



**HAL**  
open science

# Discrete-Time Trajectory-Based Stabilization Approach using Interval Observer

Frederic Mazenc, Michael Malisoff, Jackson Knox

► **To cite this version:**

Frederic Mazenc, Michael Malisoff, Jackson Knox. Discrete-Time Trajectory-Based Stabilization Approach using Interval Observer. IEEE Control Systems Letters, 2023, 10.1109/LCSYS.2023.3298068 . hal-04169026v2

**HAL Id: hal-04169026**

**<https://inria.hal.science/hal-04169026v2>**

Submitted on 9 Aug 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

# Discrete-Time Trajectory-Based Stabilization Approach using Interval Observer

Frédéric Mazenc, Michael Malisoff, and Jackson Knox

**Abstract**— We provide a discrete-time vector valued analog of recently developed continuous-time trajectory-based estimates. We use it to provide a discrete-time version of Halanay’s inequality. We combine the results with interval observers to prove exponential stability properties for discrete-time linear systems with uncertainties whose arbitrarily long input delays are compensated for by reduction model controls, and a robust global exponential stability result for observers for discrete-time linear systems.

**Index Terms**— Discrete-time, stability, delay

## I. INTRODUCTION

THE trajectory-based approach is an alternative method for proving asymptotic stability properties that has been developed for continuous-time systems [1]–[3]. In continuous-time for cases without uncertainties and using the usual Euclidean norm  $\|\cdot\|$ , the method calls for finding constants  $\rho \in (0, 1)$  and  $T > 0$  such that

$$\|x(t)\| \leq \rho \max_{s \in [t-T, t]} \|x(s)\| \quad (1)$$

holds for all  $t \geq T$ , to conclude that a continuous function  $x(t)$  exponentially converges to 0 as  $t \rightarrow +\infty$ , but see the references below for cases with uncertainties and vector versions in continuous-time. This technique is useful for the analysis of systems with delays. Halanay’s inequality results [4], based on the pioneering work [5], are also mostly for continuous-time systems, with the notable exception of [6, p. 255] containing a discrete-time version of Halanay’s inequality, and [7]. In its basic form for continuous-time systems, Halanay’s inequality requires finding positive constants  $c$ ,  $d$ , and  $T$  such that

$$\dot{w}(t) \leq -cw(t) + d \max_{s \in [t-T, t]} w(s) \quad (2)$$

for all  $t \geq T$ , to conclude that a  $C^1$  nonnegative scalar valued function  $w(t)$  exponentially converges to 0 as  $t \rightarrow +\infty$  when  $c > d$ ; see the references below for generalizations with time dependent  $c$  and  $d$  where  $c(t) \leq d(t)$  can hold for some  $t$ ’s.

Here, we establish an input-to-state stability (or ISS) type result for discrete-time systems through a vector version of the trajectory-based approach. This lets us prove a vector discrete-time Halanay’s inequality result, which differs from

the discrete-time Halanay’s inequalities in [7], [8] that were confined to scalar valued sequences. We use the theory of positive matrices. We illustrate our result by using it to solve a robust stabilization problem for delay compensation using the reduction model approach, and to prove a convergence result for a class of discrete-time observers; see, e.g., [9]–[12] for motivation for delay compensation and observers. We use interval observers for discrete-time systems, which have been developed in several works, e.g., [13] and [14].

We use standard notation, where the dimensions of our Euclidean spaces are arbitrary, unless otherwise noted, and where all matrix entries are real valued. We set  $\mathbb{Z}_0 = \{0, 1, 2, \dots\}$ . In addition to the usual Euclidean norm  $\|\cdot\|$ , which also denotes the corresponding matrix operator norm, and for a matrix  $G = [g_{ij}] \in \mathbb{R}^{r \times s}$ , we set  $|G| = [|g_{ij}|]$ , so the entries of  $|G|$  are the absolute values of the corresponding entries  $g_{ij}$  of the matrix  $G$ . We also set  $G^+ = [\max\{g_{ij}, 0\}]$  and  $G^- = G^+ - G$ . For matrices  $D = [d_{ij}]$  and  $E = [e_{ij}]$  of the same size, we write  $D < E$  (resp.,  $D \leq E$ ) provided  $d_{ij} < e_{ij}$  (resp.,  $d_{ij} \leq e_{ij}$ ) for all  $i$  and  $j$ . We call a matrix  $S$  positive (resp., nonnegative) provided  $0 < S$  (resp.,  $0 \leq S$ ), where 0 is the zero matrix. For a sequence of matrices  $M_k = [m_{ijk}]$  of the same size and a subset  $J \subseteq \mathbb{Z}_0$ , we set  $\sup_{k \in J} M_k = [m_{*ij}]$  where  $m_{*ij} = \sup\{m_{ijk} : k \in J\}$  when the suprema in the definitions of  $m_{*ij}$  are all finite, i.e., the supremum is entrywise. A square matrix is called Schur stable provided all of its eigenvalues are contained in the open unit disk in the complex plane centered at 0. We let  $\mathbf{1} \in \mathbb{R}^{n \times n}$  denote a matrix whose entries are all 1’s, and  $I$  is the identity matrix. We use standard notions of ISS for discrete-time systems [15].

## II. DISCRETE TIME TRAJECTORY-BASED RESULT

### A. Statement and Discussion of Result

Let  $S > 0$  be a Schur stable matrix. Then using the Perron-Frobenius theorem, we can fix a constant  $p \in (0, 1)$  and a vector  $U \in [1, +\infty)^n$  such that

$$SU = pU. \quad (3)$$

Let us consider any integer  $r > 0$  and any sequence

$$V_i = \begin{pmatrix} v_{1,i} \\ \vdots \\ v_{n,i} \end{pmatrix} \in [0, +\infty)^n \quad (4)$$

indexed by the set of all integers  $i \geq -r$  such that

$$V_{i+1} \leq S \begin{pmatrix} \max\{v_{1,i}, \dots, v_{1,i-r+1}\} \\ \vdots \\ \max\{v_{n,i}, \dots, v_{n,i-r+1}\} \end{pmatrix} + \Delta_i \quad (5)$$

F. Mazenc is with INRIA EPI DISCO, L2S-CNRS-CentraleSupélec, 3 rue Joliot Curie, 91192, Gif-sur-Yvette, France (e-mail: frederic.mazenc@l2s.centralesupelec.fr).

