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On finite-time observers for linear systems [★]

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Abstract: This work deals with the problem of finite-time homogeneous observer design for linear systems. Compared to other works, the stability conditions and parameters tuning are formulated using linear matrix inequalities being less conservative. The proposed results are based on the homogeneity property and Implicit Lyapunov Function method. The results are supported with simulation examples.

Keywords: Finite-time observer, state estimation, homogeneous system, Implicit Lyapunov function, finite-time stability

1. INTRODUCTION

The problem of nonlinear observer design remains to be active research domain (e.g., Afri et al., 2017; Lopez-Ramirez et al., 2018; Andrieu et al., 2009; Angulo et al., 2013; Andrieu et al., 2008, etc.). State estimation with time constraints (e.g., finite/fixed-time observers) is highly demanded specially for fast control systems, for control processes strongly restricted by time and for fault detection. Also, finite-time estimation is a simple way to realize the separation principle, when the problems of observer and control design can be considered and analyzed independently.

This paper is devoted to the well-known problem of finite-time observer design for the linear system

$$\dot{x} = Ax, \quad y = Cx, \quad x \in \mathbb{R}^n,$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad C = [1 \ 0 \ \cdots \ 0] \quad (1)$$

with the following homogeneous observer (Davila et al., 2005; Perruquetti et al., 2008)

$$\dot{\hat{x}} = A\hat{x} + \text{diag}\{l_i\}_{i=1}^n \begin{bmatrix} |\hat{x}_1 - y|^{1+\mu} \text{sign}(\hat{x}_1 - y) \\ |\hat{x}_1 - y|^{1+2\mu} \text{sign}(\hat{x}_1 - y) \\ \vdots \\ |\hat{x}_1 - y|^{1+n\mu} \text{sign}(\hat{x}_1 - y) \end{bmatrix}, \quad (2)$$

where $\mu \in (-1/n, 0)$, and the gains $l_i \in \mathbb{R}$ for $i = 1, \dots, n$ to be determined. Many studies have been devoted to the analysis of such and similar (for example, sliding-mode observers with $\mu = -1/n$) observers. However, the existing results have drawbacks, the main of which are listed below:

- most of the results are valid only for sufficiently small μ , i.e., for the case the observer (2) is close to the

standard Luenberger observer (see, e.g., Perruquetti et al., 2008);

- the proposed stability conditions of the observation error system (observer parameters tuning) are too conservative (including distributed-parameter linear matrix inequalities (LMIs) in Lopez-Ramirez et al., 2018);
- some results are presented only for a particular n (e.g., finite-time differentiators Davila et al., 2005; Angulo et al., 2013);
- some works do not provide the settling time estimation (e.g., results based on homogeneous approximations).

Compared to other results, in this paper the stability conditions and parameters tuning are formulated being less conservative. The proposed conditions are based on the use of homogeneity property and the Implicit Lyapunov Function (ILF) method.

The structure of this paper is as follows. Notations used in the work are introduced in Section 2. Section 3 considers preliminaries used in the paper. The results on finite-time observer design are presented in Section 4. Finally, conclusions are given in Section 5.

2. NOTATION

Through the paper the following notation will be used:

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$, where \mathbb{R} is the field of real numbers;
- \mathbb{R}^n denotes the n dimensional Euclidean space with vector norm $\|\cdot\|$;
- The symbol $1, m$ is used to denote a sequence of integers $1, \dots, m$;
- The order relation $P > 0$ (< 0 ; ≥ 0 ; ≤ 0) for $P \in \mathbb{R}^{n \times n}$ means that P is symmetric and positive (negative) definite (semidefinite);
- $\lambda_i(A)$ denotes i^{th} eigenvalue of a matrix $A \in \mathbb{R}^{n \times n}$;
- The set of (nonnegative) diagonal matrices with dimension $n \times n$ is defined by $(\mathbb{D}_+^n) \mathbb{D}^n$;

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- $\text{diag}\{\lambda_i\}_{i=1}^n$ is a diagonal matrix with elements λ_i , $i = \overline{1, n}$;
- $O_{m \times n}$ denotes zero matrix with dimension $m \times n$;
- $I_n \in \mathbb{R}^{n \times n}$ is the identity matrix;
- $e_s(i) = \left(\underbrace{0 \cdots 0}_{s \text{ components}} \quad \overset{i^{\text{th}}}{1} \quad 0 \cdots 0 \right)^T \in \mathbb{R}^s$, $s \geq 1$, is a vector of the canonical basis of \mathbb{R}^s .

