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Constructing annihilators for parameter estimation in nonlinearly parameterized signals

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Abstract: This paper studies the theoretical challenge of linearly representing a class of nonlinearly parameterized estimation problems. Conventional linear regression estimation methods can then be used. For illustration purposes, in our work, the DREM method is chosen to estimate the frequencies for a multi-sinusoidal uncertain signal in a fixed time. Identifying annihilators for signals generated by exponential polynomials with unknown parameters is the first step to accomplishing the linear representation. The annihilators are obtained as time-delay operators constructed with an Ore extension. The fundamental idea is to use the action of these operators on exponential polynomials to obtain the linear regression in nonlinear unknown quantities. We precisely characterize the annihilator as a principal ideal using appropriate Ore extensions, substantially simplifying further calculations. Some examples illustrate our annihilator's design.

Keywords: Keywords

1. INTRODUCTION

The problem of parameter identification for a signal or a dynamical system model is the primary step for solving many issues in engineering and natural sciences [Ljung (1999)]. Most existing identification solutions deal with linear regression: many approaches exist to represent the initial estimation problem in the required form, where the measured signal equals a weighted sum of unknown parameters with noise and known weights, which determine the regressor signal.

However, a fundamental challenge that arises in numerous real-world engineering settings, including communications [Montejo and Milstein (2010)], power systems [Ahmed et al. (2022)], and mechanics [Becedas et al. (2009)], is originated by the nonlinear dependence of the measured signals on the quantities of interest. An important and popular example is the parameter estimation of a biased sum of sinusoidal waveform signals in a noisy environment, where the frequencies are also unknown in line with other quantities, such as bias, amplitudes, and phases. An application case, where such a problem setting appears, is an out-of-the-ordinary scenario of assessing a human body's posture in the sagittal plane using just accelerometer measurements [Perruquetti et al. (2012)]. This parameter identification problem consists of estimating the triplets (amplitude, frequency, and phase) for the sum of an unknown number of complicated sinusoidal functions. Riche de Prony investigated this issue in his seminal study from 1795 [Riche de Prony (1795)], and an abundance of results in this domain followed. To mention a few: adaptive nonlinear control techniques were used in [Hsu et al. (1999); Bobtsov et al. (2012); Pin et al. (2019)]. It has also been examined in [Coluccio et al. (2009)] how elementary symmetric functions on the frequencies of multi-sinusoidal signals relate to their multiple integrals, opening up intriguing methods for estimating the parameters, also as in [Vediakova et al. (2021)].

An algebraic method based on differential algebra concepts is put forth in [Ushirobira et al. (2016)] to estimate the amplitudes,

frequencies, and phases of a biased and noisy sum of complex-valued exponential sinusoidal signals. Closed formulas built as integrals, that serve as time-varying filters of the noisy observed signal, provide the resulting parameter estimates. The proposed algebraic method yields faster and more reliable results than conventional iterative tools.

In this work, extending the algebraic framework of [Ushirobira et al. (2016)], we develop a method to transform a broad class of nonlinearly parameterized signals, which can be reduced to exponential polynomials, to a linear regression. The proposed approach uses time-delay operators and is based on an Ore extension algebra. Ore introduced the concept of skew polynomial rings in the 1930s [Ore (1933)]. In [Quadrat and Ushirobira (2022)], the ring of differential time-delay operators was constructed as an Ore extension (or skew polynomial ring), and its characteristics were thoroughly analyzed. The time-varying delay function is used to describe the Noetherianity, the existence of Gröbner bases, and other algebraic properties of this ring. In this note, we consider a particular ring: the exponential one. Thanks to the algebraic properties of a subring of an Ore extension, an annihilator ideal, (*i.e.*, *time-delay operators annihilating an exponential function*) is entirely characterized by a single minimal operator.

The issue of frequency estimation in a biased multi-sinusoidal signal, where all other parameters are also unknown as it was discussed above:

$$y(t) = A_0 + \sum_{j=1}^N A_j \sin(\omega_j t + \phi_j),$$

is used for illustration. Finally, to design a fixed-time converging identification algorithm, the DREM (dynamic regressor extension and mixing) method proposed in [Aranovskiy et al. (2017)] is utilized together with the iterative scheme from [Wang et al. (2019)]. Several analytical and one numeric examples are given to demonstrate the efficiency of our results.

2. ORE EXTENSION

This section introduces the main mathematical tool used in the sequel.

