

# Uniform boundary stabilization of a high-order finite element space discretization of the 1-d wave equation

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- <sup>1</sup> Uniform boundary stabilization of a
   <sup>2</sup> high-order finite element space
- <sup>3</sup> discretization of the 1-d wave equation
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### 9 Abstract

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The objective of this work is to propose and analyze numerical schemes for solving 10 boundary control problems or data assimilation problems by observers for wave 11 propagation problems. The efficiency of the considered control or data assimilation 12 strategy relies on the exponentially stable character of the underlying system. 13 Therefore, the aim of our work is to propose a discretization process that allows 14 preserving the exponential stability at the discrete level when using high-order spectral 15 finite element approximation. The main idea is to add a stabilizing term to the wave 16 equation that dampens the spurious oscillatory components of the solutions. This term 17 is based on a discrete multiplier analysis that gives us the exponential stability of the 18 semi-discrete problem at any order without affecting the approximation properties. 19

## <sup>21</sup> 1 Introduction

The aim of this work is to demonstrate an asymptotic exponential stability of a onedimensional wave-like system with boundary damping uniformly with respect to the time and space discretization of the system. Such a system typically arises when analyzing boundary control problems [15], data assimilation problem using boundary data [36] or stabilization property of absorbing boundary condition imposed to initially unbounded formulations [25].

Obtaining exponential stability results – through observability condition in most 28 cases – is a rather difficult result at the continuous level, which has been extensively 29 studied by the control community [18] and proved by various techniques, typically 30 multiplier techniques [2, 7, 20] or even Carleman estimates [26], spectral approaches 31 [5, 22], or microlocal analyzes [4]. However, discretizing the dynamics so that the 32 stabilization properties are preserved at the discrete level presents an additional difficulty 33 because of the presence spurious high frequencies that remain uncontrolled if one relies 34 on the classical discretization of the stabilization terms ([9, 18, 34, 37]). Several 35 strategies – either for finite element or finite difference discretizations – are presented 36 in the literature to compensate for the instabilities caused by high frequencies, namely 37 Fourier techniques ([13, 14, 17]), Tikhonov regularization [3], multi-grid finite difference 38 methods [16] or mixed finite element methods ([19] and their extension for nonuniform 39 meshes [29, 30]). 40

For dealing with high-performance wave propagation applications, high-order spectral finite elements [12] are now considered indispensable – see for instance [6, 10,

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Funding: This work was supported by ANR Projet-ANR-19-CE48-0004 ODISSE 43 35] and for seismic wave applications [28, 31] – however, the above discretization
44 strategies cannot be easily extended to such high-order discretizations without losing
45 their advantages, in particular their convergence order and their efficiency.

Here we follow a strategy of including additional consistent stabilization terms in 46 the discretization to preserve the efficiency of the methods for possible extension to 47 high dimensions. The additional vanishing stabilization terms have a similar goal as 48 those found in [38]. However, here these additional terms are constructed in such a 49 way that the classical multiplier method, which proves stabilization in the continuous 50 method, can be continued at the discrete level, achieving uniform stabilization with 51 respect to the discretization parameters. Moreover, the stabilization terms are carefully 52 designed to preserve the consistency order. In particular, our result can extend the 53 discretization result for observers proposed in [33] to high-order schemes, which states 54 that using data, one can obtain discretization schemes that do not suffer from the 55 asymptotic time drift of the error. 56

Our proof is limited to the one-dimensional case, while an implicit mid-point timescheme is chosen for the time-discretization however the principles of our approach pave the way for possible extension to higher dimensions and more efficient time schemes, typically fully explicit approaches – that are nevertheless teted in this work..

This work is organized as follows. In Section 2, the motivations for this work 61 are stated. In Section 3 we lay out the continuous configuration and the method 62 for proving exponential stability. Section 4 proves the uniform exponential in h in 63 the semi-discrete configuration using a general abstract framework. In Section 5 we 64 present a semi-discrete convergence results for a data assimilation problem. Then, 65 Section 6 presents the fully discretized framework using an implicit scheme in time. 66 The numerical results are presented in Section 7, which is split into two parts. First, we 67 consider the eigenvalues of the operator belonging to our damped system to observe the 68 uniform exponential stability property. To obtain an efficient method, in a second step 69 we introduce an explicit scheme for discretization in time that preserves the properties 70 shown in the previous sections. Using this scheme, we can then present numerical 71

#### results that illustrate the interest of adding the damping terms.

### <sup>73</sup> 2 Motivation of the observation problem

74 Let us study a model problem which shines a light on the difficulties that appear for 75 the stabilization of discretized problems.

Let  $\Omega = (0, 1)$  be the domain of propagation. We study a homogenous wave equation formulated as a first-order system

$$\begin{cases} \partial_t u(x,t) + \partial_x v(x,t) = 0, \quad (x,t) \in \Omega \times (0,T), \\ \partial_t v(x,t) + \partial_x u(x,t) = 0, \quad (x,t) \in \Omega \times (0,T). \end{cases}$$
(1)

We define the boundary conditions and the initial conditions of our wave equation. The following boundary conditions corresponds to Neumann at point x = 0 and a Robin condition at point x = 1. We exclude the degenerate case  $\gamma = 1$  which corresponds to transparent conditions.

$$\begin{cases} u(1,t) = \gamma v(1,t) & t \in (0,T), \\ v(0,t) = 0, & t \in (0,T), \end{cases} \begin{cases} u(x,0) = u_0(x), & x \in \Omega, \\ v(x,0) = v_0(x), & x \in \Omega. \end{cases}$$
(2)

The study of such a system ((1) with (2)) allows to simulate the case of the study of a system with absorbing conditions (at boundary x = 1). It can also be similar to the

so study of an error system between a target system and a simulated system (which we

would like to converge asymptotically in time toward the target trajectory). Namely, if 81 we consider having access to measurements (here v at x = 1), and we can incorporate 82 by a feedback law in the boundary conditions the difference between the solution and 83 the measurements. To study the convergence of this system called observer system (see 84

[8, 32]) towards the target system, it is equivalent to show the exponential convergence 85

towards 0 of the error system that could represent ((1) with (2)). In all cases, the 86

convergence of these problems ((1) with (2)) is related to the exponential convergence 87

of a damped PDE (at x = 1 in this case). Furthermore, at the continuous level, one

can prove that this system is exponentially stable ([5] in 1D problem, [34]). 89

#### 2.1 Parasitic waves 90

The question is to prove that such system – that is known to be exponentially stable 91 preserves its exponential stability property after standard high-order spectral finite 92 element discretization on space and explicit second order energy preserving time 93 discretization [12]. To illustrate the phenomenon of propagation, we will represent the 94 wave over time. We simulate our problem with finite elements (F.E.M.)  $\mathbb{P}_r$  the set 95

of polynomials of degree r – and an explicit time scheme and on uniform mesh. We 96

initialize u as a Gaussian (see Figure 1 (Left)) and set v to 0 as initial condition. 97



**Figure 1.** Approximation of u obtained with  $\mathbb{P}_4$  finite elements with h = 1/10 at time t = 0(Left) and at t = 2 (Right)

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The boundary conditions of the wave propagation involve two effects. At the 99 boundary x = 0, the wave bounces while on the boundary x = 1, part of the wave 100 is absorbed, we should therefore have an exponential convergence to 0. We notice 101 the oscillations of the wave in the course of time, these are due to the presence of 102 high-frequency spurious waves (see Figure 2 with a change of scale).



**Figure 2.** Approximation of u obtained with  $\mathbb{P}_4$  finite elements with h = 1/10 and dt = 1/1000 at time t = 2 (Left) and at time t = 4 (Right)

In order to show that the discretization taints this property of uniform exponential stability, we represent the evolution of the energy of the system over time in log scale (Figure 3). We notice that the decay is no longer linear after a certain time. This gives an illustration of the loss of the uniform exponential stability property caused by the high-order spectral finite elements discretization.



Figure 3. Approximation of  $\ln (||u(t)||^2 + ||v(t)||^2)^{1/2}$  obtained with  $\mathbb{P}_4$  finite elements with h = 1/10 and dt = 1/1000

#### <sup>109</sup> 2.2 Investigation of eigenvalues

As a key indicator of the behavior of the discrete system, we investigate the eigenvalues of the generator of the corresponding semi-group. We formulate (1) as follows,

$$\dot{z} = Az$$
, with  $z = \begin{pmatrix} u \\ v \end{pmatrix}$ , and  $A = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix}$ , (3)

where  $A: L^2(\Omega) \times L^2(\Omega) \to L^2(\Omega) \times L^2(\Omega)$  is an unbounded operator with domain  $D(A) \subset H^1(\Omega) \times H^1(\Omega)$ . The discrete approximation of the above systems is

$$\dot{z}_h = A_h z_h,$$

where  $A_h \in \mathcal{L}(V_h)$  is an approximation of the operator A in a finite-dimensional space  $V_h$  that is a subspace of  $H^1(\Omega) \times H^1(\Omega)$  obtained using high-order spectral finite elements ([12]). Since we look for exponential stability we aim at constructing approximations for which the eigenvalues of  $A_h$  lies in the complex plane not close to the imaginary axis.



**Figure 4.** Spectrum of  $A_h$ 

We see in Figure 4 (Left) that the eigenvalues are close to the imaginary axis, which implies that the exponential stability property is in the best case scenario, not uniform in h, or even lost. By choosing h smaller, the consistency is improved, namely the

smallest eigenvalues – in amplitude – are more and more well approximated, still the 122 uniform exponential stability property is not recovered, see Figure 4 (Right). In fact, 123 it has been shown – at least for  $\mathbb{P}_1$  finite elements, (see [9, 34]) – that the uniform 124 exponential stability property cannot be obtained at the discrete level without adding 125 additional dissipative terms. As mentioned the main contribution of this work is to 126 construct such terms that stabilized the system – in the sense where the uniform 127 exponential stability property is guaranteed – and preserves high-order accuracy that 128 is obtained with  $\mathbb{P}_r$  finite elements. 129

#### <sup>130</sup> 3 Continuous setting

This section presents standard multiplier techniques [7] to show the exponential stability property at the continuous level of System (1). We have chosen to present these classical results, first for the sake of completeness, second because they will be reproduced at the discrete level. To complete this last step the sought additional dissipative terms will be needed.

#### <sup>136</sup> 3.1 Observability of the undamped system

First, we define the variational formulation of the undamped system ( $\gamma = 0$ ),

$$\begin{cases} \int_0^1 \partial_t u \tilde{u} \, \mathrm{d}x - \int_0^1 v \partial_x \tilde{u} \, \mathrm{d}x + v(1) \tilde{u}(1) = 0, \\ \int_0^1 \partial_t v \tilde{v} \, \mathrm{d}x + \int_0^1 \partial_x u \tilde{v} \, \mathrm{d}x - u(1) \tilde{v}(1) = 0, \end{cases}$$
(4)

where  $u(t), \tilde{u} \in H^1(0, 1)$  and  $v(t), \tilde{v} \in L^2(0, 1)$ . For smooth initial data, it is standard to show that (4) has a unique solution,

$$(u,v) \in C^1([0,T]; L^2(0,1) \times L^2(0,1)) \cap C^0([0,T]; H^1(0,1) \times H^1(0,1)).$$

140 System (4) is conservative, indeed, one can easily show, the following theorem.

**Theorem 3.1.** Let (u, v) solutions of (4), we have the energy conservation

$$\frac{\mathrm{d}}{\mathrm{d}t} \left( \|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right) = 0.$$
(5)

The proof consists in choosing  $\tilde{u} = u(t)$  and  $\tilde{v} = v(t)$  in (4) and summing the two obtained equations. Note that, as an obvious consequence of the energy preservation property we have

$$\int_0^T \|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \, \mathrm{d}t = T\left(\|u(0)\|_{L^2(0,1)}^2 + \|v(0)\|_{L^2(0,1)}^2\right). \tag{6}$$

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A first step towards our goal is to derive an observability inequality (the precise
meaning is given in the theorem below). This step is not a necessity to prove exponential
stability but should rather be seen as a useful intermediate step. The strategy to prove
the observability inequality relies on multipliers [7].

**Theorem 3.2.** Let (u, v) be solutions of (4). For T large enough, there exists  $C^{st}(T)$ strictly positive such that

$$\frac{1}{2} \left( \|u(0)\|_{L^2(0,1)}^2 + \|v(0)\|_{L^2(0,1)}^2 \right) \le C^{st}(T) \int_0^T v^2(1) \,\mathrm{d}t.$$
(7)

- *Proof.* The proof given below is separated in several steps that will all be mimicked at the discrete level.
- 155 Step 1: We choose  $\tilde{u} = -xv(t)$  and  $\tilde{v} = -xu(t)$  in (4) and sum the two obtained
- equations, integrating by parts and neglecting negative terms we get, we obtain

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 xuv\,\mathrm{d}x \le -\frac{1}{2} \left( \|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right) + \frac{1}{2}v^2(1).$$

Step 2: Integrating in time and using the energy conservation property of 3.1 and its
direct consequence (6) we get

$$-\left[\int_{0}^{1} xuv \,\mathrm{d}x\right]_{0}^{T} \leq -\frac{T}{2} \left(\|u(0)\|_{L^{2}(0,1)}^{2} + \|v(0)\|_{L^{2}(0,1)}^{2}\right) + \frac{1}{2} \int_{0}^{T} v^{2}(1) \,\mathrm{d}t.$$
(8)

159 Step 3: We use Cauchy-Schwarz, Young inequalities and Theorem 3.1 to show

$$\left| \int_0^1 x u(t) v(t) \, \mathrm{d}x \right| \le \frac{1}{2} \|v(t)\|_{L^2(0,1)}^2 + \frac{1}{2} \|v(t)\|_{L^2(0,1)}^2 = \frac{1}{2} \|v(0)\|_{L^2(0,1)}^2 + \frac{1}{2} \|v(0)\|_{L^2(0,1)}^2.$$

<sup>160</sup> Finally, using this inequality together with (8) we obtain

$$(T-2)\left(\|u(0)\|_{L^{2}(0,1)}^{2}+\|v(0)\|_{L^{2}(0,1)}^{2}\right) \leq \int_{0}^{T} v^{2}(1) \,\mathrm{d}t.$$

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Inequality (7) does not directly guarantee that the boundary damped system is exponentially stable. It is however a good indicator since the damping term corresponds to the right-hand side of (7). This will be used as a clue to define the required damping terms at the discrete level.