3. PRELIMINARIES

3.1 Finite-time stability

Let us consider the system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \geq 0, \quad (3)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a nonlinear continuous vector field and $f(0) = 0$.

Definition 1 (Bhat and Bernstein, 2000; Orlov, 2004) *The origin of (3) is said to be globally finite-time stable if it is globally uniformly asymptotically stable and there exists a locally bounded function $T: \mathbb{R}^n \rightarrow [0, +\infty)$ such that any solution $x(t, x_0)$ of the system (3) for all $x_0 \in \mathbb{R}^n$ satisfies $x(t, x_0) = 0, \forall t \geq T(x_0)$. The function T is called a settling-time estimate.*

The following theorem presents the ILF method for finite-time stability analysis.

Theorem 1 (Polyakov et al., 2015) *If there exists a continuous function*

$$\begin{aligned} Q: \mathbb{R}_+ \times \mathbb{R}^n &\rightarrow \mathbb{R} \\ (V, x) &\mapsto Q(V, x) \end{aligned}$$

such that

- C1) Q is continuously differentiable outside the origin;
C2) for any $x \in \mathbb{R}^n \setminus \{0\}$ there exist $V^- \in \mathbb{R}_+$ and $V^+ \in \mathbb{R}_+$:

$$Q(V^-, x) < 0 < Q(V^+, x); \quad (4)$$

- C3) for $\Omega = \{(V, x) \in \mathbb{R}^{n+1}: Q(V, x) = 0\}$

$$\lim_{\substack{x \rightarrow 0 \\ (V, x) \in \Omega}} V = 0^+, \quad \lim_{\substack{V \rightarrow 0^+ \\ (V, x) \in \Omega}} \|x\| = 0, \quad \lim_{\substack{\|x\| \rightarrow \infty \\ (V, x) \in \Omega}} V = +\infty;$$

- C4) the inequality

$$-\infty < \frac{\partial Q(V, x)}{\partial V} < 0$$

holds for $\forall V \in \mathbb{R}_+$ and $\forall x \in \mathbb{R}^n \setminus \{0\}$;

- C5) the inequality

$$\frac{\partial Q(V, x)}{\partial x} f(x) \leq \sigma V^{1-\mu} \frac{\partial Q(V, x)}{\partial V}$$

holds $\forall (V, x) \in \Omega$, where $0 < \mu \leq 1$ and $\sigma \in \mathbb{R}_+$ are some constants.

Then the origin of the system (3) is globally finite-time stable with the following settling time estimate

$$T(x_0) \leq \frac{V_0^\mu}{\sigma \mu},$$

where $V_0 \in \mathbb{R}_+$: $Q(V_0, x_0) = 0$.

3.2 Homogeneity

The homogeneity is a symmetry property of an object with respect to a group of dilations (Zubov, 1958; Polyakov, 2020; Kawski, 1991). A map $\mathbf{d}: \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ is called a linear dilation, where $\mathbf{d}(s) = e^{G_{\mathbf{d}} s}$, and the anti-Hurwitz matrix $G_{\mathbf{d}}$ is called the generator of dilation.

Definition 2 (Kawski, 1991) *A vector field $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ (a function $g: \mathbb{R}^n \rightarrow \mathbb{R}$) is said to be \mathbf{d} -homogeneous of degree $\nu \in \mathbb{R}$ if*

$$\begin{aligned} f(\mathbf{d}(s)x) &= e^{\nu s} \mathbf{d}(s) f(x), \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \forall s \in \mathbb{R}. \\ (\text{resp. } g(\mathbf{d}(s)x) &= e^{\nu s} g(x), \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \quad \forall s \in \mathbb{R}.) \end{aligned} \quad (5)$$

The system $\dot{x} = f(x)$ is called \mathbf{d} -homogeneous if $f(x)$ is \mathbf{d} -homogeneous.

In the case when the matrix $G_{\mathbf{d}}$ is diagonal, the type of homogeneity is called weighted.

Note that the homogeneity property implies robustness to external perturbations, measurement noise and time-delays (see, for example, Bernuau et al., 2013; Efimov and Polyakov, 2021).