In the following, we consider a ring with unity A . A map $\sigma : A \rightarrow A$ is an *endomorphism* of A if it satisfies

$$\sigma(1) = 1, \quad \sigma(a+b) = \sigma(a) + \sigma(b), \quad \sigma(ab) = \sigma(a)\sigma(b),$$

for all $a, b \in A$. We denote by $\text{End}(A)$ the ring of endomorphisms of A and by $\text{id}_A \in \text{End}(A)$ the identity map of A .

Definition 1. Let $\sigma, \delta \in \text{End}(A)$. We call δ a σ -*derivation* of A if it satisfies

$$\delta(ab) = \delta(a)b + \sigma(a)\delta(b),$$

for all $a, b \in A$. We call δ a *derivation* of A if it satisfies Leibniz's rule, that is, δ is a id_A -derivation.

Definition 2. A *differential ring* (A, δ) is a ring equipped with a derivation δ .

Definition 3. Consider $\sigma, \delta \in \text{End}(A)$ with δ a σ -*derivation* of A . An *Ore extension* $A[\partial; \sigma, \delta]$ of A is the noncommutative ring formed by the elements of the polynomial ring $A[\partial]$ (hence of the form $\sum_{i=0}^n a_i \partial^i$ with $a_i \in A$), equipped with a product given by:

$$\partial a = \sigma(a)\partial + \delta(a), \quad \forall a \in A. \quad (1)$$

The Ore extension $A[\partial; \sigma, \delta]$ of A is also called a *skew polynomial ring* over A .

See [McConnell and Robson (2000)] for more details on Ore extensions.

Definition 4. The *degree* of an element $a = \sum_{i=0}^n a_i \partial^i \in A[\partial; \sigma, \delta]$ is defined as $\deg(a) = n$.

3. TIME-DELAY OPERATORS

In this section, we define particular Ore extensions related to time-delay operators on a specific ring. We refer to [Quadrat and Ushirobira (2022)] for more general constructions.

3.1 Exponential ring

Let us consider the *exponential ring* \mathcal{E} formed by the functions of the type

$$\sum_{\text{finite}} p_i(t) e^{\nu_i t}$$

where $\nu_i \in \mathbb{C}$ and $p_i \in \mathbb{C}[t]$ (polynomials with coefficients in \mathbb{C} and indeterminate t).

We wish to define an Ore extension of \mathcal{E} . Based on Definition 3, we start by specifying an endomorphism σ of \mathcal{E} and a σ -derivation of \mathcal{E} . So, let \mathbb{R}_+ denote the set of nonnegative real numbers. Given $\tau \in \mathbb{R}_+ \setminus \{0\}$, define $\sigma \in \text{End}(\mathcal{E})$ as:

$$\sigma(f(t)) = f(t-\tau), \quad \forall f \in \mathcal{E}.$$

For this particular construction, we consider the σ -derivation $\delta = 0$.

For any $f \in \mathcal{E}$, we can associate an operator $\mathcal{E} \rightarrow \mathcal{E}$ by left multiplication of $f: g \mapsto fg$, for all $g \in \mathcal{E}$. To set the corresponding *indeterminate* ∂ in Definition 3, we take the time-delay operator S (based on the definition of σ) as:

$$S : \mathcal{E} \rightarrow \mathcal{E}, S(f(t)) = f(t-\tau), \quad \forall f \in \mathcal{E}.$$

Since

$$(Sf)(g(t)) = S(f(t)g(t)) = f(t-\tau)g(t-\tau) = (\sigma(f(t))S)(g(t)),$$

then

$$Sf = \sigma(f)S, \quad (2)$$

where the composition of two operators is denoted by juxtaposition. So, the *Ore extension* $\mathcal{O} = \mathcal{E}[S; \sigma, 0]$ can be constructed. This extension is *the ring of time-delay operators with coefficients in \mathcal{E}* .

Remark 1. Clearly, from (2), if f is a constant function, then S and f are commuting operators.

We will now consider the subring $\mathcal{P} = \mathbb{C}[S; \sigma, 0]$ of \mathcal{O} .

Remark 2. Any two elements in \mathcal{P} commute. Indeed, let $A = \sum_i a_i S^i$ with $a_i \in \mathbb{C}$ and $B = \sum_i b_i S^i$ with $b_i \in \mathbb{C}$. Using Remark 1, a_i and b_i commute with S , so A and B commute. Therefore, \mathcal{P} is a commutative ring.

Definition 5. An operator $A \in \mathcal{P}$ is called an *annihilator* of $f \in \mathcal{E}$ if $A(f) = 0$. We denote the set of annihilators of f in \mathcal{P} by $\text{Ann}(f)$.