# <sup>166</sup> 3.2 Exponential stability for the damped system using Lyapunov <sup>167</sup> functional

We define the variational formulation associated with System (1) with (2)

$$\begin{cases} \int_0^1 \partial_t u \tilde{u} \, \mathrm{d}x - \int_0^1 v \partial_x \tilde{u} \, \mathrm{d}x + v(1) \tilde{u}(1) = 0, \\ \int_0^1 \partial_t v \tilde{v} \, \mathrm{d}x + \int_0^1 \partial_x u \tilde{v} \, \mathrm{d}x - \tilde{v}(1) u(1) + \gamma v(1) \tilde{v}(1) = 0, \end{cases}$$
(9)

We have existence and uniqueness of solutions for the above variational formulation, we will now establish an energy relation for this system.

**Theorem 3.3.** Let (u, v) solutions of (9), we have the energy dissipation property

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left( \|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right) = -\gamma v^2(1).$$
(10)

172 Proof. We choose  $\tilde{u} = u(t)$  and  $\tilde{v} = v(t)$  in (9). Then, adding the two equations in (9), 173 we conclude the proof.

**Theorem 3.4.** Let (u, v) solutions of (9). There exist two constants  $C_s > 0$  and  $\sigma > 0$ such that

$$\|u(t)\|_{L^{2}(0,1)}^{2} + \|v(t)\|_{L^{2}(0,1)}^{2} \le C_{s}e^{-\sigma t} \left(\|u(0)\|_{L^{2}(0,1)}^{2} + \|v(0)\|_{L^{2}(0,1)}^{2}\right).$$
(11)

- 176 *Proof.* The proof given below is separated in several steps that will all be mimicked at 177 the discrete level.
- **Step 1:** We choose  $\tilde{u} = -xv(t)$  and  $\tilde{v} = -xu(t)$  in (9).
- 179 Summing the two equations and integrating by parts, we get

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 xuv\,\mathrm{d}x = \frac{1}{2}v^2(1) - \frac{1}{2}u^2(1) + \gamma v(1)u(1) - \frac{1}{2}\left(\|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2\right).$$

Moreover, thanks to Young's inequality we have  $\gamma u(1)v(1) \leq \frac{\gamma^2}{2}v^2(1) + \frac{1}{2}u^2(1)$ , finally we obtain

$$-\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 x u v \,\mathrm{d}x \le \frac{1+\gamma^2}{2} v^2(1) - \frac{1}{2} \left( \|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right)$$

**Step 2:** We add the energy relation (3.3) multiplied by  $\alpha > 0$ .

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{\alpha}{2} \left( \|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right) - \int_0^1 x u v \,\mathrm{d}x \right) \\ &\leq \left( \frac{\gamma^2 + 1}{2} - \alpha \gamma \right) v^2(1) - \frac{1}{2} \left( \|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right). \end{aligned}$$

182 Step 3: Choosing  $\alpha$  large enough such that

$$\frac{\gamma^2 + 1}{2} - \alpha \gamma < 0,$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{\alpha}{2} \left( \|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right) - \int_0^1 x u v \,\mathrm{d}x \right) \\
\leq -\frac{1}{2} \left( \|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right). \quad (12)$$

183 Step 4: We introduce the Lyapunov functional

$$\mathcal{L}(t) = \frac{\alpha}{2} \left( \|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2 \right) - \int_0^1 x u v \, \mathrm{d}x.$$
(13)

Using Cauchy-Schwarz and Young inequalities, we prove the equivalence between theenergy functional

$$\mathcal{E}(t) := \frac{1}{2} \|u(t)\|_{L^2(0,1)}^2 + \|v(t)\|_{L^2(0,1)}^2$$

and the Lyapunov functional with  $\alpha - 1 > 0$ .

$$\frac{(\alpha - 1)}{2} \left( \|u(t)\|_{L^{2}(0,1)}^{2} + \|v(t)\|_{L^{2}(0,1)}^{2} \right) \leq \mathcal{L}(t) \\
\leq \frac{\alpha + 1}{2} \left( \|u(t)\|_{L^{2}(0,1)}^{2} + \|v(t)\|_{L^{2}(0,1)}^{2} \right). \quad (14)$$

Using this equivalence in (12), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) \le -\frac{1}{\alpha+1}\mathcal{L}(t).$$

187 Step 5: Integrating in time and using again the equivalence between the energy and
188 the Lyapunov functional we finally have

$$\left(\|u(t)\|_{L^{2}(0,1)}^{2}+\|v(t)\|_{L^{2}(0,1)}^{2}\right) \leq \frac{\alpha+1}{\alpha-1}e^{-\frac{1}{\alpha+1}t}\left(\|u(0)\|_{L^{2}(0,1)}^{2}+\|v(0)\|_{L^{2}(0,1)}^{2}\right).$$

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#### <sup>190</sup> 4 Semi-discrete setting

As mentioned we give a counterpart of Theorem 3.2 and Theorem 3.4 at the semidiscrete level. This is achieved in an abstract framework. We then show – and this is the most important part of the work – that the semi-discrete wave problem falls in the described framework.

#### <sup>195</sup> 4.1 Abstract framework

Let  $\mathcal{Z}_h$  be a family of finite-dimensional parametrized by h space associated with the scalar product  $(\cdot, \cdot)_h$ . We consider the semi-discretized abstract first-order system for  $z_h \in \mathcal{Z}_h$ ,

$$\begin{cases} \dot{z}_h = A_h z_h, & \text{in } [0, T], \\ z_h(0) = z_{h,0}, \end{cases}$$
(15)

where  $A_h \in \mathcal{L}(\mathcal{Z}_h)$  and  $||A_h||_h \to +\infty$ . We assume that the operator  $A_h$  satisfies the holowing property

$$A_h z_h, z_h)_h = 0. (16)$$

Furthermore, we have classically the existence and unicity of smooth solutions  $(u_h, v_h)$ belonging to  $C^k([0, T]; \mathcal{Z}_h)$ , for all  $k \ge 0$ . We start by presenting the discrete property of energy preservation.

Theorem 4.1. Let  $z_h \in \mathcal{Z}_h$  the solution of (15), we have the discrete conservation energy property

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z\|_h^2 = 0.$$

Proof. We multiply (15) by  $z_h$ , and since  $A_h$  satisfies (16), the result follows immediately.

Let  $\mathcal{Y}_h$  be a finite-dimensional space. We have seen that in the continuous problem, we have added a dissipative term that stabilizes the problem. By analogy, we therefore introduce a perturbation by an operator  $C_h^*C_h$ , where  $C_h \in \mathcal{L}(\mathcal{Z}_h, \mathcal{Y}_h)$ ,

$$\begin{cases} \dot{z}_h = (A_h - \gamma C_h^* C_h) \, z_h, & \text{in } [0, T], \\ z_h(0) = z_{h,0}. \end{cases}$$
(17)

<sup>211</sup> For this system, we have the discrete dissipation energy property.

**Theorem 4.2.** Let  $z_h \in \mathcal{Z}_h$  the solution of (17), we have the discrete dissipation energy property

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|z_h\|_h^2 = -\gamma \|C_h z_h\|_h^2.$$

<sup>214</sup> *Proof.* Proof similar of the proof of Theorem 4.1.

In order to demonstrate the observability property, we need a fundamental hypothesis.

**Assumption 1.**  $\exists \alpha, \beta, C_{\Pi} > 0$ , independent of h and for all h,  $\exists \Pi_h$  symmetric belonging to  $\mathcal{L}(\mathcal{Z}_h)$  and  $\|\Pi_h\|_h \leq C_{\Pi}$  with  $C_{\Pi} > 0$ , such that,  $\forall z_h \in \mathcal{Z}_h$ 

$$(\Pi_h A_h z_h, z_h)_h \le \alpha \|C_h z_h\|_h^2 - \beta \|z_h\|_h^2.$$

From this assumption, we can establish the property that interest us: the observability property.

**Theorem 4.3.** Let  $z_h$  be the solution of (15). Assuming Assumption 1, for T large enough, there exists a constant  $C^{st}(T) > 0$  independent of h such that

$$||z_{h,0}||_h^2 \le C^{st}(T) \int_0^T ||C_h z_h||_h^2 \,\mathrm{d}t.$$

Proof. The proof is separated into the same steps as for the proof of observability at the continuous level of Theorem 3.4.

**Step 1:** We multiply the first equation of (15) by  $\Pi_h z_h$ .

$$(\Pi_h \dot{z}_h, z_h)_h = (\Pi_h A_h z_h, z_h)_h.$$

<sup>226</sup> Using Assumption 1, we obtain

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} (\Pi_h z_h, z_h)_h \le \alpha \|C_h z_h\|_h^2 - \beta \|z_h\|_h^2.$$

227 Step 2: We integrate in time and use the energy conservation property

$$\frac{1}{2} \left( \Pi_h z_h(T), z_h(T) \right)_h - \frac{1}{2} \left( \Pi_h z_{h,0}, z_{0,h} \right)_h \le \alpha \int_0^T \|C_h z_h\|_h^2 \, \mathrm{d}t - \beta \int_0^T \|z_h(t)\|_h^2 \, \mathrm{d}t.$$

<sup>228</sup> The use of the energy conservation property of Theorem 4.1 leads to

$$\frac{1}{2} \left( \Pi_h z_h(T), z_h(T) \right)_h - \frac{1}{2} \left( \Pi_h z_{h,0}, z_{0,h} \right)_h \le \alpha \int_0^T \| C_h z_h \|_h^2 \, \mathrm{d}t - \beta T \| z_h(0) \|_h^2 \, \mathrm{d}t.$$

**Step 3:** Since  $\Pi_h$  is bounded, we have

$$-\frac{1}{2}\|\Pi_h\|_h\|z_h(T)\|_h^2 - \frac{1}{2}\|\Pi_h\|\|z_{h,0}\|_h^2 \le \alpha \int_0^T \|C_h z_h\|_h^2 \,\mathrm{d}t - \beta T\|z_h(0)\|_h^2 \,\mathrm{d}t.$$

<sup>230</sup> Using the energy conservation Theorem 4.1, we get

$$(\beta T - \|\Pi_h\|_h) \|z_{0,h}\|_h^2 \le \alpha \int_0^T \|C_h z_h\|_h^2 \,\mathrm{d}t.$$

For  $T > ||\Pi_h||_h \beta^{-1}$ , we obtain the result.

As for showing the observability property, we need a fundamental hypothesis to establish the exponential stability property.

Assumption 2.  $\exists \chi, \kappa > 0$ , independent of h and for all h,  $\exists \Pi_h \in \mathcal{L}(\mathcal{Z}_h)$  symmetric such that,  $\forall z_h \in \mathcal{Z}_h$ 

$$(\Pi_h A_h z_h, z_h)_h - \gamma (C_h z_h, C_h \Pi_h z_h)_h \le \chi \|C_h z_h\|_h^2 - \kappa \|z_h\|_h^2$$

We can now proceed to the main theorem of this work : the uniform exponential stability property for the damped system.

**Theorem 4.4.** Let  $z_h$  be the solution of (17). Under Assumption 2, there exist two constants  $C_s, \sigma > 0$  independent of h such that

$$\frac{1}{2} \|z_h(t)\|_h^2 \le C_s \|z_{h,0}\|_h^2 e^{-\sigma t}.$$

*Proof.* The proof is separated into the same steps as for the proof of exponential
stability at the continuous level of Theorem 3.4.

**Step 1:** We multiply the first equation of (17) and  $\Pi_h z_h$  (in the sens of the scalar product).

$$(\Pi_h \dot{z}_h, z_h)_h = (\Pi_h A_h z_h, z_h)_h - \gamma (C_h z_h, C_h \Pi_h z_h)_h.$$

<sup>244</sup> We use Assumption 2 and we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\Pi_h z_h, z_h)_h \le \chi \|C_h z_h\|_h^2 - \kappa \|z_h\|_h^2.$$

245 Step 2: Adding the dissipation energy identity of Theorem 4.2 times  $\alpha > 0$ , we get

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left( (\Pi_h z_h, z_h)_h + \alpha \|z_h\|_h^2 \right) \le (\chi - \alpha \gamma) \|C_h z_h\|_h^2 - \kappa \|z_h\|_h^2.$$

**Step 3:** We choose  $\alpha$  large enough such that  $\chi - \alpha \gamma \leq 0$ . Thus, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left((\Pi_h z_h, z_h)_h + \alpha \|z_h\|_h^2\right) \le -\kappa \|z_h\|_h^2.$$

247 Step 4: We introduce the following discrete Lyapunov functional

$$\mathcal{L}_h(t) = (\Pi_h z_h, z_h)_h + \alpha \|z_h\|_h^2.$$

Since  $\Pi_h$  is bounded, we can establish an equivalence between the Lyapunov functionaland the energy functional  $\mathcal{E}_h(t) := \frac{1}{2} ||z_h(t)||_h^2$ .

$$(-\|\Pi_h\|_h + \alpha) \|z\|_h^2 \le \mathcal{L}_h(t) \le (\|\Pi_h\|_h + \alpha) \|z_h\|_h^2.$$
(18)

Using this equivalence with  $\alpha > \|\Pi_h\|_h$ , we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_h(t) \leq -\kappa \left(\|\Pi_h\|_h + \alpha\right)^{-1}\mathcal{L}_h(t).$$

<sup>251</sup> Step 5: Integrating in time and using again (18), we finally get

$$\frac{1}{2} \|z_h(t)\|_h^2 \le \frac{\alpha + \|\Pi_h\|_h}{\alpha - \|\Pi_h\|_h} \|z_{h,0}\|_h^2 e^{-\kappa (\|\Pi_h\|_h + \alpha)^{-1}t}.$$

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#### <sup>253</sup> 4.2 Discretization with high-order spectral finite elements