3.3 Stability analysis of homogeneous systems with sector nonlinearities

Consider the system:

$$\dot{x}(t) = A_0 x(t) + \sum_{j=1}^M A_j f^j(Ux(t)), \quad t \geq 0, \quad (6)$$

where $x(t) \in \mathbb{R}^n$ is the state vector,

$$f^j(Ux) = [f_1^j(e_n^T(1)Ux) \dots f_n^j(e_n^T(n)Ux)]^T, \quad j = \overline{1, M}$$

are continuous functions ensuring existence of solutions of the system (6) in forward time, $f^j(0) = 0$, U and A_k for $k = \overline{0, M}$ are the matrices with dimensions $\mathbb{R}^{n \times n}$.

It is assumed that the system (6) satisfies the following properties:

Assumption 1 *For any $i = \overline{1, n}$, $j = \overline{1, M}$:*

$$s f_i^j(s) > 0 \quad \forall s \in \mathbb{R} \setminus \{0\}. \quad (7)$$

Assumption 2 *The system (6) is \mathbf{d} -homogeneous of degree $\mu < 0$ with the generator $G_{\mathbf{d}}$, i.e.*

$$\begin{aligned} \mathbf{d}(s) \left(A_0 x + \sum_{j=1}^M A_j f^j(Ux) \right) \\ = e^{-s\mu} \left(A_0 \mathbf{d}(s)x + \sum_{j=1}^M A_j f^j(U \mathbf{d}(s)x) \right), \quad \forall s \in \mathbb{R}, \quad \forall x \in \mathbb{R}^n; \end{aligned} \quad (8)$$

and there exists $\rho \in \{1, \dots, M\}$ such that $f^j(Ux) = e^{-s\mu} \mathbf{d}(s)^{-1} f^j(U \mathbf{d}(s)x)$ are homogeneous of degree μ for $j = \overline{1, \rho}$.

Assumption 3 *Each row of the matrix U is a left eigenvector of $G_{\mathbf{d}}$, i.e., $e_n^T(i)U = \mathbf{u}_i(G_{\mathbf{d}})$ for $i = \overline{1, n}$.*

After proper re-indexing and decomposition of a nonlinear system dynamics the sector condition (7) and homogeneity property imply that:

- for any $i = \overline{1, n}$, $j = \overline{1, \zeta}$ the nonlinearities f_i^j have unbounded integrals, i.e.,

$$\lim_{s \rightarrow \pm\infty} \int_0^s f_i^j(\sigma) d\sigma = +\infty, \quad (9)$$

where $\zeta \geq \rho$;

- due to negative homogeneity degree μ for any $i = \overline{1, n}$, $j = \overline{1, \rho}$:

$$s f_i^j(s) > \psi_i^j s^2 \quad \forall s \in [\underline{s}_i, \bar{s}_i] \setminus \{0\}, \quad (10)$$

where $\psi_i^j \in \mathbb{R}_+$ and $-\infty < \underline{s}_i < 0 < \bar{s}_i < +\infty$.

Define $\Psi^j = \text{diag}\{\psi_i^j\}_{i=1}^n \in \mathbb{D}_+^n$, $j = \overline{1, \rho}$ and denote $\mathbf{u}_i = \mathbf{u}_i(G_{\mathbf{d}})$ for compactness. Define $\mathcal{I} \in \mathbb{R}^{n \times n} : \mathcal{I}_{i,j} = \begin{cases} \text{sign}(\mathbf{u}_i \mathbf{u}_j^T), & \text{if } \mathbf{u}_j, \mathbf{u}_i \text{ are parallel} \\ 0, & \text{otherwise} \end{cases}$ for $i, j = \overline{1, n}$.

Introduce an ILF candidate in the form

$$Q(V, x) = x^T \mathbf{d}^T (-\ln V) P \mathbf{d} (-\ln V) x + 2 \sum_{j=1}^M \sum_{i=1}^n \Lambda_i^j \int_0^{\mathbf{u}_i \mathbf{d} (-\ln V) x} f_i^j(s) ds - \chi, \quad (11)$$

where $\Lambda^j = \text{diag}\{\Lambda_i^j\}_{i=1}^n \in \mathbb{D}_+^n$, $P \in \mathbb{R}^{n \times n}$ is a symmetric positive semi-definite matrix, and $\chi \in \mathbb{R}_+$.