J. Knox and M. Malisoff are with Department of Mathematics, 303 Lockett Hall, 175 Field House Drive, Louisiana State University, Baton Rouge, LA 70803, USA (e-mails: jknox9@lsu.edu, malisoff@lsu.edu).

Supported by US National Science Foundation Grant 2009659 and Office of Naval Research Grant N00014-22-1-2135.

holds for each  $i \in \mathbb{Z}_0$ , where

$$\Delta_i = \begin{pmatrix} \delta_{1,i} \\ \vdots \\ \delta_{n,i} \end{pmatrix} \quad (6)$$

is a nondecreasing nonnegative sequence, meaning

$$0 \leq \Delta_i \leq \Delta_j \quad (7)$$

when  $i \leq j$ . We use the sequences

$$\Omega_i = (I - S)^{-1} \Delta_i \text{ and } \Gamma_i = cp^{\frac{i}{r}} U + \Omega_i \quad (8)$$

where  $p$  and  $r$  are as above and  $c$  is any constant such that

$$\frac{1}{p} \max_{i \in \{-r, \dots, 0\}} \{\max\{v_{1,i}, \dots, v_{n,i}\}\} < c, \quad (9)$$

and where the invertibility of  $I - S$  follows since  $S$  is Schur stable. We prove:

*Theorem 1:* With the preceding notation, the inequality

$$V_i < \Gamma_i \quad (10)$$

is satisfied for all  $i \in \mathbb{Z}_0$ .

*Remark 1:* Since  $p \in (0, 1)$  implies that  $p^{\frac{1}{r}} \in (0, 1)$ , we can use (8) and (10) to obtain an ISS type of inequality for the sequence  $V_i$  with an exponential decay rate of  $-\ln(p)/r$ . One cannot expect to get a better estimated convergence rate than this rate of convergence for the sequence  $V_i$ . To see why, consider the scalar valued sequence

$$v_i = e^{\frac{\ln(p)}{r}i} \quad (11)$$

whose rate of exponential convergence to 0 is  $-\ln(p)/r$ . Then

$$v_{i+1} = pe^{\frac{\ln(p)}{r}(i+1-r)} = pv_{i+1-r}$$

for all  $i \in \mathbb{Z}_0$ . Thus, for all  $i \in \mathbb{Z}_0$ , we have

$$v_{i+1} \leq p \max\{v_i, \dots, v_{i-r+1}\}. \quad (12)$$

Hence, there is a sequence satisfying the assumptions of Theorem 1 and whose rate of convergence equals the estimate of the rate of convergence given by Theorem 1. The nondecreasing property (7) is not restrictive at all, because any nonnegative sequence  $\Delta_{ai}$  can be replaced by the new sequence that is defined by  $\Delta_i = \sup_{0 \leq j \leq i} \Delta_{aj}$ , which is nondecreasing as a function of  $i$ .

*Remark 2:* Using Theorem 1, one can straightforwardly establish a vector discrete-time Halanay's inequality result. Indeed, consider a nonnegative valued sequence  $V_i$  such that

$$V_{i+1} \leq S_a V_i + S_b \begin{pmatrix} \max\{v_{1,i}, \dots, v_{1,i-r+1}\} \\ \vdots \\ \max\{v_{n,i}, \dots, v_{n,i-r+1}\} \end{pmatrix} + \Delta_i \quad (13)$$

for all  $i \in \mathbb{Z}_0$ , where  $S_a \geq 0$ ,  $S_b \geq 0$ , and  $S_a + S_b > 0$  is Schur stable and  $r > 0$ . Then the Halanay inequality (13) yields

$$V_{i+1} \leq (S_a + S_b) \begin{pmatrix} \max\{v_{1,i}, \dots, v_{1,i-r+1}\} \\ \vdots \\ \max\{v_{n,i}, \dots, v_{n,i-r+1}\} \end{pmatrix} + \Delta_i \quad (14)$$

for all  $i \in \mathbb{Z}_0$ , which is of the form (5) from Theorem 1. This lets us establish an ISS type of inequality using Theorem 1.

## B. Proof of Theorem 1

From (9), we deduce that  $v_{j,i} < pc$  for all  $j \in \{1, \dots, n\}$  and all  $i \in \{-r, \dots, 0\}$ . Since  $U \in [1, +\infty)^n$ , it follows that

$$v_{j,i} < pc u_j \text{ for all } j \in \{1, \dots, n\} \quad (15)$$

for all  $i \in \{-r, \dots, 0\}$ , where  $u_j$  is the  $j$ th entry of  $u$ . Since  $p \in (0, 1)$  and  $\frac{i}{r} \leq 1$ , it follows from (15) that

$$v_{j,i} < p^{\frac{i}{r}} c u_j \text{ for all } j \in \{1, \dots, n\} \quad (16)$$

for all  $i \in \{-r, \dots, 0\}$ . Since the geometric series gives

$$(I - S)^{-1} = I + S + S^2 + \dots > 0, \quad (17)$$

we have  $\Omega_i \geq 0$  for all  $i$ , so the inequalities (16) imply that  $V_i < \Gamma_i$  for all  $i \in \{-r, \dots, 0\}$ . We next prove that

$$V_i < \Gamma_i \quad (18)$$

for all  $i \geq 1$ . We proceed by contradiction.