The last definition provides the next straightforward lemma on the annihilators of the sum of two functions:

Lemma 1. Let $A, B \in \mathcal{P}$ be two operators on \mathcal{E} and let f and $g \in \mathcal{E}$. Suppose that A is an annihilator of f and B is an annihilator of g . Then AB is an annihilator of $f+g$.

Recall that a ring is *Noetherian* if every ideal is finitely generated. Since \mathbb{C} is Noetherian ring and σ is endomorphism of $\mathbb{C} \subset \mathcal{E}$ (actually, $\sigma|_{\mathbb{C}} \equiv \text{id}_{\mathbb{C}}$), then \mathcal{P} is also Noetherian, see McConnell and Robson (2000), subsection 1.2.9. Moreover, since \mathcal{P} is an Ore extension of a field, then \mathcal{P} is also a *principal ideal domain*. That means that every left (or right) ideal of \mathcal{P} is generated by a single element of \mathcal{P} .

An immediate consequence of Definition 5 is the following Lemma; its proof is omitted for brevity:

Lemma 2. Let $f \in \mathcal{E}$. The annihilator $\text{Ann}(f)$ is an ideal of \mathcal{P} .

Corollary 1. For any $f \in \mathcal{E}$, there exists $A \in \mathcal{P}$ such that $\text{Ann}(f) = (A)^1$.

Definition 6. An annihilator $A \in \text{Ann}(f)$ is *minimal* if A is nonzero and has the smallest degree in $\text{Ann}(f)$.

3.2 Some annihilators

Several little results follow concerning elements in \mathcal{E} and their annihilator, and we omit their proofs for brevity.

Lemma 3. Consider $g \in \mathcal{E}$ given by

$$g(t) = \mu t^n, \quad n \in \mathbb{N}, \mu \in \mathbb{C}.$$

Then the operator $(S-1)^{n+1} \in \mathcal{P}$ annihilates μt^n .

Lemma 4. Consider $g \in \mathcal{E}$ given by

$$g(t) = \mu e^{\nu t}, \quad \mu, \nu \in \mathbb{C}.$$

The time-delay operator $S - e^{-\nu\tau} \in \mathcal{P}$ annihilates g .

Corollary 2. Consider $g, h : \mathbb{R}_+ \rightarrow \mathbb{R}$ given by

$$g(t) = \mu \cos(\omega t + \varphi), \quad h(t) = \eta \sin(\omega t + \varphi), \quad \mu, \eta \in \mathbb{C}.$$

The operator $S^2 - 2\cos(\omega\tau)S + 1 \in \mathcal{P}$ annihilates g and h .

¹ The notation (A) means that any element in this ideal is of the form BA with $B \in \mathcal{P}$.

Corollary 3. Consider $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ given by

$$g(t) = \mu e^{\nu t} \cos(\omega t + \varphi), \nu \in \mathbb{C}.$$

The operator $S^2 - 2\cos(\omega\tau)e^{-\nu\tau}S + e^{-2\nu\tau}$ annihilates g .

Lemma 5. Consider $g \in \mathcal{E}$ given by

$$g(t) = \mu t^n e^{\nu t}, n \in \mathbb{N}, \mu, \nu \in \mathbb{C}$$

The operator $(S - e^{-\nu\tau})^{n+1} \in \mathcal{P}$ annihilates g .

Corollary 4. Consider $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ given by

$$g(t) = \mu t^n \cos(\omega t + \varphi), n \in \mathbb{N}.$$

The operator $(S^2 - 2\cos(\omega\tau)S + 1)^{n+1} = (S - e^{i\omega\tau})^{n+1}(S - e^{-i\omega\tau})^{n+1} \in \mathcal{P}$ annihilates g .

The following table summarizes the generators for the annihilator ideal in \mathcal{P} :

function	annihilator
$t^n (n \in \mathbb{N})$	$(S - 1)^{n+1}$
$e^{\nu t} (\nu \in \mathbb{C})$	$S - e^{-\nu\tau}$
$\cos(\omega t + \varphi)$	$S^2 - 2\cos(\omega\tau)S + 1$
$\sin(\omega t + \varphi)$	$S^2 - 2\cos(\omega\tau)S + 1$
$e^{\nu t} \cos(\omega t + \varphi)$	$S^2 - 2\cos(\omega\tau)e^{-\nu\tau}S + e^{-2\nu\tau}$
$t^n e^{\nu t}$	$(S - e^{-\nu\tau})^{n+1}$
$t^n \cos(\omega t + \varphi)$	$(S^2 - 2\cos(\omega\tau)S + 1)^{n+1}$

It is important to stress that all annihilators in the previous results are minimal, following the decomposition into prime polynomials as defined in [Ore (1933)]. The reader may refer to [McConnell and Robson (2000)] for many results on this topic. Evidently, having annihilators of a minimal degree may reduce the computation complexity.