Theorem 2 (Zimenko et al., 2021) *Let Assumptions 1, 2, 3 be satisfied. Let $\alpha, \iota_1, \iota_2 \in \mathbb{R}_+$ be chosen such that the LMI system*

$$P \geq 0; \quad \Gamma > 0; \quad \iota_1 P + U^T \sum_{j=1}^{\zeta} \Lambda^j U > 0; \quad (12a)$$

$$P G_{\mathbf{d}} + G_{\mathbf{d}}^T P \geq 0; \quad (12b)$$

$$U^T \sum_{j=1}^M \Lambda^j U + \iota_2 (P G_{\mathbf{d}} + G_{\mathbf{d}}^T P) > 0; \quad (12c)$$

$$Q_1 + Q_3 + \alpha Q_2 \leq 0, \quad (12d)$$

is feasible for some $P = P^T \in \mathbb{R}^{n \times n}$; $\Lambda^j \in \mathbb{D}_+^n$, $j = \overline{1, M}$, $0 \leq \Xi = \Xi^T \in \mathbb{R}^{(M+1)n \times (M+1)n}$, where for $X = P + \sum_{j=1}^{\rho} U^T \Lambda^j \Psi^j U$ and $\tilde{A}_0 = A_0 + \sum_{j=1}^{\rho} A_j \Psi^j U$

$$Q_1 = \begin{bmatrix} X \tilde{A}_0 + \tilde{A}_0^T X & X A_1 + \tilde{A}_0^T U^T \Lambda^1 & \dots & X A_M + \tilde{A}_0^T U^T \Lambda^M \\ \Lambda^1 U \tilde{A}_0 + \tilde{A}_0^T X & \Lambda^1 U A_1 + \tilde{A}_0^T U^T \Lambda^1 & \dots & \Lambda^1 U A_M + \tilde{A}_0^T U^T \Lambda^M \\ \vdots & \vdots & \ddots & \vdots \\ \Lambda^M U \tilde{A}_0 + \tilde{A}_0^T X & \Lambda^M U A_1 + \tilde{A}_0^T U^T \Lambda^1 & \dots & \Lambda^M U A_M + \tilde{A}_0^T U^T \Lambda^M \end{bmatrix};$$

$$Q_2 = \begin{bmatrix} X G_{\mathbf{d}} + G_{\mathbf{d}}^T X & U^T \Lambda^1 H & \dots & U^T \Lambda^M H \\ H \Lambda^1 U & & & \\ \vdots & \mathbf{O}_{Mn \times Mn} & & \\ H \Lambda^M U & & & \end{bmatrix};$$

$$Q_3 = \Xi + \begin{bmatrix} \Upsilon_{0,0} & U^T \Upsilon_{0,1} & \dots & U^T \Upsilon_{0,M} \\ \Upsilon_{0,1} U & \Upsilon_{1,1} & \dots & \Upsilon_{1,M} \\ \vdots & \vdots & \ddots & \vdots \\ \Upsilon_{0,M} U & \Upsilon_{1,M} & \dots & \Upsilon_{M,M} \end{bmatrix},$$

$H = \text{diag}\{\lambda_i(G_{\mathbf{d}})\}_{i=1}^n$, $\Upsilon_{0,0} = \mathbf{O}_{n \times n}$, $\Upsilon_{0,i} \in \mathbb{D}_+^n$, $\Upsilon_{i,i} = \Upsilon_{i,i}^T \in \mathbb{R}^{n \times n}$,

$$\Upsilon_{i,j} \in \mathbb{R}^{n \times n} : e_n^T(\eta) \Upsilon_{i,j} e_n(l) \begin{cases} \geq 0, & \text{if } e_n^T(\eta) \mathcal{I} e_n(l) = 1 \\ \leq 0, & \text{if } e_n^T(\eta) \mathcal{I} e_n(l) = -1 \\ = 0, & \text{otherwise} \end{cases}$$

for $i = \overline{1, M}$, $j = \overline{i, M}$, $\eta, l = \overline{1, n}$.

Then the origin is globally finite-time stable and

$$T(x_0) \leq -\frac{V_0^{-\mu}}{\alpha \mu},$$

where $Q(V_0, x_0) = 0$ for

$$\chi < n \min_{i \in \overline{1, n}} \{\underline{s}_i^2, \bar{s}_i^2\} \lambda_{\min}(X) \|U\|^{-2}.$$

As shown in the following section, the error equation for the observer (2) can be represented in the form (6). However, in this case the LMI system from Theorem 2 for $n \geq 3$ is unfeasible. The present paper extends the result of Theorem 2 for stability analysis of the observation error in (2) for any $n \geq 1$.