Let us suppose that there were an integer  $k_c \geq 1$  such that

$$V_j < \Gamma_j \quad (19)$$

when  $j \in \{-r, \dots, k_c - 1\}$  and an  $m \in \{1, \dots, n\}$  such that  $v_{m,k_c} \geq \gamma_{m,k_c}$ , where  $\gamma_{m,k_c}$  is the  $m$ th entry of  $\Gamma_{k_c}$ . Then our choice of  $\Gamma_i$  in (8), (19), and the choice of  $k_c$  imply that

$$v_{j,k_c-\ell} < \gamma_{j,k_c-\ell} = cp^{\frac{k_c-\ell}{r}} u_j + \omega_{j,k_c-\ell} \quad (20)$$

for  $j = 1, \dots, n$  and  $\ell = 1, \dots, r$ , where  $\omega_{j,i}$  is the  $j$ th entry of  $\Omega_i$  for each  $i$ . Hence, (5) applied with  $i = k_c - 1$  gives

$$\begin{aligned} \gamma_{m,k_c} &\leq v_{m,k_c} < \\ &\hat{s}_m \begin{pmatrix} \max\left\{cp^{\frac{k_c-1}{r}} u_1 + \omega_{1,k_c-1}, \dots, cp^{\frac{k_c-r}{r}} u_1 + \omega_{1,k_c-r}\right\} \\ \vdots \\ \max\left\{cp^{\frac{k_c-1}{r}} u_n + \omega_{n,k_c-1}, \dots, cp^{\frac{k_c-r}{r}} u_n + \omega_{n,k_c-r}\right\} \end{pmatrix} \\ &+ \delta_{m,k_c-1}, \end{aligned} \quad (21)$$

where  $\hat{s}_m$  denotes the  $m$ th row of  $S$ , and where the strict inequality in (21) came from using the strict inequalities in (20) to bound each of the terms in the maxima in (5) when  $i = k_c - 1$ . Again using (17), we deduce from (7) that each  $\omega_{j,i}$  is nondecreasing as a function of  $i$  for  $j = 1$  to  $n$ . Hence,

$$\begin{aligned} \gamma_{m,k_c} &< \hat{s}_m \begin{pmatrix} \max\left\{cp^{\frac{k_c-1}{r}} u_1, \dots, cp^{\frac{k_c-r}{r}} u_1\right\} \\ \vdots \\ \max\left\{cp^{\frac{k_c-1}{r}} u_n, \dots, cp^{\frac{k_c-r}{r}} u_n\right\} \end{pmatrix} \\ &+ \hat{s}_m \begin{pmatrix} \omega_{1,k_c-1} \\ \vdots \\ \omega_{n,k_c-1} \end{pmatrix} + \delta_{m,k_c-1}. \end{aligned} \quad (22)$$

Since  $p \in (0, 1)$ , we conclude that

$$\gamma_{m,k_c} < \hat{s}_m \begin{pmatrix} cp^{\frac{k_c-r}{r}} u_1 \\ \vdots \\ cp^{\frac{k_c-r}{r}} u_n \end{pmatrix} + \hat{s}_m \begin{pmatrix} \omega_{1,k_c-1} \\ \vdots \\ \omega_{n,k_c-1} \end{pmatrix} + \delta_{m,k_c-1}. \quad (23)$$

Since  $SU = pU$ , our formula for  $\Gamma_i$  in (8) and (23) give

$$cp^{\frac{k_c}{r}} u_m + \omega_{m,k_c} = \gamma_{m,k_c} < cp^{\frac{k_c}{r}} u_m + \hat{s}_m \Omega_{k_c-1} + \delta_{m,k_c-1}. \quad (24)$$

By cancelling  $cp^{\frac{k_c}{r}}u_m$  from both sides of (24), we get

$$\omega_{m,k_c} < \hat{s}_m \Omega_{k_c-1} + \delta_{m,k_c-1}. \quad (25)$$

Now, observe that

$$\Omega_i = [(I - S)^{-1} - I]\Delta_i + \Delta_i = S(I - S)^{-1}\Delta_i + \Delta_i \quad (26)$$

for all  $i \in \mathbb{Z}_0$ . Consequently, since  $S(I - S)^{-1} \geq 0$  (again by the geometric sum formula (17)) and  $\Delta_i$  is nondecreasing, we get

$$\Omega_i \geq S(I - S)^{-1}\Delta_{i-1} + \Delta_i = S\Omega_{i-1} + \Delta_i \quad (27)$$

for all  $i \geq 1$ . In particular, this inequality implies that  $\omega_{m,k_c} \geq \hat{s}_m \Omega_{k_c-1} + \delta_{m,k_c}$ , which we can combine with (25) to get  $\hat{s}_m \Omega_{k_c-1} + \delta_{m,k_c-1} > \hat{s}_m \Omega_{k_c-1} + \delta_{m,k_c}$ . Hence,  $\delta_{m,k_c-1} > \delta_{m,k_c}$ , which contradicts the fact that  $\delta_{m,\ell}$  is nondecreasing with respect to  $\ell$ . This concludes the proof.

### III. STABILIZATION OF SYSTEM WITH DELAY IN INPUT

#### A. Statement of Result and Remarks

Consider the system

$$x_{i+1} = (A + \delta_i)x_i + B\mu_{i-h} + \lambda_i \quad (28)$$

indexed by  $i \in \mathbb{Z}_0$ , where  $x_i$  is valued in  $\mathbb{R}^n$ , the input  $\mu_i$  is valued in  $\mathbb{R}^q$  and has the initial values  $\mu_i = 0$  for all  $i < 0$ , the unknown bounded sequence  $\delta_i$  is valued in  $\mathbb{R}^{n \times n}$ , the unknown bounded sequence  $\lambda_i$  is valued in  $\mathbb{R}^n$ ,  $A \in \mathbb{R}^{n \times n}$  and  $B \in \mathbb{R}^{n \times q}$  are known matrices, and  $h \in \mathbb{Z}_0$  is a known positive constant that models a delay. We assume:

*Assumption 1:* The matrix  $A$  is invertible, and there is a matrix  $K$  such that

$$H = A + BK \quad (29)$$

is nonnegative. Also, there are known compact sets  $\Delta_i \subseteq \mathbb{R}^{n \times n}$  such that  $\delta_i \in \Delta_i$  for each  $i$  and such that the matrix

$$S_0 = H + \sup_{i \in \mathbb{Z}_0} \sup_{M \in \Delta_i} \left[ |A^h M A^{-h}| + \sum_{m=1}^h |A^h M A^{m-1-h} B K| \right] \quad (30)$$

is Schur stable.