In this section, we show how the semi-discretization in space using high-order spectral finite elements of the wave propagation problem (1) enters the abstract framework presented above. We introduce two discrete spaces  $U_h^r$  and  $V_h^r$  – where r is the order of the finite elements used – defined by

$$U_h^r = \{ u_h \in C^0[0,1] \mid u_{h|[x_i,x_{i+1}]} \in \mathbb{P}_r \},\$$

258 and

$$V_h^r = \{ v_h \in L^2(0,1) \mid v_h|_{[x_i, x_{i+1}]} \in \mathbb{P}_r \}.$$

Note that  $U_h^r \subset V_h^r$  and we denote  $\mathcal{Z}_h = U_h^r \times V_h^r$ . We consider a quasi-uniform subdivision  $[x_i, x_{i+1}]$ , for  $i \in \{0, N\}$  of the domain [0, 1], namely we denote  $h_i = |x_{i+1} - x_i|$  for  $i \in \{0, N-1\}$ , and

$$h = \sup_{i \in \{0, N-1\}} h_i,$$

<sup>262</sup> and there exists  $\rho > 0$  such that

$$\min_{i \in \{0, N-1\}} h_i \le \rho h.$$

- <sup>263</sup> The efficiency of high-order spectral finite elements stems from the fact that mass-
- <sup>264</sup> lumping can be achieved without loss of consistency using a quadrature rule based on
- Gauss-Lobatto points. We represent the use of this quadrature by the symbol

$$\oint_0^1 f(x)dx \sim \int_0^1 f(x)dx \text{ with } \oint_0^1 f(x)dx := \sum_{i=0}^N \sum_{k=0}^r h_i w_k f(x_i + h_i \xi_k),$$

- where and  $w_k$  are the quadrature coefficients and  $\xi_k \in [0, 1]$  the Gauss-Lobatto interpolation nodes.
- <sup>268</sup> Note that we have the following bounds (see [21]),

$$\exists C_q, c_q > 0, \text{ such that } \forall u \in \mathbb{P}_r, \quad c_q \int_0^1 u^2 \, \mathrm{d}x \le \oint_0^1 u^2 \, \mathrm{d}x \le C_q \int_0^1 u^2 \, \mathrm{d}x.$$
(19)

The semi-discrete formulation of the damped system (9) reads: Find  $(u_h, v_h)(t) \in \mathcal{Z}_h$ such that for  $(\tilde{u}_h, \tilde{v}_h) \in \mathcal{Z}_h$ ,

$$\begin{cases} \int_{0}^{1} \partial_{t} u_{h} \tilde{u}_{h} \, \mathrm{d}x - \int_{0}^{1} v_{h} \partial_{x} \tilde{u}_{h} \, \mathrm{d}x + v_{h}(1) \tilde{u}_{h}(1) = 0, \\ \int_{0}^{1} \partial_{t} v_{h} \tilde{v}_{h} \, \mathrm{d}x + \int_{0}^{1} \partial_{x} u_{h} \tilde{v}_{h} \, \mathrm{d}x - u_{h}(1) \tilde{v}_{h}(1) + \gamma v_{h}(1) \tilde{v}_{h}(1) = 0, \end{cases}$$
(20)

271 with initial data

$$u_h(0) = u_h^0, \quad v_h(0) = v_h^0.$$

Note that when  $\gamma = 0$  one recovers the semi-discrete variational formulation of the undamped system (4). In order to relate to the abstract theory presented in Section 4.1, we reformulate the variational formulation (20) as an operator equation. To do so, we introduce the scalar product in  $V_h^r$  by

$$\forall (v_h, \tilde{v}_h) \in V_h^r \times V_h^r, \quad (v_h, \tilde{v}_h)_h := \oint_0^1 v_h \tilde{v}_h \, \mathrm{d}x, \; . \tag{21}$$

The scalar product in  $U_h^r \subset V_h^r$  is defined similarly, finally we extend this notation for  $z_{h} = (u_h, v_h)^t$ ,  $\tilde{z}_h = (\tilde{u}_h, \tilde{v}_h)^t \in U_h^r \times V_h^r$ ,

$$(z_h, \tilde{z}_h)_h = \oint_0^1 u_h \tilde{u}_h \, \mathrm{d}x + \oint_0^1 v_h \tilde{v}_h \, \mathrm{d}x$$

<sup>278</sup> The variational formulation (20) is equivalent to

$$\begin{cases} \partial_t u_h - R_h^* v_h = 0, \\ \partial_t v_h + R_h u_h + \gamma B_h v_h = 0, \end{cases} \quad (u_h, v_h)(0) = (u_h^0, v_h^0) := z_{h,0}, \quad (22)$$

where, for all  $(u_h, v_h, \tilde{v}_h)^t \in U_h^r \times V_h^r \times V_h^r$ ,

$$(R_h u_h, \tilde{v}_h)_h = \int_0^1 \partial_x u_h \tilde{v}_h \, \mathrm{d}x - u_h(1)\tilde{v}_h(1) \quad \text{and} \quad (B_h v_h, \tilde{v}_h)_h = v_h(1)\tilde{v}_h(1).$$
(23)

280 Denoting  $z_h = (u_h, v_h)^t$ , we can rewrite System (22) as

$$\begin{cases} \dot{z}_h = A_h z_h - \gamma \begin{pmatrix} 0 & 0 \\ 0 & B_h \end{pmatrix} z_h, & \text{where } A_h = \begin{pmatrix} 0 & -R_h^* \\ R_h & 0 \end{pmatrix}. \tag{24}$$

Note that the operator  $A_h \in \mathcal{L}(\mathcal{Z}_h)$  satisfies, for all  $z_h \in \mathcal{Z}_h$ ,  $(A_h z_h, z_h)_h = 0$ , i.e., it is skew-adjoint.

#### 283 4.3 Discrete observability

- Our objective is in a preliminary step to prove an observability property for the system (24). To do so, Theorem 4.3, is used and therefore Assumption 1 is satisfied in what follows by introducing an appropriate operator  $\Pi_h \in \mathcal{L}(\mathcal{Z}_h)$ .
- <sup>287</sup> The strategy to define  $\Pi_h$  is based upon our analysis presented at the continuous level.
- <sup>288</sup> More precisely one want to have (see the proof of Theorem 3.2 in Section 3)

$$\Pi_h z_h \sim \begin{pmatrix} -xv_h \\ -xu_h \end{pmatrix}$$

However, this later equation can not be an equality since  $xv_h \notin U_h^r$  and  $xu_h \notin V_h^r$ . A simple remedy is to introduce the following interpolation operator:  $\chi_h(\cdot): U_h^r \to V_h^r$  by

$$\oint_0^1 \chi_h(u_h) v_h \, \mathrm{d}x = \oint_0^1 x u_h v_h \, \mathrm{d}x, \ \forall u_h \in U_h^r, v_h \in V_h^r,$$
(25)

and its adjoint  $\chi_h^*(\cdot): V_h^r \to U_h^r$  is given by

$$\oint_0^1 \chi_h^*(v_h) u_h \, \mathrm{d}x = \oint_0^1 x v_h u_h \, \mathrm{d}x, \ \forall u_h \in U_h^r, v_h \in V_h^r.$$
(26)

<sup>292</sup> We define, for  $z_h = (u_h, v_h)^t$  and  $\tilde{z}_h = (\tilde{u}_h, \tilde{v}_h)^t$  the operator  $\Pi_h$  by

$$\Pi_h z_h = \begin{pmatrix} -\chi_h^*(v_h) \\ -\chi_h(u_h) \end{pmatrix}.$$
(27)

Note that it is easy to show that  $\Pi_h$  is self-adjoint and bounded using (19) uniformly with respect to h. More precisely, there exists  $C_{\Pi}$  independent of h such that  $\forall z_h \in \mathcal{Z}_h$ 

$$\|\Pi_h z_h\|_h \le C_{\Pi} \|z_h\|_h.$$
(28)

We are interested in the discrepancy between the multiplier that we would like to use (but cannot) and the one we are going to use, namely in the following discrepancy

$$\Pi_h z_h - \begin{pmatrix} -xu_h \\ -xv_h \end{pmatrix}.$$

It turns out that the operator  $C_h$  – which appears in the fundamental assumption 1 –, involves the local semi-norm  $H_1$  of this discrepancy. That is why, for all  $u_h, \tilde{u}_h \in U_h^r$ and  $\forall v_h, \tilde{v}_h \in V_h^r$ , we define two bilinear forms  $d_h$  and  $d_{h,*}$  as

$$d_h(u_h, \tilde{u}_h) := \sum_{i=0}^N \int_{x_i}^{x_{i+1}} \partial_x \left( x u_h - \chi_h(u_h) \right) \partial_x \left( x \tilde{u}_h - \chi_h(\tilde{u}_h) \right) \, \mathrm{d}x, \tag{29}$$

300 and

$$d_{h,*}(v_h, \tilde{v}_h) := \sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} \partial_x \left( xv_h - \chi_h^*(v_h) \right) \partial_x \left( x\tilde{v}_h - \chi_h^*(\tilde{v}_h) \right) \, \mathrm{d}x.$$
(30)

**Proposition 4.5.** Let  $d_h$  and  $d_{h,*}$  be the positive and symmetric bilinear forms defined by (29) and (30). Then, there exists a constant  $C_d > 0$  depend only on r such that  $\forall (u_h, v_h) \in U_h^r \times V_h^r$ 

$$d_h(u_h, u_h) + d_{h,*}(v_h, v_h) \le C_d \int_0^1 u_h^2 + v_h^2 \, \mathrm{d}x.$$

Proof. Let us show first, that  $d_h$  is bounded. We use an interpolation property [24, Chapter 4, Section 4, Theorem 4.4.4]. Since  $xu_h$  belongs to  $P_{r+1}(K)$  with  $K = [x_i, x_{i+1}]$ for some  $i \in \{0, N-1\}$ , we have

$$|xu_h - \chi_h(u_h)|_{H^1(K_i)} \le h_i^r C_r |xu_h|_{H^{r+1}(K_i)},$$

where  $C_r > 0$  depends only on r. However, as  $x \in P_1(K)$  and  $u_h \in P_r(K)$ , the general Leibniz rule leads to

$$(xu_h)^{r+1} = \sum_{j=0}^{r+1} \binom{r+1}{j} x^{(j)} u_h^{(r+1-j)} = (r+1)(u_h)^{(r)}.$$

<sup>309</sup> Injecting this result in the equation above, we get

$$|xu_h - \chi_h(u_h)|_{H^1(K_i)} \le h_i^r C_r(r+1) |u_h|_{H^r(K_i)}.$$

Thus, using an inverse estimate [24, Chapter 4, Section 5, Lemma 4.5.3], we have with  $C_r > 0$ , another scalar depending only on r,

$$|xu_h - \chi_h(u_h)|_{H^1(K_i)} \le C_r ||u_h||_{L^2(K_i)}.$$

312 Hence, finally we have

$$d_h(u_h, u_h) \le C_r^2 \|u_h\|_{L^2(0,1)}^2.$$

313 Since we can reproduce this inequality for  $d_{h,*}$ , the proof is ended.

We define the following operators  $D_h \in \mathcal{L}(U_h^r, U_h^r)$  and  $D_{h,*} \in \mathcal{L}(V_h^r, V_h^r)$  associated with the bilinear forms  $d_h$  and  $d_{h,*}$ ,

$$\begin{cases} (D_h u_h, \tilde{u}_h)_h = d_h(u_h, \tilde{u}_h), & \forall u_h, \tilde{u}_h \in U_h^r, \\ (D_{h,*} v_h, \tilde{v}_h)_h = d_{h,*}(v_h, \tilde{v}_h), & \forall v_h, \tilde{v}_h \in V_h^r. \end{cases}$$
(31)

Unfortunately, the definition of the operator  $C_h$  – appearing in Assumption 1 – requires, in this case, to define the jump of the variable  $v_h$ . For this purpose, we introduce the notation, for all  $i \in \{1, N-1\}$ ,

$$\llbracket v_i \rrbracket := v_i^+ - v_i^-, \text{ where } v_i^+ = v_h(x_i^+) \text{ and } v_i^- = v_h(x_i^-).$$
 (32)

319 Then, we define the jump operator

$$(J_h v_h, \tilde{v}_h)_h = \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} [\![v_i]\!] [\![\tilde{v}_i]\!], \text{ where } \tilde{h}_i = \min(h_{i-1}, h_i).$$
(33)

In the theorem below, we check that the multiplier  $\Pi_h$  satisfies the Assumption 1. We can then apply Theorem 4.3 and prove the observability inequality. Note that since the operators  $J_h, B_h, D_h$  and  $D_{h,*}$  are self-adjoint and positive, it is possible to consider their roots. We define the operator  $C_h \in \mathcal{L}(\mathcal{Z}_h, U_h^r \times [V_h^r]^3)$  – equipped with the scalar product  $(\cdot, \cdot)_h$  inherited from  $V_h^r$  – by

$$C_{h} = \begin{pmatrix} D_{h}^{1/2} & 0\\ 0 & J_{h}^{1/2}\\ 0 & B_{h}^{1/2}\\ 0 & D_{h,*}^{1/2} \end{pmatrix}.$$
(34)

- **Theorem 4.6.** Let  $z_h = (u_h, v_h) \in \mathcal{Z}_h = U_h^r \times V_h^r$  be the solutions of (24) with  $\gamma = 0$ .
- 326 Then,  $\Pi_h$  defined by (27) satisfies Assumption 1 namely

$$(\Pi_h A_h z_h, z_h)_h \le \alpha \|C_h z_h\|_h^2 - \beta \|z_h\|_h^2, \ \alpha, \beta > 0 \text{ independent of } h,$$

327 with the operator  $C_h$  is given by (34).