Remark 3. Efficient approximations by exponential polynomials (elements of \mathcal{E}) of a given signal y exist. These algorithms can be found, for instance, in [Beylkin and Monzón (2005)].

The following general construction can be proposed to summarize the above results, and we skip the proof:

Proposition 1. Let $f \in \mathcal{E}$,

$$f = \sum_{i=1}^n p_i(t) e^{\nu_i t} \text{ with } p_i(t) = \sum_{k=0}^{d_i} a_{k,i} t^k \in \mathbb{C}[t]$$

and $\nu_i \in \mathbb{C}$, for all $i = 1, \dots, n$. Then the time-delay operator

$$A = \sum_{i=1}^n \sum_{k=0}^{d_i} (S - e^{-\nu_i \tau})^{k+1} \in \mathcal{P}$$

annihilates f .

Remark 4. Note that $\deg(A) \leq \max_{1 \leq i \leq n} \{d_i\}$.

3.3 Idea of utilization

Annihilators allow estimating parameters, which nonlinearly enter in a signal g , by using linear regression. For example, for $g(t) = A \sin(\omega t + \phi)$, where $A, \omega, \phi \in \mathbb{R}$ are unknown parameters, applying the annihilator from Corollary 2, we get:

$$g(t - 2\tau) + g(t) = 2\cos(\omega\tau)g(t - \tau),$$

and for given delay $\tau > 0$ the values of $g(t - \tau)$ and $g(t - 2\tau)$ can be reconstructed from the measurements of $g(t)$, then we obtained a linear regression with respect to $\cos(\omega\tau)$ (without influence of unknown quantities A and ϕ), whose knowledge leads to the evaluation of ω .

Remark 5. The approach can also be extended to more general cases by considering rational fractions of exponential polynomials. For example, for $g_1(t) = \frac{\psi}{\mu + t}$ or $g_2(t) = \frac{e^{\nu t}}{1 + e^{\nu t}}$, where $\psi, \mu, \nu \in \mathbb{C}$, the qualities $\psi(S - 1)g_1(t) - \tau S g_1^2(t) = 0$ and $e^{-\nu t}(S - 1)g_2(t) + (1 - e^{-\nu t})S g_2^2(t) = 0$ are satisfied, respectively.

4. SUM OF SINUSOIDAL SIGNALS

In this section, we provide an application of the annihilators found in Subsection 3.2 working with a class of palindromic operators.

Here, we study a biased finite sum of sinusoidal signals:

$$y(t) = A_0 + \sum_{j=1}^N A_j \sin(\omega_j t + \phi_j), \quad (3)$$

then we can consider the time-delay operator

$$(S - 1) \prod_{j=1}^n (S^2 - 2\cos(\omega_j \tau)S + 1)$$

which annihilates y by Corollary 2.

Recall that a polynomial P of degree n is a *palindromic polynomial* if the coefficient of its k th power is equal to the coefficient of its $(n - k)$ th power, for all $0 \leq k \leq n$. If these coefficients have opposite signs, a polynomial is an *anti-palindromic polynomial*. An equivalent definition of a palindromic polynomial is that $P(X) = X^{\deg(P)} P(X^{-1})$.

Notice that the polynomial $\prod_{j=1}^n (S^2 - 2\cos(\omega_j \tau)S + 1)$ is then palindromic, since:

$$\prod_{j=1}^n (S^2 - 2\cos(\omega_j \tau)S + 1) = S^{2N} \prod_{j=1}^n (1 - 2\cos(\omega_j \tau)S^{-1} + S^{-2}).$$

Hence $(S - 1) \prod_{j=1}^n (S^2 - 2\cos(\omega_j \tau)S + 1)$ is anti-palindromic (since an anti-palindromic polynomial is a multiple of $S - 1$, and its quotient is palindromic).