We prove the following result. In this result, the sufficiently small constant  $\epsilon_0 > 0$  is used to transform (30) (which is not necessarily positive) into a Schur stable positive matrix  $S$  in order to apply the Perron-Frobenius Theorem. Also, the existence of such an  $\epsilon_0$  follows from the continuity of eigenvalues of a matrix as functions of the entries of the matrix (which follows, e.g., from the corresponding argument from [16], and which ensures that  $S > 0$  is Schur stable when  $S_0$  is Schur stable and when  $\epsilon_0 > 0$  is small enough), so the existence of  $\epsilon_0$  is not at all a restrictive condition. Also, we cannot pick  $\epsilon_0 = 0$  in general, because some entries of  $S_0$  could be zero, and because we need  $S$  to be positive.

*Theorem 2:* Let (28) satisfy Assumption 1, and fix constants  $\epsilon_0 > 0$  and  $p \in (0, 1)$  and a vector  $U \in [1, +\infty)^n$  such that  $SU = pU$  is satisfied with  $S = S_0 + \epsilon_0 \mathbf{1}$ , where  $S_0$  is defined in (30), and where  $\epsilon_0 > 0$  is chosen small enough such that  $S$  is positive and Schur stable. Choose the control

$$\mu_i = K \left[ A^h x_i + \sum_{\ell=0}^{h-1} A^{h-\ell-1} B \mu_{i+\ell-h} \right] \quad (31)$$

for all  $i \in \mathbb{Z}_0$  where  $\mu_i = 0$  for all  $i < 0$  and the other notation is as above. Choose any constant  $\omega_0 > 1$  and the constant

$$C_* = \max \left\{ \|A^{-h}\|, \sum_{\ell=0}^{h-1} \|A^{-\ell-1} B K\| \right\}. \quad (32)$$

Then all solutions  $x : \mathbb{Z}_0 \rightarrow \mathbb{R}^n$  of (28) in closed loop with (31) are such that

$$\|x_i\| \leq \frac{4\omega_0 C_*}{p} \|A^h\| \|U\| \|x_0\| p^{\frac{i-h}{h+1}} + n C_* \|(I - S)^{-1}\| \|A^h\| \max\{\|\lambda_j\| : 0 \leq j \leq i\} \quad (33)$$

holds for all  $i \in \mathbb{Z}_0$ .

*Remark 3:* We can satisfy Assumption 1 after a change of coordinates (and extend Theorem 2 to the case where  $H$  is not nonnegative) by assuming that  $(A, B)$  is controllable with  $A$  invertible, and then using changes of coordinates to get a system for which the corresponding  $H$  is nonnegative as was done in [14]. This entails choosing  $K$  such that the eigenvalues of  $A + BK$  are distinct real numbers on  $(0, 1)$  (which can be done by the Pole-Shifting Theorem [17], or the StateFeedbackGains command in Mathematica), and then replacing  $A$ ,  $B$ , and  $K$  by  $P^{-1}AP$ ,  $P^{-1}B$  and  $KP$  respectively, where  $P$  is such that  $P^{-1}(A + BK)P \geq 0$  is diagonal and Schur, so the nonnegativity of  $H$  holds after a change of coordinates (given by a similarity transformation [17]), and then  $S_0$  will be Schur when the  $\Delta_i$ 's are small enough sets containing the 0 matrix. The sets  $\Delta_i$  can be chosen to satisfy this smallness requirement (and with all of the  $\Delta_i$ 's being equal) when the  $|\delta_i|$ 's have small enough entries. Our Assumption 1 implies that the  $\delta_i$ 's do not preclude global exponential stabilization when the  $|\delta_i|$ 's are small enough.

*Remark 4:* The Schur stability and structure of  $S_0$  imply that  $H$  is Schur stable (e.g., because for any solutions  $w_i$  and  $q_i$  of  $w_{i+1} = Hw_i$  and  $q_{i+1} = S_0 q_i$  respectively having any standard basis element as the same initial state, it follows from induction on  $i$  and nonnegativity of  $H$ ,  $S_0$ ,  $S_0 - H$ , and the  $q_i$ 's that  $q_{i+1} - w_{i+1} = H(q_i - w_i) + (S_0 - H)q_i \geq H(q_i - w_i) \geq 0$  for all  $i \in \mathbb{Z}_0$ , so the  $w_i$  dynamics are exponentially stable to 0 because its dynamics are linear and because the  $q_i$  dynamics are exponentially stable to 0). Invertibility of  $A$  is unrestrictive, since it is generic in the sense of [17, p.96], i.e., the set of all noninvertible matrices in  $\mathbb{R}^{n \times n}$  is a set of measure 0 in  $\mathbb{R}^{n^2}$  by using an isomorphism to view  $\mathbb{R}^{n \times n}$  as  $\mathbb{R}^{n^2}$  arranged as rows and columns (because the set of all noninvertible matrices are the roots of the determinant function from this space into  $\mathbb{R}$ , and the set of all roots of any polynomial function has measure 0). Our constant  $\omega_0$  is used to ensure that (9) holds when we apply Theorem 1 in the next subsection. See Remark 5 for relationships between our assumptions and bounds and those of other predictor results and the need for invertibility of  $A$ . Formula (31) is the nominal prediction of  $x(i+h)$ , neglecting uncertainties and forcing terms, and is found by a change of variable in the sum.