*Proof.* By definition, we have

$$(\Pi_h A_h z_h, z_h)_h = (A_h z_h, \Pi_h z_h)_h$$
  
=  $-\int_0^1 v_h \partial_x \chi_h^*(v_h) \, \mathrm{d}x + v_h^2(1) + \int_0^1 \partial_x u_h \chi_h^*(u_h) \, \mathrm{d}x - u_h^2(1).$ 

Inserting  $xv_h - xv_h = 0$  in the first term of the right-hand side, we get

$$(A_h z_h, \Pi_h z_h)_h = -\sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x \left( x v_h - \chi_h^*(v_h) \right) \, \mathrm{d}x - \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x (x v_h) \, \mathrm{d}x + v_h^2(1) + \int_0^1 \partial_x u_h \chi_h(u_h) \, \mathrm{d}x - u_h^2(1).$$
(35)

328 Let us focus on the second term in the right-hand side.

$$-\sum_{i=0}^{N} \int_{x_{i}}^{x_{i+1}} v_{h} \partial_{x}(xv_{h}) \, \mathrm{d}x = -\sum_{i=0}^{N} \int_{x_{i}}^{x_{i+1}} v_{h}^{2} \, \mathrm{d}x - \frac{1}{2} \sum_{i=0}^{N} \int_{x_{i}}^{x_{i+1}} x \partial_{x}\left(v_{h}^{2}\right) \, \mathrm{d}x$$

Integrating by parts, we get

$$-\sum_{i=0}^{N} \int_{x_{i}}^{x_{i+1}} v_{h} \partial_{x}(xv_{h}) \, \mathrm{d}x = -\sum_{i=0}^{N} \int_{x_{i}}^{x_{i+1}} v_{h}^{2} \, \mathrm{d}x + \frac{1}{2} \sum_{i=0}^{N} \int_{x_{i}}^{x_{i+1}} v_{h}^{2} \, \mathrm{d}x \\ -\frac{1}{2} v_{h}^{2}(1) - \frac{1}{2} \sum_{i=1}^{N-1} x_{i} \left(v_{i}^{+}\right)^{2} - x_{i} \left(v_{i}^{-}\right)^{2}.$$

Using the algebraic identity  $a^2 - b^2 = (a - b)(a + b)$ , we obtain

$$-\sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} v_h \partial_x(xv_h) \, \mathrm{d}x = -\frac{1}{2} \sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} v_h^2 \, \mathrm{d}x - \frac{1}{2} v_h^2(1) \\ -\frac{1}{2} \sum_{i=1}^{N-1} x_i (v_i^+ - v_i^-) (v_i^+ + v_i^-).$$

329 We define the average of  $v_h$  for  $\in \{1, N-1\}$  by

$$\{\!\!\{v_i\}\!\!\} := \frac{v_i^+ + v_i^-}{2}, \text{ where } v_i^+ = v_h(x_i^+) \text{ and } v_i^- = v_h(x_i^-).$$

<sup>330</sup> Using the jump definition (32) and for the last term, we obtain

$$-\sum_{i=0}^{N}\int_{x_{i}}^{x_{i+1}}v_{h}\partial_{x}(xv_{h})\,\mathrm{d}x = -\frac{1}{2}\sum_{i=0}^{N}\int_{x_{i}}^{x_{i+1}}v_{h}^{2}\,\mathrm{d}x - \frac{1}{2}v_{h}^{2}(1) - \sum_{i=1}^{N-1}x_{i}\{\!\{v_{i}\}\!\}[\![v_{i}]\!].$$

Injecting this result into (35), we get

$$(A_h z_h, \Pi_h z_h)_h = \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x \left( x v_h - \chi_h^*(v_h) \right) \, \mathrm{d}x - \frac{1}{2} \int_0^1 v_h^2 \, \mathrm{d}x \\ + \frac{1}{2} v_h^2(1) - \sum_{i=1}^{N-1} x_i \{\!\!\{v_i\}\!\} [\![v_i]\!] + \int_0^1 \partial_x u_h \chi_h(u_h) \, \mathrm{d}x - u_h^2(1).$$

Injecting  $xu_h - xu_h = 0$  in the penultimate term in the right-hand side, then we get

$$(A_h z_h, \Pi_h z_h)_h = \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x \left( x v_h - \chi_h^*(v_h) \right) \, \mathrm{d}x - \frac{1}{2} \int_0^1 v_h^2 \, \mathrm{d}x + \frac{1}{2} v_h^2(1) \\ - \sum_{i=1}^{N-1} x_i \{\!\!\{v_i\}\!\} [\![v_i]\!] - \sum_{i=0}^N \int_{x_i}^{x_{i+1}} \left( x u_h - \chi_h(u_h) \right) \partial_x u_h \, \mathrm{d}x \\ + \frac{1}{2} \int_0^1 x \partial_x (u_h^2) \, \mathrm{d}x - u_h^2(1).$$

We integrate by parts, we have

$$(A_h z_h, \Pi_h z_h)_h = \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x \left( x v_h - \chi_h^*(v_h) \right) \, \mathrm{d}x - \frac{1}{2} \int_0^1 v_h^2 \, \mathrm{d}x + \frac{1}{2} v_h^2(1) \\ - \sum_{i=1}^{N-1} x_i \{\!\!\{v_i\}\!\} [\![v_i]\!] - \sum_{i=0}^N \int_{x_i}^{x_{i+1}} \left( x u_h - \chi_h(u_h) \right) \partial_x u_h \, \mathrm{d}x \\ - \frac{1}{2} \int_0^1 u_h^2 \, \mathrm{d}x - \frac{1}{2} u_h^2(1).$$

We integrate by part and use the property that the terms involving the difference between  $xu_h$  and its interpolation are zero at the boundary. This leads to the following important inequality

$$(A_h z_h, \Pi_h z_h)_h = \sum_{i=0}^N \int_{x_i}^{x_{i+1}} v_h \partial_x \left( x v_h - \chi_h^*(v_h) \right) \, \mathrm{d}x - \frac{1}{2} \int_0^1 v_h^2 \, \mathrm{d}x + \frac{1}{2} v_h^2(1) \\ - \sum_{i=1}^{N-1} x_i \{\!\!\{v_i\}\!\} [\![v_i]\!] + \sum_{i=0}^N \int_{x_i}^{x_{i+1}} u_h \partial_x \left( x u_h - \chi_h(u_h) \right) \, \mathrm{d}x \\ - \frac{1}{2} \int_0^1 u_h^2 \, \mathrm{d}x - \frac{1}{2} u_h^2(1).$$
(36)

Using Cauchy-Schwarz and Young inequalities, we obtain with  $\lambda > 0$ 

$$(A_{h}z_{h},\Pi_{h}z_{h})_{h} \leq -\frac{1}{2}(1-\lambda)\int_{0}^{1}u_{h}^{2} + v_{h}^{2}\,\mathrm{d}x + \frac{1}{2\lambda}\sum_{i=0}^{N}\int_{x_{i}}^{x_{i+1}}\left[\partial_{x}\left(xv_{h} - \chi_{h}^{*}(v_{h})\right)\right]^{2}\,\mathrm{d}x + \frac{1}{2\lambda}\sum_{i=0}^{N}\int_{x_{i}}^{x_{i+1}}\left[\partial_{x}\left(xu_{h} - \chi_{h}(u_{h})\right)\right]^{2}\,\mathrm{d}x + \frac{1}{2}v_{h}^{2}(1) - \sum_{i=1}^{N-1}x_{i}\{\!\{v_{i}\}\!\}[\![v_{i}]\!].$$
 (37)

Let us focus on the last term. Using Cauchy-Schwarz and Young inequalities and bounding  $x_i$  by 1, we get with  $\mu > 0$ 

$$-\sum_{i=1}^{N-1} x_i \{\!\!\{v_i\}\!\} [\![v_i]\!] \le \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} [\![v_i]\!]^2 + \frac{\mu}{2} \sum_{i=1}^{N-1} \tilde{h}_i \{\!\!\{v_i\}\!\}^2.$$

Using the definition of the average and using  $(a+b)^2 \leq 2(a^2+b^2)$ , we obtain

$$-\sum_{i=1}^{N-1} x_i \{\!\!\{v_i\}\!\} [\![v_i]\!] \le \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} [\![v_i]\!]^2 + \frac{\mu}{4} \sum_{i=1}^{N-1} \tilde{h}_i \left( (v_i^+)^2 + (v_i^-)^2 \right).$$
(38)

On the segment [0, h], we classically have the trace inequality

$$|v_h^2(0)|^2 \le \|v_h\|_{L^2(0,h)} \left(\frac{1}{h} \|v_h\|_{L^2(0,h)} + 2\|\partial_x v_h\|_{L^2(0,h)}\right).$$

Using an inverse estimate [24, Chapter 4, Section 5, Lemma 4.5.3], we get, with  $C_r > 0$ depending only on r

$$|v_h^2(0)|^2 \le \left(\frac{1+2C_r}{h}\right) \|v_h\|_{L^2(0,h)}^2.$$

 $_{\rm 337}$   $\,$  This inequality is generalized on each mesh, so we have

$$\begin{cases} |v_i^+|^2 \le \left(\frac{1+2C_r}{h_i}\right) \|v_h\|_{L^2([x_i, x_{i+1}])}^2, & i \in [1, N-1], \\ |v_i^-|^2 \le \left(\frac{1+2C_r}{h_{i-1}}\right) \|v_h\|_{L^2([x_{i-1}, x_i])}^2, & i \in [1, N-1]. \end{cases}$$

Summing the two equations and using the definition of  $\tilde{h}_i$ , we get

$$\tilde{h}_{i}\left((v_{i}^{+})^{2}+(v_{i}^{-})^{2}\right) \leq \tilde{h}_{i}\frac{1+2C_{r}}{h_{i}}\|v_{h}\|_{L^{2}\left([x_{i},x_{i+1}]\right)}^{2} + \tilde{h}_{i}\frac{1+2C_{r}}{h_{i-1}}\|v_{h}\|_{L^{2}\left([x_{i-1},x_{i}]\right)}^{2},$$
$$\leq \left(1+2C_{r}\right)\left(\|v_{h}\|_{L^{2}\left([x_{i},x_{i+1}]\right)}^{2} + \|v_{h}\|_{L^{2}\left([x_{i-1},x_{i}]\right)}^{2}\right).$$

338 We use the above inequalities in (38) to get

$$-\sum_{i=1}^{N-1} x_i \{\!\!\{v_i\}\!\}[\![v_i]\!] \le \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i}[\![v_i]\!]^2 + \frac{\mu(1+2C_r)}{2} \|v_h\|_{L^2(0,1)}^2$$

Injecting this result in (37) we obtain

$$(A_h z_h, \Pi_h z_h)_h \leq -\frac{1}{2} (1 - \lambda) \int_0^1 u_h^2 \, \mathrm{d}x - \frac{1}{2} (1 - \lambda - \mu (1 + 2C_r)) \int_0^1 v_h^2 \, \mathrm{d}x + \frac{1}{2\lambda} \sum_{i=0}^N \int_{x_i}^{x_{i+1}} \left[ \partial_x \left( x v_h - \chi_h^*(v_h) \right) \right]^2 \, \mathrm{d}x + \frac{1}{2\lambda} \sum_{i=0}^N \int_{x_i}^{x_{i+1}} \left[ \partial_x \left( x u_h - \chi_h(u_h) \right) \right]^2 \, \mathrm{d}x + \frac{1}{2} v_h^2(1) + \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} [\![v_i]\!]^2.$$

We recognize the expression of the bilinear forms  $d_h$  and  $d_{h,*}$  defined by (29) and (30). Using (19) for the first and the second terms in the right-hand side, we get

$$(A_h z_h, \Pi_h z_h)_h \le -\frac{1}{2} (1 - \lambda) \frac{1}{C_q} \oint_0^1 u_h^2 \, \mathrm{d}x - \frac{1}{2} \frac{1}{C_q} (1 - \lambda - \mu (1 + 2C_r)) \oint_0^1 v_h^2 \, \mathrm{d}x \\ + \frac{1}{2\lambda} d_{h,*}(v_h, v_h) + \frac{1}{2\lambda} d_h(u_h, u_h) + \frac{1}{2} v_h^2(1) + \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} \llbracket v_i \rrbracket^2.$$

We choose  $\lambda$  and  $\mu$  small enough such that

$$1 - \lambda > 0$$
 and  $1 - \lambda - \mu(1 + 2C_r) > 0$ .

We obtain the fundamental multiplier inequality

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$$(A_h z_h, \Pi_h z_h)_h \leq -\frac{1}{2C_q} \left(1 - \lambda - \mu (1 + 2C_r)\right) \|z_h\|_{\mathcal{Z}}^2 + \frac{1}{2\lambda} d_h(u_h, u_h) + \frac{1}{2\lambda} d_{h,*}(v_h, v_h) + \frac{1}{2} v_h^2(1) + \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} [v_i]^2.$$
(39)

340 Thus,

$$\Pi_h A_h z_h, z_h)_h \le \alpha \|C_h z_h\|_h^2 - \beta \|z_h\|_h^2,$$
(40)

341 where

$$\beta = \frac{1}{2C_q} \left( 1 - \lambda - \mu (1 + 2C_r) \right) \text{ and } \alpha = \max\left(\frac{1}{2\lambda}, \frac{1}{2\mu}, \frac{1}{2}\right).$$

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**Corollary 4.7.** Let  $z_h = (u_h, v_h)$  belonging to  $U_h^r \times V_h^r$  be the solutions of (24). For T large enough, there exists  $C^{st}(T)$  independent of h, such that

$$\begin{aligned} \|u_h(0)\|_h^2 + \|v_h(0)\|_h^2 &\leq C^{st}(T) \left[ \int_0^T d_h(u_h, u_h) \,\mathrm{d}t + \int_0^T d_{h,*}(v_h, v_h) \,\mathrm{d}t \\ &+ \int_0^T v_h^2(1) \,\mathrm{d}t + \int_0^T \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} [\![v_i]\!]^2 \,\mathrm{d}t \right] = C^{st}(T) \int_0^T \|C_h z_h\|_h^2 \,\mathrm{d}t, \end{aligned} \tag{41}$$

where  $d_h$  and  $d_{h,*}$  are defined respectively by (29) and (30).

Proof. Since  $\Pi_h$  defined by (27) satisfies Assumption 1 by Theorem 4.6, we can apply Theorem 4.3 where  $C_h$  and  $A_h$  are defined by (34) and (24).