4. MAIN RESULT

Consider the problem of state estimation for the system (1) with the observer in the form (2).

For $e = \hat{x} - x$ the error system takes the form

$$\dot{e} = Ae + \text{diag}\{l_i\}_{i=1}^n \begin{bmatrix} |e_1|^{1+\mu} \text{sign}(e_1) \\ |e_1|^{1+2\mu} \text{sign}(e_1) \\ \vdots \\ |e_1|^{1+n\mu} \text{sign}(e_1) \end{bmatrix}, \quad e(0) = e_0. \quad (13)$$

Note that the system (13) is homogeneous of degree $\mu < 0$ with $G_{\mathbf{d}} = \text{diag}\{1+(i-1)\mu\}_{i=1}^n$, i.e., the system is weighted homogeneous.

Note that a qualitative robustness analysis for such observer based on the homogeneity property is provided by Lopez-Ramirez et al., 2018 (the observer is Input-to-State Stable with respect to exogenous disturbances and measurement noises). Thus, in this paper we consider the disturbance free case.

Remark 1 For $A_0 = A$, $M = 1$, $A_1 = \text{diag}\{l_i\}_{i=1}^n$, $d \equiv 0$,

$$U = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & 0 \end{bmatrix},$$

$$f^1(Ue) = \begin{bmatrix} |e_1|^{1+\mu} \text{sign}(e_1) \\ |e_1|^{1+2\mu} \text{sign}(e_1) \\ \vdots \\ |e_1|^{1+n\mu} \text{sign}(e_1) \end{bmatrix} \quad (14)$$

the system (13) corresponds to (6), i.e.,

$$\dot{e} = Ae + A_1 f^1(Ue). \quad (15)$$

However, for the system (15) the LMIs (12) are not feasible for $n \geq 3$, i.e., Theorem 2 is not applicable for stability analysis of (15). Compensation of this drawback is the main result of this paper.

Denote $\hat{f}_i(s) = f_i^1(s) - s$, $i = \overline{1, n}$ and $\hat{f}(s) = [\hat{f}_1(s), \dots, \hat{f}_n(s)]^T$. Note that according to (14) we have $0 \leq \hat{f}_i(s) \leq \hat{f}_{i+1}(s)$ for any $s \in [0, 1]$, $i = \overline{1, n-1}$. Then, taking it into account, define the matrices $\Upsilon_{0,1} \in \mathbb{D}_+^n$, $\Upsilon_{1,1} = \Upsilon_{1,1}^T \in \mathbb{R}^{n \times n}$ such that

$$\begin{bmatrix} s \\ \hat{f}(s) \end{bmatrix}^T \begin{bmatrix} \mathbf{O}_{n \times n} & U^T \Upsilon_{0,1} \\ \Upsilon_{0,1} U & \Upsilon_{1,1} \end{bmatrix} \begin{bmatrix} s \\ \hat{f}(s) \end{bmatrix} \geq 0, \quad \forall s \in [0, 1], \quad (16)$$

i.e., some elements $e_n^T(\eta_1)\Upsilon_{i,j}e_n(\eta_2)$, $\eta_1, \eta_2 \in \overline{1, n}$ of the matrices $\Upsilon_{i,j}$, $i \in \overline{0, 1}$, $j = 1$ can be selected to be negative if there are positive elements of $\Upsilon_{l,q}$, $l, q \in \overline{0, 1}$ that correspond to multiplication of terms $\hat{f}_{\eta_3}(s)$ and $\hat{f}_{\eta_4}(s)$, $\eta_3, \eta_4 \in \overline{1, n}$ with the same multipliers, where

$$\hat{f}_{\eta_3}(s)\hat{f}_{\eta_4}(s) \geq \hat{f}_{\eta_1}(s)\hat{f}_{\eta_2}(s);$$

and the sum of these elements is greater or equal than the sum of absolute values of negative counterparts that correspond to terms with slower dynamics. For example, for $n = 3$ the matrix $\begin{bmatrix} \text{O}_{n \times n} & U^T \Upsilon_{0,1} \\ \Upsilon_{0,1} U & \Upsilon_{1,1} \end{bmatrix}$ can take the form

