficient condition for hyperexponential stability.

Theorem 2 *Suppose there exist positive definite C^1 functions $V_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i \in \mathbb{N}$ defined on D and a corresponding sequence of open sets $\{D_i\}_{i=1}^{\infty} : D = D_1 \supset D_2 \supset \dots \supset D_i \supset \dots \ni \{0\}$ such that $D_i = \{x \in \mathbb{R}^n : V_i(x) < v_i, v_i \in \mathbb{R}_+\}$, $V_i(x) \geq k_i \|x\|^a$, $a \in \mathbb{R}_+$, $k_i \in \mathbb{R}_+ : \frac{v_i}{k_i} \leq C$ for some $C \in \mathbb{R}_+$ and*

$$\dot{V}_i(x) \leq -c_i V_i(x), \quad \forall x \in D_i \setminus D_{i+1}, \quad (5)$$

where $c_{i+1} > c_i \in \mathbb{R}_+$ for any $i \in \mathbb{N}$ and $c_i \rightarrow +\infty$ as $i \rightarrow +\infty$. Then the origin of system (1) is hyperexponentially stable. If $D = \mathbb{R}^n$, then the system (1) admits this property globally.

All proofs are skipped due to space limitations.

4.1 ILF-based finite-time control under sampled-time realization

The following application shows that sampled-time realization of ILF-based finite-time control implies hyperexponential stability.

Example 1 Consider the system

$$\dot{x} = Ax + Bu, \quad (6)$$

where $x \in \mathbb{R}^n$ is a state, $u \in \mathbb{R}$ is a control input,

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad (7)$$

$$B = [0 \ 0 \ \dots \ 0 \ 1]^\top \in \mathbb{R}^{n \times 1}.$$

Introduce the ILF function

$$Q(V, x) = x^\top D(V^{-1})PD(V^{-1})x - 1,$$

where $V \in \mathbb{R}_+$,

$$D(\lambda) = \begin{bmatrix} \lambda^{q_1} & 0 & \dots & 0 \\ 0 & \lambda^{q_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \lambda^{q_n} \end{bmatrix}, \quad \lambda \in \mathbb{R}_+,$$

$$q_i = 1 + (n - i)\mu, \quad i = \overline{1, n},$$

$\mu \in (0, 1]$ and $0 < P \in \mathbb{R}^{n \times n}$. Denote $H = \text{diag}\{q_i\}_{i=1}^n$.

According to (Polyakov et al., 2015b, Theorem 6) the control

$$u(V, x) = V^{1-\mu}KD(V^{-1})x \quad (8)$$

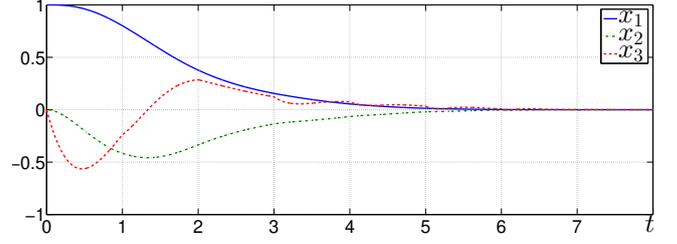


Fig. 2. State transients for the system (6) with sampled-time realization of the control (8)

stabilizes the system (6) in a finite time, where $V \in \mathbb{R}_+ : Q(V, x) = 0$ and $K = YX^{-1}$, where $X = P^{-1} \in \mathbb{R}^{n \times n}$, $Y \in \mathbb{R}^{1 \times n}$ satisfy the system

$$AX + XA^T + BY + Y^T B^T + \alpha X \leq 0, \\ XH + HX > 0, \quad X > 0,$$

for some $\alpha \in \mathbb{R}_+$.

Denote an arbitrary sequence of time instances $\{t_i\}_{i=1}^{+\infty}$ such that $0 = t_0 < t_1 < t_2 < \dots$, $\lim_{i \rightarrow +\infty} t_i = +\infty$. In (Polyakov et al., 2015b, Corollary 8) it is shown that sampled-time realization of the control (8) (i.e. $u = u(V_i, x)$ on each time interval $[t_i, t_{i+1})$, where $V_i \in \mathbb{R}_+ : Q(V_i, x(t_i)) = 0$) provides asymptotic stabilization of the closed loop system independent of the length of the sampling interval. Let us show that sampled-time realization implies *hyperexponential stability* of the closed loop system.