#### B. Proof of Theorem 2

The proof uses the theory of the interval observers. We set

$$z_i = A^h x_i + \sum_{\ell=0}^{h-1} A^{h-\ell-1} B \mu_{i+\ell-h} \quad (34)$$

for all  $i \in \mathbb{Z}_0$ , and  $z_i = 0$  for all  $i < 0$ . Then (28) gives

$$\begin{aligned}
z_{i+1} &= A^h x_{i+1} + \sum_{\ell=0}^{h-1} A^{h-\ell-1} B \mu_{i+1+\ell-h} \\
&= A^{h+1} x_i + A^h B \mu_{i-h} + A^h \delta_i x_i \\
&\quad + \sum_{m=1}^h A^{h-m} B \mu_{i+m-h} + A^h \lambda_i \\
&= A^{h+1} x_i + B \mu_i + A^h B \mu_{i-h} \\
&\quad + \sum_{m=1}^{h-1} A^{h-m} B \mu_{i+m-h} + A^h \delta_i x_i + A^h \lambda_i \\
&= A^{h+1} x_i + \sum_{m=0}^{h-1} A^{h-m} B \mu_{i+m-h} \\
&\quad + B \mu_i + A^h \delta_i x_i + A^h \lambda_i
\end{aligned} \tag{35}$$

by the change of index  $m = \ell + 1$ , where the inequalities and equalities in the proof are for all  $i \in \mathbb{Z}_0$  unless indicated otherwise. By (34) and (35), we get  $z_{i+1} = A z_i + B \mu_i + A^h \delta_i x_i + A^h \lambda_i$ , and (31) gives  $\mu_i = K z_i$ . Hence,  $z_{i+1} = H z_i + A^h \delta_i x_i + A^h \lambda_i$ , where  $H$  is from (29), so (34) gives:

$$\begin{cases} z_{i+1} = H z_i + A^h \delta_i x_i + A^h \lambda_i \\ z_i = A^h x_i + \sum_{\ell=0}^{h-1} A^{h-\ell-1} B K z_{i+\ell-h} \end{cases} \tag{36}$$

Since  $A$  is invertible, we combine the equalities (36) to get

$$z_{i+1} = H z_i + A^h \delta_i \left( A^{-h} z_i - \sum_{\ell=0}^{h-1} A^{-\ell-1} B K z_{i+\ell-h} \right) + A^h \lambda_i. \tag{37}$$

Thus, since the change of index  $m = h - \ell$  gives

$$\sum_{\ell=0}^{h-1} A^{h-\ell-1} B K z_{i+\ell-h} = \sum_{m=1}^h A^{m-1} B K z_{i-m}, \tag{38}$$

we get

$$z_{i+1} = H z_i + \delta_{*i} z_i - \delta_{*i} \sum_{m=1}^h A^{m-1} B K z_{i-m} + A^h \lambda_i \tag{39}$$

where  $\delta_{*i} = A^h \delta_i A^{-h}$  for each  $i \in \mathbb{Z}_0$ .

Bearing in mind that Assumption 1 requires that  $H$  is nonnegative, we next use the following interval observer:

$$\begin{cases} \bar{z}_{i+1} = H \bar{z}_i + \delta_{*i}^+ \bar{z}_i - \delta_{*i}^- \bar{z}_i \\ \quad + \sum_{m=1}^h (-\delta_{*i} A^{m-1} B K)^+ \bar{z}_{i-m} \\ \quad - \sum_{m=1}^h (-\delta_{*i} A^{m-1} B K)^- \bar{z}_{i-m} + (A^h \lambda_i)^+ \\ \underline{z}_{i+1} = H \underline{z}_i + \delta_{*i}^+ \underline{z}_i - \delta_{*i}^- \underline{z}_i \\ \quad + \sum_{m=1}^h (-\delta_{*i} A^{m-1} B K)^+ \underline{z}_{i-m} \\ \quad - \sum_{m=1}^h (-\delta_{*i} A^{m-1} B K)^- \underline{z}_{i-m} - (A^h \lambda_i)^- \end{cases} \tag{40}$$

The interval observer (40) was obtained by splitting factors of the last three right side terms in (39) into their positive and negative parts (which are indicated by superscripts  $+$  and  $-$  respectively), to provide dynamics for  $\bar{z}_i$  and  $\underline{z}_i$  that ensure that the framing property that we describe next is satisfied.

Choosing initial conditions for (40) such that  $\underline{z}_{-m} \leq 0 \leq \bar{z}_{-m}$  and  $-2|z_{-m}| \leq \underline{z}_{-m} \leq z_{-m} \leq \bar{z}_{-m} \leq 2|z_{-m}|$  hold for

all  $m \in \{0, \dots, h+1\}$ , and rewriting  $\delta_{*i}$  in (39) as  $(\delta_{*i})^+ - (\delta_{*i})^-$  and similarly for  $-\delta_{*i} A^{m-1} B K$  and  $A^h \lambda_i$  in (39), we deduce from the structure of (39)-(40) that  $-\underline{z}_i \geq 0$ ,  $\bar{z}_i \geq 0$ ,  $\bar{z}_i - z_i \geq 0$ , and  $z_i - \underline{z}_i \geq 0$  hold for all  $i \in \mathbb{Z}_0$ , and therefore that  $\bar{z}_i - \underline{z}_i \geq z_i \geq \underline{z}_i - \bar{z}_i$  and

$$|z_i| \leq \bar{z}_i - \underline{z}_i \tag{41}$$

hold for all  $i \in \mathbb{Z}_0$ , by arguing by induction on  $i$  to show that  $[0, +\infty)^{2n}$  is a forwardly invariant set for the dynamics for  $(\bar{z}_i, -\underline{z}_i)$  and for the dynamics for  $(\bar{z}_i - z_i, z_i - \underline{z}_i)$ . Hence, (40) is a framer for  $z_i$ . We next use the bound (41) to prove that the dynamics for  $z_i$  satisfy a suitable exponential stability condition. This will allow us to use (34) and the relation  $\mu_i = K z_i$  to obtain the conclusion of the theorem.