Lemma 6. [Lachaud (2014)] For $\alpha = (\alpha_1 \dots \alpha_N)^T \in \mathbb{C}^N$, the following equality holds

$$\prod_{j=1}^N (S^2 - 2\alpha_j S + 1) = \sum_{m=0}^{2N} (-1)^m a_m(\alpha) S^{2N-m} \text{ with } (4)$$

$$a_m(\alpha) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{N + 2k - m}{k} \sigma_{m-2k}(\alpha), \forall 1 \leq m \leq N$$

where $\sigma_0 \equiv 1$ and $\sigma_\ell(\alpha) = \sum_{i_1 < \dots < i_\ell} \alpha_{i_1} \dots \alpha_{i_\ell}$ denotes the m th elementary symmetric polynomial in α , for all $1 \leq \ell \leq N$.

Corollary 5. Using the notation in Lemma 6, we have

$$(S - 1) \prod_{j=1}^N (S^2 - 2\alpha_j S + 1) = \sum_{m=0}^{2N+1} c_m(\alpha) S^m,$$

where $c_{2N+1}(\alpha) = -c_0(\alpha) = 1$ and

$$c_m(\alpha) = (-1)^m (a_m(\alpha) - a_{m-1}(\alpha)),$$

for all $1 \leq m \leq 2N$.

Example 1. Particular expressions of the coefficients $c_m(\alpha)$ are:

- when $N = 1$, $c_1(\alpha_1) = \alpha_1 - 1$;
- when $N = 2$, $c_1(\alpha) = \alpha_1 + \alpha_2 - 1$ and $c_2(\alpha) = \alpha_1\alpha_2 - \alpha_1 - \alpha_2 + 2$;
- when $N = 3$, $c_1(\alpha) = \alpha_1 + \alpha_2 + \alpha_3 - 1$, $c_2(\alpha) = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 - \alpha_1 - \alpha_2 - \alpha_3 + 3$, and $c_3(\alpha) = \alpha_1\alpha_2\alpha_3 - \alpha_1\alpha_2 - \alpha_1\alpha_3 - \alpha_2\alpha_3 + 2\alpha_1 + 2\alpha_2 + 2\alpha_3 - 3$.

Corollary 6. Using the notation in Lemma 6, we have

$$(S-1) \prod_{j=1}^N (S^2 - 2\alpha_j S + 1) = \sum_{m=0}^N c_m(\alpha) (S^{2N-m+1} - S^m).$$

In the sequel, we set $\alpha_j := \cos(\omega_j \tau)$ in (3). Since $(S-1) \prod_{j=1}^N (S^2 - 2\alpha_j S + 1)$ annihilates y , we have

$$(S^{2N+1} - 1)(y) = \sum_{m=1}^N c_m(\alpha) (S^{2N-m+1} - S^m)(y). \quad (5)$$

Setting $\mathbf{s} = (S \dots S^{2N})^\top \in \mathcal{P}^{2N}$ (the set of $2N$ -vectors with entries in \mathcal{P}) and $\theta = (c_1(\alpha) \dots c_N(\alpha))^\top \in \mathbb{C}^N$, we may rewrite (5) as:

$$(S^{2N+1} - 1)(y) = \mathbf{s}^\top(y) \mathbf{A} \theta \quad (6)$$

where $\mathbf{A} = \begin{pmatrix} -I \\ B \end{pmatrix}$ is a block matrix in $\mathbb{R}^{2N \times N}$, with I denoting

the N -identity matrix and $B = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix} \in \mathbb{R}^{N \times N}$ (for

$N = 1$ it becomes $B = 1$).

Remark 6. It is an obvious observation that the matrix \mathbf{A} has rank N . Also, $\text{rank}(\mathbf{s}(y)) = 1$.

A regression expression is then obtained from (5):

$$\mathbf{y}(t) = \varphi^\top(t) \theta, \forall t \in \mathbb{R}_+ \quad (7)$$

where $\mathbf{y}(t) = y(t - (2N+1)\tau) - y(t)$ and $\varphi^\top(t) = \mathbf{s}^\top(y) \mathbf{A}$ for all $t \in \mathbb{R}_+$.