$$\begin{bmatrix} \text{O}_{n \times n} & U^T \Upsilon_{0,1} \\ \Upsilon_{0,1} U & \Upsilon_{1,1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \xi_1 & \xi_2 & \xi_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \xi_1 & 0 & 0 & -2(\xi_2 + \xi_3 + \xi_5 + \xi_6) & \xi_5 & \xi_6 \\ \xi_2 & 0 & 0 & \xi_5 & -2\xi_4 & \xi_4 \\ \xi_3 & 0 & 0 & \xi_6 & \xi_4 & 0 \\ \xi_i \geq 0, & i = \overline{1, 6}, \end{bmatrix},$$

where the following inequalities $\hat{f}_i(s) \leq \hat{f}_{i+1}(s)$ for $i = \overline{1, 2}$ and $s\hat{f}_j(s) \geq [\hat{f}_1(s)]^2$ for $j = \overline{2, 3}$ are taken into account.

In line with (11), let us introduce an ILF candidate in the form $V \in \mathbb{R}_+$: $\tilde{Q}(V, e) = 0$, where

$$\begin{aligned} \tilde{Q}(V, e) &= e^T \mathbf{d}^T(-\ln V) P \mathbf{d}(-\ln V) e \\ &+ 2 \sum_{i=1}^n \frac{\Lambda_i}{2 + i\mu} |V^{-1} e_1|^{2+i\mu} - \chi, \end{aligned} \quad (17)$$

$$0 \leq P \in \mathbb{R}^{n \times n}, \Lambda \in \mathbb{D}_+^n, \chi \in \mathbb{R}_+.$$

Now we are ready to present our main results.

Theorem 3 Let $\alpha, \iota_1, \iota_2, \in \mathbb{R}_+$ and $\beta_i \geq 0$, $i \in \overline{1, n}$ be chosen such that the LMI system

$$P \geq \tilde{P} \geq 0; \quad (18a)$$

$$\iota_1 P + U^T \Lambda U > 0; \quad (18b)$$

$$P G_{\mathbf{d}} + G_{\mathbf{d}}^T P \geq 0; \quad (18c)$$

$$U^T \Lambda U + \iota_2 (P G_{\mathbf{d}} + G_{\mathbf{d}}^T P) > 0; \quad (18d)$$

$$Q_1 + Q_3 + Q_4 + \alpha Q_2 \leq 0, \quad (18e)$$

is feasible for $\tilde{P}, \Lambda \in \mathbb{D}_+^n$, $P, \Xi \geq 0$, where for $\tilde{A}_0 = A_0 + A_1 U$, $A_1 = \text{diag}\{\iota_i\}_{i=1}^n$, $\tilde{P}_1 = \lambda_1(\tilde{P})$ and $X = P + U^T \Lambda U$:

$$Q_1 = \begin{bmatrix} X \tilde{A}_0 + \tilde{A}_0^T X & X A_1 + \tilde{A}_0^T U^T \Lambda \\ \Lambda U \tilde{A}_0 + A_1 X & \Lambda U A_1 + A_1 U^T \Lambda \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} X G_{\mathbf{d}} + G_{\mathbf{d}}^T X & U^T \Lambda G_{\mathbf{d}} \\ G_{\mathbf{d}} \Lambda U & \text{O}_{n \times n} \end{bmatrix},$$

$$Q_3 = \Xi + \begin{bmatrix} \text{O}_{n \times n} & U^T \Upsilon_{0,1} \\ \Upsilon_{0,1} U & \Upsilon_{1,1} \end{bmatrix},$$

$$Q_4 = \begin{bmatrix} Q_{41} & \text{O}_{n \times n} \\ \text{O}_{n \times n} & Q_{42} \end{bmatrix},$$

$$Q_{41} = \sum_{i=1}^n -\beta_i i\mu \left(\frac{1}{1+i\mu} \right)^{1+\frac{1}{i\mu}} \left(P - \frac{1+i\mu}{2} R \tilde{P} \right),$$

$$Q_{42} = - \left(\tilde{P}_1 + 2 \sum_{j=1}^n \frac{\Lambda_j}{2+j\mu} \right) \text{diag}\{\beta_i\}_{i=1}^n.$$

and $\Upsilon_{0,1} \in \mathbb{D}_+^n$, $\Upsilon_{1,1} = \Upsilon_{1,1}^T \in \mathbb{R}^{n \times n}$ are in accordance with (16). Then the observer (2) provides finite-time convergence of the observation error and

$$T(e_0) \leq -\frac{V_0^{-\mu}}{\alpha\mu},$$

where $V_0 \in \mathbb{R}_+$: $\tilde{Q}(V_0, e_0) = 0$ for $\chi = \tilde{P}_1 + 2 \sum_{i=1}^n \frac{\Lambda_i}{2+i\mu}$.

