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1 Uniform boundary stabilization of a 2 high-order finite element space 3 discretization of the 1-d wave equation

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9 Abstract

10 The objective of this work is to propose and analyze numerical schemes for solving
11 boundary control problems or data assimilation problems by observers for wave
12 propagation problems. The efficiency of the considered control or data assimilation
13 strategy relies on the exponentially stable character of the underlying system.
14 Therefore, the aim of our work is to propose a discretization process that allows
15 preserving the exponential stability at the discrete level when using high-order spectral
16 finite element approximation. The main idea is to add a stabilizing term to the wave
17 equation that dampens the spurious oscillatory components of the solutions. This term
18 is based on a discrete multiplier analysis that gives us the exponential stability of the
19 semi-discrete problem at any order without affecting the approximation properties.


20

21 1 Introduction

22 The aim of this work is to demonstrate an asymptotic exponential stability of a one-
23 dimensional wave-like system with boundary damping uniformly with respect to the
24 time and space discretization of the system. Such a system typically arises when
25 analyzing boundary control problems [15], data assimilation problem using boundary
26 data [36] or stabilization property of absorbing boundary condition imposed to initially
27 unbounded formulations [25].

28 Obtaining exponential stability results – through observability condition in most
29 cases – is a rather difficult result at the continuous level, which has been extensively
30 studied by the control community [18] and proved by various techniques, typically
31 multiplier techniques [2, 7, 20] or even Carleman estimates [26], spectral approaches
32 [5, 22], or microlocal analyzes [4]. However, discretizing the dynamics so that the
33 stabilization properties are preserved at the discrete level presents an additional difficulty
34 because of the presence spurious high frequencies that remain uncontrolled if one relies
35 on the classical discretization of the stabilization terms ([9, 18, 34, 37]). Several
36 strategies – either for finite element or finite difference discretizations – are presented
37 in the literature to compensate for the instabilities caused by high frequencies, namely
38 Fourier techniques ([13, 14, 17]), Tikhonov regularization [3], multi-grid finite difference
39 methods [16] or mixed finite element methods ([19] and their extension for nonuniform
40 meshes [29, 30]).

41 For dealing with high-performance wave propagation applications, high-order spec-
42 tral finite elements [12] are now considered indispensable – see for instance [6, 10,

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35] and for seismic wave applications [28, 31] – however, the above discretization
 44 strategies cannot be easily extended to such high-order discretizations without losing
 45 their advantages, in particular their convergence order and their efficiency.

46 Here we follow a strategy of including additional consistent stabilization terms in
 47 the discretization to preserve the efficiency of the methods for possible extension to
 48 high dimensions. The additional vanishing stabilization terms have a similar goal as
 49 those found in [38]. However, here these additional terms are constructed in such a
 50 way that the classical multiplier method, which proves stabilization in the continuous
 51 method, can be continued at the discrete level, achieving uniform stabilization with
 52 respect to the discretization parameters. Moreover, the stabilization terms are carefully
 53 designed to preserve the consistency order. In particular, our result can extend the
 54 discretization result for observers proposed in [33] to high-order schemes, which states
 55 that using data, one can obtain discretization schemes that do not suffer from the
 56 asymptotic time drift of the error.

57 Our proof is limited to the one-dimensional case, while an implicit mid-point time-
 58 scheme is chosen for the time-discretization however the principles of our approach pave
 59 the way for possible extension to higher dimensions and more efficient time schemes,
 60 typically fully explicit approaches – that are nevertheless tested in this work..

61 This work is organized as follows. In Section 2, the motivations for this work
 62 are stated. In Section 3 we lay out the continuous configuration and the method
 63 for proving exponential stability. Section 4 proves the uniform exponential in h in
 64 the semi-discrete configuration using a general abstract framework. In Section 5 we
 65 present a semi-discrete convergence results for a data assimilation problem. Then,
 66 Section 6 presents the fully discretized framework using an implicit scheme in time.
 67 The numerical results are presented in Section 7, which is split into two parts. First, we
 68 consider the eigenvalues of the operator belonging to our damped system to observe the
 69 uniform exponential stability property. To obtain an efficient method, in a second step
 70 we introduce an explicit scheme for discretization in time that preserves the properties
 71 shown in the previous sections. Using this scheme, we can then present numerical
 72 results that illustrate the interest of adding the damping terms.

73 2 Motivation of the observation problem

74 Let us study a model problem which shines a light on the difficulties that appear for
 75 the stabilization of discretized problems.

76 Let $\Omega = (0, 1)$ be the domain of propagation. We study a homogenous wave equation
 77 formulated as a first-order system

$$\begin{cases} \partial_t u(x, t) + \partial_x v(x, t) = 0, & (x, t) \in \Omega \times (0, T), \\ \partial_t v(x, t) + \partial_x u(x, t) = 0, & (x, t) \in \Omega \times (0, T). \end{cases} \quad (1)$$

We define the boundary conditions and the initial conditions of our wave equation. The
 following boundary conditions corresponds to Neumann at point $x = 0$ and a Robin
 condition at point $x = 1$. We exclude the degenerate case $\gamma = 1$ which corresponds to
 transparent conditions.

$$\begin{cases} u(1, t) = \gamma v(1, t) & t \in (0, T), \\ v(0, t) = 0, & t \in (0, T), \end{cases} \quad \begin{cases} u(x, 0) = u_0(x), & x \in \Omega, \\ v(x, 0) = v_0(x), & x \in \Omega. \end{cases} \quad (2)$$

78 The study of such a system ((1) with (2)) allows to simulate the case of the study of a
 79 system with absorbing conditions (at boundary $x = 1$). It can also be similar to the
 80 study of an error system between a target system and a simulated system (which we

81 would like to converge asymptotically in time toward the target trajectory). Namely, if
 82 we consider having access to measurements (here v at $x = 1$), and we can incorporate
 83 by a feedback law in the boundary conditions the difference between the solution and
 84 the measurements. To study the convergence of this system called observer system (see
 85 [8, 32]) towards the target system, it is equivalent to show the exponential convergence
 86 towards 0 of the error system that could represent ((1) with (2)). In all cases, the
 87 convergence of these problems ((1) with (2)) is related to the exponential convergence
 88 of a damped PDE (at $x = 1$ in this case). Furthermore, at the continuous level, one
 89 can prove that this system is exponentially stable ([5] in 1D problem, [34]).

90 **2.1 Parasitic waves**

91 The question is to prove that such system – that is known to be exponentially stable
 92 preserves its exponential stability property after standard high-order spectral finite
 93 element discretization on space and explicit second order energy preserving time
 94 discretization [12]. To illustrate the phenomenon of propagation, we will represent the
 95 wave over time. We simulate our problem with finite elements (F.E.M.) \mathbb{P}_r – the set
 96 of polynomials of degree r – and an explicit time scheme and on uniform mesh. We
 97 initialize u as a Gaussian (see Figure 1 (Left)) and set v to 0 as initial condition.

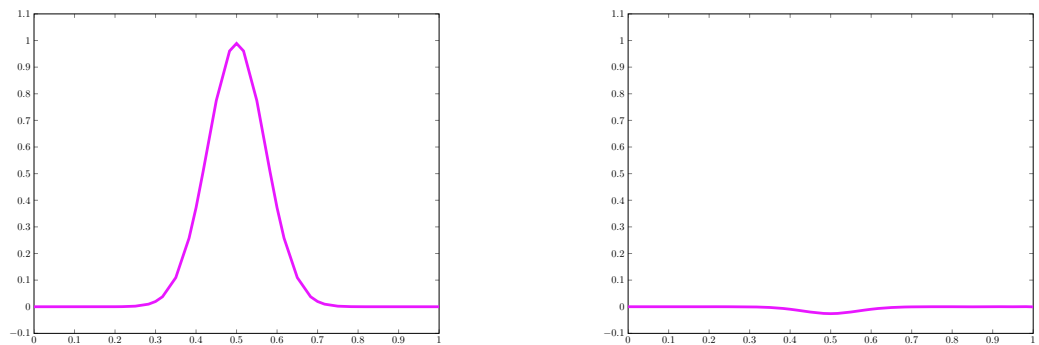


Figure 1. Approximation of u obtained with \mathbb{P}_4 finite elements with $h = 1/10$ at time $t = 0$ (Left) and at $t = 2$ (Right)

98
 99 The boundary conditions of the wave propagation involve two effects. At the
 100 boundary $x = 0$, the wave bounces while on the boundary $x = 1$, part of the wave
 101 is absorbed, we should therefore have an exponential convergence to 0. We notice
 102 the oscillations of the wave in the course of time, these are due to the presence of
 high-frequency spurious waves (see Figure 2 with a change of scale).

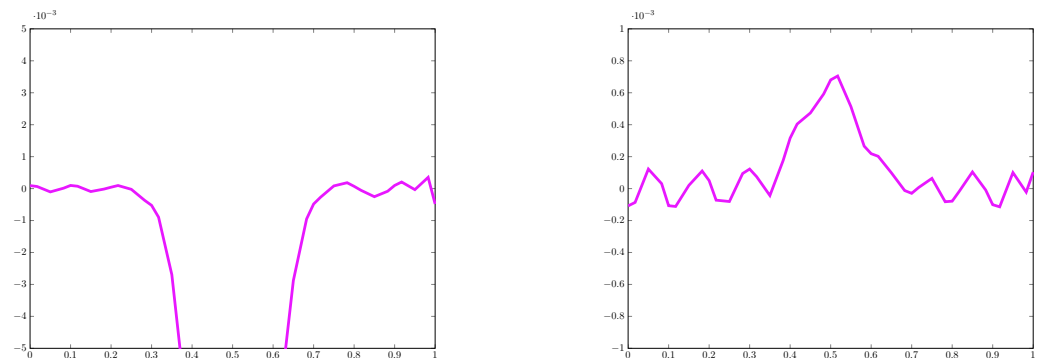


Figure 2. Approximation of u obtained with \mathbb{P}_4 finite elements with $h = 1/10$ and $dt = 1/1000$ at time $t = 2$ (Left) and at time $t = 4$ (Right)

104 In order to show that the discretization taints this property of uniform exponential
 105 stability, we represent the evolution of the energy of the system over time in log scale
 106 (Figure 3). We notice that the decay is no longer linear after a certain time. This gives
 107 an illustration of the loss of the uniform exponential stability property caused by the
 108 high-order spectral finite elements discretization.

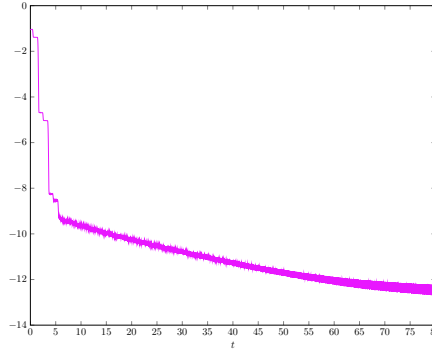


Figure 3. Approximation of $\ln(\|u(t)\|^2 + \|v(t)\|^2)^{1/2}$ obtained with \mathbb{P}_4 finite elements with $h = 1/10$ and $dt = 1/1000$

109 **2.2 Investigation of eigenvalues**

110 As a key indicator of the behavior of the discrete system, we investigate the eigenvalues
 111 of the generator of the corresponding semi-group. We formulate (1) as follows,

$$\dot{z} = Az, \text{ with } z = \begin{pmatrix} u \\ v \end{pmatrix}, \text{ and } A = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix}, \quad (3)$$

112 where $A : L^2(\Omega) \times L^2(\Omega) \rightarrow L^2(\Omega) \times L^2(\Omega)$ is an unbounded operator with domain
 113 $D(A) \subset H^1(\Omega) \times H^1(\Omega)$. The discrete approximation of the above systems is

$$\dot{z}_h = A_h z_h,$$

114 where $A_h \in \mathcal{L}(V_h)$ is an approximation of the operator A in a finite-dimensional
 115 space V_h that is a subspace of $H^1(\Omega) \times H^1(\Omega)$ obtained using high-order spectral
 116 finite elements ([12]). Since we look for exponential stability we aim at constructing
 117 approximations for which the eigenvalues of A_h lies in the complex plane not close to
 118 the imaginary axis.

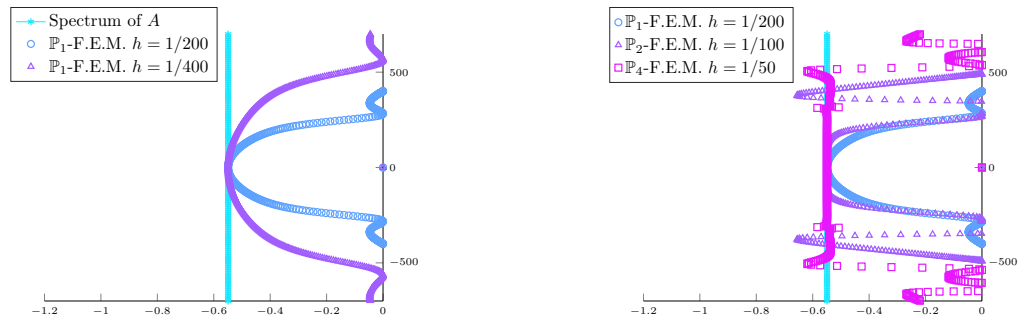


Figure 4. Spectrum of A_h

119 We see in Figure 4 (Left) that the eigenvalues are close to the imaginary axis, which
 120 implies that the exponential stability property is in the best case scenario, not uniform
 121 in h , or even lost. By choosing h smaller, the consistency is improved, namely the

122 smallest eigenvalues – in amplitude – are more and more well approximated, still the
 123 uniform exponential stability property is not recovered, see Figure 4 (Right). In fact,
 124 it has been shown – at least for \mathbb{P}_1 finite elements, (see [9, 34]) – that the uniform
 125 exponential stability property cannot be obtained at the discrete level without adding
 126 additional dissipative terms. As mentioned the main contribution of this work is to
 127 construct such terms that stabilized the system – in the sense where the uniform
 128 exponential stability property is guaranteed – and preserves high-order accuracy that
 129 is obtained with \mathbb{P}_r finite elements.

130 3 Continuous setting

131 This section presents standard multiplier techniques [7] to show the exponential stability
 132 property at the continuous level of System (1). We have chosen to present these classical
 133 results, first for the sake of completeness, second because they will be reproduced at
 134 the discrete level. To complete this last step the sought additional dissipative terms
 135 will be needed.