To this end, we use the sequence  $P_i = \bar{z}_i - \underline{z}_i$ . Then, by (40),

$$P_{i+1} = (H + |\delta_{*i}|) P_i + \sum_{m=1}^h |\delta_{*i} A^{m-1} B K| P_{i-m} + |A^h \lambda_i| \tag{42}$$

because  $|M| = M^+ + M^-$  holds for each matrix  $M$ , and the preceding paragraph ensures that  $P_i \in [0, +\infty)^n$  for all  $i$ . Recalling our choice  $S = S_0 + \epsilon_0 \mathbf{1}$ , we can use (42) to get

$$P_{i+1} \leq S \begin{pmatrix} \max_{m \in \{0, \dots, h\}} p_{1, i-m} \\ \vdots \\ \max_{m \in \{0, \dots, h\}} p_{n, i-m} \end{pmatrix} + \sup_{0 \leq j \leq i} |A^h \lambda_j| \tag{43}$$

where  $p_{j, i-m}$  is the  $j$ th entry of  $P_{i-m}$  for all  $m$ . Choosing

$$C_0 = \frac{\omega_0}{p} \max\{\max\{p_{1, i}, \dots, p_{n, i}\} : i = -h-1, \dots, 0\}, \tag{44}$$

we now apply Theorem 1 with  $r = h+1$  and  $c = C_0$  to get

$$P_i \leq C_0 p^{\frac{i}{h+1}} U + (I - S)^{-1} \sup_{0 \leq j \leq i} |A^h \lambda_j| \tag{45}$$

for all  $i \in \mathbb{Z}_0$ . Also, since  $z_m = 0$  and  $\mu_m = 0$  hold for all  $m < 0$ , and since (34) gives  $z_0 = A^h x_0$ , our initial conditions for (40) give  $P_i = \bar{z}_i - \underline{z}_i = 0$  for all  $i < 0$  and  $P_0 = \bar{z}_0 - \underline{z}_0 \leq 4|z_0| = 4|A^h x_0|$ . Therefore, our choice (44) gives

$$C_0 \leq \frac{4\omega_0}{p} \|A^h\| \|x_0\|. \tag{46}$$

Since (41) holds, we can combine (45) and (46) to obtain

$$\begin{aligned}
\|z_i\| &\leq \|\bar{z}_i - \underline{z}_i\| \\
&\leq \frac{4\omega_0}{p} \|A^h\| \|x_0\| p^{\frac{i}{h+1}} \|U\| \\
&\quad + S^\# \max\{\|\lambda_j\| : 0 \leq j \leq \max\{i, 0\}\} \end{aligned} \tag{47}$$

for all  $i \geq -h-1$ , where  $S^\# = n\|(I - S)^{-1}\| \|A^h\|$ . Also, (34) and the formula  $\mu_i = K z_i$  give

$$x_i = A^{-h} z_i - \sum_{\ell=0}^{h-1} A^{-\ell-1} B K z_{i+\ell-h} \tag{48}$$

for all  $i \in \mathbb{Z}_0$ . Therefore,

$$\begin{aligned}
\|x_i\| &\leq \|A^{-h}\| \|z_i\| \\
&\quad + \sum_{\ell=0}^{h-1} \|A^{-\ell-1} B K\| \max\{\|z_j\| : i-h \leq j \leq i-1\} \end{aligned} \tag{49}$$

so our choice of the constant  $C_*$  in (32) gives

$$\|x_i\| \leq C_* \max\{\|z_j\| : i-h \leq j \leq i\} \tag{50}$$

for all  $i \in \mathbb{Z}_0$ . Theorem 2 follows by using (47) to upper bound the right side of (50).

*Remark 5:* The exponential convergence rate  $-\ln(p)/(h+1)$  in (33) gives a trade-off, where longer delays  $h$  can reduce the convergence rate of  $x_i$  to zero, but a smaller  $p \in (0, 1)$  can increase this rate. This contrasts significantly with prior treatments of predictors, which did not use the nonnegative and Schur conditions on  $S_0$  from (30) or matrices of absolute values as we used them in Assumption 1. Hence, our use of  $S_0$  is significantly different from prior treatments of predictive control. The invertibility of  $A$  was used in (48). It was also used in (37), for solving for  $x_i$  in the second equation in (36) which can be written as  $A^h x_i = b_i$ , where  $b_i = z_i - S_i$  and  $S_i$  is the summation on the right side of the second equality in (36). Since we place no additional restrictions on  $x_i$  and  $b_i$ , we therefore require invertibility of  $A^h$  (since being able to solve  $A^h \bar{x} = \bar{b}$  for  $\bar{x}$  for any  $\bar{b}$  is equivalent to invertibility of  $A^h$ ). On the other hand, invertibility of  $A^h$  is equivalent to invertibility of  $A$ , e.g., because  $\det(A^h) = (\det(A))^h$ , and because a square matrix is invertible if and only if its determinant is nonzero.