Let us define the set of nested ellipsoids $D_i := \{x \in \mathbb{R}^n : x^\top P_i x \leq 1\}$, where $P_i = D(V_i^{-1})PD(V_i^{-1})$ and the corresponding set of quadratic functions $\tilde{V}_i = x^\top P_i x$. Note that $\tilde{V}_i = V_i$ at time instant t_i . Then for any $i \in \mathbb{N}$ on $D_i \setminus D_{i+1}$ by (Polyakov et al., 2015b, Corollary 7) we have

$$\dot{\tilde{V}}_i = 2x^\top P_i(Ax + Bu(V_i, x)) \\ \leq -\alpha V_i^{-\mu} \tilde{V}_i$$

Due to $V_i > V_{i+1}$ and $V_i \rightarrow 0$ as $i \rightarrow \infty$ (see Polyakov et al. (2015b) for more details), then applying Theorem 2 with $c_i = \alpha V_i^{-\mu}$, $v_i = 1$, $k_i = C = \lambda_{\min}(P) \min\{V_1^{-2}, V_1^{-2-2(n-1)\mu}\}$, $a = 2$ the closed loop system is hyperexponentially stable. The results of simulation are shown in Fig. 2 for $n = 3$, $\mu = 0.5$,

$$P = \begin{bmatrix} 3.3119 & 3.2943 & 0.8806 \\ 3.2943 & 5.2366 & 1.4426 \\ 0.8806 & 1.4426 & 0.9682 \end{bmatrix}, \quad K^T = \begin{bmatrix} -4.2618 \\ -7.7818 \\ -3.2635 \end{bmatrix}$$

and sampling period $t_{i+1} - t_i = 1$.

The same outcomes can be obtained for finite/fixed-time controls proposed in (Polyakov et al., 2015b, 2016; Zimenko et al., 2020), where finite/fixed-time controls are proposed for linear controllable MIMO systems of the form (6):

Proposition 1 Let $\{t_i\}_{i=0}^{\infty}$ be a strictly increasing sequence of arbitrary time instants, $0 = t_0 < t_1 < t_2 < \dots$ such that $\lim_{i \rightarrow \infty} t_i = +\infty$. Let for (6) all conditions proposed in (Polyakov et al., 2015b) (or in Polyakov et al., 2016; Zimenko et al., 2020) for ILF-based finite/fixed-time control $u(V, x)$ design be satisfied. Then sampled-time realization $u(t) = u(V_i, x(t))$ for $t \in [t_i, t_{i+1})$, where $V_i \in \mathbb{R}_+ : Q(V_i, x(t_i)) = 0$ provides hyperexponential stability of the closed loop system.

Proposition 1 shows that finite/fixed-time controls in (Polyakov et al., 2015b, 2016; Zimenko et al., 2020) under sampled-time realization provide hyperexponential stability of the closed loop system.

4.2 Explicit Lyapunov function method

Since the hyperexponential stability assumes increasing of exponential rates as $\Phi(t, x_0) \rightarrow 0$, then the following result can be obtained:

Theorem 3 Let $\beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be positive nondecreasing function such that $\beta(s) \rightarrow \infty$ as $s \rightarrow \infty$. Suppose there exists a positive definite C^1 function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ defined on an open neighborhood of the origin $D \subset \mathbb{R}^n$, such that $V(x) \geq k\|x\|^a$, for all $x \in D$, $a, k \in \mathbb{R}_+$ and the following condition is true for the system (1)

$$\dot{V}(x) \leq -\beta(V^{-1}(x))V(x), \quad x \in D \setminus \{0\}.$$

Then the origin of system (1) is globally hyperexponentially stable. If $D = \mathbb{R}^n$, then the system (1) admits this property globally.

4.3 ILF method

Let us introduce the functions $\sigma_i^\alpha : (0, 1] \rightarrow \mathbb{R}$, $i = \overline{1, r}$ by the following recursive formula

$$\begin{aligned} \sigma_1^\alpha(s) &= -\ln s + \alpha_r, \\ \sigma_i^\alpha(s) &= \ln \left(\frac{1}{\alpha_{r-i+2}} \sigma_{i-1}^\alpha(s) \right) + \alpha_{r-i+1}, \quad i = \overline{2, r}. \end{aligned}$$

Obviously $\sigma_i^\alpha(1) = \alpha_{r-i+1}$. Note that $\sigma_i^\alpha(s) > 0$ for $s \leq 1$ and $\sigma_i^\alpha(s) \rightarrow +\infty$ as $s \rightarrow 0$.

The next theorem presents the ILF method for *rated hyperexponential stability*.