136 3.1 Observability of the undamped system

137 First, we define the variational formulation of the undamped system ($\gamma = 0$),

$$\begin{cases} \int_0^1 \partial_t u \tilde{u} \, dx - \int_0^1 v \partial_x \tilde{u} \, dx + v(1) \tilde{u}(1) = 0, \\ \int_0^1 \partial_t v \tilde{v} \, dx + \int_0^1 \partial_x u \tilde{v} \, dx - u(1) \tilde{v}(1) = 0, \end{cases} \quad (4)$$

138 where $u(t), \tilde{u} \in H^1(0, 1)$ and $v(t), \tilde{v} \in L^2(0, 1)$. For smooth initial data, it is standard
 139 to show that (4) has a unique solution,

$$(u, v) \in C^1([0, T]; L^2(0, 1) \times L^2(0, 1)) \cap C^0([0, T]; H^1(0, 1) \times H^1(0, 1)).$$

140 System (4) is conservative, indeed, one can easily show, the following theorem.

141 **Theorem 3.1.** *Let (u, v) solutions of (4), we have the energy conservation*

$$\frac{d}{dt} \left(\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right) = 0. \quad (5)$$

142 The proof consists in choosing $\tilde{u} = u(t)$ and $\tilde{v} = v(t)$ in (4) and summing the two
 143 obtained equations. Note that, as an obvious consequence of the energy preservation
 144 property we have

$$\int_0^T \|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \, dt = T \left(\|u(0)\|_{L^2(0,1)}^2 + \|v(0)\|_{L^2(0,1)}^2 \right). \quad (6)$$

145

146
 147 A first step towards our goal is to derive an observability inequality (the precise
 148 meaning is given in the theorem below). This step is not a necessity to prove exponential
 149 stability but should rather be seen as a useful intermediate step. The strategy to prove
 150 the observability inequality relies on multipliers [7].

151 **Theorem 3.2.** *Let (u, v) be solutions of (4). For T large enough, there exists $C^{st}(T)$
 152 strictly positive such that*

$$\frac{1}{2} \left(\|u(0)\|_{L^2(0,1)}^2 + \|v(0)\|_{L^2(0,1)}^2 \right) \leq C^{st}(T) \int_0^T v^2(1) \, dt. \quad (7)$$

153 *Proof.* The proof given below is separated in several steps that will all be mimicked at
 154 the discrete level.

155 **Step 1:** We choose $\tilde{u} = -xv(t)$ and $\tilde{v} = -xu(t)$ in (4) and sum the two obtained
 156 equations, integrating by parts and neglecting negative terms we get, we obtain

$$-\frac{d}{dt} \int_0^1 xuv \, dx \leq -\frac{1}{2} (\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2) + \frac{1}{2} v^2(1).$$

157 **Step 2:** Integrating in time and using the energy conservation property of 3.1 and its
 158 direct consequence (6) we get

$$-\left[\int_0^1 xuv \, dx \right]_0^T \leq -\frac{T}{2} (\|u(0)\|_{L^2(0,1)}^2 + \|v(0)\|_{L^2(0,1)}^2) + \frac{1}{2} \int_0^T v^2(1) \, dt. \quad (8)$$

159 **Step 3:** We use Cauchy-Schwarz, Young inequalities and Theorem 3.1 to show

$$\left| \int_0^1 xu(t)v(t) \, dx \right| \leq \frac{1}{2} \|v(t)\|_{L^2(0,1)}^2 + \frac{1}{2} \|v(t)\|_{L^2(0,1)}^2 = \frac{1}{2} \|v(0)\|_{L^2(0,1)}^2 + \frac{1}{2} \|v(0)\|_{L^2(0,1)}^2.$$

160 Finally, using this inequality together with (8) we obtain

$$(T - 2) \left(\|u(0)\|_{L^2(0,1)}^2 + \|v(0)\|_{L^2(0,1)}^2 \right) \leq \int_0^T v^2(1) \, dt.$$

161 □

162 Inequality (7) does not directly guarantee that the boundary damped system is
 163 exponentially stable. It is however a good indicator since the damping term corresponds
 164 to the right-hand side of (7). This will be used as a clue to define the required damping
 165 terms at the discrete level.

166 3.2 Exponential stability for the damped system using Lyapunov 167 functional

168 We define the variational formulation associated with System (1) with (2)

$$\begin{cases} \int_0^1 \partial_t u \tilde{u} \, dx - \int_0^1 v \partial_x \tilde{u} \, dx + v(1) \tilde{u}(1) = 0, \\ \int_0^1 \partial_t v \tilde{v} \, dx + \int_0^1 \partial_x u \tilde{v} \, dx - \tilde{v}(1) u(1) + \gamma v(1) \tilde{v}(1) = 0, \end{cases} \quad (9)$$

169 We have existence and uniqueness of solutions for the above variational formulation,
 170 we will now establish an energy relation for this system.

171 **Theorem 3.3.** *Let (u, v) solutions of (9), we have the energy dissipation property*

$$\frac{1}{2} \frac{d}{dt} \left(\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right) = -\gamma v^2(1). \quad (10)$$

172 *Proof.* We choose $\tilde{u} = u(t)$ and $\tilde{v} = v(t)$ in (9). Then, adding the two equations in (9),
 173 we conclude the proof. □

174 **Theorem 3.4.** *Let (u, v) solutions of (9). There exist two constants $C_s > 0$ and $\sigma > 0$
 175 such that*

$$\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \leq C_s e^{-\sigma t} \left(\|u(0)\|_{L^2(0,1)}^2 + \|v(0)\|_{L^2(0,1)}^2 \right). \quad (11)$$

176 *Proof.* The proof given below is separated in several steps that will all be mimicked at
177 the discrete level.

178 **Step 1:** We choose $\tilde{u} = -xv(t)$ and $\tilde{v} = -xu(t)$ in (9).

179 Summing the two equations and integrating by parts, we get

$$-\frac{d}{dt} \int_0^1 xuv \, dx = \frac{1}{2}v^2(1) - \frac{1}{2}u^2(1) + \gamma v(1)u(1) - \frac{1}{2} \left(\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right).$$

180 Moreover, thanks to Young's inequality we have $\gamma u(1)v(1) \leq \frac{\gamma^2}{2}v^2(1) + \frac{1}{2}u^2(1)$, finally
181 we obtain

$$-\frac{d}{dt} \int_0^1 xuv \, dx \leq \frac{1+\gamma^2}{2}v^2(1) - \frac{1}{2} \left(\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right).$$

Step 2: We add the energy relation (3.3) multiplied by $\alpha > 0$.

$$\begin{aligned} \frac{d}{dt} \left(\frac{\alpha}{2} \left(\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right) - \int_0^1 xuv \, dx \right) \\ \leq \left(\frac{\gamma^2 + 1}{2} - \alpha\gamma \right) v^2(1) - \frac{1}{2} \left(\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right). \end{aligned}$$

182 **Step 3:** Choosing α large enough such that

$$\frac{\gamma^2 + 1}{2} - \alpha\gamma < 0,$$

we get

$$\begin{aligned} \frac{d}{dt} \left(\frac{\alpha}{2} \left(\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right) - \int_0^1 xuv \, dx \right) \\ \leq -\frac{1}{2} \left(\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right). \quad (12) \end{aligned}$$

183 **Step 4:** We introduce the Lyapunov functional

$$\mathcal{L}(t) = \frac{\alpha}{2} \left(\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right) - \int_0^1 xuv \, dx. \quad (13)$$

184 Using Cauchy-Schwarz and Young inequalities, we prove the equivalence between the
185 energy functional

$$\mathcal{E}(t) := \frac{1}{2} \|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2$$

and the Lyapunov functional with $\alpha - 1 > 0$.

$$\begin{aligned} \frac{(\alpha - 1)}{2} \left(\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right) &\leq \mathcal{L}(t) \\ &\leq \frac{\alpha + 1}{2} \left(\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right). \quad (14) \end{aligned}$$

186 Using this equivalence in (12), we get

$$\frac{d}{dt} \mathcal{L}(t) \leq -\frac{1}{\alpha + 1} \mathcal{L}(t).$$

187 **Step 5:** Integrating in time and using again the equivalence between the energy and
188 the Lyapunov functional we finally have

$$\left(\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right) \leq \frac{\alpha + 1}{\alpha - 1} e^{-\frac{1}{\alpha + 1}t} \left(\|u(0)\|_{L^2(0,1)}^2 + \|v(0)\|_{L^2(0,1)}^2 \right).$$

190 **4 Semi-discrete setting**

191 As mentioned we give a counterpart of Theorem 3.2 and Theorem 3.4 at the semi-
 192 discrete level. This is achieved in an abstract framework. We then show – and this is
 193 the most important part of the work – that the semi-discrete wave problem falls in the
 194 described framework.

195 **4.1 Abstract framework**

196 Let \mathcal{Z}_h be a family of finite-dimensional parametrized by h space associated with the
 197 scalar product $(\cdot, \cdot)_h$. We consider the semi-discretized abstract first-order system for
 198 $z_h \in \mathcal{Z}_h$,

$$\begin{cases} \dot{z}_h = A_h z_h, & \text{in } [0, T], \\ z_h(0) = z_{h,0}, \end{cases} \quad (15)$$

199 where $A_h \in \mathcal{L}(\mathcal{Z}_h)$ and $\|A_h\|_h \xrightarrow{h \rightarrow 0} +\infty$. We assume that the operator A_h satisfies the
 200 following property

$$(A_h z_h, z_h)_h = 0. \quad (16)$$

201 Furthermore, we have classically the existence and unicity of smooth solutions (u_h, v_h)
 202 belonging to $C^k([0, T]; \mathcal{Z}_h)$, for all $k \geq 0$. We start by presenting the discrete property
 203 of energy preservation.

204 **Theorem 4.1.** *Let $z_h \in \mathcal{Z}_h$ the solution of (15), we have the discrete conservation*
 205 *energy property*

$$\frac{1}{2} \frac{d}{dt} \|z\|_h^2 = 0.$$

206 *Proof.* We multiply (15) by z_h , and since A_h satisfies (16), the result follows immediately.
 207 □

208 Let \mathcal{Y}_h be a finite-dimensional space. We have seen that in the continuous problem,
 209 we have added a dissipative term that stabilizes the problem. By analogy, we therefore
 210 introduce a perturbation by an operator $C_h^* C_h$, where $C_h \in \mathcal{L}(\mathcal{Z}_h, \mathcal{Y}_h)$,

$$\begin{cases} \dot{z}_h = (A_h - \gamma C_h^* C_h) z_h, & \text{in } [0, T], \\ z_h(0) = z_{h,0}. \end{cases} \quad (17)$$

211 For this system, we have the discrete dissipation energy property.

212 **Theorem 4.2.** *Let $z_h \in \mathcal{Z}_h$ the solution of (17), we have the discrete dissipation*
 213 *energy property*

$$\frac{d}{dt} \frac{1}{2} \|z_h\|_h^2 = -\gamma \|C_h z_h\|_h^2.$$

214 *Proof.* Proof similar of the proof of Theorem 4.1. □

215 In order to demonstrate the observability property, we need a fundamental hypoth-
 216 esis.

217 **Assumption 1.** $\exists \alpha, \beta, C_\Pi > 0$, independent of h and for all h , $\exists \Pi_h$ symmetric
 218 belonging to $\mathcal{L}(\mathcal{Z}_h)$ and $\|\Pi_h\|_h \leq C_\Pi$ with $C_\Pi > 0$, such that, $\forall z_h \in \mathcal{Z}_h$

$$(\Pi_h A_h z_h, z_h)_h \leq \alpha \|C_h z_h\|_h^2 - \beta \|z_h\|_h^2.$$

219 From this assumption, we can establish the property that interest us: the observ-
 220 ability property.

221 **Theorem 4.3.** Let z_h be the solution of (15). Assuming Assumption 1, for T large
 222 enough, there exists a constant $C^{st}(T) > 0$ independent of h such that

$$\|z_{h,0}\|_h^2 \leq C^{st}(T) \int_0^T \|C_h z_h\|_h^2 dt.$$

223 *Proof.* The proof is separated into the same steps as for the proof of observability at
 224 the continuous level of Theorem 3.4.

225 **Step 1:** We multiply the first equation of (15) by $\Pi_h z_h$.

$$(\Pi_h \dot{z}_h, z_h)_h = (\Pi_h A_h z_h, z_h)_h.$$

226 Using Assumption 1, we obtain

$$\frac{1}{2} \frac{d}{dt} (\Pi_h z_h, z_h)_h \leq \alpha \|C_h z_h\|_h^2 - \beta \|z_h\|_h^2.$$

227 **Step 2:** We integrate in time and use the energy conservation property

$$\frac{1}{2} (\Pi_h z_h(T), z_h(T))_h - \frac{1}{2} (\Pi_h z_{h,0}, z_{0,h})_h \leq \alpha \int_0^T \|C_h z_h\|_h^2 dt - \beta \int_0^T \|z_h(t)\|_h^2 dt.$$

228 The use of the energy conservation property of Theorem 4.1 leads to

$$\frac{1}{2} (\Pi_h z_h(T), z_h(T))_h - \frac{1}{2} (\Pi_h z_{h,0}, z_{0,h})_h \leq \alpha \int_0^T \|C_h z_h\|_h^2 dt - \beta T \|z_h(0)\|_h^2 dt.$$

229 **Step 3:** Since Π_h is bounded, we have

$$-\frac{1}{2} \|\Pi_h\|_h \|z_h(T)\|_h^2 - \frac{1}{2} \|\Pi_h\| \|z_{h,0}\|_h^2 \leq \alpha \int_0^T \|C_h z_h\|_h^2 dt - \beta T \|z_h(0)\|_h^2 dt.$$

230 Using the energy conservation Theorem 4.1, we get

$$(\beta T - \|\Pi_h\|_h) \|z_{0,h}\|_h^2 \leq \alpha \int_0^T \|C_h z_h\|_h^2 dt.$$

231 For $T > \|\Pi_h\|_h \beta^{-1}$, we obtain the result. □

232 As for showing the observability property, we need a fundamental hypothesis to
 233 establish the exponential stability property.

234 **Assumption 2.** $\exists \chi, \kappa > 0$, independent of h and for all h , $\exists \Pi_h \in \mathcal{L}(\mathcal{Z}_h)$ symmetric
 235 such that, $\forall z_h \in \mathcal{Z}_h$

$$(\Pi_h A_h z_h, z_h)_h - \gamma (C_h z_h, C_h \Pi_h z_h)_h \leq \chi \|C_h z_h\|_h^2 - \kappa \|z_h\|_h^2.$$

236 We can now proceed to the main theorem of this work : the uniform exponential
 237 stability property for the damped system.

238 **Theorem 4.4.** Let z_h be the solution of (17). Under Assumption 2, there exist two
 239 constants $C_s, \sigma > 0$ independent of h such that

$$\frac{1}{2} \|z_h(t)\|_h^2 \leq C_s \|z_{h,0}\|_h^2 e^{-\sigma t}.$$

240 *Proof.* The proof is separated into the same steps as for the proof of exponential
241 stability at the continuous level of Theorem 3.4.