Following what was done in Section 3, (41) is a guideline to build an exponentially stable problem. Using the operator  $C_h$  – defined in the above section by (34) – to damp the dynamics (24), we construct an exponentially stable dynamics uniformly in h.

#### 350 4.4 Exponential stability

We incorporate in the system (24) the dissipative terms  $C_h$  in the observability conditions. In other words, we introduce a perturbation of system (24) by the operator  $\gamma C_h^* C_h$ , which gives us the following system

$$\begin{cases} \dot{z}_h = A_h z_h - \gamma C_h^* C_h z_h, & \text{in } [0, T], \\ z_h(0) = z_{h,0}, \end{cases}$$
(42)

where  $A_h$  and  $C_h$  are respectively defined by (24) and (34). We will be able to expose our main result, namely the uniform exponential stability property in the semi-discrete case. First, as for the observability property, we will check that our multiplier  $\Pi_h$ satisfies the fundamental Assumption 2.

**Theorem 4.8.** Let  $C_h$  be defined by (34). Let  $z_h = (u_h, v_h) \in \mathcal{Z}_h$  be the solutions of (42). Then,  $\Pi_h$  defined by (27) satisfies Assumption 2 namely

$$(\Pi_h A_h z_h, z_h)_h - \gamma (C_h z_h, C_h \Pi_h z_h)_h \le \chi \|C_h z_h\|_h^2 - \kappa \|z_h\|_h^2$$

- 360 where  $\chi, \kappa > 0$  are independent of h.
- <sup>361</sup> Proof. Let clarify the expression of  $(C_h z_h, C_h \Pi_h z_h)_h$ .

$$-(C_h z_h, C_h \Pi_h z_h)_h = -\sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} \llbracket v_i \rrbracket \llbracket \chi_h(u_h) \rrbracket + v_h(1)u_h(1) + d_h(u_h, \chi_h^*(v_h)) + d_{h,*}(v_h, \chi_h(u_h)).$$

At a point,  $xu_h$  has the same value as its interpolation. Therefore, as  $u_h$  is continuous (and consequently its jump is zero), then  $\chi_h(u_h)$  has a zero jump too. Then, thanks to (36), we have

$$(\Pi_h A_h z_h, z_h)_h - \gamma (C_h z_h, C_h \Pi_h z_h)_h$$
  
=  $-\int_0^1 v_h \partial_x \chi_h^*(v_h) \, \mathrm{d}x + \int_0^1 \partial_x u_h \chi_h(u_h) \, \mathrm{d}x - u_h^2(1) + v_h^2(1)$   
+  $\gamma v_h(1) u_h(1) + \gamma d_h(u_h, \chi_h^*(v_h)) + \gamma d_{h,*}(v_h, \chi_h(u_h)).$ 

Following the proof of Theorem 4.7, one can show

$$\begin{aligned} (\Pi_h A_h z_h, z_h)_h &- \gamma (C_h z_h, C_h \Pi_h z_h)_h \\ &\leq -\frac{1}{2C_q} \left( 1 - \lambda - \mu (1 + 2C_r) \right) \|z_h\|_h^2 + \frac{1}{2\lambda} d_h(u_h, u_h) + \frac{1}{2\lambda} d_{h,*}(v_h, v_h) \\ &+ \frac{\gamma^2 + 1}{2} v_h^2(1) + \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} \llbracket v_i \rrbracket^2 + \gamma d_h(u_h, \chi_h^*(v_h)) + \gamma d_{h,*}(v_h, \chi_h(u_h)), \end{aligned}$$

where  $\lambda$  and  $\mu$  are parameters that can be chosen as small desired. Thanks to Cauchy-Schwarz and Young inequalities for last terms, we get with  $\delta > 0$ 

$$\begin{aligned} (\Pi_h A_h z_h, z_h)_h &- \gamma (C_h z_h, C_h \Pi_h z_h)_h \\ &\leq -\frac{1}{C_q} \left( 1 - \lambda - \mu (1 + 2C_r) \right) \|z_h\|_h^2 + \left( \frac{1}{2\lambda} + \frac{\gamma}{2\delta} \right) \left( d_h(u_h, u_h) + d_{h,*}(v_h, v_h) \right) \\ &+ \frac{\gamma^2 + 1}{2} v_h^2(1) + \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} [\![v_i]\!]^2 + \frac{\delta\gamma}{2} d_h(\chi_h^*(v_h), \chi_h^*(v_h)) + \frac{\delta\gamma}{2} d_{h,*}(\chi_h(u_h)\chi_h(u_h)). \end{aligned}$$

Thanks to Proposition 4.5 and (28), we obtain

$$\begin{aligned} (\Pi_h A_h z_h, z_h)_h &- \gamma (C_h z_h, C_h \Pi_h z_h)_h \\ &\leq - \left( \frac{1}{C_q} - \frac{\lambda}{C_q} - \frac{\mu (1 + 2C_r)}{C_q} - \frac{\gamma \delta C_d C_\Pi^2}{2} \right) \|z_h\|_h^2 \\ &+ \left( \frac{1}{2\lambda} + \frac{\gamma}{2\delta} \right) (d_h (u_h, u_h) + d_{h,*} (v_h, v_h)) \\ &+ \frac{\gamma^2 + 1}{2} v_h^2 (1) + \frac{1}{2\mu} \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} [\![v_i]\!]^2. \end{aligned}$$

We choose  $\lambda, \mu, \delta$  sufficiently small so that 362

$$\frac{1}{C_q} - \frac{\lambda}{C_q} - \frac{\mu(1 + 2C_r)}{C_q} - \frac{\gamma \delta C_d C_{\text{II}}^2}{2} > 0.$$

Thus, finally we show that the Assumption 2 is satisfied

$$(\Pi_h A_h z_h, z_h)_h - \gamma (C_h z_h, C_h \Pi_h z_h)_h \le \chi \|C_h z_h\|_h^2 - \kappa \|z_h\|_h^2,$$

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with 363

and

$$\chi = \max\left(\frac{1}{2\lambda} + \frac{\gamma}{2\delta}, \frac{\gamma^2}{2} + \frac{1}{2}, \frac{1}{2\mu}\right),$$
$$\kappa = \frac{1}{2\lambda} - \frac{\lambda}{2\lambda} - \frac{\mu(1 + 2C_r)}{2\lambda} - \frac{\gamma\delta C_d C_{\Pi}^2}{2\lambda}$$

$$\kappa = \frac{1}{C_q} - \frac{\lambda}{C_q} - \frac{\mu(1+2C_r)}{C_q} - \frac{\gamma \delta C_d C_{\mathrm{II}}^2}{2}.$$

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We can deduce from this theorem the uniform exponential stability property for 366 the damped wave equation (42). 367

**Corollary 4.9.** Let  $z_h = (u_h, v_h)$  belonging to  $\in U_h^r \times V_h^r$  be the solutions of (42). 368 There exist two constants  $C_s > 0$  and  $\sigma > 0$  independent of h, such that 369

$$||u_h(t)||_h^2 + ||v_h(t)||_h^2 \le C_s e^{-\sigma t} \left( ||u_h(0)||_h^2 + ||v_h(0)||_h^2 \right).$$

*Proof.* Using Theorem 4.8, we have that the multiplier  $\Pi_h$  defined by (27) satisfies 370 Assumption 2. Therefore, we apply Theorem 4.4 where  $C_h$  and is defined by (34).  $\Box$ 371

#### 4.5 Further properties of the stabilization operators 372

It is possible to interpret that the dissipation term  $d_h$  defined by (29) corresponds to 373 element-wise r-laplacians operators. In what follows we introduce the notation 374

$$x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2}.$$

**Proposition 4.10.** For  $u_h \in U_h^r$  then 375

$$d_h(u_h, u_h) = C_r \sum_{i=0}^{N-1} h_i^{2r+1} \left| u_h^{(r)} \left( x_{i+\frac{1}{2}} \right) \right|^2,$$

where  $C_r$  is a positive scalar depending only on the finite element order r. 376

*Proof.* To show that stated result, it is sufficient to study the following term for  $u_h \in \mathbb{P}_r$ , 377

$$\frac{d}{dx}\left(xu_h - \chi_h(u_h)\right)$$

Since  $u_h \in U_h^r$ , we have that  $u_i := u_h|_{[x_i, x_{i+1}]} \in \mathbb{P}_r$ , hence 378

$$u_i(x) = \sum_{k=0}^{r-1} \frac{x^k}{k!} u_i^{(k)}(x_i) + r_i(x) \quad \text{where} \quad r_i(x) := \frac{x^r}{r!} u_i^{(r)}.$$

Moreover, one can show that 379

$$\chi_h(u_h)|_{[x_i, x_{i+1}]} = I_i(xu_h)$$

- where,  $I_i$  is the interpolation operator on the Gauss-Lobatto points  $\{x_i + h_i \xi_k\}_{k=0}^r$ , it 380 381
  - is a projection operator over  $\mathbb{P}_r$ . Therefore, one can deduce that

$$(xu_h - \chi_h(u_h))_{[x_i, x_{i+1}]} = xr_i(x) - I_i(xr_i(x)) = \frac{u_i^{(r)}}{r!} \left(x^{r+1} - I_i(x^{r+1})\right)$$
(43)

Using this result and a simple change of variable one can show that 382

$$\int_{x_i}^{x_{i+1}} |\partial_x \left( x u_h - \chi_h(u_h) \right)|^2 = \frac{\left( u_i^{(r)} \right)^2}{r!^2} \int_0^{h_i} \left| \frac{d}{dx} \left( x^{r+1} - \mathcal{I}_{h_i}(x^{r+1}) \right) \right|^2 \, \mathrm{d}x, \quad (44)$$

where  $\mathcal{I}_{h_i}$  is the interpolation operator on the Gauss-Lobatto points  $\{h_i \xi_k\}_{k=0}^r$ . We 383 introduce now the following change of variable, 384

$$\phi_i: \quad \begin{vmatrix} [0,1] \to [0,h_i], \\ \xi \to x = h_i \xi \end{vmatrix}$$

Thus, we have 385

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$$(I_{h_i}(x^{r+1}) \circ \phi_i)(\xi) = \sum_{k=0}^r \psi_k(\xi)\xi_k^{r+1}h_i^{r+1}$$

where  $\{\psi_k(\xi)\}_{k=0}^r$  are a basis for  $\mathbb{P}_r$  such that  $\psi_k(\xi_\ell) = \delta_{k,\ell}$ . The use of the above 386 identity, the composition of (43) and  $\phi_i$  leads to 387

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{r+1}-I_{h_i}\left(x^{r+1}\right)\right)\right)\circ\phi_i=\frac{1}{h_i}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(h^{r+1}\xi^{r+1}-\sum_{k=0}^r\psi_k(\xi)\xi_k^{r+1}h_i^{r+1}\right).$$

Finally, (44) is rewritten as follows, 388

$$\int_{x_i}^{x_{i+1}} |\partial_x \left( x u_h - \chi_h(u_h) \right)|^2 = C_r \left( u_i^{(r)} \right)^2 \frac{1}{h_i} h_i^{2r+2},$$
  
where  $C_r = \frac{1}{r!^2} \int_0^1 \left| \frac{\mathrm{d}}{\mathrm{d}\xi} \left( \xi^{r+1} - \sum_{k=0}^r \psi_k(\xi) \xi_k^{r+1} \right) \right|^2 \mathrm{d}\xi.$ 

Of note, in the work of [23] a Laplace operator us used as a viscous term for 390 stabilization of finite elements of order 1. Therefore, in some sense our work generalizes 391 these results by showing that an element-wise r-laplace operator – scaled with the 392 appropriate power of h – can be used for stabilization of high order finite elements. 393

The same result cannot be obtained straightforwardly for  $d_{h,*}$ . In fact, due to the 394 discontinuous nature of the space  $V_h^r$ , jump terms are involved. 395

**Proposition 4.11.** For  $v_h \in V_h^r$  then 396

$$d_{h,*}(v_h, v_h) \le C_r^* \left( \sum_{i=0}^{N-1} h_i^{2r+1} \left| v_h^{(r)} \left( x_{i+\frac{1}{2}} \right) \right|^2 + h \sum_{i=1}^{N-1} \llbracket v_i \rrbracket^2 \right),$$

where  $C_r^*$  is a positive scalars depending only on the finite element order r. 397

*Proof.* The proof is based on an adequate decomposition of any function  $v_h \in V_h^r$ . More 398 precisely we introduce  $w_h \in V_h^1$  defined by 399

$$w_h(0) = w_h(1) = 0, \quad w_i^{\pm} = \pm \frac{\llbracket v_i \rrbracket}{2}, \ i \in \{1, \cdots, N-1\} \text{ and } w_{h|[x_i, x_{i+1}]} \in \mathbb{P}^1.$$

400 We then defined  $u_h = v_h - w_h$ . It can be verified that  $u_h \in U_h^r$ . We have

$$d_{h,*}(v_h, v_h) \le 2d_{h,*}(u_h, u_h) + 2d_{h,*}(w_h, w_h).$$

401 Using Proposition 4.5 and the property that  $u_h \in U_h^r$  we obtain the estimation

$$d_{h,*}(v_h, v_h) \le 2d_h(u_h, u_h) + 2C_d \int_0^1 w_h^2 \, \mathrm{d}x,$$

402 hence, using Proposition 4.10,

$$d_{h,*}(v_h, v_h) \le 2C_r \sum_{i=0}^{N-1} h_i^{2r+1} \left| u_h^{(r)} \left( x_{i+\frac{1}{2}} \right) \right|^2 + 2C_d \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} w_h^2 \, \mathrm{d}x.$$

403 It is then possible to show that

$$\sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} w_h^2 \, \mathrm{d}x \le h \sum_{i=1}^{N-1} \llbracket v_i \rrbracket^2,$$

finally we observe that  $u_h^{(r)}(x_{i+\frac{1}{2}}) = (u_h + w_h)^{(r)}(x_{i+\frac{1}{2}}) = v_h^{(r)}(x_{i+\frac{1}{2}})$  by construction.