Theorem 4 Let there exist two functions Q_1 and Q_2 satisfying the conditions C1)-C4) of Theorem 1 and C5) $Q_1(1, x) = Q_2(1, x)$ for all $x \in \mathbb{R}^n \setminus \{0\}$; C6) there exist $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_r)^\top \in \mathbb{R}_+^{r+1}$, $r \in \mathbb{N}$, $c_1 \in \mathbb{R}_+$, such that the inequality

$$\frac{\partial Q_1(V, x)}{\partial x} f(x) \leq c_1 V \prod_{i=1}^r \sigma_i^\alpha(V) \frac{\partial Q_1(V, x)}{\partial V},$$

holds for all $V \in (0, 1]$ and $x \in \mathbb{R}^n \setminus \{0\}$ satisfying the equation $Q_1(V, x) = 0$;

C7) there exists $c_2 \in \mathbb{R}_+$ such that the inequality

$$\frac{\partial Q_2(V, x)}{\partial x} f(x) \leq c_2 V \frac{\partial Q_2(V, x)}{\partial V},$$

holds for all $V \geq 1$ and $x \in \mathbb{R}^n \setminus \{0\}$ satisfying the equation $Q_2(V, x) = 0$;

C8) for some $k_1, k_2, a \in \mathbb{R}_+$ the inequality

$$k_1 \|x\|^a \leq V \leq k_2 \|x\|^a,$$

holds for all $V \in \mathbb{R}_+$ and $x \in \mathbb{R}^n \setminus \{0\}$ satisfying the equation $Q_1(V, x) = 0$ for $V \leq 1$ and $Q_2(V, x) = 0$ for $V > 1$, then the origin of the system (1) is globally hyperexponentially stable with degree r .

Note that due to the condition C3) the function V is radially unbounded and it can be continuously extended to the origin by $V(0) = 0$.

The condition C8) in Theorem 4 may be relaxed as it is shown in the following corollary.

Corollary 1 Let there exist two functions Q_1 and Q_2 that satisfy the conditions C1)-C4) of Theorem 1, the conditions C5)-C7) of Theorem 4 with $r \geq 2$ and C9) for some $k \in \mathbb{R}_+$ the inequality

$$\frac{\alpha_r}{-\ln(V) + 1} \geq k \|x\| \quad (9)$$

holds for all $V \leq 1$ and $x \in \mathbb{R}^n \setminus \{0\}$ satisfying the equation $Q_1(V, x) = 0$;

C10) for some $k_1, k_2, a \in \mathbb{R}_+$ the inequality

$$k_1 \|x\|^a \leq V \leq k_2 \|x\|^a, \quad (10)$$

holds for all $V \geq 1$ and $x \in \mathbb{R}^n \setminus \{0\}$ satisfying the equation $Q_2(V, x) = 0$,

then the origin of the system (1) is globally hyperexponentially stable with degree $r - 1$.

Applying this result to the system (6), (7), let us introduce the ILF functions

$$Q_1(V, x) = x^T D(\varrho(V)) P D(\varrho(V)) x - 1, \quad (11)$$

$$Q_2(V, x) = \frac{1}{V^2} x^T P x - 1,$$

where $V \in \mathbb{R}_+$, $\varrho(V) = \ln \left(\frac{V+e-1}{V} \right)$, $D(\lambda) = \text{diag}\{\lambda^{q_i}\}_{i=1}^n$, $q_i = 1 + (n-i)\mu$, $\mu \in (0, 1]$, $\lambda \in \mathbb{R}_+$ and $0 < P \in \mathbb{R}^{n \times n}$. Denote $H = \text{diag}\{q_i\}_{i=1}^n$. The following theorem is on rated hyperexponential control design for the linear system (6), (7).

Theorem 5 Let the system of matrix inequalities

$$\begin{aligned} AX + XA^T + BY + Y^T B^T + \gamma(XH + HX) &\leq 0, \\ XH + HX &> 0, \quad X > 0 \end{aligned} \quad (12)$$

be feasible for some $\gamma \in \mathbb{R}_+$ and $X \in \mathbb{R}^{n \times n}$, $Y \in \mathbb{R}^{1 \times n}$. Let

$$u(V, x) = \begin{cases} \varrho^{\mu-1}(V) K D(\varrho(V)) x & \text{for } x^T P x < 1, \\ K x & \text{for } x^T P x \geq 1, \end{cases} \quad (13)$$

where $K = YX^{-1}$, $X = P^{-1} \in \mathbb{R}^{n \times n}$, $Y \in \mathbb{R}^{1 \times n}$ and $V \in \mathbb{R}_+ : Q_1(V, x) = 0$ for $x^T P x < 1$. Then the closed-loop system (6), (13) is hyperexponentially stable with degree $r = 1$.