242 **Step 1:** We multiply the first equation of (17) and $\Pi_h z_h$ (in the sens of the scalar
243 product).

$$(\Pi_h \dot{z}_h, z_h)_h = (\Pi_h A_h z_h, z_h)_h - \gamma (C_h z_h, C_h \Pi_h z_h)_h.$$

244 We use Assumption 2 and we obtain

$$\frac{1}{2} \frac{d}{dt} (\Pi_h z_h, z_h)_h \leq \chi \|C_h z_h\|_h^2 - \kappa \|z_h\|_h^2.$$

245 **Step 2:** Adding the dissipation energy identity of Theorem 4.2 times $\alpha > 0$, we get

$$\frac{1}{2} \frac{d}{dt} ((\Pi_h z_h, z_h)_h + \alpha \|z_h\|_h^2) \leq (\chi - \alpha\gamma) \|C_h z_h\|_h^2 - \kappa \|z_h\|_h^2.$$

246 **Step 3:** We choose α large enough such that $\chi - \alpha\gamma \leq 0$. Thus, we have

$$\frac{1}{2} \frac{d}{dt} ((\Pi_h z_h, z_h)_h + \alpha \|z_h\|_h^2) \leq -\kappa \|z_h\|_h^2.$$

247 **Step 4:** We introduce the following discrete Lyapunov functional

$$\mathcal{L}_h(t) = (\Pi_h z_h, z_h)_h + \alpha \|z_h\|_h^2.$$

248 Since Π_h is bounded, we can establish an equivalence between the Lyapunov functional–
249 and the energy functional $\mathcal{E}_h(t) := \frac{1}{2} \|z_h(t)\|_h^2$.

$$(-\|\Pi_h\|_h + \alpha) \|z\|_h^2 \leq \mathcal{L}_h(t) \leq (\|\Pi_h\|_h + \alpha) \|z_h\|_h^2. \quad (18)$$

250 Using this equivalence with $\alpha > \|\Pi_h\|_h$, we obtain

$$\frac{1}{2} \frac{d}{dt} \mathcal{L}_h(t) \leq -\kappa (\|\Pi_h\|_h + \alpha)^{-1} \mathcal{L}_h(t).$$

251 **Step 5:** Integrating in time and using again (18), we finally get

$$\frac{1}{2} \|z_h(t)\|_h^2 \leq \frac{\alpha + \|\Pi_h\|_h}{\alpha - \|\Pi_h\|_h} \|z_{h,0}\|_h^2 e^{-\kappa(\|\Pi_h\|_h + \alpha)^{-1}t}.$$

252

□

253 4.2 Discretization with high-order spectral finite elements

254 In this section, we show how the semi-discretization in space using high-order spectral
255 finite elements of the wave propagation problem (1) enters the abstract framework
256 presented above. We introduce two discrete spaces U_h^r and V_h^r – where r is the order
257 of the finite elements used – defined by

$$U_h^r = \{u_h \in C^0[0, 1] \mid u_h|_{[x_i, x_{i+1}]} \in \mathbb{P}_r\},$$

258 and

$$V_h^r = \{v_h \in L^2(0, 1) \mid v_h|_{[x_i, x_{i+1}]} \in \mathbb{P}_r\}.$$

259 Note that $U_h^r \subset V_h^r$ and we denote $\mathcal{Z}_h = U_h^r \times V_h^r$. We consider a quasi-uniform
260 subdivision $[x_i, x_{i+1}]$, for $i \in \{0, N\}$ of the domain $[0, 1]$, namely we denote $h_i =$
261 $|x_{i+1} - x_i|$ for $i \in \{0, N-1\}$, and

$$h = \sup_{i \in \{0, N-1\}} h_i,$$

262 and there exists $\rho > 0$ such that

$$\min_{i \in \{0, N-1\}} h_i \leq \rho h.$$

263 The efficiency of high-order spectral finite elements stems from the fact that mass-
264 lumping can be achieved without loss of consistency using a quadrature rule based on
265 Gauss-Lobatto points. We represent the use of this quadrature by the symbol

$$\oint_0^1 f(x) dx \sim \int_0^1 f(x) dx \quad \text{with} \quad \oint_0^1 f(x) dx := \sum_{i=0}^N \sum_{k=0}^r h_i w_k f(x_i + h_i \xi_k),$$

266 where and w_k are the quadrature coefficients and $\xi_k \in [0, 1]$ the Gauss-Lobatto interpo-
267 lation nodes.

268 Note that we have the following bounds (see [21]),

$$\exists C_q, c_q > 0, \quad \text{such that} \quad \forall u \in \mathbb{P}_r, \quad c_q \int_0^1 u^2 dx \leq \oint_0^1 u^2 dx \leq C_q \int_0^1 u^2 dx. \quad (19)$$

269 The semi-discrete formulation of the damped system (9) reads: Find $(u_h, v_h)(t) \in \mathcal{Z}_h$
270 such that for $(\tilde{u}_h, \tilde{v}_h) \in \mathcal{Z}_h$,

$$\begin{cases} \int_0^1 \partial_t u_h \tilde{u}_h dx - \int_0^1 v_h \partial_x \tilde{u}_h dx + v_h(1) \tilde{u}_h(1) = 0, \\ \int_0^1 \partial_t v_h \tilde{v}_h dx + \int_0^1 \partial_x u_h \tilde{v}_h dx - u_h(1) \tilde{v}_h(1) + \gamma v_h(1) \tilde{v}_h(1) = 0, \end{cases} \quad (20)$$

271 with initial data

$$u_h(0) = u_h^0, \quad v_h(0) = v_h^0.$$

272 Note that when $\gamma = 0$ one recovers the semi-discrete variational formulation of the
273 undamped system (4). In order to relate to the abstract theory presented in Section
274 4.1, we reformulate the variational formulation (20) as an operator equation. To do so,
275 we introduce the scalar product in V_h^r by

$$\forall (v_h, \tilde{v}_h) \in V_h^r \times V_h^r, \quad (v_h, \tilde{v}_h)_h := \oint_0^1 v_h \tilde{v}_h dx, \quad (21)$$

276 The scalar product in $U_h^r \subset V_h^r$ is defined similarly, finally we extend this notation for
277 $z_h = (u_h, v_h)^t$, $\tilde{z}_h = (\tilde{u}_h, \tilde{v}_h)^t \in U_h^r \times V_h^r$,

$$(z_h, \tilde{z}_h)_h = \oint_0^1 u_h \tilde{u}_h dx + \oint_0^1 v_h \tilde{v}_h dx.$$

278 The variational formulation (20) is equivalent to

$$\begin{cases} \partial_t u_h - R_h^* v_h = 0, \\ \partial_t v_h + R_h u_h + \gamma B_h v_h = 0, \end{cases} \quad (u_h, v_h)(0) = (u_h^0, v_h^0) := z_{h,0}, \quad (22)$$

279 where, for all $(u_h, v_h, \tilde{v}_h)^t \in U_h^r \times V_h^r \times V_h^r$,

$$(R_h u_h, \tilde{v}_h)_h = \int_0^1 \partial_x u_h \tilde{v}_h dx - u_h(1) \tilde{v}_h(1) \quad \text{and} \quad (B_h v_h, \tilde{v}_h)_h = v_h(1) \tilde{v}_h(1). \quad (23)$$

280 Denoting $z_h = (u_h, v_h)^t$, we can rewrite System (22) as

$$\begin{cases} \dot{z}_h = A_h z_h - \gamma \begin{pmatrix} 0 & 0 \\ 0 & B_h \end{pmatrix} z_h, \\ z_h(0) = z_{h,0}, \end{cases} \quad \text{where} \quad A_h = \begin{pmatrix} 0 & -R_h^* \\ R_h & 0 \end{pmatrix}. \quad (24)$$

281 Note that the operator $A_h \in \mathcal{L}(\mathcal{Z}_h)$ satisfies, for all $z_h \in \mathcal{Z}_h$, $(A_h z_h, z_h)_h = 0$, i.e., it is
282 skew-adjoint.

283 4.3 Discrete observability

284 Our objective is – in a preliminary step – to prove an observability property for the
285 system (24). To do so, Theorem 4.3, is used and therefore Assumption 1 is satisfied in
286 what follows by introducing an appropriate operator $\Pi_h \in \mathcal{L}(\mathcal{Z}_h)$.

287 The strategy to define Π_h is based upon our analysis presented at the continuous level.
288 More precisely one want to have (see the proof of Theorem 3.2 in Section 3)

$$\Pi_h z_h \sim \begin{pmatrix} -xv_h \\ -xu_h \end{pmatrix}.$$

289 However, this later equation can not be an equality since $xv_h \notin U_h^r$ and $xu_h \notin V_h^r$. A
290 simple remedy is to introduce the following interpolation operator: $\chi_h(\cdot) : U_h^r \rightarrow V_h^r$ by

$$\oint_0^1 \chi_h(u_h)v_h \, dx = \oint_0^1 xu_hv_h \, dx, \quad \forall u_h \in U_h^r, v_h \in V_h^r, \quad (25)$$

291 and its adjoint $\chi_h^*(\cdot) : V_h^r \rightarrow U_h^r$ is given by

$$\oint_0^1 \chi_h^*(v_h)u_h \, dx = \oint_0^1 xv_hu_h \, dx, \quad \forall u_h \in U_h^r, v_h \in V_h^r. \quad (26)$$

292 We define, for $z_h = (u_h, v_h)^t$ and $\tilde{z}_h = (\tilde{u}_h, \tilde{v}_h)^t$ the operator Π_h by

$$\Pi_h z_h = \begin{pmatrix} -\chi_h^*(v_h) \\ -\chi_h(u_h) \end{pmatrix}. \quad (27)$$

293 Note that it is easy to show that Π_h is self-adjoint and bounded using (19) uniformly
294 with respect to h . More precisely, there exists C_Π independent of h such that $\forall z_h \in \mathcal{Z}_h$

$$\|\Pi_h z_h\|_h \leq C_\Pi \|z_h\|_h. \quad (28)$$

295 We are interested in the discrepancy between the multiplier that we would like to use
296 (but cannot) and the one we are going to use, namely in the following discrepancy

$$\Pi_h z_h - \begin{pmatrix} -xu_h \\ -xv_h \end{pmatrix}.$$

297 It turns out that the operator C_h – which appears in the fundamental assumption 1 –,
298 involves the local semi-norm H_1 of this discrepancy. That is why, for all $u_h, \tilde{u}_h \in U_h^r$
299 and $\forall v_h, \tilde{v}_h \in V_h^r$, we define two bilinear forms d_h and $d_{h,*}$ as

$$d_h(u_h, \tilde{u}_h) := \sum_{i=0}^N \int_{x_i}^{x_{i+1}} \partial_x(xu_h - \chi_h(u_h)) \partial_x(x\tilde{u}_h - \chi_h(\tilde{u}_h)) \, dx, \quad (29)$$

300 and

$$d_{h,*}(v_h, \tilde{v}_h) := \sum_{i=0}^N \int_{x_i}^{x_{i+1}} \partial_x(xv_h - \chi_h^*(v_h)) \partial_x(x\tilde{v}_h - \chi_h^*(\tilde{v}_h)) \, dx. \quad (30)$$

301 **Proposition 4.5.** *Let d_h and $d_{h,*}$ be the positive and symmetric bilinear forms defined*
302 *by (29) and (30). Then, there exists a constant $C_d > 0$ depend only on r such that*
303 $\forall (u_h, v_h) \in U_h^r \times V_h^r$

$$d_h(u_h, u_h) + d_{h,*}(v_h, v_h) \leq C_d \int_0^1 u_h^2 + v_h^2 \, dx.$$

304 *Proof.* Let us show first, that d_h is bounded. We use an interpolation property [24,
305 Chapter 4, Section 4, Theorem 4.4.4]. Since xu_h belongs to $P_{r+1}(K)$ with $K = [x_i, x_{i+1}]$
306 for some $i \in \{0, N - 1\}$, we have

$$|xu_h - \chi_h(u_h)|_{H^1(K_i)} \leq h_i^r C_r |xu_h|_{H^{r+1}(K_i)},$$

307 where $C_r > 0$ depends only on r . However, as $x \in P_1(K)$ and $u_h \in P_r(K)$, the general
308 Leibniz rule leads to

$$(xu_h)^{r+1} = \sum_{j=0}^{r+1} \binom{r+1}{j} x^{(j)} u_h^{(r+1-j)} = (r+1)(u_h)^{(r)}.$$

309 Injecting this result in the equation above, we get

$$|xu_h - \chi_h(u_h)|_{H^1(K_i)} \leq h_i^r C_r (r+1) |u_h|_{H^r(K_i)}.$$

310 Thus, using an inverse estimate [24, Chapter 4, Section 5, Lemma 4.5.3], we have with
311 $C_r > 0$, another scalar depending only on r ,

$$|xu_h - \chi_h(u_h)|_{H^1(K_i)} \leq C_r \|u_h\|_{L^2(K_i)}.$$

312 Hence, finally we have

$$d_h(u_h, u_h) \leq C_r^2 \|u_h\|_{L^2(0,1)}^2.$$

313 Since we can reproduce this inequality for $d_{h,*}$, the proof is ended. \square

314 We define the following operators $D_h \in \mathcal{L}(U_h^r, U_h^r)$ and $D_{h,*} \in \mathcal{L}(V_h^r, V_h^r)$ associated
315 with the bilinear forms d_h and $d_{h,*}$,

$$\begin{cases} (D_h u_h, \tilde{u}_h)_h = d_h(u_h, \tilde{u}_h), & \forall u_h, \tilde{u}_h \in U_h^r, \\ (D_{h,*} v_h, \tilde{v}_h)_h = d_{h,*}(v_h, \tilde{v}_h), & \forall v_h, \tilde{v}_h \in V_h^r. \end{cases} \quad (31)$$

316 Unfortunately, the definition of the operator C_h – appearing in Assumption 1 – requires,
317 in this case, to define the jump of the variable v_h . For this purpose, we introduce the
318 notation, for all $i \in \{1, N - 1\}$,

$$[[v_i]] := v_i^+ - v_i^-, \quad \text{where } v_i^+ = v_h(x_i^+) \text{ and } v_i^- = v_h(x_i^-). \quad (32)$$

319 Then, we define the jump operator

$$(J_h v_h, \tilde{v}_h)_h = \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} [[v_i]] [[\tilde{v}_i]], \quad \text{where } \tilde{h}_i = \min(h_{i-1}, h_i). \quad (33)$$

320 In the theorem below, we check that the multiplier Π_h satisfies the Assumption 1. We
321 can then apply Theorem 4.3 and prove the observability inequality. Note that since the
322 operators J_h, B_h, D_h and $D_{h,*}$ are self-adjoint and positive, it is possible to consider
323 their roots. We define the operator $C_h \in \mathcal{L}(\mathcal{Z}_h, U_h^r \times [V_h^r]^3)$ – equipped with the scalar
324 product $(\cdot, \cdot)_h$ inherited from V_h^r – by

$$C_h = \begin{pmatrix} D_h^{1/2} & 0 \\ 0 & J_h^{1/2} \\ 0 & B_h^{1/2} \\ 0 & D_{h,*}^{1/2} \end{pmatrix}. \quad (34)$$

325 **Theorem 4.6.** Let $z_h = (u_h, v_h) \in \mathcal{Z}_h = U_h^r \times V_h^r$ be the solutions of (24) with $\gamma = 0$.
 326 Then, Π_h defined by (27) satisfies Assumption 1 namely

$$(\Pi_h A_h z_h, z_h)_h \leq \alpha \|C_h z_h\|_h^2 - \beta \|z_h\|_h^2, \quad \alpha, \beta > 0 \text{ independent of } h,$$

327 with the operator C_h is given by (34).