**Remark 4.12.** In the numerical results, we used the following simplified expression instead of the term  $d_{h,*}$ 

$$\tilde{d}_{h,*}(v_h, \tilde{v}_h) := C_r \sum_{i=0}^{N-1} h_i^{2r+1} v_h^{(r)} \left( x_{i+\frac{1}{2}} \right) \tilde{v}_h^{(r)} \left( x_{i+\frac{1}{2}} \right).$$
(45)

408 Proposition 4.11 allows us to show that Theorem 4.6 and Theorem 4.8 hold with

$$\widetilde{C}_{h} = \begin{pmatrix} D_{h}^{1/2} & 0\\ 0 & J_{h}^{1/2}\\ 0 & B_{h}^{1/2}\\ 0 & \widetilde{D}_{h,*}^{1/2} \end{pmatrix}.$$
(46)

where  $\widetilde{D}_{h,*}$  is associated with  $\widetilde{d}_{h,*}$ , thus showing the exponential decay behavior of the system

$$\begin{cases} \dot{z}_h = A_h z_h - \gamma \widetilde{C}_h^* \widetilde{C}_h z_h, & in \ [0, T], \\ z_h(0) = z_{h, 0}. \end{cases}$$

#### 411 5 Observer system: Semi-discrete convergence result

In this section we follow the approach of [33] and aim at defining at the semi-discrete level an observer of the solution  $z = (u, v)^t$  of the system (4) with the assumption that, for all T > 0

$$z = (u, v) \in C^1([0, T]; H^{r+1}(0, 1)^2).$$

It is shown below that the constructed observer as the property to converge asymptotically in time to the solution of z up to an order related to the space discretization – typically in  $O(h^r)$  – but bounded in time. For the sake of simplicity and conciseness we directly introduce the interpolation on the of z denoted  $\bar{z}_h(t) = I_h z(t) = (I_{U_h} u(t), I_{V_h} v(t))$ , it is solution of the system

$$\dot{\bar{z}}_h = A_h \bar{z}_h + \varepsilon_h, \quad \bar{z}_h(0) = z_{0,h} = I_h z_0,$$

420 where  $\varepsilon_h$  is the consistency error and is assumed to be smooth in time and space and

$$|||\varepsilon_h||| := \sup_{t \in [0,T]} \|\varepsilon_h(t)\|_{L^2(0,1)^2} \le C_{\varepsilon} h^r, \tag{47}$$

where  $C_{\varepsilon} > 0$  is independent of T. Moreover, the interpolation operator satisfies, for  $0 \le \ell \le k$  and  $0 \le k \le r+1$ ,

$$\forall w \in H^{r+1}(0,1)^2, \quad \left(\sum_{i=0}^{N-1} |I_h w - w|^2_{H^{\ell}(x_i, x_{i+1})^2}\right)^{\frac{1}{2}} \le C_I h^{k-\ell} \|w\|_{H^k(0,1)}.$$
(48)

We refer to [11, 21, 24] for more details on how these properties can be obtained. The system for the observer is constructed as follow: we assume v(1,t) is available/measured for  $t \in [0, T]$ - which is equivalent to having access to  $(I_h^V v)(1, t)$  – and solve,

$$\dot{\hat{z}}_h = A_h \hat{z}_h - \gamma \begin{pmatrix} \widetilde{D}_h & 0\\ 0 & \widetilde{D}_{h,*} + J_h \end{pmatrix} \hat{z}_h - \gamma \begin{pmatrix} 0 & 0\\ 0 & B_h \end{pmatrix} (\hat{z}_h - \bar{z}_h), \quad \hat{z}_h(0) = 0.$$

Note that observer systems – and more precisely, state estimator – are very well studied at the continuous level, we refer the reader to [1, 32, 39]. In short, one expect that  $\hat{z}_h(t) \rightarrow \bar{z}_h(t) \sim z(t)$  asymptotically in time, thus alleviating the mismatch in the initial data between the observer and the interpolation of the solution z(t) of (4). The error

 $e_h = \bar{z}_h - \hat{z}_h,$ 

431 satisfies, with  $A_{\gamma,h} = A_h - \gamma \widetilde{C}_h^* \widetilde{C}_h$ ,

$$\dot{e}_h = A_{\gamma,h} e_h + \gamma \begin{pmatrix} D_h & 0\\ 0 & \widetilde{D}_{h,*} + J_h \end{pmatrix} \bar{z}_h + \varepsilon_h, \quad e_h(0) = z_{0,h}.$$

432 Lemma 5.1. There exists a constant  $C_D > 0$  independent of h, such that

$$\forall u \in H^{r+1}(0,1), \quad \|D_h I_{U_h} u\|_{L^2(0,1)} \le C_D h^r \|u\|_{H^r(0,1)},$$

433 and

$$\forall v \in H^{r+1}(0,1), \quad J_h I_{V_h} v = 0, \quad \|\widetilde{D}_{h,*} I_{V_h} v\|_{L^2(0,1)} \le C_D h^r \|v\|_{H^r(0,1)}.$$

*Proof.* We show the first identity. From the Proposition 4.10, we have

$$d_{h}(I_{U_{h}}u, I_{U_{h}}u) = C_{r} \sum_{i=0}^{N-1} h_{i}^{2r+1} \left| (I_{U_{h}}u)^{(r)} \left(x_{i+\frac{1}{2}}\right) \right|^{2}$$
  
$$= C_{r} \sum_{i=0}^{N-1} \int_{x_{i}}^{x_{i+1}} h_{i}^{2r} \left| (I_{U_{h}}u)^{(r)} \left(x_{i+\frac{1}{2}}\right) \right|^{2}$$
  
$$\leq C_{r}h^{2r} \sum_{i=0}^{N-1} |I_{U_{h}}u|^{2}_{H^{r}(x_{i}, x_{i+1})}.$$

434 Using (48) and a triangular inequality, we get the existence of a constant  $\tilde{C}_r$  such that

$$d_h(I_{U_h}u, I_{U_h}u) \le \tilde{C}_r h^{2r} \|u\|_{H^r(0,1)}^2.$$

The analysis of  $\|\widetilde{D}_{h,*}I_{V_h}v\|_{L^2(0,1)}$  is the same as the previous one. Finally, note that since v is smooth, we have that  $I_{V_h}v$  is continuous and therefore  $J_hI_{V_h}v = 0$ . 437 Theorem 5.2. There exists a positive scalar  $C_{st} > 0$  such that, for any T > 0, we have

$$\sup_{t \in [0,T]} \|z(t) - \hat{z}_h(t)\|_{L^2(0,1)^2} \le C_{st} \left( e^{-\sigma t} \|z_{0,h}\|_h + h^r \right)$$

*Proof.* By triangular inequality and using (19), we have that

$$\begin{aligned} \|z(t) - \hat{z}_h(t)\|_{L^2(0,1)^2} &= \|z(t) - \bar{z}_h + \bar{z}_h - \hat{z}_h(t)\|_{L^2(0,1)^2} \\ &\leq \|z(t) - \bar{z}_h\|_{L^2(0,1)^2} + c_q^{-1/2} \|\bar{z}_h - \hat{z}_h(t)\|_h \end{aligned}$$

Using the interpolation property (48), we obtain that there exists a positive scalar  $C_{st}$ such that

$$\|z(t) - \hat{z}_h(t)\|_{L^2(0,1)^2} \le C_{st}h^r + c_q^{-1/2} \|\bar{z}_h - \hat{z}_h(t)\|_h.$$
(49)

440 Let us focus on the error  $e_h = \bar{z}_h - \hat{z}_h$ , it satisfies

$$\dot{e}_h = A_{\gamma,h} e_h + \bar{\varepsilon}_h,$$

441 where

$$\bar{\varepsilon}_h = \gamma \begin{pmatrix} D_h & 0\\ 0 & \widetilde{D}_{h,*} + J_h \end{pmatrix} \bar{z}_h + \varepsilon_h.$$

Using Lemma 5.1, (47) combined with (19), we have that there exists  $C_{st}$  independent 443 of h such that

$$|||\bar{\varepsilon}_h||| \leq C_{st}h^r.$$

444 From the Duhamel formula, we have

$$e_h(t) = e^{tA_{\gamma,h}} z_{0,h} + \int_0^t e^{(t-s)A_{\gamma,h}} \bar{\varepsilon}_h(s) \,\mathrm{d}s.$$

Using the exponential stability of the operator  $A_{\gamma,h}$  (Corollary 4.9), we have

$$||e_h(t)||_h \le C_{st} \left( e^{-\sigma t} ||z_{0,h}||_h + \sup_{s \in [0,t]} ||\bar{e}_h(s)||_h \int_0^t e^{-\sigma(t-s)} \, \mathrm{d}s \right).$$

As a consequence, we finally have that there exist a constant  $C_{st} > 0$  independent of hand t such that

$$||e_h(t)||_h \le C_{st} \left( e^{-\sigma t} ||z_{0,h}||_h + \frac{1}{\sigma} (1 - e^{-\sigma t}) h^r \right),$$

<sup>448</sup> which combining with (49) proves the desired result.

It is worth noting that the preceding theorem states, in particular, that an observer  $\hat{z}_h$  initialized from an interpolated initial condition with an accuracy of order  $h^r$  then remains at a distance  $h^r$  from the true solution without error drift with respect to time. This is due to the fact that the constant  $C_{st}$  does not depend on time. Therefore, in the context of numerical analysis by semi-discretization, this result extends [33] to high order schemes.

### 455 6 Time discretization

#### 456 6.1 Abstract results

In this section, we are interested in the discretization of System (17) with an implicit midpoint scheme. The operator  $C_h$  or  $\widetilde{C}_h$  can be used indifferently (we have chosen to use  $C_h$  in what follows for the sake of clarity). We verify that the use of this time discretization does not alter the stabilization properties.

$$\begin{cases} \frac{z_h^{n+1} - z_h^n}{\Delta t} = A_h z_h^{n+1/2} - \gamma C_h^* C_h z_h^{n+1/2}, \\ z_h^0 = z_{h,0}. \end{cases}$$
(50)

It is easy to show that the energy conservation property is preserved in the fully discretecase.

**Theorem 6.1.** Let  $\{z_h^n\}_{n\geq 0} \in \mathcal{Z}_h$  be the solutions of (50), we have the discrete dissipation energy property

$$\frac{1}{2} \frac{\|z_h^{n+1}\|_h^2 - \|z_h^n\|_h^2}{\Delta t} = -\gamma \|C_h z_h^{n+1/2}\|_h^2, \ \forall n \ge 0.$$
(51)

465 Proof. We multiply (50) by  $z_h^{n+1/2}$ . Since  $A_h$  satisfies (16) and using the difference of 466 square formula, we obtain the result

Following the same reasoning as in the semi-discrete case, we start by demonstrating the observability property for the undamped case  $\gamma = 0$ .

**Theorem 6.2.** Let  $\{z_h^n\}_{n\geq 0} \in \mathcal{Z}_h$  be the solutions of (50) with  $\gamma = 0$ . Assuming Aro Assumption 1, for T large enough, there exists a constant  $C^{st}(T) > 0$ , independent of h, such that

$$||z_{h,0}||_h^2 \le C^{st}(T)\Delta t \sum_{n=0}^N ||C_h z_h^{n+1/2}||_h^2.$$

472 Proof. The proof of this theorem is very similar to that of Theorem 4.3 with  $\gamma = 0$ , in 473 particular, it follows the same steps.

**Step 1:** We multiply the first equation of (50) (with  $\gamma = 0$ ) by  $\Pi_h z_h^{n+1/2}$ , and we get

$$\left(\Pi_h \frac{z_h^{n+1} - z_h^n}{\Delta t}, z_h^{n+1/2}\right)_h = \left(\Pi_h A_h z_h^{n+1/2}, z_h^{n+1/2}\right)_h.$$

We decompose the half-sum in the left hand-side to find, using Assumption 1,

$$\frac{1}{2\Delta t} \left( \Pi_h z_h^{n+1}, z_h^{n+1} \right)_h - \frac{1}{2\Delta t} \left( \Pi_h z_h^n, z_h^n \right)_h = \left( \Pi_h A_h z_h^{n+1/2}, z_h^{n+1/2} \right)_h, \qquad (52)$$

$$\leq \alpha \| C_h z_h^{n+1/2} \|_h^2 - \beta \| z_h^{n+1/2} \|_h^2.$$

475 Step 2: Summing (52) over n we get,

$$\frac{1}{2\Delta t} \left( \Pi_h z_h^{N+1}, z_h^{N+1} \right)_h - \frac{1}{2\Delta t} \left( \Pi_h z_h^0, z_h^0 \right)_h \le \alpha \sum_{n=0}^N \|C_h z_h^{n+1/2}\|_h^2 - \beta \sum_{n=0}^N \|z_h^{n+1/2}\|_h^2.$$

<sup>476</sup> The fully discrete energy conservation property of Theorem 6.1 with  $\gamma = 0$  gives

$$\frac{1}{2\Delta t} \left( \Pi_h z_h^{N+1}, z_h^{N+1} \right)_h - \frac{1}{2\Delta t} \left( \Pi_h z_h^0, z_h^0 \right)_h \le \alpha \sum_{n=0}^N \|C_h z_h^{n+1/2}\|_h^2 - \beta (N+1) \|z_h^0\|_h^2.$$

**Step 3:** Since  $\Pi_h$  is bounded, by Cauchy-Schwarz theorem and using again the energy 477 conservation theorem 6.1 with  $\gamma = 0$ , we have 478

$$\left(\beta(N+1)\Delta t - \|\Pi_h\|_h\right) \|z_{0,h}\|_h^2 \le \alpha \sum_{n=0}^N \Delta t \|C_h z_h^{n+1/2}\|_h^2.$$
  
using  $(N+1)\Delta t = T > \|\Pi_h\|_h \beta^{-1}$ , we obtain the result.