Example 2. For $N = 1$, we have $y(t) = A_0 + A_1 \sin(\omega_1 t + \phi_1)$. Then the regression equation is

$$\mathbf{y}(t) = \varphi^\top(t) \theta, \forall t \in \mathbb{R}_+$$

with $y(t) = A_1 (\sin(\omega_1(t-3\tau) + \phi_1) - \sin(\omega_1 t + \phi_1))$,

$$\theta = c_1(\alpha) = \cos(\omega_1 \tau) - 1,$$

$$\varphi(t) = A_1 (\sin(\omega_1(t-2\tau) + \phi_1) - \sin(\omega_1(t-\tau) + \phi_1)).$$

Example 3. For $N = 2$, we have

$$y(t) = A_0 + A_1 \sin(\omega_1 t + \phi_1) + A_2 \sin(\omega_2 t + \phi_2).$$

Then the regression equation is as in (7), with

$$y(t) = \sum_{j=1}^2 A_j (\sin(\omega_j(t-5\tau) + \phi_j) - \sin(\omega_j t + \phi_j)),$$

$$\varphi(t) = \begin{pmatrix} \sum_{j=1}^2 A_j (\sin(\omega_j(t-4\tau) + \phi_j) - \sin(\omega_j(t-\tau) + \phi_j)) \\ \sum_{j=1}^2 A_j (\sin(\omega_j(t-3\tau) + \phi_j) - \sin(\omega_j(t-2\tau) + \phi_j)) \end{pmatrix},$$

$$\theta = \begin{pmatrix} \cos(\omega_1 \tau) + \cos(\omega_2 \tau) - 1 \\ \cos(\omega_1 \tau) \cos(\omega_2 \tau) - \cos(\omega_1 \tau) - \cos(\omega_2 \tau) + 2 \end{pmatrix}.$$

Example 4. For $N = 3$, we have $y(t) = A_0 + \sum_{j=1}^3 A_j \sin(\omega_j t + \phi_j)$. Then the regression equation (7) yields

$$y(t) = \sum_{j=1}^3 A_j (\sin(\omega_j(t-7\tau) + \phi_j) - \sin(\omega_j t + \phi_j)),$$

$$\varphi(t) = \begin{pmatrix} \sum_{j=1}^3 A_j (\sin(\omega_j(t-6\tau) + \phi_j) - \sin(\omega_j(t-\tau) + \phi_j)) \\ \sum_{j=1}^3 A_j (\sin(\omega_j(t-5\tau) + \phi_j) - \sin(\omega_j(t-2\tau) + \phi_j)) \\ \sum_{j=1}^3 A_j (\sin(\omega_j(t-4\tau) + \phi_j) - \sin(\omega_j(t-3\tau) + \phi_j)) \end{pmatrix},$$

$$\theta = \begin{pmatrix} \alpha_1 + \alpha_2 + \alpha_3 - 1 \\ \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 - \alpha_1 - \alpha_2 - \alpha_3 + 3 \\ \alpha_1\alpha_2\alpha_3 - \alpha_1\alpha_2 - \alpha_1\alpha_3 - \alpha_2\alpha_3 + 2\alpha_1 + 2\alpha_2 + 2\alpha_3 - 3 \end{pmatrix}.$$

5. FIXED-TIME ESTIMATION OF FREQUENCIES

Our goal is to estimate the frequencies in (3) uniformly in time using the annihilators presented in the previous section.

5.1 Applying the DREM method

In the sequel, the DREM method Aranovskiy et al. (2017) is applied first to decouple (7) into N separate equations. To realize this procedure, we apply N times the operator S to the expression (7) to obtain an *extended* regression equation.

In matrix form, we set $\mathcal{Y} = (S \ S^2 \ \dots \ S^{N-1})^\top(\mathbf{y}) \in \mathbb{R}^N$ and the block matrix $\mathbf{S} = ((S\mathbf{s})^\top \ (S^2\mathbf{s})^\top \ \dots \ (S^N\mathbf{s})^\top)^\top \in \mathbb{R}^{N \times N}$ and it follows:

$$\mathcal{Y}(t) = \Phi(t) \theta \quad (8)$$

with $\Phi := \mathbf{S}(y) \otimes \mathbf{A} \in \mathbb{R}^{2N^2 \times N^2}$.

Recall that for an arbitrary matrix M , it holds $M \text{adj}(M) = \det(M)I$. Now, at the *mixing* step of DREM, the use of the adjugate matrix leads to a (yet) new equation:

$$\mathbf{Y}(t) = \Delta(t) \theta \quad (9)$$

where $\mathbf{Y} = \text{adj}(\Phi) \mathcal{Y} \in \mathbb{R}^N$ and $\Delta = \det(\Phi)$, or equivalently, for $\mathbf{Y} = (\mathbf{Y}_1 \ \dots \ \mathbf{Y}_N)^\top$:

$$\mathbf{Y}_m(t) = \Delta(t) \theta_m \quad (10)$$

for all $1 \leq m \leq N$.