Proof. By definition, we have

$$\begin{aligned} (\Pi_h A_h z_h, z_h)_h &= (A_h z_h, \Pi_h z_h)_h \\ &= - \int_0^1 v_h \partial_x \chi_h^*(v_h) \, dx + v_h^2(1) + \int_0^1 \partial_x u_h \chi_h^*(u_h) \, dx - u_h^2(1). \end{aligned}$$

Inserting $xv_h - xv_h = 0$ in the first term of the right-hand side, we get

$$\begin{aligned} (A_h z_h, \Pi_h z_h)_h &= - \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x (xv_h - \chi_h^*(v_h)) \, dx - \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x (xv_h) \, dx \\ &\quad + v_h^2(1) + \int_0^1 \partial_x u_h \chi_h(u_h) \, dx - u_h^2(1). \quad (35) \end{aligned}$$

328 Let us focus on the second term in the right-hand side.

$$- \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x (xv_h) \, dx = - \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h^2 \, dx - \frac{1}{2} \sum_{i=0}^N \int_{x_i}^{x_{i+1}} x \partial_x (v_h^2) \, dx.$$

Integrating by parts, we get

$$\begin{aligned} - \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x (xv_h) \, dx &= - \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h^2 \, dx + \frac{1}{2} \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h^2 \, dx \\ &\quad - \frac{1}{2} v_h^2(1) - \frac{1}{2} \sum_{i=1}^{N-1} x_i (v_i^+)^2 - x_i (v_i^-)^2. \end{aligned}$$

Using the algebraic identity $a^2 - b^2 = (a - b)(a + b)$, we obtain

$$\begin{aligned} - \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x (xv_h) \, dx &= - \frac{1}{2} \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h^2 \, dx - \frac{1}{2} v_h^2(1) \\ &\quad - \frac{1}{2} \sum_{i=1}^{N-1} x_i (v_i^+ - v_i^-)(v_i^+ + v_i^-). \end{aligned}$$

329 We define the average of v_h for $i \in \{1, N - 1\}$ by

$$\{\!\{v_i\}\!\} := \frac{v_i^+ + v_i^-}{2}, \text{ where } v_i^+ = v_h(x_i^+) \text{ and } v_i^- = v_h(x_i^-).$$

330 Using the jump definition (32) and for the last term, we obtain

$$- \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x (xv_h) \, dx = - \frac{1}{2} \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h^2 \, dx - \frac{1}{2} v_h^2(1) - \sum_{i=1}^{N-1} x_i \{\!\{v_i\}\!\} [v_i].$$

Injecting this result into (35), we get

$$\begin{aligned} (A_h z_h, \Pi_h z_h)_h &= \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x (x v_h - \chi_h^*(v_h)) \, dx - \frac{1}{2} \int_0^1 v_h^2 \, dx \\ &\quad + \frac{1}{2} v_h^2(1) - \sum_{i=1}^{N-1} x_i \{v_i\} [v_i] + \int_0^1 \partial_x u_h \chi_h(u_h) \, dx - u_h^2(1). \end{aligned}$$

Injecting $xu_h - xu_h = 0$ in the penultimate term in the right-hand side, then we get

$$\begin{aligned} (A_h z_h, \Pi_h z_h)_h &= \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x (x v_h - \chi_h^*(v_h)) \, dx - \frac{1}{2} \int_0^1 v_h^2 \, dx + \frac{1}{2} v_h^2(1) \\ &\quad - \sum_{i=1}^{N-1} x_i \{v_i\} [v_i] - \sum_{i=0}^N \int_{x_i}^{x_{i+1}} (x u_h - \chi_h(u_h)) \partial_x u_h \, dx \\ &\quad + \frac{1}{2} \int_0^1 x \partial_x (u_h^2) \, dx - u_h^2(1). \end{aligned}$$

We integrate by parts, we have

$$\begin{aligned} (A_h z_h, \Pi_h z_h)_h &= \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x (x v_h - \chi_h^*(v_h)) \, dx - \frac{1}{2} \int_0^1 v_h^2 \, dx + \frac{1}{2} v_h^2(1) \\ &\quad - \sum_{i=1}^{N-1} x_i \{v_i\} [v_i] - \sum_{i=0}^N \int_{x_i}^{x_{i+1}} (x u_h - \chi_h(u_h)) \partial_x u_h \, dx \\ &\quad - \frac{1}{2} \int_0^1 u_h^2 \, dx - \frac{1}{2} u_h^2(1). \end{aligned}$$

We integrate by part and use the property that the terms involving the difference between xu_h and its interpolation are zero at the boundary. This leads to the following important inequality

$$\begin{aligned} (A_h z_h, \Pi_h z_h)_h &= \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x (x v_h - \chi_h^*(v_h)) \, dx - \frac{1}{2} \int_0^1 v_h^2 \, dx + \frac{1}{2} v_h^2(1) \\ &\quad - \sum_{i=1}^{N-1} x_i \{v_i\} [v_i] + \sum_{i=0}^N \int_{x_i}^{x_{i+1}} u_h \partial_x (x u_h - \chi_h(u_h)) \, dx \\ &\quad - \frac{1}{2} \int_0^1 u_h^2 \, dx - \frac{1}{2} u_h^2(1). \quad (36) \end{aligned}$$

Using Cauchy-Schwarz and Young inequalities, we obtain with $\lambda > 0$

$$\begin{aligned} (A_h z_h, \Pi_h z_h)_h &\leq -\frac{1}{2} (1 - \lambda) \int_0^1 u_h^2 + v_h^2 \, dx + \frac{1}{2\lambda} \sum_{i=0}^N \int_{x_i}^{x_{i+1}} [\partial_x (x v_h - \chi_h^*(v_h))]^2 \, dx \\ &\quad + \frac{1}{2\lambda} \sum_{i=0}^N \int_{x_i}^{x_{i+1}} [\partial_x (x u_h - \chi_h(u_h))]^2 \, dx \\ &\quad + \frac{1}{2} v_h^2(1) - \sum_{i=1}^{N-1} x_i \{v_i\} [v_i]. \quad (37) \end{aligned}$$

331 Let us focus on the last term. Using Cauchy-Schwarz and Young inequalities and
 332 bounding x_i by 1, we get with $\mu > 0$

$$-\sum_{i=1}^{N-1} x_i \{v_i\} [v_i] \leq \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} [v_i]^2 + \frac{\mu}{2} \sum_{i=1}^{N-1} \tilde{h}_i \{v_i\}^2.$$

333 Using the definition of the average and using $(a + b)^2 \leq 2(a^2 + b^2)$, we obtain

$$-\sum_{i=1}^{N-1} x_i \{v_i\} [v_i] \leq \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} [v_i]^2 + \frac{\mu}{4} \sum_{i=1}^{N-1} \tilde{h}_i ((v_i^+)^2 + (v_i^-)^2). \quad (38)$$

334 On the segment $[0, h]$, we classically have the trace inequality

$$|v_h^2(0)|^2 \leq \|v_h\|_{L^2(0,h)} \left(\frac{1}{h} \|v_h\|_{L^2(0,h)} + 2 \|\partial_x v_h\|_{L^2(0,h)} \right).$$

335 Using an inverse estimate [24, Chapter 4, Section 5, Lemma 4.5.3], we get, with $C_r > 0$
 336 depending only on r

$$|v_h^2(0)|^2 \leq \left(\frac{1 + 2C_r}{h} \right) \|v_h\|_{L^2(0,h)}^2.$$

337 This inequality is generalized on each mesh, so we have

$$\begin{cases} |v_i^+|^2 \leq \left(\frac{1 + 2C_r}{h_i} \right) \|v_h\|_{L^2([x_i, x_{i+1}])}^2, & i \in [1, N - 1], \\ |v_i^-|^2 \leq \left(\frac{1 + 2C_r}{h_{i-1}} \right) \|v_h\|_{L^2([x_{i-1}, x_i])}^2, & i \in [1, N - 1]. \end{cases}$$

Summing the two equations and using the definition of \tilde{h}_i , we get

$$\begin{aligned} \tilde{h}_i ((v_i^+)^2 + (v_i^-)^2) &\leq \tilde{h}_i \frac{1 + 2C_r}{h_i} \|v_h\|_{L^2([x_i, x_{i+1}])}^2 + \tilde{h}_i \frac{1 + 2C_r}{h_{i-1}} \|v_h\|_{L^2([x_{i-1}, x_i])}^2, \\ &\leq (1 + 2C_r) \left(\|v_h\|_{L^2([x_i, x_{i+1}])}^2 + \|v_h\|_{L^2([x_{i-1}, x_i])}^2 \right). \end{aligned}$$

338 We use the above inequalities in (38) to get

$$-\sum_{i=1}^{N-1} x_i \{v_i\} [v_i] \leq \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} [v_i]^2 + \frac{\mu(1 + 2C_r)}{2} \|v_h\|_{L^2(0,1)}^2.$$

Injecting this result in (37) we obtain

$$\begin{aligned} (A_h z_h, \Pi_h z_h)_h &\leq -\frac{1}{2} (1 - \lambda) \int_0^1 u_h^2 \, dx - \frac{1}{2} (1 - \lambda - \mu(1 + 2C_r)) \int_0^1 v_h^2 \, dx \\ &\quad + \frac{1}{2\lambda} \sum_{i=0}^N \int_{x_i}^{x_{i+1}} [\partial_x (x v_h - \chi_h^*(v_h))]^2 \, dx \\ &\quad + \frac{1}{2\lambda} \sum_{i=0}^N \int_{x_i}^{x_{i+1}} [\partial_x (x u_h - \chi_h(u_h))]^2 \, dx \\ &\quad + \frac{1}{2} v_h^2(1) + \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} [v_i]^2. \end{aligned}$$

We recognize the expression of the bilinear forms d_h and $d_{h,*}$ defined by (29) and (30). Using (19) for the first and the second terms in the right-hand side, we get

$$(A_h z_h, \Pi_h z_h)_h \leq -\frac{1}{2}(1-\lambda) \frac{1}{C_q} \int_0^1 u_h^2 dx - \frac{1}{2} \frac{1}{C_q} (1-\lambda-\mu(1+2C_r)) \int_0^1 v_h^2 dx \\ + \frac{1}{2\lambda} d_{h,*}(v_h, v_h) + \frac{1}{2\lambda} d_h(u_h, u_h) + \frac{1}{2} v_h^2(1) + \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} \llbracket v_i \rrbracket^2.$$

339 We choose λ and μ small enough such that

$$1 - \lambda > 0 \text{ and } 1 - \lambda - \mu(1 + 2C_r) > 0.$$

We obtain the fundamental multiplier inequality

$$(A_h z_h, \Pi_h z_h)_h \leq -\frac{1}{2C_q} (1 - \lambda - \mu(1 + 2C_r)) \|z_h\|_{\mathcal{Z}}^2 + \frac{1}{2\lambda} d_h(u_h, u_h) \\ + \frac{1}{2\lambda} d_{h,*}(v_h, v_h) + \frac{1}{2} v_h^2(1) + \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} \llbracket v_i \rrbracket^2. \quad (39)$$

340 Thus,

$$(\Pi_h A_h z_h, z_h)_h \leq \alpha \|C_h z_h\|_h^2 - \beta \|z_h\|_h^2, \quad (40)$$

341 where

$$\beta = \frac{1}{2C_q} (1 - \lambda - \mu(1 + 2C_r)) \text{ and } \alpha = \max\left(\frac{1}{2\lambda}, \frac{1}{2\mu}, \frac{1}{2}\right).$$

342

□

Corollary 4.7. *Let $z_h = (u_h, v_h)$ belonging to $U_h^r \times V_h^r$ be the solutions of (24). For T large enough, there exists $C^{st}(T)$ independent of h , such that*

$$\|u_h(0)\|_h^2 + \|v_h(0)\|_h^2 \leq C^{st}(T) \left[\int_0^T d_h(u_h, u_h) dt + \int_0^T d_{h,*}(v_h, v_h) dt \quad (41) \right. \\ \left. + \int_0^T v_h^2(1) dt + \int_0^T \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} \llbracket v_i \rrbracket^2 dt \right] = C^{st}(T) \int_0^T \|C_h z_h\|_h^2 dt,$$

343 where d_h and $d_{h,*}$ are defined respectively by (29) and (30).

344 *Proof.* Since Π_h defined by (27) satisfies Assumption 1 by Theorem 4.6, we can apply
345 Theorem 4.3 where C_h and A_h are defined by (34) and (24). □

346 Following what was done in Section 3, (41) is a guideline to build an exponentially
347 stable problem. Using the operator C_h – defined in the above section by (34) – to
348 damp the dynamics (24), we construct an exponentially stable dynamics uniformly in
349 h .

350 4.4 Exponential stability

351 We incorporate in the system (24) the dissipative terms C_h in the observability con-
352 ditions. In other words, we introduce a perturbation of system (24) by the operator
353 $\gamma C_h^* C_h$, which gives us the following system

$$\begin{cases} \dot{z}_h = A_h z_h - \gamma C_h^* C_h z_h, & \text{in } [0, T], \\ z_h(0) = z_{h,0}, \end{cases} \quad (42)$$

354 where A_h and C_h are respectively defined by (24) and (34). We will be able to expose
355 our main result, namely the uniform exponential stability property in the semi-discrete
356 case. First, as for the observability property, we will check that our multiplier Π_h
357 satisfies the fundamental Assumption 2.