Note that due to $\varrho(V) = 1$ on the set $\{x \in \mathbb{R}^n : x^T P x = 1\}$, the control function $u = u(V, x)$ is continuous in $\mathbb{R}_+ \times \mathbb{R}^n$.

Remark 2 Theorem 5 can be easily extended to linear MIMO systems using block decomposition like in (Polyakov et al., 2016).

Remark 3 In order to calculate the function $V(x)$ implicitly defined by (11) the bisection method may be utilized (see, e.g., Polyakov et al., 2013):

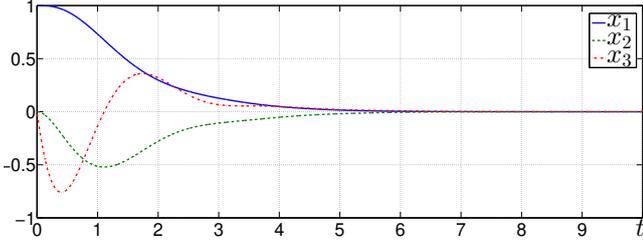


Fig. 3. State transients for the system (6), (13)

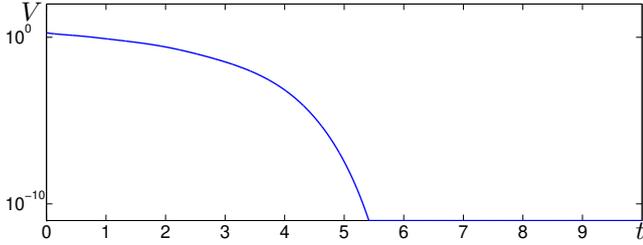


Fig. 4. Lyapunov function for the system (6), (13)

Algorithm

INITIALIZATION: $V_0 = 1$; $a = V_{\min}$; $b = 1$;

STEP:

```

If  $x_i^T D(\varrho(b)) P D(\varrho(b)) x_i > 1$  then
     $a = b$ ;  $b = 2b$ ;
elseif  $x_i^T D(\varrho(a)) P D(\varrho(a)) x_i < 1$  then
     $b = a$ ;  $a = \max\{\frac{a}{2}, V_{\min}\}$ ;
else
     $c = \frac{a+b}{2}$ 
    If  $x_i^T D(\varrho(c)) P D(\varrho(c)) x_i < 1$  then
         $b = c$ ;
    else  $a = \max\{V_{\min}, c\}$ ;
endif;
endif;
 $V_i = b$ ;

```

If STEP is applied recurrently many times to the same vector x_i then it allows to localize the unique positive root of the equation $x^T D(\varrho(V)) P D(\varrho(V)) x - 1 = 0$.

Example 2 Consider the system (6) for $n = 3$. Let us choose the same P, K as in Example 1, and $\mu = 0.2$ that satisfy (12) with $X = P^{-1}, Y = KX$. The results of simulation are shown in Fig. 3, Fig. 4. Fig. 5 demonstrates plots of the state norm $\|x\|$ for the hyperexponential control (13) (red line) and finite-time control (8) (blue line). Fig. 6 demonstrates plots of $\|x\|$ for the hyperexponential and finite-time controls with measurement band limited noise of power 10^{-5} . Fig. 7 shows plots of the state norm $\|x\|$ in the presence of delay $\tau = 0.05$ in the control channel. It is easy to see that the control (13) has better robustness properties than the finite-time control (8). A detailed study of the presented control algorithms on robustness analysis with respect to disturbances, uncertainties, delays and extension of these results on a wider class of systems goes beyond the scope of the paper providing the subjects for a future research.

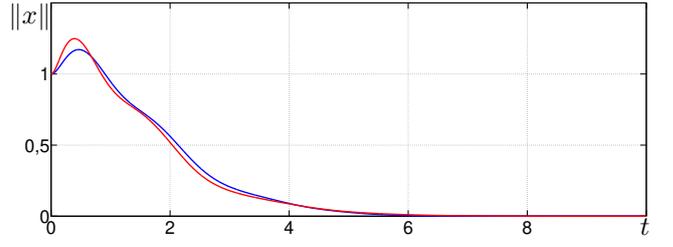


Fig. 5. Simulation plot of $\|x\|$ for the hyperexponential (red line) and finite-time (blue line) control with initial conditions $x(0) = [1 \ 0 \ 0]^T$