Remark 7. Notice that the regressor function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ is an element of the Banach space $\mathcal{L}^\infty(\mathbb{R}, \mathbb{R}^n)$ of measurable functions bounded almost everywhere with respect to the norm $\|\cdot\|_\infty = \text{esssup}|\cdot|$.

In the next part, we estimate frequencies using (10).

5.2 Parameter estimation

In this section, to approach the reality, we consider the noisy case of (3) when we measure the signal

$$\mathbf{y}(t) = y(t) + \mathbf{w}(t),$$

where $\mathbf{w} \in \mathcal{L}^\infty(\mathbb{R}, \mathbb{R})$ is the perturbation. That results in a noisy regression equation obtained from (7):

$$\mathbf{y}(t) = \varphi^\top(t) \theta + w(t), \forall t \in \mathbb{R}_+ \quad (11)$$

where $w(t) = \mathbf{w}(t - (2N+1)\tau) - \mathbf{w}(t) - \mathbf{s}^\top(\mathbf{w}) \mathbf{A} \theta$ denotes an external disturbance, and by abuse of notation, we denote by

$\mathbf{y}(t) = \mathbf{y}(t - (2N + 1)\tau) - \mathbf{y}(t)$ the disturbed version of (7). The equivalent version of the DREM equation (9) is:

$$\mathbf{Y}(t) = \Delta(t) \theta + \mathbf{W}(t) \quad (12)$$

where $\mathbf{W} = \text{adj}(\Phi) (S \ S^2 \ \dots \ S^{N-1})^\top (w) \in \mathbb{R}^N$, and similarly to (10):

$$\mathbf{Y}_m(t) = \Delta(t) \theta_m + \mathbf{W}_m \quad (13)$$

for all $1 \leq m \leq N$.

To calculate an estimate $\hat{\theta}$ of θ from (12) or (13), an adaptive parameter estimation technique has been developed in Wang et al. (2019):

Algorithm 1. [Wang et al. (2019)]²

$$\begin{aligned} \hat{\theta}(t) = \text{sgn}(\Delta(t)) & \left(\gamma_1 \left[\mathbf{Y}(t) - \Delta(t) \hat{\theta}(t) \right]^{\varrho(t)} \right. \\ & \left. + \gamma_2 \left[\mathbf{Y}(t) - \Delta(t) \hat{\theta}(t) \right]^{\zeta + \varrho(t)} \right) \end{aligned}$$

for $\gamma_1 > 0$, $\gamma_2 > 0$, $\zeta > 1$, and $\varrho(t) = \frac{|\Delta(t)|}{1 + |\Delta(t)|}$.

In this version, the power ϱ is approaching zero together with the regressor Δ , the contribution of the regressor in the adaptation rate is proportional to $|\Delta(t)|^{\varrho(t)}$, $\forall t \in \mathbb{R}$. Given some T_0 and T_f , the algorithm (1) provides short-fixed-time stability of the estimation error $e := \theta - \hat{\theta}$ dynamics, when $\|\mathbf{W}\|_\infty = 0$, as well as the short-fixed-time ISS property when $\|\mathbf{W}\|_\infty \neq 0$. The reader may refer to Wang et al. (2019) for the corresponding definitions.

5.3 Formulas for the estimates

From Lemma 6, it follows a matrix representation for the coefficients $a_m(\alpha)$:

$$\mathbf{a} = \mathbf{M} \sigma \quad (14)$$

where

$$\begin{aligned} \mathbf{a} &= (a_1(\alpha) \ \dots \ a_N(\alpha))^\top, \\ \sigma &= (\sigma_0(\alpha) \ \sigma_1(\alpha) \ \dots \ \sigma_N(\alpha))^\top, \\ \mathbf{M} &= \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ N & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \dots & \dots & \dots & \dots & \vdots & \vdots & \vdots \\ 0 & \binom{N}{\lceil \frac{N}{2} \rceil} & 0 & \binom{N}{\lceil \frac{N}{2} \rceil - 1} & \dots & 2 & 0 & 1 \end{pmatrix}, \text{ for } N \text{ even and} \\ \mathbf{M} &= \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ N & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \dots & \dots & \dots & \dots & \vdots & \vdots & \vdots \\ \binom{N}{\frac{N}{2}} & 0 & \binom{N}{\frac{N}{2} - 1} & 0 & \dots & 2 & 0 & 1 \end{pmatrix}, \text{ for } N \text{ odd.} \end{aligned}$$

Recalling now $\theta = (c_1(\alpha) \ \dots \ c_N(\alpha))^\top$, we obtain a formula for the coefficients c_m :

$$\theta = G \mathbf{a} + b \quad (15)$$

where $b = (1 \ 0 \ \dots \ 0)^\top$ and the matrix G has columns $G = (G_1 \ G_2 \ \dots \ G_N)$ with $G_i = (-1)^i (0 \ 0 \ \dots \ 1 \ 1 \ \dots \ 0 \ 0)^\top$ with the ones at the i th and $(i + 1)$ th entries.