By Implicit Function Theorem (Courant and John, 2000) we have

$$\begin{aligned} \dot{V} &= - \left(\frac{\partial \tilde{Q}}{\partial V} \right)^{-1} \frac{\partial \tilde{Q}}{\partial x} \dot{e} \\ &= 2V \left(e^T \mathbf{d}(-\ln V) (P G_{\mathbf{d}} + G_{\mathbf{d}} P) \mathbf{d}(-\ln V) e \right. \\ &\quad \left. + 2u \sum_{i=1}^n \Lambda_i \lambda_i (G_{\mathbf{d}}) V^{-1} e_1 f_i^1(u V^{-1} e_1) \right)^{-1} \\ &\quad \times \left[e^T \mathbf{d}(-\ln V) X \mathbf{d}(-\ln V) \dot{e} + \sum_{i=1}^n \Lambda_i \hat{f}_i(u V^{-1} e_1) u V^{-1} \dot{e}_1 \right] \\ &= 2V^{1+\mu} \left(e^T \mathbf{d}(-\ln V) (X G_{\mathbf{d}} + G_{\mathbf{d}}^T X) \mathbf{d}(-\ln V) e \right. \\ &\quad \left. + 2e^T \mathbf{d}(-\ln V) U^T \Lambda G_{\mathbf{d}} \hat{f}(U \mathbf{d}(-\ln V) e) \right)^{-1} \\ &\quad \times \left[e^T \mathbf{d}(-\ln V) X \tilde{A}_0 \mathbf{d}(-\ln V) e \right. \\ &\quad \left. + e^T \mathbf{d}(-\ln V) X A_1 \hat{f}(U \mathbf{d}(-\ln V) e) \right. \\ &\quad \left. + V^{-\mu} \dot{e}^T \mathbf{d}(-\ln V) U^T \Lambda \hat{f}(U \mathbf{d}(-\ln V) e) \right] \\ &= 2V^{1+\mu} (y^T Q_2 y)^{-1} \left[e^T \mathbf{d}(-\ln V) X \tilde{A}_0 \mathbf{d}(-\ln V) e \right. \\ &\quad \left. + e^T \mathbf{d}(-\ln V) X A_1 \hat{f}(U \mathbf{d}(-\ln V) e) \right. \\ &\quad \left. + e^T \mathbf{d}(-\ln V) \tilde{A}_0^T U^T \Lambda \hat{f}(U \mathbf{d}(-\ln V) e) \right. \\ &\quad \left. + \hat{f}(U \mathbf{d}(-\ln V) e)^T A_1 U^T \Lambda \hat{f}(U \mathbf{d}(-\ln V) e) \right] \\ &= V^{1+\mu} \frac{y^T Q_1 y}{y^T Q_2 y}, \end{aligned} \quad (19)$$

where

$$y = \begin{bmatrix} \mathbf{d}(-\ln V) e \\ \hat{f}(U \mathbf{d}(-\ln V) e) \end{bmatrix},$$

and according to Remark 1, the matrix Q_1 is not negative definite. The main technical step that allowed us to extend the result of Theorem 2 to the system (15) with $n \geq 3$ is the use of the inequality

$$\begin{aligned} &[\hat{f}_i(U \mathbf{d}(-\ln V) e)]^2 \\ &\leq \frac{-i\mu \left(\frac{1}{1+i\mu} \right)^{1+\frac{1}{i\mu}} u^2}{\tilde{P}_1 + 2u^2 \sum_{i=1}^n \frac{\Lambda_i}{2+i\mu}} e^T \mathbf{d}^T(-\ln V) \left(P - \frac{1+i\mu}{2} R \tilde{P} \right) \mathbf{d}(-\ln V) e. \end{aligned} \quad (20)$$

Indeed, using Q_4 based on (20) (i.e., $y^T Q_4 y \geq 0$ hold), one can obtain in $Q_1 + Q_3 + Q_4$ a stable matrix in the lower right quadrant (due to the form of the matrix U the lower right quadrant $Q_1 + Q_3$ has non-negative eigenvalues), while the positive definite matrix Q_{41} is added to the upper left quadrant. Since $-i\mu \left(\frac{1}{1+i\mu} \right)^{1+\frac{1}{i\mu}} \rightarrow 0$ as $\mu \rightarrow 0$ (i.e., the undesirable effect of the matrix Q_{41} decreases as $|\mu|$ decreases) the inequality (18d) is feasible for sufficiently small $|\mu|$. As shown in the following example, the system (18) may be feasible for μ close to the limit value $-1/n$.