358 **Theorem 4.8.** *Let C_h be defined by (34). Let $z_h = (u_h, v_h) \in \mathcal{Z}_h$ be the solutions of*
359 *(42). Then, Π_h defined by (27) satisfies Assumption 2 namely*

$$(\Pi_h A_h z_h, z_h)_h - \gamma(C_h z_h, C_h \Pi_h z_h)_h \leq \chi \|C_h z_h\|_h^2 - \kappa \|z_h\|_h^2,$$

360 where $\chi, \kappa > 0$ are independent of h .

361 *Proof.* Let clarify the expression of $(C_h z_h, C_h \Pi_h z_h)_h$.

$$\begin{aligned} - (C_h z_h, C_h \Pi_h z_h)_h &= - \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} \llbracket v_i \rrbracket \llbracket \chi_h(u_h) \rrbracket + v_h(1) u_h(1) \\ &\quad + d_h(u_h, \chi_h^*(v_h)) + d_{h,*}(v_h, \chi_h(u_h)). \end{aligned}$$

At a point, xu_h has the same value as its interpolation. Therefore, as u_h is continuous (and consequently its jump is zero), then $\chi_h(u_h)$ has a zero jump too. Then, thanks to (36), we have

$$\begin{aligned} (\Pi_h A_h z_h, z_h)_h - \gamma(C_h z_h, C_h \Pi_h z_h)_h &= - \int_0^1 v_h \partial_x \chi_h^*(v_h) \, dx + \int_0^1 \partial_x u_h \chi_h(u_h) \, dx - u_h^2(1) + v_h^2(1) \\ &\quad + \gamma v_h(1) u_h(1) + \gamma d_h(u_h, \chi_h^*(v_h)) + \gamma d_{h,*}(v_h, \chi_h(u_h)). \end{aligned}$$

Following the proof of Theorem 4.7, one can show

$$\begin{aligned} (\Pi_h A_h z_h, z_h)_h - \gamma(C_h z_h, C_h \Pi_h z_h)_h &\leq - \frac{1}{2C_q} (1 - \lambda - \mu(1 + 2C_r)) \|z_h\|_h^2 + \frac{1}{2\lambda} d_h(u_h, u_h) + \frac{1}{2\lambda} d_{h,*}(v_h, v_h) \\ &\quad + \frac{\gamma^2 + 1}{2} v_h^2(1) + \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} \llbracket v_i \rrbracket^2 + \gamma d_h(u_h, \chi_h^*(v_h)) + \gamma d_{h,*}(v_h, \chi_h(u_h)), \end{aligned}$$

where λ and μ are parameters that can be chosen as small desired. Thanks to Cauchy-Schwarz and Young inequalities for last terms, we get with $\delta > 0$

$$\begin{aligned} (\Pi_h A_h z_h, z_h)_h - \gamma(C_h z_h, C_h \Pi_h z_h)_h &\leq - \frac{1}{C_q} (1 - \lambda - \mu(1 + 2C_r)) \|z_h\|_h^2 + \left(\frac{1}{2\lambda} + \frac{\gamma}{2\delta} \right) (d_h(u_h, u_h) + d_{h,*}(v_h, v_h)) \\ &\quad + \frac{\gamma^2 + 1}{2} v_h^2(1) + \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} \llbracket v_i \rrbracket^2 + \frac{\delta\gamma}{2} d_h(\chi_h^*(v_h), \chi_h^*(v_h)) + \frac{\delta\gamma}{2} d_{h,*}(\chi_h(u_h), \chi_h(u_h)). \end{aligned}$$

Thanks to Proposition 4.5 and (28), we obtain

$$\begin{aligned} (\Pi_h A_h z_h, z_h)_h - \gamma(C_h z_h, C_h \Pi_h z_h)_h &\leq - \left(\frac{1}{C_q} - \frac{\lambda}{C_q} - \frac{\mu(1 + 2C_r)}{C_q} - \frac{\gamma\delta C_d C_\Pi^2}{2} \right) \|z_h\|_h^2 \\ &\quad + \left(\frac{1}{2\lambda} + \frac{\gamma}{2\delta} \right) (d_h(u_h, u_h) + d_{h,*}(v_h, v_h)) \\ &\quad + \frac{\gamma^2 + 1}{2} v_h^2(1) + \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} \llbracket v_i \rrbracket^2. \end{aligned}$$

362 We choose λ, μ, δ sufficiently small so that

$$\frac{1}{C_q} - \frac{\lambda}{C_q} - \frac{\mu(1 + 2C_r)}{C_q} - \frac{\gamma\delta C_d C_\Pi^2}{2} > 0.$$

Thus, finally we show that the Assumption 2 is satisfied

$$(\Pi_h A_h z_h, z_h)_h - \gamma(C_h z_h, C_h \Pi_h z_h)_h \leq \chi \|C_h z_h\|_h^2 - \kappa \|z_h\|_h^2,$$

363 with

$$\chi = \max\left(\frac{1}{2\lambda} + \frac{\gamma}{2\delta}, \frac{\gamma^2}{2} + \frac{1}{2}, \frac{1}{2\mu}\right),$$

364 and

$$\kappa = \frac{1}{C_q} - \frac{\lambda}{C_q} - \frac{\mu(1 + 2C_r)}{C_q} - \frac{\gamma\delta C_d C_\Pi^2}{2}.$$

365

□

366 We can deduce from this theorem the uniform exponential stability property for
367 the damped wave equation (42).

368 **Corollary 4.9.** *Let $z_h = (u_h, v_h)$ belonging to $\in U_h^r \times V_h^r$ be the solutions of (42).
369 There exist two constants $C_s > 0$ and $\sigma > 0$ independent of h , such that*

$$\|u_h(t)\|_h^2 + \|v_h(t)\|_h^2 \leq C_s e^{-\sigma t} (\|u_h(0)\|_h^2 + \|v_h(0)\|_h^2).$$

370 *Proof.* Using Theorem 4.8, we have that the multiplier Π_h defined by (27) satisfies
371 Assumption 2. Therefore, we apply Theorem 4.4 where C_h and is defined by (34). □

372 4.5 Further properties of the stabilization operators

373 It is possible to interpret that the dissipation term d_h defined by (29) corresponds to
374 element-wise r -laplacians operators. In what follows we introduce the notation

$$x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2}.$$

375 **Proposition 4.10.** *For $u_h \in U_h^r$ then*

$$d_h(u_h, u_h) = C_r \sum_{i=0}^{N-1} h_i^{2r+1} \left| u_h^{(r)} \left(x_{i+\frac{1}{2}} \right) \right|^2,$$

376 where C_r is a positive scalar depending only on the finite element order r .

377 *Proof.* To show that stated result, it is sufficient to study the following term for $u_h \in \mathbb{P}_r$,

$$\frac{d}{dx} (x u_h - \chi_h(u_h)).$$

378 Since $u_h \in U_h^r$, we have that $u_i := u_h|_{[x_i, x_{i+1}]} \in \mathbb{P}_r$, hence

$$u_i(x) = \sum_{k=0}^{r-1} \frac{x^k}{k!} u_i^{(k)}(x_i) + r_i(x) \quad \text{where} \quad r_i(x) := \frac{x^r}{r!} u_i^{(r)}.$$

379 Moreover, one can show that

$$\chi_h(u_h)|_{[x_i, x_{i+1}]} = I_i(x u_h)$$

380 where, I_i is the interpolation operator on the Gauss-Lobatto points $\{x_i + h_i \xi_k\}_{k=0}^r$, it
 381 is a projection operator over \mathbb{P}_r . Therefore, one can deduce that

$$(xu_h - \chi_h(u_h))_{[x_i, x_{i+1}]} = xr_i(x) - I_i(xr_i(x)) = \frac{u_i^{(r)}}{r!} (x^{r+1} - I_i(x^{r+1})) \quad (43)$$

382 Using this result and a simple change of variable one can show that

$$\int_{x_i}^{x_{i+1}} |\partial_x (xu_h - \chi_h(u_h))|^2 = \frac{(u_i^{(r)})^2}{r!^2} \int_0^{h_i} \left| \frac{d}{dx} (x^{r+1} - \mathcal{I}_{h_i}(x^{r+1})) \right|^2 dx, \quad (44)$$

383 where \mathcal{I}_{h_i} is the interpolation operator on the Gauss-Lobatto points $\{h_i \xi_k\}_{k=0}^r$. We
 384 introduce now the following change of variable,

$$\phi_i : \begin{cases} [0, 1] \rightarrow [0, h_i], \\ \xi \rightarrow x = h_i \xi. \end{cases}$$

385 Thus, we have

$$(I_{h_i}(x^{r+1}) \circ \phi_i)(\xi) = \sum_{k=0}^r \psi_k(\xi) \xi_k^{r+1} h_i^{r+1},$$

386 where $\{\psi_k(\xi)\}_{k=0}^r$ are a basis for \mathbb{P}_r such that $\psi_k(\xi_\ell) = \delta_{k,\ell}$. The use of the above
 387 identity, the composition of (43) and ϕ_i leads to

$$\left(\frac{d}{dx} (x^{r+1} - I_{h_i}(x^{r+1})) \right) \circ \phi_i = \frac{1}{h_i} \frac{d}{d\xi} \left(h^{r+1} \xi^{r+1} - \sum_{k=0}^r \psi_k(\xi) \xi_k^{r+1} h_i^{r+1} \right).$$

388 Finally, (44) is rewritten as follows,

$$\int_{x_i}^{x_{i+1}} |\partial_x (xu_h - \chi_h(u_h))|^2 = C_r (u_i^{(r)})^2 \frac{1}{h_i} h_i^{2r+2},$$

389 where $C_r = \frac{1}{r!^2} \int_0^1 \left| \frac{d}{d\xi} \left(\xi^{r+1} - \sum_{k=0}^r \psi_k(\xi) \xi_k^{r+1} \right) \right|^2 d\xi$. □

390 Of note, in the work of [23] a Laplace operator us used as a viscous term for
 391 stabilization of finite elements of order 1. Therefore, in some sense our work generalizes
 392 these results by showing that an element-wise r -laplace operator – scaled with the
 393 appropriate power of h – can be used for stabilization of high order finite elements.

394 The same result cannot be obtained straightforwardly for $d_{h,*}$. In fact, due to the
 395 discontinuous nature of the space V_h^r , jump terms are involved.

396 **Proposition 4.11.** *For $v_h \in V_h^r$ then*

$$d_{h,*}(v_h, v_h) \leq C_r^* \left(\sum_{i=0}^{N-1} h_i^{2r+1} \left| v_h^{(r)} \left(x_{i+\frac{1}{2}} \right) \right|^2 + h \sum_{i=1}^{N-1} \llbracket v_i \rrbracket^2 \right),$$

397 where C_r^* is a positive scalars depending only on the finite element order r .

398 *Proof.* The proof is based on an adequate decomposition of any function $v_h \in V_h^r$. More
 399 precisely we introduce $w_h \in V_h^1$ defined by

$$w_h(0) = w_h(1) = 0, \quad w_i^\pm = \pm \frac{\llbracket v_i \rrbracket}{2}, \quad i \in \{1, \dots, N-1\} \quad \text{and} \quad w_{h|[x_i, x_{i+1}]} \in \mathbb{P}^1.$$

400 We then defined $u_h = v_h - w_h$. It can be verified that $u_h \in U_h^r$. We have

$$d_{h,*}(v_h, v_h) \leq 2d_{h,*}(u_h, u_h) + 2d_{h,*}(w_h, w_h).$$

401 Using Proposition 4.5 and the property that $u_h \in U_h^r$ we obtain the estimation

$$d_{h,*}(v_h, v_h) \leq 2d_h(u_h, u_h) + 2C_d \int_0^1 w_h^2 dx,$$

402 hence, using Proposition 4.10,

$$d_{h,*}(v_h, v_h) \leq 2C_r \sum_{i=0}^{N-1} h_i^{2r+1} \left| u_h^{(r)} \left(x_{i+\frac{1}{2}} \right) \right|^2 + 2C_d \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} w_h^2 dx.$$

403 It is then possible to show that

$$\sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} w_h^2 dx \leq h \sum_{i=1}^{N-1} [v_i]^2,$$

404 finally we observe that $u_h^{(r)}(x_{i+\frac{1}{2}}) = (u_h + w_h)^{(r)}(x_{i+\frac{1}{2}}) = v_h^{(r)}(x_{i+\frac{1}{2}})$ by construction. □

406 **Remark 4.12.** *In the numerical results, we used the following simplified expression*
 407 *instead of the term $d_{h,*}$*

$$\tilde{d}_{h,*}(v_h, \tilde{v}_h) := C_r \sum_{i=0}^{N-1} h_i^{2r+1} v_h^{(r)} \left(x_{i+\frac{1}{2}} \right) \tilde{v}_h^{(r)} \left(x_{i+\frac{1}{2}} \right). \quad (45)$$

408 *Proposition 4.11 allows us to show that Theorem 4.6 and Theorem 4.8 hold with*

$$\tilde{C}_h = \begin{pmatrix} D_h^{1/2} & 0 \\ 0 & J_h^{1/2} \\ 0 & B_h^{1/2} \\ 0 & \tilde{D}_{h,*}^{1/2} \end{pmatrix}. \quad (46)$$

409 *where $\tilde{D}_{h,*}$ is associated with $\tilde{d}_{h,*}$, thus showing the exponential decay behavior of the*
 410 *system*

$$\begin{cases} \dot{z}_h = A_h z_h - \gamma \tilde{C}_h^* \tilde{C}_h z_h, & \text{in } [0, T], \\ z_h(0) = z_{h,0}. \end{cases}$$

411 5 Observer system: Semi-discrete convergence result

412 In this section we follow the approach of [33] and aim at defining at the semi-discrete
 413 level an observer of the solution $z = (u, v)^t$ of the system (4) with the assumption that,
 414 for all $T > 0$

$$z = (u, v) \in C^1([0, T]; H^{r+1}(0, 1)^2).$$

415 It is shown below that the constructed observer as the property to converge asymptoti-
 416 cally in time to the solution of z up to an order related to the space discretization – typ-
 417 ically in $O(h^r)$ – but bounded in time. For the sake of simplicity and conciseness we di-
 418 rectly introduce the interpolation on the of z denoted $\bar{z}_h(t) = I_h z(t) = (I_{U_h} u(t), I_{V_h} v(t))$,
 419 it is solution of the system

$$\dot{\bar{z}}_h = A_h \bar{z}_h + \varepsilon_h, \quad \bar{z}_h(0) = z_{0,h} = I_h z_0,$$

420 where ε_h is the consistency error and is assumed to be smooth in time and space and

$$\|\varepsilon_h\| := \sup_{t \in [0, T]} \|\varepsilon_h(t)\|_{L^2(0,1)^2} \leq C_\varepsilon h^r, \quad (47)$$