Choo 479  $\|\mathbf{11}_{h}\|_{h}$ 

In the same way as in the semi-discrete case, we obtain the exponential stability 480 property for the damped system. 481

**Theorem 6.3.** Let  $\{z_h^n\}_{n\geq 0} \in \mathcal{Z}_h$  be the solutions of (50). We assume that there exists 482  $C_c > 0$  and  $0 < \beta < 1$  such that for all h > 0,  $\Delta t$  is chosen so that 483

$$\Delta t \| C_h \|_h \le C_c, \tag{53}$$

and 484

$$\frac{\Delta t}{2} \|A_h\|_h \le \beta < 1.$$
(54)

With this assumption and Assumption 2, there exist two constants  $C_s, \sigma > 0$  independent 485 of h such that 486

$$||z_h^n||_h^2 \le C_s ||z_{h,0}||_h^2 e^{-\sigma(1-\beta^2)n\Delta t}.$$

*Proof.* The proof is separated into the same steps as for the proof of exponential 487 stability at the continuous level of Theorem 4.4. 488

**Step 1:** We multiply the first equation by (50) and  $\Pi_h z_h^{n+1/2}$ , and we get 489

$$\left(\Pi_h \frac{z_h^{n+1} - z_h^n}{\Delta t}, z_h^{n+1/2}\right)_h = (\Pi_h A_h z_h^{n+1/2}, z_h^{n+1/2})_h - \gamma (C_h z_h^{n+1/2}, C_h \Pi_h z_h^{n+1/2})_h.$$

Using  $(a + b)(a - b) = a^2 - b^2$  and Assumption 2, we have 400

$$\frac{1}{2\Delta t} \left( \Pi_h z_h^{n+1}, z_h^{n+1} \right)_h - \frac{1}{2\Delta t} \left( \Pi_h z_h^n, z_h^n \right)_h \le \chi \| C_h z_h^{n+1/2} \|_h^2 - \kappa \| z_h^{n+1/2} \|_h^2.$$

**Step 2:** Using the energy identity of Theorem 6.1, we get with  $\alpha > 0$ ,

$$\frac{1}{2\Delta t} \left( \Pi_h z_h^{n+1}, z_h^{n+1} \right)_h - \frac{1}{2\Delta t} \left( \Pi_h z_h^n, z_h^n \right)_h + \alpha \frac{\|z_h^{n+1}\|_h^2 - \|z_h^n\|_h^2}{2\Delta t} \\
\leq \left( \chi - \alpha \gamma \right) \|C_h z_h^{n+1/2}\|_h^2 - \kappa \|z_h^{n+1/2}\|_h^2. \tag{55}$$

We intend to bound the last term by  $-\|z_h^{n+1}\|_h^2 - \|z_h^n\|_h^2$ . Note that this intermediate 491 necessary step is specific to the time discretization. Decomposing  $z_h^{n+1/2}$  in a well-chose 492 way, we have 493

$$-\|z_h^{n+1/2}\|_h^2 = -\frac{1}{2}\|z_h^{n+1}\|_h^2 - \frac{1}{2}\|z_h^n\|_h^2 + \frac{(\Delta t)^2}{4} \left\|\frac{z_h^{n+1} - z_h^n}{\Delta t}\right\|_h^2.$$

Using the scheme (50) and Cauchy-Schwarz and Young inequalities, we obtain with  $\epsilon > 0$ ,

$$- \|z_h^{n+1/2}\|_h^2 \le -\frac{1}{2} \|z_h^{n+1}\|_h^2 - \frac{1}{2} \|z_h^n\|_h^2 + \frac{(\Delta t)^2}{4} (1+\epsilon) \left\|A_h z_h^{n+1/2}\right\|_h^2 \\ + \frac{\gamma(\Delta t)^2}{4} (1+\epsilon^{-1}) \|C_h^*\|_h^2 \|C_h z_h^{n+1/2}\|_h^2.$$

By triangular inequality and Cauchy-Schwarz inequality and using hypothesis (53), we get

$$- \|z_h^{n+1/2}\|_h^2 \le -\frac{1}{2} \left( 1 - \frac{(\Delta t)^2}{4} (1+\epsilon) \|A_h\|_h^2 \right) \left( \|z_h^{n+1}\|_h^2 + \|z_h^n\|_h^2 \right) \\ + \frac{\gamma C_c^2}{4} (1+\epsilon^{-1}) \|C_h z_h^{n+1/2}\|_h^2.$$

We choose  $\epsilon = \frac{1}{2\beta^2} - \frac{1}{2} > 0$ , using the condition (54), we obtain

$$- \|z_h^{n+1/2}\|_h^2 \le -\frac{1}{4} \left(1 - \beta^2\right) \left(\|z_h^{n+1}\|_h^2 + \|z_h^n\|_h^2\right) \\ + \frac{\gamma C_c^2}{4} \left(1 + \frac{2\beta^2}{1 - \beta^2}\right) \|C_h z_h^{n+1/2}\|_h^2.$$

Using the above equation in (55), we get

$$\frac{1}{2\Delta t} \left( \Pi_h z_h^{n+1}, z_h^{n+1} \right)_h - \frac{1}{2\Delta t} \left( \Pi_h z_h^n, z_h^n \right)_h + \alpha \frac{\|z_h^{n+1}\|_h^2 - \|z_h^n\|_h^2}{2\Delta t} \\
\leq \left( \frac{\gamma C_c^2}{4} \left( 1 + \frac{2\beta^2}{1 - \beta^2} \right) + \chi - \alpha \gamma \right) \|C_h z_h^{n+1/2}\|_h^2 - \frac{\kappa}{4} \left( 1 - \beta^2 \right) \left( \|z_h^{n+1}\|_h^2 + \|z_h^n\|_h^2 \right).$$

**Step 3:** We choose  $\alpha$  large enough such that  $\alpha \geq \frac{\chi}{\gamma} + \frac{C_c^2}{4} \left(1 + \frac{2\beta^2}{1 - \beta^2}\right)$ . Thus, we have

$$\frac{1}{2\Delta t} \left( \Pi_h z_h^{n+1}, z_h^{n+1} \right)_h - \frac{1}{2\Delta t} \left( \Pi_h z_h^n, z_h^n \right)_h + \alpha \frac{\|z_h^{n+1}\|_h^2 - \|z_h^n\|_h^2}{2\Delta t} \\ \leq -\frac{\kappa}{4} \left( 1 - \beta^2 \right) \left( \|z_h^{n+1}\|_h^2 + \|z_h^n\|_h^2 \right).$$
(56)

498 Step 4: We introduce the fully discrete Lyapunov functional

$$\mathcal{L}_h^n = (\Pi_h z_h^n, z_h^n)_h + \alpha \|z_h^n\|_h^2,$$

and we can also show that it is equivalent to the energy functional  $\mathcal{E}_h^n := \frac{1}{2} \|z_h^n\|_h^2$  using the bounded nature of  $\Pi_h$  and choosing  $\alpha$  sufficiently large.

$$(\alpha - \|\Pi_h\|_h) \|z_h^n\|_h^2 \le \mathcal{L}_h^n \le (\alpha + \|\Pi_h\|_h) \|z_h^n\|_h^2.$$
(57)

<sup>501</sup> Using the above equivalence between the Lyapunov functional and the energy, we <sup>502</sup> obtain

$$\frac{1}{2\Delta t} \left( \mathcal{L}_h^{n+1} - \mathcal{L}_h^n \right) \leq -\frac{\kappa}{4} \left( 1 - \beta^2 \right) \left( \| \Pi_h \|_h + \alpha \right)^{-1} \left( \mathcal{L}_h^{n+1} + \mathcal{L}_h^n \right).$$

503 Rearranging the terms, we have

$$\mathcal{L}_{h}^{n+1} \leq \delta(\Delta t, \alpha) \mathcal{L}_{h}^{n}.$$
504 We denote  $\delta(\Delta t, \alpha) = \left(1 - \frac{\Delta t \frac{\kappa}{2} (1 - \beta^{2})}{\|\Pi_{h}\|_{h} + \alpha}\right) / \left(1 + \frac{\Delta t \frac{\kappa}{2} (1 - \beta^{2})}{\|\Pi_{h}\|_{h} + \alpha}\right).$ 
505

<sup>506</sup> Step 5: By induction and using again (18), we obtain

$$\|z_{h}^{n}\|_{h}^{2} \leq \frac{\alpha + \|\Pi_{h}\|_{h}}{\alpha - \|\Pi_{h}\|_{h}} \|z_{h,0}\|_{h}^{2} e^{n \ln(\delta(\Delta t, \alpha))}$$

We must now specify the constant  $\ln(\delta)$ . Bounding from below  $\Delta$  by 0 and using a series expansion we get

$$\ln\left(\delta(\Delta t,\alpha)\right) = \ln\left(1 - \frac{\frac{\Delta t \kappa \left(1 - \beta^2\right)}{\|\Pi_h\|_h + \alpha}}{1 + \frac{\Delta t \frac{\kappa}{2} \left(1 - \beta^2\right)}{\|\Pi_h\|_h + \alpha}}\right) \le \ln\left(1 - \Delta t \frac{\kappa \left(1 - \beta^2\right)}{\|\Pi_h\|_h + \alpha}\right)$$
$$\underset{\alpha \to +\infty}{\sim} - \frac{\Delta t}{\alpha} \kappa \left(1 - \beta^2\right) + o\left(\frac{1}{\alpha}\right).$$

507 Finally, we can conclude,

$$\|z_{h}^{n}\|_{h}^{2} \leq \left(\frac{\alpha + \|\Pi_{h}\|_{h}}{\alpha - \|\Pi_{h}\|_{h}}\right) \|z_{h,0}\|_{h}^{2} e^{-\sigma n\Delta t},$$

so with  $\sigma(1-\beta^2) = \frac{\kappa(1-\beta^2)}{\alpha} > 0.$ 

**Remark 6.4.** In many situations, when we consider an unbounded operator C, we can write  $C = \tilde{C}A$  where  $\tilde{C}$  is bounded. So for all h, there exists  $C_A$  independent of h such that,

$$\|C_h\|_h \le C_A \|A_h\|_h, \tag{58}$$

and in this case, we can get rid of condition (53) and condition (54) is sufficient for Theorem 6.3. We recognize "CFL" condition usually encounter with explicit scheme. Nevertheless, it allows us to obtain the exponential stability property.

<sup>516</sup> Under the same two assumptions as in the semi-discrete case, we obtain the <sup>517</sup> properties of observability and exponential stability.

#### 518 6.2 Application: 1D fully discrete wave problem

<sup>519</sup> We can now present the observability theorem in a totally discrete framework.

**Theorem 6.5.** Let  $\{u_h^n, v_h^n\}_{n\geq 0}$  belonging to  $U_h^r \times V_h^r$  be the solutions of (50) with  $\gamma = 0$ . For T large enough, there exists  $C^{st}(T)$  independent of h such that

$$\begin{aligned} \|u_{h,0}\|_{h}^{2} + \|v_{h,0}\|_{h}^{2} &\leq C^{st}(T) \sum_{n=1}^{N} \Delta t \left[ d_{h} \left( u_{h}^{n+1/2}, u_{h}^{n+1/2} \right) + d_{h,*} \left( v_{h}^{n+1/2}, v_{h}^{n+1/2} \right) \right. \\ &\left. + \left( v_{h}^{n+1/2}(1) \right)^{2} \, \mathrm{d}t + \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_{i}} \left[ v_{i}^{n+1/2} \right]^{2} \right], \end{aligned}$$

so where  $d_h$  and  $d_h^*$  are defined by (29) and by (30).

<sup>521</sup> Proof. Since  $\Pi_h$  satisfies Assumption 1, we can apply Theorem 6.2 and the result <sup>522</sup> follows immediately. **Theorem 6.6.** Let  $\{u_h^n, v_h^n\}_{n\geq 0}$  belonging to  $U_h^r \times V_h^r$  be the solutions of (50). We assume that  $\exists \beta > 0$  independent of h and  $\Delta t$ ,  $\forall h$ ,  $\Delta t$  is chosen small enough such

$$\frac{\Delta t}{2} \|R_h\|_h \le \beta < 1.$$
<sup>(59)</sup>

<sup>525</sup> There exists two constants  $C_s > 0$  and  $\sigma > 0$  independent of h such that

$$||u_h^n||_h^2 + ||v_h^n||_h^2 \le C_s e^{-n\Delta t\sigma(\Delta t)} \left( ||u_{h,0}||_h^2 + ||v_{h,0}||_h^2 \right).$$

*Proof.* In order to prove Theorem 6.3, we need to show that the conditions (54) holds.
Since we have the following property

$$||A_h||_h \le C_A ||R_h||_h, \ C_A > 0,$$

 $_{528}$  if (59) holds, then (54) is satisfied.

In order to apply Remark 6.4 – and thus to get rid of the condition (53) –, it only remains to show that

$$||C_h||_h \le C_c ||R_h||_h, \ C_c > 0.$$

<sup>531</sup> It is well known that we have the following inequality

$$\frac{\alpha}{h} \le \|R_h\|_h, \ \alpha > 0$$

so we only need to show that  $||C_h||_h \leq \frac{C^{\text{st}}}{h}$  with  $C^{\text{st}} > 0$ . Let us compute the operator norm of  $C_h$ . By definition of the operator  $C_h$ , we have for  $z_h \in \mathcal{Z}_h$ ,

$$||C_h z_h||_h^2 = d_h(u_h, u_h) + d_{h'*}(v_h, v) + v_h^2(1) + \sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} [\![v_i]\!]^2.$$

We only treat the last term because it is the most difficult. Using Proposition 4.5 and the same kind of computations as in the proof of Theorem 6.3, one can show that

$$\sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} [\![v_i]\!]^2 \le 2 \sum_{i=0}^{N-1} \frac{1+2C_r}{\tilde{h}_i h_i} |\!|v_h|\!|_{L^2([x_i,x_{i+1}])}^2 + 2 \sum_{i=1}^N \frac{1+2C_r}{\tilde{h}_i h_{i-1}} |\!|v_h|\!|_{L^2([x_{i-1},x_i])}^2.$$

536 Since we have a quasi-uniform discretization, we obtain

$$\sum_{i=1}^{N-1} \frac{1}{\tilde{h}_i} [\![v_i]\!]^2 \le 4 \frac{1+2C_r}{\rho^2 h^2} \|v_h\|_{L^2(0,1)}^2.$$

537 We can therefore bound  $||C_h||_h$  by 1/h,

$$\|C_h\|_h \le \left(\sqrt{C_d} + \frac{\sqrt{1+2C_r}}{\sqrt{\rho h}} + \frac{2\sqrt{1+2C_r}}{\rho h}\right).$$

538 Furthermore,  $\forall h, \Delta t$  chosen small enough such that

$$\Delta t \| R_h \|_h \le \beta < 1,$$

satisfies the condition 54 of Theorem 6.3. Since  $\Pi_h$  satisfies Assumption 2 and (58) and (54) holds, we can apply Theorem 6.3 and the result follows immediately.