Example 1 Consider the 3rd order system (1) and the observer (2) with $\mu = -0.3$. Let us choose $A_1 = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & -0.3 & 0 \\ 0 & 0 & -0.025 \end{bmatrix}$ (the matrix \tilde{A}_0 is Hurwitz). The system (18) is feasible with $\alpha = 0.005$, $\beta_1 = 0$, $\beta_2 = 0.06$, $\beta_3 = 0.03$ and

$$Q_3 = \begin{bmatrix} 0 & 0 & 0 & 14.7643 & 6.4945 & 3.2112 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 14.7643 & 0 & 0 & -189.9629 & 68.2185 & 17.0571 \\ 6.4945 & 0 & 0 & 68.2185 & -24.3751 & 12.1875 \\ 3.2112 & 0 & 0 & 17.0571 & 12.1875 & 0 \end{bmatrix}.$$

The results of simulation are shown in Fig. 1.

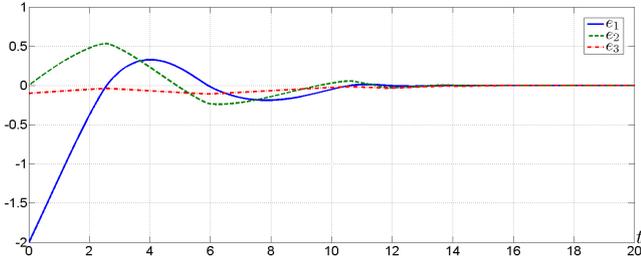


Fig. 1. Simulation plot of e

Remark 2 Note that the term $2 \sum_{i=1}^n \frac{\Lambda_i}{2+i\mu} (uV^{-1}e_1)^{2+i\mu}$ in (17) allows to weaken inequalities. Indeed, for Example 1, where $\Lambda = 0$ (in this case ILF takes the same form as considered by Lopez-Ramirez et al., 2018), the system (18) is feasible only for $\mu \approx -0.1$.

To accelerate the convergence rate one can refer to the following result.

Theorem 4 (Accelerated convergence) *Let all conditions of Theorem 3 be satisfied. Then the observer*

$$\dot{\hat{x}} = A_0 \hat{x} + \text{diag}\{l_i\}_{i=1}^n \text{diag}\{u^{i\mu}\}_{i=1}^n \begin{bmatrix} |\hat{x}_1 - y|^{1+\mu} \text{sign}(\hat{x}_1 - y) \\ |\hat{x}_1 - y|^{1+2\mu} \text{sign}(\hat{x}_1 - y) \\ \vdots \\ |\hat{x}_1 - y|^{1+n\mu} \text{sign}(\hat{x}_1 - y) \end{bmatrix}, \quad (21)$$

provides finite-time convergence of the observation error to 0 for any $u \in \mathbb{R}_+$. Moreover, a smaller $u \in \mathbb{R}_+$ implies a faster convergence.

Example 2 Consider the observer (21), where A_1 is the same as in Example 1. The results of simulation with accelerated convergence are shown in Fig. 2.

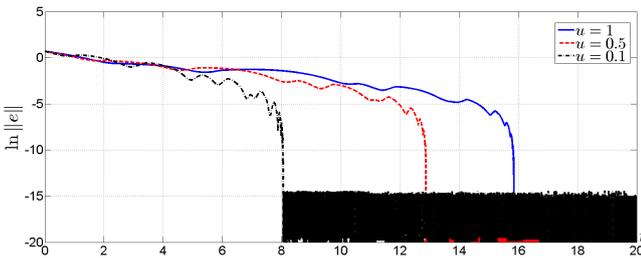


Fig. 2. Simulation plot of $\ln \|e\|$

5. CONCLUSIONS

This paper is devoted to stability analysis and parameters tuning for the finite-time homogeneous observer (2). Compared to other works, the stability conditions and parameters tuning are less conservative. The obtained results are based on the use of homogeneity property and the ILF method. Extension for MIMO systems can be considered as an important problem for further research.

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