## IV. ILLUSTRATIONS

### A. Scalar Example

Consider the scalar system

$$x_{i+1} = (2 + \delta_i)x_i + \mu_{i-h} \quad (51)$$

with  $\delta_i \geq 0$  for all  $i \in \mathbb{Z}_0$  and  $h > 1$ . Let us select  $K = -\frac{3}{2}$ . Then, with the notation of Section III,  $H = \frac{1}{2}$  and

$$H + |\delta_{*i}| + \sum_{m=1}^h |\delta_{*i} A^{m-1} B K| = \frac{1}{2} + \delta_i + \delta_i \sum_{m=1}^h 2^{m-1} \frac{3}{2}. \quad (52)$$

It follows from the geometric sum formula  $1 + a + \dots + a^n = \frac{1-a^{n+1}}{1-a}$  applied with  $a = 2$  that

$$H + |\delta_{*i}| + \sum_{m=1}^h |\delta_{*i} A^{m-1} B K| = \frac{1}{2} + \left[1 + \frac{3(2^h - 1)}{2}\right] \delta_i. \quad (53)$$

From Theorem 2, we deduce that the condition

$$\sup_{i \geq 0} \delta_i < \frac{1}{2+3(2^h-1)} \quad (54)$$

is sufficient for the exponential stabilization. See also Section IV-C for a two-dimensional example with simulations.

### B. Observer Example

Consider this system with an output indexed by  $i \in \mathbb{Z}_0$ :

$$x_{i+1} = A x_i + \delta_i x_i, \quad y_i = C x_i \quad (55)$$

where the  $\delta_i$ 's are known, and  $A \in \mathbb{R}^{n \times n}$  and  $C \in \mathbb{R}^{q \times n}$  are known matrices where  $A$  is invertible and  $n \geq 2$ . We assume there is a  $\bar{\delta} \geq 0$  in  $\mathbb{R}^{n \times n}$  such that  $|\delta_i| \leq \bar{\delta}$  for all  $i \geq 0$ , and we place conditions on  $\bar{\delta} \in \mathbb{R}^{n \times n}$  ensuring that an observation error for an observer exponentially converges to 0.

We assume that the pair  $(A, C)$  is observable. Therefore, by the equivalences of the Hautus Criterion (e.g., [17, p.94]), the pair  $((A^{-1})^\top, C^\top)$  is controllable, so the observability matrix

$$\mathcal{M} = [C^\top, (CA^{-1})^\top, \dots, (CA^{-n+1})^\top]^\top \quad (56)$$

has full rank  $n$ . Also, by induction on  $j \in \{2, \dots, n\}$  and the relation  $x_{i-j} = A^{-1}(x_{i-j+1} - \delta_{i-j} x_{i-j})$  for  $i \geq j$ , we have  $x_{i-j+1} = A^{-j+1} x_i - A^{-j+1} \delta_{i-1} x_{i-1} - \dots - A^{-1} \delta_{i-j+1} x_{i-j+1}$  for each  $j \in \{2, \dots, n\}$  when  $i \geq n$ , so

$$\begin{aligned} y_i &= C x_i \\ y_{i-1} &= CA^{-1} x_i - CA^{-1} \delta_{i-1} x_{i-1} \\ &\vdots \\ y_{i-n+1} &= CA^{-n+1} x_i - CA^{-n+1} \delta_{i-1} x_{i-1} \\ &\quad - \dots - CA^{-1} \delta_{i-n+1} x_{i-n+1} \end{aligned} \quad (57)$$

for each  $i \geq n$ . Using the simplifying notation  $Y_i = [y_i, y_{i-1}, \dots, y_{i-n+1}]^\top$ , and letting  $\mathcal{N}$  denote the left inverse  $\mathcal{N} = (\mathcal{M}^\top \mathcal{M})^{-1} \mathcal{M}^\top$  of  $\mathcal{M}$ , we rewrite (57) as

$$x_i = \mathcal{N} Y_i + \mathcal{N} \begin{pmatrix} 0 \\ CA^{-1} \delta_{i-1} x_{i-1} \\ \vdots \\ C \sum_{j=1}^{n-1} A^{-n+j} \delta_{i-j} x_{i-j} \end{pmatrix}. \quad (58)$$

In terms of our known sequence  $\delta_i$ , we introduce the observer

$$\hat{x}_i = \mathcal{N} Y_i + \mathcal{N} \begin{pmatrix} 0 \\ CA^{-1} \delta_{i-1} \hat{x}_{i-1} \\ \vdots \\ C \sum_{j=1}^{n-1} A^{-n+j} \delta_{i-j} \hat{x}_{i-j} \end{pmatrix}. \quad (59)$$

Let us introduce the error variable:  $\tilde{x}_i = x_i - \hat{x}_i$ . It satisfies

$$|\tilde{x}_i| \leq \underbrace{|\mathcal{N}| \begin{pmatrix} 0 \\ |CA^{-1}| \bar{\delta} \\ \vdots \\ \sum_{j=1}^{n-1} |CA^{-n+j}| \bar{\delta} \end{pmatrix}}_{\mathcal{N}^\sharp} \max_{\ell \in \{i-n+1, \dots, i-1\}} |\tilde{x}_\ell|. \quad (60)$$

Then we apply Theorem 1 to  $v_i = |\tilde{x}_{i+n}|$ , to get a condition on  $\bar{\delta}$  ensuring that  $\mathcal{N}^\sharp$  in (60) is Schur stable so  $\lim_{i \rightarrow +\infty} \tilde{x}_i = 0$ . Therefore, a sufficient condition for state estimation for (55) is that the following conditions both hold: (i) the pair  $(A, C)$  is observable and (ii) there is a  $\bar{\delta} \in \mathbb{R}^{n \times n}$  such that  $|\delta_i| \leq \bar{\delta}$  for all  $i \in \mathbb{Z}_0$  and such that  $\mathcal{N}^\sharp$  in (60) is a Schur stable matrix in  $\mathbb{R}^{n \times n}$ . This is a new sufficient condition for the state estimation, as compared with earlier observer designs, and since (55) is a time-varying system, it is not amenable to standard treatments involving the observability matrix.