In perspective, adapting the previous result to the observer analysis, as we did for the semi-discretization in space, should allow us to extend the fully discretized numerical analysis proposed in [33] to high order schemes. This would then imply that we can propose an observer that avoids the asymptotic temporal error drift by using data while remaining compatible with the state of the art of high order discretization.

#### 546 7 Numerical results

We compare the numerical results of Section 2 and those obtained with our method. 547 First we make an analysis of the eigenvalues by comparing our method and other 548 existing methods in the literature. In a second step, for the sake of efficiency of the 549 method, we will introduce a leap-frog scheme which preserves the energy dissipation 550 property of the previous sections. Then we compare the shape of the solution of the 551 damped system with the solution of the undamped system of Section 2. Note that 552 in all the section the term  $D_{h,*}$  is the operator associated with the bilinear form  $d_{h,*}$ 553 defined by (45) namely 554

$$(\tilde{D}_{h,*}v_h, \tilde{v}_h)_h = \tilde{d}_{h,*}(v_h, \tilde{v}_h), \ \forall v_h, \tilde{v}_h \in V_h^r.$$

$$(60)$$

#### 555 7.1 Eigenvalues analysis

<sup>556</sup> We look at the effect of adding the new dissipative terms on the spectrum. We start <sup>557</sup> by doing a comparison with methods used in the literature. A first idea is to penalize <sup>558</sup> the jump as seen in [27] for Maxwell's equations. Thus, we add a dissipation term <sup>559</sup>  $J_h$  corresponding to this penalization of the jump of the second variable which is <sup>560</sup> discontinuous.



**Figure 5.** Spectrum of  $A_h - J_h$ 

We see in Figure 5 (Left) that this method is adapted for finite elements of  $\mathbb{P}_1$ , but that the decay rate is not as good as the continuous one. As soon as we go to higher finite elements in Figure 5 (Right), we find eigenvalues close to the imaginary axis. We also represent the spectrum obtained viscosity is added – as a classical stabilization strategy, (see [23]). We denote by  $V_h$  the discrete Laplace operator on the first component.



Figure 6. Spectrum of  $A_h - J_h - h^2 V_h$  (Left) and of  $A_h - J_h - h^{r+1} V_h$ (Right), r is the order of the method

566

Thus, as we can see in Figure 6 (Left), a second order perturbation gives the exponential stability but with a wrong decay for finite elements of order greater than  $\mathbb{P}_1$ .

Furthermore, the consistency is really deteriorated. If we soften the impact of the viscous term – Figure 6 (Right) –, for instance by replacing the scaling  $h^2$  by  $h^{r+1}$  with the order r of the finite elements, then the exponential stability is not satisfied anymore.



**Figure 7.** Spectrum of  $A_h - J_h - D_h - \tilde{D}_{h,*}$ 

<sup>573</sup> We now use our method. Note that we penalize the jump with the addition of the <sup>574</sup> term  $J_h$ , as in Figure 5.

We preserve the consistency, and we have the uniform exponential stability with a constant which has the same order as what we had at the continuous level as we can see in Figure 7 (Left). When refining – see Figure 7 (Right)–, the consistency is improved. In the end, we found dissipative terms that exponentially stabilize the system while preserving consistency.

#### 580 7.2 A numerical example with an explicit leap-frog scheme

One of the objectives being to have an effective method, we present the leap-frog scheme used in the numerical simulations that guarantees the energy conservation and dissipation properties for damped and undamped systems respectively. We consider the damped wave equation (42) discretized with an explicit scheme where the dissipation terms are off-centered.

$$\begin{cases} \frac{u_h^{n+1} - u_h^n}{\Delta t} - R_h^* v_h^{n+1/2} + \gamma D_h u_h^n = 0\\ \frac{v_h^{n+3/2} - v_h^{n+1/2}}{\Delta t} - R_h u_h^{n+1} + \gamma B_h v_h^{n+1} \\ + \gamma J_h \frac{v_h^{n+3/2} + v_h^{n+1/2}}{2} + \gamma \tilde{D}_{h,*} v_h^{n+1/2} = 0. \end{cases}$$
(61)

The scheme is almost fully explicit thanks to the mass-lumping strategy and off-centering of the newly added dissipative terms  $D_h$  and  $\tilde{D}_{h,*}$ . Note that:

• the centering of the jump term involves, at the algebraic level, to solve (N-1)linear system of size 2 × 2 at each iteration, where N is the number of elements

 off-centering does not imply here loss of consistency in time since it is used on terms that are small when apply to interpolation of smooth solution of the problem. To show the energy identity, we rewrite the scheme using the following trick

$$\begin{cases} D_h u_h^n = D_h u_h^{n+1/2} - \frac{\Delta t}{2} D_h \frac{u_h^{n+1} - u_h^n}{\Delta t}, \\ \tilde{D}_{h,*} v_h^{n+1/2} = \tilde{D}_{h,*} v_h^{n+1} - \frac{\Delta t}{2} \tilde{D}_{h,*} \frac{v_h^{n+3/2} - v_h^{n+1/2}}{\Delta t}, \end{cases}$$

594 thus, we obtain

$$\begin{cases} \tilde{M}_{h} \frac{u_{h}^{n+1} - u_{h}^{n}}{\Delta t} - R_{h}^{*} v_{h}^{n+1/2} + \gamma D_{h} u_{h}^{n+1/2} = 0 \\ \tilde{M}_{h,*} \frac{v_{h}^{n+3/2} - v_{h}^{n+1/2}}{\Delta t} + R_{h} u_{h}^{n+1} + \gamma B_{h} v_{h}^{n+1} + \gamma J_{h} v_{h}^{n+1} + \gamma \tilde{D}_{h,*} v_{h}^{n+1} = 0, \end{cases}$$
(62)

595 where

$$\tilde{M}_h = \mathbb{1} - \gamma \frac{\Delta t}{2} D_h$$
 and  $\tilde{M}_{h,*} = \mathbb{1} - \gamma \frac{\Delta t}{2} \tilde{D}_{h,*}$ .

The energy identity for System (61) involves a new energy functional that is given below: for  $u_h^n \in U_h^r$  and  $v_h^{n+1/2} \in V_h^r$ ,

$$\mathcal{E}_{h,ex}^{n} = \frac{1}{2} \left( \tilde{M}_{h} u_{h}^{n}, u_{h}^{n} \right)_{h} + \frac{1}{2} \left( \tilde{M}_{h,*} v_{h}^{n+1/2}, v_{h}^{n+1/2} \right)_{h} - \frac{\Delta t}{2} \left( R_{h} u_{h}^{n}, v_{h}^{n+1/2} \right)_{h}.$$
 (63)

**Theorem 7.1.** Let  $\{u_h^n, v_h^n\}_{n\geq 0}$  belonging to  $U_h^r \times V_h^r$  be the solutions of (62). The following energy identity holds,

$$\frac{\mathcal{E}_{h,ex}^{n+1} - \mathcal{E}_{h,ex}^{n}}{\Delta t} = -\gamma \left( B_{h} v_{h}^{n+1}, v_{h}^{n+1} \right)_{h} - \gamma \left( J_{h} v_{h}^{n+1}, v_{h}^{n+1} \right)_{h} - \gamma \left( D_{h} u_{h}^{n+1/2}, u_{h}^{n+1/2} \right)_{h} + \gamma \left( \tilde{D}_{h,*} v_{h}^{n+1}, v_{h}^{n+1} \right)_{h} - \gamma \left( D_{h} u_{h}^{n+1/2}, u_{h}^{n+1/2} \right)_{h} + \gamma \left( \tilde{D}_{h,*} v_{h}^{n+1}, v_{h}^{n+1} \right)_{h} - \gamma \left( D_{h} u_{h}^{n+1/2}, u_{h}^{n+1/2} \right)_{h} + \gamma \left( \tilde{D}_{h,*} v_{h}^{n+1}, v_{h}^{n+1} \right)_{h} - \gamma \left( \tilde{D}_{h,*} v_{h}^{n+1}, v_{h}^{n+1} \right)_{h} + \gamma \left( \tilde{D}_{h,*} v_{h}^{n+1} \right)_{h} + \gamma \left( \tilde$$

598 Moreover if  $\forall h, \Delta t$  is chosen small enough such that

$$1 - \gamma \frac{\Delta t}{2} \|D_h\|_h > 0 \quad and \ 1 - \gamma \frac{\Delta t}{2} \|\tilde{D}_{h,*}\|_h > 0, \tag{64}$$

599 and

$$1 - \frac{\Delta t}{2} \|R_h\|_h > 0, \tag{65}$$

600 then  $\mathcal{E}_{h,ex}^n \geq 0$  and the scheme is stable.

*Proof.* We only show here the energy identity. The positivity of the energy and related stability of the scheme are well-known results. We multiply the two equations of (62) by  $u_h^{n+1/2}$  and  $v_h^{n+1}$  respectively and we obtain,

$$\begin{pmatrix} \tilde{M}_{h} \frac{u_{h}^{n+1} - u_{h}^{n}}{\Delta t}, \frac{u_{h}^{n+1} + u_{h}^{n}}{2} \end{pmatrix}_{h} - \begin{pmatrix} R_{h}^{*} v_{h}^{n+1/2}, \frac{u_{h}^{n+1} + u_{h}^{n}}{2} \end{pmatrix}_{h} + \gamma \left( D_{h} u_{h}^{n+1/2}, u_{h}^{n+1/2} \right)_{h} \\ + \left( \tilde{M}_{h,*} \frac{v_{h}^{n+3/2} - v_{h}^{n+1/2}}{\Delta t}, \frac{v_{h}^{n+3/2} + v_{h}^{n+1/2}}{2} \right)_{h} + \left( R_{h} u_{h}^{n+1}, \frac{v_{h}^{n+3/2} + v_{h}^{n+1/2}}{2} \right)_{h} \\ + \gamma \left( B_{h} v_{h}^{n+1}, v_{h}^{n+1} \right)_{h} \gamma \left( J_{h} v_{h}^{n+1}, v_{h}^{n+1} \right)_{h} + \gamma \left( \tilde{D}_{h,*} v_{h}^{n+1}, v_{h}^{n+1} \right)_{h} = 0.$$

We use the follow identity

$$- \left( R_h^* v_h^{n+1/2}, \frac{u_h^{n+1} + u_h^n}{2} \right)_h + \left( R_h u_h^{n+1}, \frac{v_h^{n+3/2} + v_h^{n+1/2}}{2} \right)_h \\ = -\frac{1}{2} \left( R_h^* v_h^{n+1/2}, u_h^n \right)_h + \frac{1}{2} \left( R_h^* v_h^{n+3/2}, u_h^{n+1} \right)_h,$$

to obtain

$$\begin{aligned} \frac{1}{2\Delta t} \left( \tilde{M}_{h} u_{h}^{n+1}, u_{h}^{n+1} \right)_{h} &- \frac{1}{2\Delta t} \left( \tilde{M}_{h} u_{h}^{n}, u_{h}^{n} \right)_{h} + \frac{1}{2\Delta t} \left( \tilde{M}_{h,*} v_{h}^{n+3/2}, v_{h}^{n+3/2} \right)_{h} \\ &- \frac{1}{2\Delta t} \left( \tilde{M}_{h,*} v_{h}^{n+1/2}, v_{h}^{n+1/2} \right)_{h} + \frac{\Delta t}{2} \left( R_{h} u_{h}^{n+1}, v_{h}^{n+3/2} \right)_{h} - \frac{\Delta t}{2} \left( R_{h} u_{h}^{n}, v_{h}^{n+1/2} \right)_{h} \\ &= -\gamma \left( B_{h} v_{h}^{n+1}, v_{h}^{n+1} \right)_{h} + \gamma \left( J_{h} v_{h}^{n+1}, v_{h}^{n+1} \right)_{h} \\ &- \gamma \left( \tilde{D}_{h,*} v_{h}^{n+1}, v_{h}^{n+1} \right)_{h} - \gamma \left( D_{h} u_{h}^{n+1/2}, u_{h}^{n+1/2} \right)_{h}. \end{aligned}$$

We recognize the expression of the explicit energy (63). This concludes the proof.  $\Box$ 

**Remark 7.2.** Note that since  $D_h$  and  $D_{h,*}$  are bounded, condition (64) is not a CFL condition. On the other hand, condition (65) corresponds to the classical CFL condition for explicit schemes.

As in Section 2, we now look at the evolution of the shape of the solution and of the energy over time, with the added damped terms. The numerical pollution due to parasitic waves has completely vanished. We can observe on Figure 8 a totally smooth wave that is not oscillating anymore.



Figure 8. Solution u discretized with  $\mathbb{P}_4$  F.E.M. with h = 1/10 and dt = 1/1000 at time t = 2 (Left) and at time t = 4 (Right)

While the system without the damping terms had an energy that became constant in long time (Figure 9 in pink), we notice that with the addition of these dissipative terms, we obtain a linearly decreasing energy (in logarithm scale) as can be seen on Figure 9 in blue. This confirms the result of Theorem 6.6, we succeeded in making the system uniformly exponentially stable.

### 614 8 Conclusion

<sup>615</sup> By penalizing the jump and introducing an additive dissipative term consistent with <sup>616</sup> high-order finite element schemes, we can adapt multiplier techniques at the fully



discrete level to obtain exponential stabilization results that are uniform with respect to the discretization. The main perspective is to extend the results for 2-dimensional dynamics. The main obstacle lies in the difficulties observed at the continuous level, namely that the variable  $v_h$  must be defined as a gradient on each element, which may not be the case for a weak formulation of the equation. Moreover, we need the discrete energy uniform distribution property for the first order formulation. Finally, it would be necessary to introduce a numerical treatment of the boundary conditions.

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