421 where $C_\varepsilon > 0$ is independent of T . Moreover, the interpolation operator satisfies, for
422 $0 \leq \ell \leq k$ and $0 \leq k \leq r + 1$,

$$\forall w \in H^{r+1}(0, 1)^2, \quad \left(\sum_{i=0}^{N-1} |I_h w - w|_{H^\ell(x_i, x_{i+1})}^2 \right)^{\frac{1}{2}} \leq C_I h^{k-\ell} \|w\|_{H^k(0,1)}. \quad (48)$$

423 We refer to [11, 21, 24] for more details on how these properties can be obtained. The
424 system for the observer is constructed as follow: we assume $v(1, t)$ is available/measured
425 for $t \in [0, T]$ – which is equivalent to having access to $(I_h^V v)(1, t)$ – and solve,

$$\dot{\hat{z}}_h = A_h \hat{z}_h - \gamma \begin{pmatrix} \tilde{D}_h & 0 \\ 0 & \tilde{D}_{h,*} + J_h \end{pmatrix} \hat{z}_h - \gamma \begin{pmatrix} 0 & 0 \\ 0 & B_h \end{pmatrix} (\hat{z}_h - \bar{z}_h), \quad \hat{z}_h(0) = 0.$$

426 Note that observer systems – and more precisely, state estimator – are very well studied
427 at the continuous level, we refer the reader to [1, 32, 39]. In short, one expect that
428 $\hat{z}_h(t) \rightarrow \bar{z}_h(t) \sim z(t)$ asymptotically in time, thus alleviating the mismatch in the
429 initial data between the observer and the interpolation of the solution $z(t)$ of (4). The
430 error

$$e_h = \bar{z}_h - \hat{z}_h,$$

431 satisfies, with $A_{\gamma,h} = A_h - \gamma \tilde{C}_h^* \tilde{C}_h$,

$$\dot{e}_h = A_{\gamma,h} e_h + \gamma \begin{pmatrix} D_h & 0 \\ 0 & \tilde{D}_{h,*} + J_h \end{pmatrix} \bar{z}_h + \varepsilon_h, \quad e_h(0) = z_{0,h}.$$

432 **Lemma 5.1.** *There exists a constant $C_D > 0$ independent of h , such that*

$$\forall u \in H^{r+1}(0, 1), \quad \|D_h I_{U_h} u\|_{L^2(0,1)} \leq C_D h^r \|u\|_{H^r(0,1)},$$

433 and

$$\forall v \in H^{r+1}(0, 1), \quad J_h I_{V_h} v = 0, \quad \|\tilde{D}_{h,*} I_{V_h} v\|_{L^2(0,1)} \leq C_D h^r \|v\|_{H^r(0,1)}.$$

Proof. We show the first identity. From the Proposition 4.10, we have

$$\begin{aligned} d_h(I_{U_h} u, I_{U_h} u) &= C_r \sum_{i=0}^{N-1} h_i^{2r+1} \left| (I_{U_h} u)^{(r)} \left(x_{i+\frac{1}{2}} \right) \right|^2 \\ &= C_r \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} h_i^{2r} \left| (I_{U_h} u)^{(r)} \left(x_{i+\frac{1}{2}} \right) \right|^2 \\ &\leq C_r h^{2r} \sum_{i=0}^{N-1} |I_{U_h} u|_{H^r(x_i, x_{i+1})}^2. \end{aligned}$$

434 Using (48) and a triangular inequality, we get the existence of a constant \tilde{C}_r such that

$$d_h(I_{U_h} u, I_{U_h} u) \leq \tilde{C}_r h^{2r} \|u\|_{H^r(0,1)}^2.$$

435 The analysis of $\|\tilde{D}_{h,*} I_{V_h} v\|_{L^2(0,1)}$ is the same as the previous one. Finally, note that
436 since v is smooth, we have that $I_{V_h} v$ is continuous and therefore $J_h I_{V_h} v = 0$. \square

437 **Theorem 5.2.** *There exists a positive scalar $C_{st} > 0$ such that, for any $T > 0$, we have*

$$\sup_{t \in [0, T]} \|z(t) - \hat{z}_h(t)\|_{L^2(0,1)^2} \leq C_{st} (e^{-\sigma t} \|z_{0,h}\|_h + h^r).$$

Proof. By triangular inequality and using (19), we have that

$$\begin{aligned} \|z(t) - \hat{z}_h(t)\|_{L^2(0,1)^2} &= \|z(t) - \bar{z}_h + \bar{z}_h - \hat{z}_h(t)\|_{L^2(0,1)^2} \\ &\leq \|z(t) - \bar{z}_h\|_{L^2(0,1)^2} + c_q^{-1/2} \|\bar{z}_h - \hat{z}_h(t)\|_h. \end{aligned}$$

438 Using the interpolation property (48), we obtain that there exists a positive scalar C_{st}
439 such that

$$\|z(t) - \hat{z}_h(t)\|_{L^2(0,1)^2} \leq C_{st} h^r + c_q^{-1/2} \|\bar{z}_h - \hat{z}_h(t)\|_h. \tag{49}$$

440 Let us focus on the error $e_h = \bar{z}_h - \hat{z}_h$, it satisfies

$$\dot{e}_h = A_{\gamma,h} e_h + \bar{\varepsilon}_h,$$

441 where

$$\bar{\varepsilon}_h = \gamma \begin{pmatrix} D_h & 0 \\ 0 & \tilde{D}_{h,*} + J_h \end{pmatrix} \bar{z}_h + \varepsilon_h.$$

442 Using Lemma 5.1, (47) combined with (19), we have that there exists C_{st} independent
443 of h such that

$$\|\bar{\varepsilon}_h\| \leq C_{st} h^r.$$

444 From the Duhamel formula, we have

$$e_h(t) = e^{tA_{\gamma,h}} z_{0,h} + \int_0^t e^{(t-s)A_{\gamma,h}} \bar{\varepsilon}_h(s) \, ds.$$

445 Using the exponential stability of the operator $A_{\gamma,h}$ (Corollary 4.9), we have

$$\|e_h(t)\|_h \leq C_{st} \left(e^{-\sigma t} \|z_{0,h}\|_h + \sup_{s \in [0,t]} \|\bar{\varepsilon}_h(s)\|_h \int_0^t e^{-\sigma(t-s)} \, ds \right).$$

446 As a consequence, we finally have that there exist a constant $C_{st} > 0$ independent of h
447 and t such that

$$\|e_h(t)\|_h \leq C_{st} \left(e^{-\sigma t} \|z_{0,h}\|_h + \frac{1}{\sigma} (1 - e^{-\sigma t}) h^r \right),$$

448 which combining with (49) proves the desired result. □

449 It is worth noting that the preceding theorem states, in particular, that an observer
450 \hat{z}_h initialized from an interpolated initial condition with an accuracy of order h^r then
451 remains at a distance h^r from the true solution without error drift with respect to time.
452 This is due to the fact that the constant C_{st} does not depend on time. Therefore, in
453 the context of numerical analysis by semi-discretization, this result extends [33] to high
454 order schemes.

455 6 Time discretization

456 6.1 Abstract results

457 In this section, we are interested in the discretization of System (17) with an implicit
 458 midpoint scheme. The operator C_h or \tilde{C}_h can be used indifferently (we have chosen
 459 to use C_h in what follows for the sake of clarity). We verify that the use of this time
 460 discretization does not alter the stabilization properties.

$$\begin{cases} \frac{z_h^{n+1} - z_h^n}{\Delta t} = A_h z_h^{n+1/2} - \gamma C_h^* C_h z_h^{n+1/2}, \\ z_h^0 = z_{h,0}. \end{cases} \quad (50)$$

461 It is easy to show that the energy conservation property is preserved in the fully discrete
 462 case.

463 **Theorem 6.1.** *Let $\{z_h^n\}_{n \geq 0} \in \mathcal{Z}_h$ be the solutions of (50), we have the discrete*
 464 *dissipation energy property*

$$\frac{1}{2} \frac{\|z_h^{n+1}\|_h^2 - \|z_h^n\|_h^2}{\Delta t} = -\gamma \|C_h z_h^{n+1/2}\|_h^2, \quad \forall n \geq 0. \quad (51)$$

465 *Proof.* We multiply (50) by $z_h^{n+1/2}$. Since A_h satisfies (16) and using the difference of
 466 square formula, we obtain the result □

467 Following the same reasoning as in the semi-discrete case, we start by demonstrating
 468 the observability property for the undamped case $\gamma = 0$.

469 **Theorem 6.2.** *Let $\{z_h^n\}_{n \geq 0} \in \mathcal{Z}_h$ be the solutions of (50) with $\gamma = 0$. Assuming*
 470 *Assumption 1, for T large enough, there exists a constant $C^{st}(T) > 0$, independent of*
 471 *h , such that*

$$\|z_{h,0}\|_h^2 \leq C^{st}(T) \Delta t \sum_{n=0}^N \|C_h z_h^{n+1/2}\|_h^2.$$

472 *Proof.* The proof of this theorem is very similar to that of Theorem 4.3 with $\gamma = 0$, in
 473 particular, it follows the same steps.

474 **Step 1:** We multiply the first equation of (50) (with $\gamma = 0$) by $\Pi_h z_h^{n+1/2}$, and we get

$$\left(\Pi_h \frac{z_h^{n+1} - z_h^n}{\Delta t}, z_h^{n+1/2} \right)_h = \left(\Pi_h A_h z_h^{n+1/2}, z_h^{n+1/2} \right)_h.$$

We decompose the half-sum in the left hand-side to find, using Assumption 1,

$$\begin{aligned} \frac{1}{2\Delta t} \left(\Pi_h z_h^{n+1}, z_h^{n+1/2} \right)_h - \frac{1}{2\Delta t} \left(\Pi_h z_h^n, z_h^{n+1/2} \right)_h &= \left(\Pi_h A_h z_h^{n+1/2}, z_h^{n+1/2} \right)_h, \\ &\leq \alpha \|C_h z_h^{n+1/2}\|_h^2 - \beta \|z_h^{n+1/2}\|_h^2. \end{aligned} \quad (52)$$

475 **Step 2:** Summing (52) over n we get,

$$\frac{1}{2\Delta t} \left(\Pi_h z_h^{N+1}, z_h^{N+1/2} \right)_h - \frac{1}{2\Delta t} \left(\Pi_h z_h^0, z_h^{1/2} \right)_h \leq \alpha \sum_{n=0}^N \|C_h z_h^{n+1/2}\|_h^2 - \beta \sum_{n=0}^N \|z_h^{n+1/2}\|_h^2.$$

476 The fully discrete energy conservation property of Theorem 6.1 with $\gamma = 0$ gives

$$\frac{1}{2\Delta t} \left(\Pi_h z_h^{N+1}, z_h^{N+1/2} \right)_h - \frac{1}{2\Delta t} \left(\Pi_h z_h^0, z_h^{1/2} \right)_h \leq \alpha \sum_{n=0}^N \|C_h z_h^{n+1/2}\|_h^2 - \beta(N+1) \|z_h^0\|_h^2.$$

477 **Step 3:** Since Π_h is bounded, by Cauchy-Schwarz theorem and using again the energy
478 conservation theorem 6.1 with $\gamma = 0$, we have

$$\left(\beta(N+1)\Delta t - \|\Pi_h\|_h \right) \|z_{0,h}\|_h^2 \leq \alpha \sum_{n=0}^N \Delta t \|C_h z_h^{n+1/2}\|_h^2.$$

479 Choosing $(N+1)\Delta t = T > \|\Pi_h\|_h \beta^{-1}$, we obtain the result. \square

480 In the same way as in the semi-discrete case, we obtain the exponential stability
481 property for the damped system.

482 **Theorem 6.3.** Let $\{z_h^n\}_{n \geq 0} \in \mathcal{Z}_h$ be the solutions of (50). We assume that there exists
483 $C_c > 0$ and $0 < \beta < 1$ such that for all $h > 0$, Δt is chosen so that

$$\Delta t \|C_h\|_h \leq C_c, \quad (53)$$

484 and

$$\frac{\Delta t}{2} \|A_h\|_h \leq \beta < 1. \quad (54)$$

485 With this assumption and Assumption 2, there exist two constants $C_s, \sigma > 0$ independent
486 of h such that

$$\|z_h^n\|_h^2 \leq C_s \|z_{h,0}\|_h^2 e^{-\sigma(1-\beta^2)n\Delta t}.$$

487 *Proof.* The proof is separated into the same steps as for the proof of exponential
488 stability at the continuous level of Theorem 4.4.