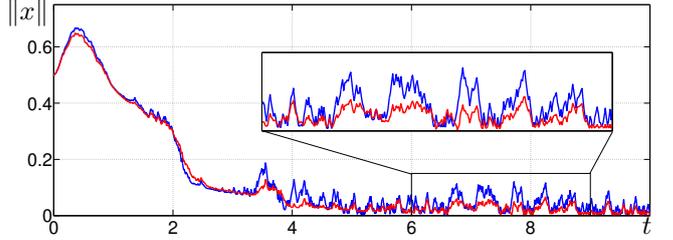


Fig. 6. Simulation plot of $\|x\|$ for the hyperexponential (red line) and finite-time (blue line) control with measurement band limited noise of power 10^{-5} and initial conditions $x(0) = [0.5 \ 0 \ 0]^T$

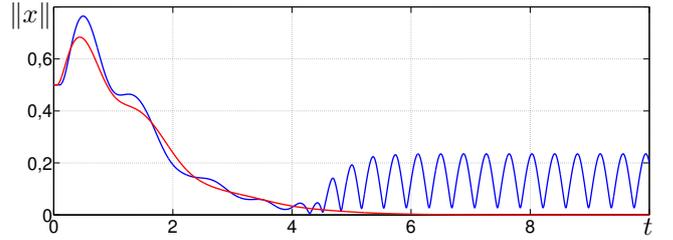


Fig. 7. Simulation plot of $\|x\|$ for the hyperexponential (red line) and finite-time (blue line) control in the presence of delay $\tau = 0.05$ in the control channel for initial conditions $x(0) = [0.5 \ 0 \ 0]^T$

In order to ensure fast convergence outside the vicinity of the origin one can combine a nearly fixed-time control with hyperexponential one.

Define

$$\bar{Q}_2(V, x) = x^T \bar{D}(V^{-1}) P \bar{D}(V^{-1}) x - 1,$$

where $\bar{D}(\lambda) = \text{diag}\{\lambda^{p_i}\}_{i=1}^n$, $p_i = 1 + (i-1)\nu$, $\nu \in \mathbb{R}_+$, $\lambda \in \mathbb{R}_+$ and $0 < P \in \mathbb{R}^{n \times n}$. Denote $\bar{H} = \text{diag}\{p_i\}_{i=1}^n$.

Corollary 2 Let the system of matrix inequalities (12) and

$$X \bar{H} + \bar{H} X > 0$$

be feasible for some $\gamma \in \mathbb{R}_+$ and $X \in \mathbb{R}^{n \times n}, Y \in \mathbb{R}^{1 \times n}$. Let

$$u(V, x) = \begin{cases} KD(\varrho(V))x & \text{for } x^T P x < 1, \\ V^{1+\nu} K \bar{D}(V^{-1})x & \text{for } x^T P x \geq 1, \end{cases} \quad (14)$$

where $K = YX^{-1}$, $X = P^{-1} \in \mathbb{R}^{n \times n}$, $Y \in \mathbb{R}^{1 \times n}$ and

$$V \in \mathbb{R}_+ : \begin{cases} Q_1(V, x) = 0 & \text{for } x^T P x < 1, \\ Q_2(V, x) = 0 & \text{for } x^T P x \geq 1. \end{cases}$$

Then the closed-loop system (6), (14) is globally nearly fixed-time and hyperexponentially stable with degree 1.

Using the same arguments as in Proposition 1 it is easy to show that under sampled-time realization the ILF-based control (13) preserves hyperexponential stability:

Corollary 3 *Let $\{t_i\}_{i=0}^{\infty}$ be a strictly increasing sequence of arbitrary time instants, $0 = t_0 < t_1 < t_2 < \dots$ such that $\lim_{i \rightarrow \infty} t_i = +\infty$. Let for (6) all conditions of Theorem 5 (Corollary 2) be satisfied. Then sampled-time realization $u(t) = u(V_i, x(t))$ for $t \in [t_i, t_{i+1})$, where $V_i \in \mathbb{R}_+ : Q(V_i, x(t_i)) = 0$ provides hyperexponential stability of the closed-loop system.*

5. CONCLUSIONS

In the paper sufficient Lyapunov characterizations of hyperexponential stability are presented for the system (1) using explicit and implicit Lyapunov function approaches. Firstly, it is shown that ILF-based finite/fixed-time control methods provide hyperexponential stability under sampled-time realization. Next, the hyperexponential control (13) was proposed for the linear system (6). The preliminary numeric experiments indicate that this control is less sensitive with respect to noises than its finite-time analog. In addition, the hyperexponential control demonstrates better performance in the presence of delays as well. Tuning of control parameters is presented in the form of linear matrix inequalities. The performance of the proposed control is illustrated through simulations.

The presented results open a lot of topics for future research. For example, development of ILF-based hyperexponential controls and observers for nonlinear and MIMO systems, detailed study of the presented control on robustness analysis with respect to disturbances, uncertainties, delays, etc.

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