² For $z \in \mathbb{R}$ and $\xi > 0$, denote $[z]^\xi := |z|^\xi \text{sign}(z)$.

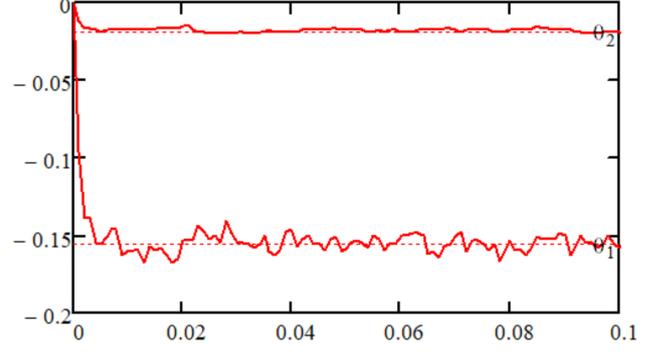


Fig. 1. The results of simulation for $N = 2$

Considering the estimates $\hat{\theta}$, we find the relations with the estimates $\hat{\sigma}$:

$$\hat{\sigma} = (GM)^{-1} (\hat{\theta} - b) \quad (16)$$

Recall that the elementary symmetric polynomials represent the coefficients of a polynomial having exactly $\alpha_1, \dots, \alpha_N$ as roots. A suitable numerical root-finding procedure provides the frequencies in (3):

$$\omega_m = \frac{1}{\tau} \arccos(\alpha_m), \forall m.$$

Finally, the following algorithm can be formulated:

Algorithm 2. Given a signal y .

1. Compute \mathbf{M}
2. Compute \mathbf{G}
3. Derive θ and estimate $\hat{\theta}$
4. Compute $\hat{\sigma}$ with (16)
5. Find (numerically) the roots $\alpha_1, \dots, \alpha_N$ of the polynomial $\sum_{m=0}^N \sigma_m(\alpha) X^m$
6. $\omega_m = \arccos(\alpha_m)$, $\forall m$

Remark 8. Based on Proposition 1, it is important to stress that given a signal y , an approximation by a function in \mathcal{E} can be found. It follows that Algorithm 2 can be applied for other signals having annihilators as in subsection 3.2, as long as the DREM applicability conditions are satisfied.

6. EXAMPLES

Consider the case with $N = 2$, where $\omega_1 = 95\pi$, $\omega_2 = 285\pi$, $A_0 = 1$, $A_1 = 2$, $A_2 = 1$, $\phi_1 = 2$, $\phi_2 = 1$. The noise $\|\mathbf{w}\|_\infty \leq 0.01$ is a uniformly distributed random signal. The sampling frequency is $f = 1\text{kHz}$ and $\tau = \frac{5}{f}$, with $\theta_1 = -0.155$ and $\theta_2 = -0.018$. The results of estimation for $\gamma_1 = \gamma_2 = 10$ and $\zeta = 1.1$ are shown in Fig. 1.

7. CONCLUSION

This paper has studied the problem of linear representing a class of nonlinear regressions by finding annihilators of signals given by exponential polynomials containing unknown parameters. The obtained annihilators are time-delay operators constructed with an Ore extension. Thanks to properly expressed Ore extensions, we have obtained a precise characterization of the annihilators as a principal ideal, greatly simplifying further computations. The problem of frequency estimation for a signal composed of multiple biased sinusoidal signals of unknown

amplitudes and phases was studied, demonstrating the efficiency of the proposed decomposition approach. The parameters were identified in fixed time using the DREM method. In future work, we will generalize these results to other types of signals.

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8. APPENDIX

8.1 Combinatorial identity

The following proposition is used in the proof of Lemma 3. The argumentation follows Ruiz (1996).

Proposition 2. For all $x \in \mathbb{R}$ and all $n \in \mathbb{N}^*$, let

$$f_n(x) = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (x-k)^{n-1}.$$

Then $f_n \equiv 0, \forall n \in \mathbb{N}^*$.