489 **Step 1:** We multiply the first equation by (50) and $\Pi_h z_h^{n+1/2}$, and we get

$$\left(\Pi_h \frac{z_h^{n+1} - z_h^n}{\Delta t}, z_h^{n+1/2} \right)_h = (\Pi_h A_h z_h^{n+1/2}, z_h^{n+1/2})_h - \gamma (C_h z_h^{n+1/2}, C_h \Pi_h z_h^{n+1/2})_h.$$

490 Using $(a+b)(a-b) = a^2 - b^2$ and Assumption 2, we have

$$\frac{1}{2\Delta t} (\Pi_h z_h^{n+1}, z_h^{n+1})_h - \frac{1}{2\Delta t} (\Pi_h z_h^n, z_h^n)_h \leq \chi \|C_h z_h^{n+1/2}\|_h^2 - \kappa \|z_h^{n+1/2}\|_h^2.$$

Step 2: Using the energy identity of Theorem 6.1, we get with $\alpha > 0$,

$$\begin{aligned} \frac{1}{2\Delta t} (\Pi_h z_h^{n+1}, z_h^{n+1})_h - \frac{1}{2\Delta t} (\Pi_h z_h^n, z_h^n)_h + \alpha \frac{\|z_h^{n+1}\|_h^2 - \|z_h^n\|_h^2}{2\Delta t} \\ \leq (\chi - \alpha\gamma) \|C_h z_h^{n+1/2}\|_h^2 - \kappa \|z_h^{n+1/2}\|_h^2. \end{aligned} \quad (55)$$

491 We intend to bound the last term by $-\|z_h^{n+1}\|_h^2 + \|z_h^n\|_h^2$. Note that this intermediate
492 necessary step is specific to the time discretization. Decomposing $z_h^{n+1/2}$ in a well-chose
493 way, we have

$$-\|z_h^{n+1/2}\|_h^2 = -\frac{1}{2} \|z_h^{n+1}\|_h^2 - \frac{1}{2} \|z_h^n\|_h^2 + \frac{(\Delta t)^2}{4} \left\| \frac{z_h^{n+1} - z_h^n}{\Delta t} \right\|_h^2.$$

Using the scheme (50) and Cauchy-Schwarz and Young inequalities, we obtain with
 $\epsilon > 0$,

$$\begin{aligned} -\|z_h^{n+1/2}\|_h^2 &\leq -\frac{1}{2} \|z_h^{n+1}\|_h^2 - \frac{1}{2} \|z_h^n\|_h^2 + \frac{(\Delta t)^2}{4} (1 + \epsilon) \|A_h z_h^{n+1/2}\|_h^2 \\ &\quad + \frac{\gamma(\Delta t)^2}{4} (1 + \epsilon^{-1}) \|C_h^* z_h^{n+1/2}\|_h^2. \end{aligned}$$

By triangular inequality and Cauchy-Schwarz inequality and using hypothesis (53), we get

$$\begin{aligned}
 -\|z_h^{n+1/2}\|_h^2 &\leq -\frac{1}{2} \left(1 - \frac{(\Delta t)^2}{4} (1 + \epsilon) \|A_h\|_h^2\right) (\|z_h^{n+1}\|_h^2 + \|z_h^n\|_h^2) \\
 &\quad + \frac{\gamma C_c^2}{4} (1 + \epsilon^{-1}) \|C_h z_h^{n+1/2}\|_h^2.
 \end{aligned}$$

We choose $\epsilon = \frac{1}{2\beta^2} - \frac{1}{2} > 0$, using the condition (54), we obtain

$$\begin{aligned}
 -\|z_h^{n+1/2}\|_h^2 &\leq -\frac{1}{4} (1 - \beta^2) (\|z_h^{n+1}\|_h^2 + \|z_h^n\|_h^2) \\
 &\quad + \frac{\gamma C_c^2}{4} \left(1 + \frac{2\beta^2}{1 - \beta^2}\right) \|C_h z_h^{n+1/2}\|_h^2.
 \end{aligned}$$

Using the above equation in (55), we get

$$\begin{aligned}
 &\frac{1}{2\Delta t} (\Pi_h z_h^{n+1}, z_h^{n+1})_h - \frac{1}{2\Delta t} (\Pi_h z_h^n, z_h^n)_h + \alpha \frac{\|z_h^{n+1}\|_h^2 - \|z_h^n\|_h^2}{2\Delta t} \\
 &\leq \left(\frac{\gamma C_c^2}{4} \left(1 + \frac{2\beta^2}{1 - \beta^2}\right) + \chi - \alpha\gamma\right) \|C_h z_h^{n+1/2}\|_h^2 - \frac{\kappa}{4} (1 - \beta^2) (\|z_h^{n+1}\|_h^2 + \|z_h^n\|_h^2).
 \end{aligned}$$

494

495

496 **Step 3:** We choose α large enough such that $\alpha \geq \frac{\chi}{\gamma} + \frac{C_c^2}{4} \left(1 + \frac{2\beta^2}{1 - \beta^2}\right)$. Thus, we
 497 have

$$\begin{aligned}
 &\frac{1}{2\Delta t} (\Pi_h z_h^{n+1}, z_h^{n+1})_h - \frac{1}{2\Delta t} (\Pi_h z_h^n, z_h^n)_h + \alpha \frac{\|z_h^{n+1}\|_h^2 - \|z_h^n\|_h^2}{2\Delta t} \\
 &\leq -\frac{\kappa}{4} (1 - \beta^2) (\|z_h^{n+1}\|_h^2 + \|z_h^n\|_h^2). \quad (56)
 \end{aligned}$$

498 **Step 4:** We introduce the fully discrete Lyapunov functional

$$\mathcal{L}_h^n = (\Pi_h z_h^n, z_h^n)_h + \alpha \|z_h^n\|_h^2,$$

499 and we can also show that it is equivalent to the energy functional $\mathcal{E}_h^n := \frac{1}{2} \|z_h^n\|_h^2$ using
 500 the bounded nature of Π_h and choosing α sufficiently large.

$$(\alpha - \|\Pi_h\|_h) \|z_h^n\|_h^2 \leq \mathcal{L}_h^n \leq (\alpha + \|\Pi_h\|_h) \|z_h^n\|_h^2. \quad (57)$$

501 Using the above equivalence between the Lyapunov functional and the energy, we
 502 obtain

$$\frac{1}{2\Delta t} (\mathcal{L}_h^{n+1} - \mathcal{L}_h^n) \leq -\frac{\kappa}{4} (1 - \beta^2) (\|\Pi_h\|_h + \alpha)^{-1} (\mathcal{L}_h^{n+1} + \mathcal{L}_h^n).$$

503 Rearranging the terms, we have

$$\mathcal{L}_h^{n+1} \leq \delta(\Delta t, \alpha) \mathcal{L}_h^n.$$

504 We denote $\delta(\Delta t, \alpha) = \left(1 - \frac{\Delta t \frac{\kappa}{2} (1 - \beta^2)}{\|\Pi_h\|_h + \alpha}\right) / \left(1 + \frac{\Delta t \frac{\kappa}{2} (1 - \beta^2)}{\|\Pi_h\|_h + \alpha}\right)$.
 505

506 **Step 5:** By induction and using again (18), we obtain

$$\|z_h^n\|_h^2 \leq \frac{\alpha + \|\Pi_h\|_h}{\alpha - \|\Pi_h\|_h} \|z_{h,0}\|_h^2 e^{n \ln(\delta(\Delta t, \alpha))}.$$

We must now specify the constant $\ln(\delta)$. Bounding from below Δ by 0 and using a series expansion we get

$$\begin{aligned} \ln(\delta(\Delta t, \alpha)) &= \ln \left(1 - \frac{\frac{\Delta t \kappa (1 - \beta^2)}{\|\Pi_h\|_h + \alpha}}{1 + \frac{\Delta t \frac{\kappa}{2} (1 - \beta^2)}{\|\Pi_h\|_h + \alpha}} \right) \leq \ln \left(1 - \Delta t \frac{\kappa (1 - \beta^2)}{\|\Pi_h\|_h + \alpha} \right) \\ &\underset{\alpha \rightarrow +\infty}{\sim} -\frac{\Delta t}{\alpha} \kappa (1 - \beta^2) + o\left(\frac{1}{\alpha}\right). \end{aligned}$$

507 Finally, we can conclude,

$$\|z_h^n\|_h^2 \leq \left(\frac{\alpha + \|\Pi_h\|_h}{\alpha - \|\Pi_h\|_h} \right) \|z_{h,0}\|_h^2 e^{-\sigma n \Delta t},$$

508 with $\sigma(1 - \beta^2) = \frac{\kappa(1 - \beta^2)}{\alpha} > 0$.

509

□

510 **Remark 6.4.** *In many situations, when we consider an unbounded operator C , we can*
 511 *write $C = \tilde{C}A$ where \tilde{C} is bounded. So for all h , there exists C_A independent of h such*
 512 *that,*

$$\|C_h\|_h \leq C_A \|A_h\|_h, \tag{58}$$

513 *and in this case, we can get rid of condition (53) and condition (54) is sufficient for*
 514 *Theorem 6.3. We recognize "CFL" condition usually encounter with explicit scheme.*
 515 *Nevertheless, it allows us to obtain the exponential stability property.*

516 Under the same two assumptions as in the semi-discrete case, we obtain the
 517 properties of observability and exponential stability.

518 6.2 Application: 1D fully discrete wave problem

519 We can now present the observability theorem in a totally discrete framework.

Theorem 6.5. *Let $\{u_h^n, v_h^n\}_{n \geq 0}$ belonging to $U_h^r \times V_h^r$ be the solutions of (50) with $\gamma = 0$. For T large enough, there exists $C^{st}(T)$ independent of h such that*

$$\begin{aligned} \|u_{h,0}\|_h^2 + \|v_{h,0}\|_h^2 &\leq C^{st}(T) \sum_{n=1}^N \Delta t \left[d_h \left(u_h^{n+1/2}, u_h^{n+1/2} \right) + d_{h,*} \left(v_h^{n+1/2}, v_h^{n+1/2} \right) \right. \\ &\quad \left. + \left(v_h^{n+1/2}(1) \right)^2 dt + \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} \left[v_i^{n+1/2} \right]^2 \right], \end{aligned}$$

520 where d_h and d_h^* are defined by (29) and by (30).

521 *Proof.* Since Π_h satisfies Assumption 1, we can apply Theorem 6.2 and the result
 522 follows immediately. □

523 **Theorem 6.6.** Let $\{u_h^n, v_h^n\}_{n \geq 0}$ belonging to $U_h^r \times V_h^r$ be the solutions of (50). We
524 assume that $\exists \beta > 0$ independent of h and Δt , $\forall h$, Δt is chosen small enough such

$$\frac{\Delta t}{2} \|R_h\|_h \leq \beta < 1. \quad (59)$$

525 There exists two constants $C_s > 0$ and $\sigma > 0$ independent of h such that

$$\|u_h^n\|_h^2 + \|v_h^n\|_h^2 \leq C_s e^{-n\Delta t\sigma(\Delta t)} (\|u_{h,0}\|_h^2 + \|v_{h,0}\|_h^2).$$

526 *Proof.* In order to prove Theorem 6.3, we need to show that the conditions (54) holds.
527 Since we have the following property

$$\|A_h\|_h \leq C_A \|R_h\|_h, \quad C_A > 0,$$

528 if (59) holds, then (54) is satisfied.

529 In order to apply Remark 6.4 – and thus to get rid of the condition (53) –, it only
530 remains to show that

$$\|C_h\|_h \leq C_c \|R_h\|_h, \quad C_c > 0.$$

531 It is well known that we have the following inequality

$$\frac{\alpha}{h} \leq \|R_h\|_h, \quad \alpha > 0,$$

532 so we only need to show that $\|C_h\|_h \leq \frac{C^{\text{st}}}{h}$ with $C^{\text{st}} > 0$. Let us compute the operator
533 norm of C_h . By definition of the operator C_h , we have for $z_h \in \mathcal{Z}_h$,

$$\|C_h z_h\|_h^2 = d_h(u_h, u_h) + d_{h' *}(v_h, v) + v_h^2(1) + \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} \llbracket v_i \rrbracket^2.$$

534 We only treat the last term because it is the most difficult. Using Proposition 4.5 and
535 the same kind of computations as in the proof of Theorem 6.3, one can show that

$$\sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} \llbracket v_i \rrbracket^2 \leq 2 \sum_{i=0}^{N-1} \frac{1 + 2C_r}{\tilde{h}_i h_i} \|v_h\|_{L^2([x_i, x_{i+1}])}^2 + 2 \sum_{i=1}^N \frac{1 + 2C_r}{\tilde{h}_i h_{i-1}} \|v_h\|_{L^2([x_{i-1}, x_i])}^2.$$

536 Since we have a quasi-uniform discretization, we obtain

$$\sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} \llbracket v_i \rrbracket^2 \leq 4 \frac{1 + 2C_r}{\rho^2 h^2} \|v_h\|_{L^2(0,1)}^2.$$

537 We can therefore bound $\|C_h\|_h$ by $1/h$,

$$\|C_h\|_h \leq \left(\sqrt{C_d} + \frac{\sqrt{1 + 2C_r}}{\sqrt{\rho h}} + \frac{2\sqrt{1 + 2C_r}}{\rho h} \right).$$

538 Furthermore, $\forall h$, Δt chosen small enough such that

$$\Delta t \|R_h\|_h \leq \beta < 1,$$

539 satisfies the condition 54 of Theorem 6.3. Since Π_h satisfies Assumption 2 and (58)
540 and (54) holds, we can apply Theorem 6.3 and the result follows immediately. \square

541 In perspective, adapting the previous result to the observer analysis, as we did
542 for the semi-discretization in space, should allow us to extend the fully discretized
543 numerical analysis proposed in [33] to high order schemes. This would then imply that
544 we can propose an observer that avoids the asymptotic temporal error drift by using
545 data while remaining compatible with the state of the art of high order discretization.

546 **7 Numerical results**

547 We compare the numerical results of Section 2 and those obtained with our method.
 548 First we make an analysis of the eigenvalues by comparing our method and other
 549 existing methods in the literature. In a second step, for the sake of efficiency of the
 550 method, we will introduce a leap-frog scheme which preserves the energy dissipation
 551 property of the previous sections. Then we compare the shape of the solution of the
 552 damped system with the solution of the undamped system of Section 2. Note that
 553 in all the section the term $\tilde{D}_{h,*}$ is the operator associated with the bilinear form $\tilde{d}_{h,*}$
 554 defined by (45) namely

$$(\tilde{D}_{h,*}v_h, \tilde{v}_h)_h = \tilde{d}_{h,*}(v_h, \tilde{v}_h), \forall v_h, \tilde{v}_h \in V_h^r. \tag{60}$$

555 **7.1 Eigenvalues analysis**

556 We look at the effect of adding the new dissipative terms on the spectrum. We start
 557 by doing a comparison with methods used in the literature. A first idea is to penalize
 558 the jump as seen in [27] for Maxwell’s equations. Thus, we add a dissipation term
 559 J_h corresponding to this penalization of the jump of the second variable which is
 560 discontinuous.

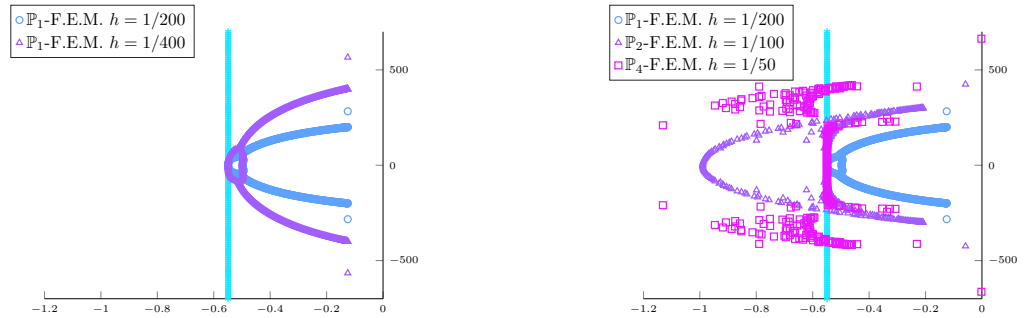


Figure 5. Spectrum of $A_h - J_h$

561 We see in Figure 5 (Left) that this method is adapted for finite elements of \mathbb{P}_1 ,
 562 but that the decay rate is not as good as the continuous one. As soon as we go to
 563 higher finite elements in Figure 5 (Right), we find eigenvalues close to the imaginary
 564 axis. We also represent the spectrum obtained viscosity is added – as a classical
 565 stabilization strategy, (see [23]). We denote by V_h the discrete Laplace operator on the
 first component.

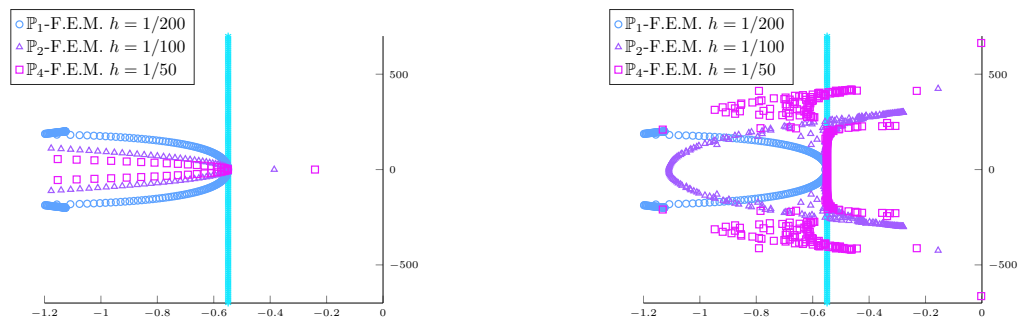


Figure 6. Spectrum of $A_h - J_h - h^2 V_h$ (Left) and of $A_h - J_h - h^{r+1} V_h$ (Right), r is the order of the method

566 Thus, as we can see in Figure 6 (Left), a second order perturbation gives the expo-
 567 nential stability but with a wrong decay for finite elements of order greater than \mathbb{P}_1 .
 568

569 Furthermore, the consistency is really deteriorated. If we soften the impact of the
 570 viscous term – Figure 6 (Right) –, for instance by replacing the scaling h^2 by h^{r+1} with
 571 the order r of the finite elements, then the exponential stability is not satisfied anymore.
 572

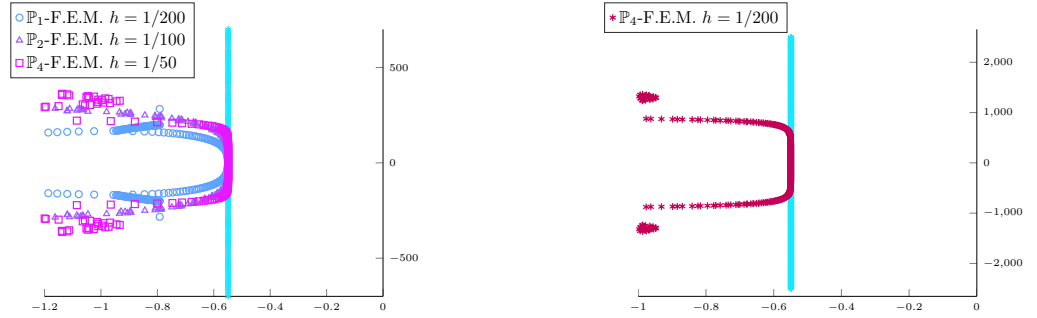


Figure 7. Spectrum of $A_h - J_h - D_h - \tilde{D}_{h,*}$

573 We now use our method. Note that we penalize the jump with the addition of the
 574 term J_h , as in Figure 5.

575 We preserve the consistency, and we have the uniform exponential stability with
 576 a constant which has the same order as what we had at the continuous level as we
 577 can see in Figure 7 (Left). When refining – see Figure 7 (Right)–, the consistency
 578 is improved. In the end, we found dissipative terms that exponentially stabilize the
 579 system while preserving consistency.

580 7.2 A numerical example with an explicit leap-frog scheme

581 One of the objectives being to have an effective method, we present the leap-frog
 582 scheme used in the numerical simulations that guarantees the energy conservation and
 583 dissipation properties for damped and undamped systems respectively. We consider the
 584 damped wave equation (42) discretized with an explicit scheme where the dissipation
 585 terms are off-centered.

$$\begin{cases} \frac{u_h^{n+1} - u_h^n}{\Delta t} - R_h^* v_h^{n+1/2} + \gamma D_h u_h^n = 0 \\ \frac{v_h^{n+3/2} - v_h^{n+1/2}}{\Delta t} - R_h u_h^{n+1} + \gamma B_h v_h^{n+1} \\ \qquad \qquad \qquad + \gamma J_h \frac{v_h^{n+3/2} + v_h^{n+1/2}}{2} + \gamma \tilde{D}_{h,*} v_h^{n+1/2} = 0. \end{cases} \quad (61)$$

586 The scheme is almost fully explicit thanks to the mass-lumping strategy and off-centering
 587 of the newly added dissipative terms D_h and $\tilde{D}_{h,*}$.

588 Note that:

- 589 • the centering of the jump term involves, at the algebraic level, to solve $(N - 1)$
 590 linear system of size 2×2 at each iteration, where N is the number of elements
- 591 • off-centering does not imply here loss of consistency in time since it is used
 592 on terms that are small when apply to interpolation of smooth solution of the
 593 problem.

To show the energy identity, we rewrite the scheme using the following trick

$$\begin{cases} D_h u_h^n = D_h u_h^{n+1/2} - \frac{\Delta t}{2} D_h \frac{u_h^{n+1} - u_h^n}{\Delta t}, \\ \tilde{D}_{h,*} v_h^{n+1/2} = \tilde{D}_{h,*} v_h^{n+1} - \frac{\Delta t}{2} \tilde{D}_{h,*} \frac{v_h^{n+3/2} - v_h^{n+1/2}}{\Delta t}, \end{cases}$$

594 thus, we obtain

$$\begin{cases} \tilde{M}_h \frac{u_h^{n+1} - u_h^n}{\Delta t} - R_h^* v_h^{n+1/2} + \gamma D_h u_h^{n+1/2} = 0 \\ \tilde{M}_{h,*} \frac{v_h^{n+3/2} - v_h^{n+1/2}}{\Delta t} + R_h u_h^{n+1} + \gamma B_h v_h^{n+1} + \gamma J_h v_h^{n+1} + \gamma \tilde{D}_{h,*} v_h^{n+1} = 0, \end{cases} \quad (62)$$

595 where

$$\tilde{M}_h = \mathbb{1} - \gamma \frac{\Delta t}{2} D_h \quad \text{and} \quad \tilde{M}_{h,*} = \mathbb{1} - \gamma \frac{\Delta t}{2} \tilde{D}_{h,*}.$$

596 The energy identity for System (61) involves a new energy functional that is given
597 below: for $u_h^n \in U_h^r$ and $v_h^{n+1/2} \in V_h^r$,

$$\mathcal{E}_{h,ex}^n = \frac{1}{2} \left(\tilde{M}_h u_h^n, u_h^n \right)_h + \frac{1}{2} \left(\tilde{M}_{h,*} v_h^{n+1/2}, v_h^{n+1/2} \right)_h - \frac{\Delta t}{2} \left(R_h u_h^n, v_h^{n+1/2} \right)_h. \quad (63)$$

Theorem 7.1. *Let $\{u_h^n, v_h^n\}_{n \geq 0}$ belonging to $U_h^r \times V_h^r$ be the solutions of (62). The following energy identity holds,*

$$\begin{aligned} \frac{\mathcal{E}_{h,ex}^{n+1} - \mathcal{E}_{h,ex}^n}{\Delta t} &= -\gamma \left(B_h v_h^{n+1}, v_h^{n+1} \right)_h - \gamma \left(J_h v_h^{n+1}, v_h^{n+1} \right)_h \\ &\quad - \gamma \left(\tilde{D}_{h,*} v_h^{n+1}, v_h^{n+1} \right)_h - \gamma \left(D_h u_h^{n+1/2}, u_h^{n+1/2} \right)_h. \end{aligned} \quad .+$$

598 Moreover if $\forall h, \Delta t$ is chosen small enough such that

$$1 - \gamma \frac{\Delta t}{2} \|D_h\|_h > 0 \quad \text{and} \quad 1 - \gamma \frac{\Delta t}{2} \|\tilde{D}_{h,*}\|_h > 0, \quad (64)$$

599 and

$$1 - \frac{\Delta t}{2} \|R_h\|_h > 0, \quad (65)$$

600 then $\mathcal{E}_{h,ex}^n \geq 0$ and the scheme is stable.

Proof. We only show here the energy identity. The positivity of the energy and related stability of the scheme are well-known results. We multiply the two equations of (62) by $u_h^{n+1/2}$ and v_h^{n+1} respectively and we obtain,

$$\begin{aligned} &\left(\tilde{M}_h \frac{u_h^{n+1} - u_h^n}{\Delta t}, \frac{u_h^{n+1} + u_h^n}{2} \right)_h - \left(R_h^* v_h^{n+1/2}, \frac{u_h^{n+1} + u_h^n}{2} \right)_h + \gamma \left(D_h u_h^{n+1/2}, u_h^{n+1/2} \right)_h \\ &+ \left(\tilde{M}_{h,*} \frac{v_h^{n+3/2} - v_h^{n+1/2}}{\Delta t}, \frac{v_h^{n+3/2} + v_h^{n+1/2}}{2} \right)_h + \left(R_h u_h^{n+1}, \frac{v_h^{n+3/2} + v_h^{n+1/2}}{2} \right)_h \\ &+ \gamma \left(B_h v_h^{n+1}, v_h^{n+1} \right)_h + \gamma \left(J_h v_h^{n+1}, v_h^{n+1} \right)_h + \gamma \left(\tilde{D}_{h,*} v_h^{n+1}, v_h^{n+1} \right)_h = 0. \end{aligned}$$

We use the follow identity

$$\begin{aligned}
 & - \left(R_h^* v_h^{n+1/2}, \frac{u_h^{n+1} + u_h^n}{2} \right)_h + \left(R_h u_h^{n+1}, \frac{v_h^{n+3/2} + v_h^{n+1/2}}{2} \right)_h \\
 & = -\frac{1}{2} \left(R_h^* v_h^{n+1/2}, u_h^n \right)_h + \frac{1}{2} \left(R_h^* v_h^{n+3/2}, u_h^{n+1} \right)_h,
 \end{aligned}$$

to obtain

$$\begin{aligned}
 & \frac{1}{2\Delta t} \left(\tilde{M}_h u_h^{n+1}, u_h^{n+1} \right)_h - \frac{1}{2\Delta t} \left(\tilde{M}_h u_h^n, u_h^n \right)_h + \frac{1}{2\Delta t} \left(\tilde{M}_{h,*} v_h^{n+3/2}, v_h^{n+3/2} \right)_h \\
 & - \frac{1}{2\Delta t} \left(\tilde{M}_{h,*} v_h^{n+1/2}, v_h^{n+1/2} \right)_h + \frac{\Delta t}{2} \left(R_h u_h^{n+1}, v_h^{n+3/2} \right)_h - \frac{\Delta t}{2} \left(R_h u_h^n, v_h^{n+1/2} \right)_h \\
 & = -\gamma \left(B_h v_h^{n+1}, v_h^{n+1} \right)_h + \gamma \left(J_h v_h^{n+1}, v_h^{n+1} \right)_h \\
 & \quad - \gamma \left(\tilde{D}_{h,*} v_h^{n+1}, v_h^{n+1} \right)_h - \gamma \left(D_h u_h^{n+1/2}, u_h^{n+1/2} \right)_h.
 \end{aligned}$$

601 We recognize the expression of the explicit energy (63). This concludes the proof. \square

602 **Remark 7.2.** Note that since D_h and $\tilde{D}_{h,*}$ are bounded, condition (64) is not a CFL
 603 condition. On the other hand, condition (65) corresponds to the classical CFL condition
 604 for explicit schemes.

605 As in Section 2, we now look at the evolution of the shape of the solution and of
 606 the energy over time, with the added damped terms. The numerical pollution due to
 607 parasitic waves has completely vanished. We can observe on Figure 8 a totally smooth
 608 wave that is not oscillating anymore.

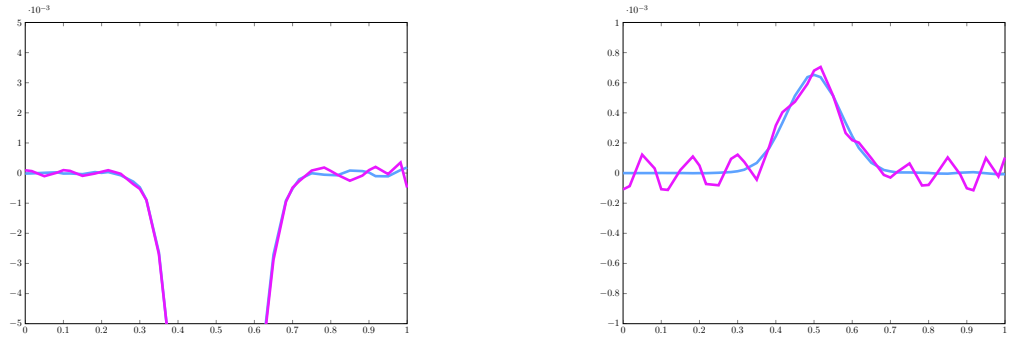


Figure 8. Solution u discretized with \mathbb{P}_4 F.E.M. with $h = 1/10$ and $dt = 1/1000$ at time
 $t = 2$ (Left) and at time $t = 4$ (Right)

609 While the system without the damping terms had an energy that became constant
 610 in long time (Figure 9 in pink), we notice that with the addition of these dissipative
 611 terms, we obtain a linearly decreasing energy (in logarithm scale) as can be seen on
 612 Figure 9 in blue. This confirms the result of Theorem 6.6, we succeeded in making the
 613 system uniformly exponentially stable.

614 8 Conclusion

615 By penalizing the jump and introducing an additive dissipative term consistent with
 616 high-order finite element schemes, we can adapt multiplier techniques at the fully

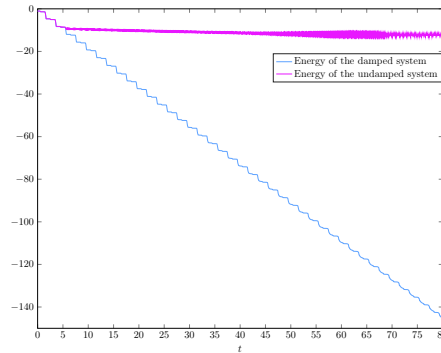


Figure 9. $\ln(\|u(t)\|^2 + \|v(t)\|^2)^{1/2}$

617 discrete level to obtain exponential stabilization results that are uniform with respect
 618 to the discretization. The main perspective is to extend the results for 2-dimensional
 619 dynamics. The main obstacle lies in the difficulties observed at the continuous level,
 620 namely that the variable v_h must be defined as a gradient on each element, which may
 621 not be the case for a weak formulation of the equation. Moreover, we need the discrete
 622 energy uniform distribution property for the first order formulation. Finally, it would
 623 be necessary to introduce a numerical treatment of the boundary conditions.

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