In Fig. 1, we illustrate the preceding observer by plotting the unmeasured second component  $x_{i2}$  of the state  $x_i = (x_{i1}, x_{i2})$ , the estimator value  $\hat{x}_{i2}$ , and the estimation error  $\tilde{x}_{i2}$  when

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^\top. \quad (61)$$

The red trajectory in Fig. 1 was generated using the initial state  $x_0 = [5, 5]$  for (55) and with a  $\delta_i$  whose entries were chosen at each step to be random values in the interval  $[0, 1]$ . The blue trajectory was generated using the initial state  $x_0 = [-5, -5]$  for (55) and with the cyclic sequence of

diagonal matrices  $\delta_i$  given by  $\{D_1, D_2, D_3, D_1, D_2, D_3, \dots\}$ , where  $D_1 = \text{diag}\{0.8, 0.2\}$ ,  $D_2 = \text{diag}\{0.7, 0.3\}$ , and  $D_3 = \text{diag}\{0.6, 0.4\}$ . Finally, the green trajectory was generated using  $x_0 = [2, -2]^T$  for (55) and the fixed choice  $\delta_i = \text{diag}\{0.2, 0.5\}$ . In each case, we chose the initial values  $\hat{x}_{-m} = 0$  for all  $m \geq 0$  for the observer. The required Schur stability condition was satisfied when the lower right entry of each  $\delta_i$  is bounded in absolute value by 1. The plots were generated using Python. Since they exhibit desirable rapid convergence of the observation error to zero, they help illustrate our theory, in the special case of (61). We also repeated the preceding simulations in the cyclic case with  $D_1 = \text{diag}\{0.8, 1\}$ ,  $D_2 = \text{diag}\{0.7, 0.9\}$ , and  $D_3 = \{0.6, 1.1\}$  replacing the choices above, and in the fixed case with  $\delta_i = \text{diag}\{0.2, 1.1\}$ , but with all other choices the same as before, and in both of these cases,  $\tilde{x}_2$  diverged away from 0, which suggests that our bound 1 is not too conservative.

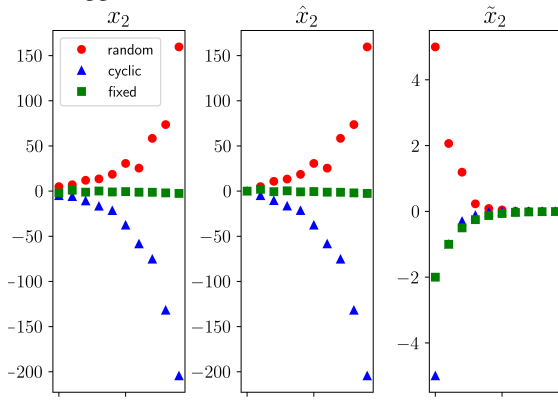


Fig. 1. Three trajectories for  $x_{i2}$  (left),  $\hat{x}_{i2}$  (middle), and  $\tilde{x}_{i2}$  (right)

### C. Feedback Control Example

We apply Theorem 2 in the special case where  $h = 2$ ,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ and } K = \begin{bmatrix} -1 & 1 \\ & 2 \end{bmatrix} \quad (62)$$

and  $\delta_i = (-1)^i(0.16)I$  which satisfy the assumptions of Theorem 2 with  $\Delta_i = \{M \in \mathbb{R}^{2 \times 2} : M = RI, -0.16 \leq R \leq 0.16\}$  for each  $i$ . In Fig. 2, we plot the closed loop trajectory for the initial state  $x_0 = [10, -10]^T$ , again using Python. Since  $x_i$  converges to 0 with a small undershoot corresponding to the  $\delta_i$ 's, it illustrates Theorem 2. It shows oscillations that end by time  $t = 10$ . The transient behavior of these oscillations does not follow from Theorem 2. Instead, it can be attributed to convergence of the state to 0 that mitigates the effects of the oscillations by damping them. We also repeated the preceding simulation with 0.16 replaced by 0.24, 0.26, 0.36, and 0.46 and all other values the same as before, and in each of these four cases, the solution instead diverged away from 0. This suggests that our bound 0.16 is not too conservative.

## V. CONCLUSION

We proposed a discrete-time vector trajectory-based approach, which provided a discrete-time analog of previous continuous-time trajectory-based methods. Novel techniques in our analysis included positive systems and discrete-time interval observer approaches. We illustrated our work by

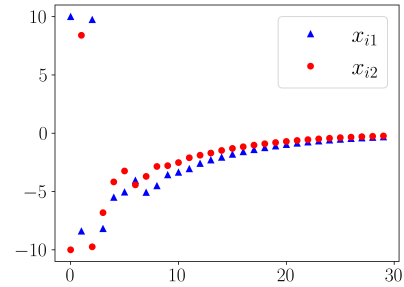


Fig. 2. Closed-loop solutions from Theorem 2 using (62)

proving a new ISS type of estimate for a linear discrete-time system with an uncertain coefficient matrix and a reduction model controller. We also proved an exponential convergence property for an observer, which led to a new condition on the magnitude of the allowable time-varying portion of a vector field. We aim to prove generalizations that provide observers for nonlinear discrete-time systems with disturbances or poorly known parameters, which can be viewed as unknown, or poorly known, uncertainties.

## REFERENCES

- [1] F. Mazenc and M. Malisoff, "Trajectory based approach for the stability analysis of nonlinear systems with time delays," *IEEE Transactions on Automatic Control*, vol. 60, no. 6, pp. 1716–1721, 2015.
- [2] F. Mazenc, M. Malisoff, and M. Krstic, "Stability analysis using generalized sup-delay inequalities," *IEEE Control Systems Letters*, vol. 5, no. 4, pp. 1411–1416, 2020.
- [3] F. Mazenc, M. Malisoff, and S.-I. Niculescu, "Stability and control design for time-varying systems with time-varying delays using a trajectory-based approach," *SIAM Journal on Control and Optimization*, vol. 55, no. 1, pp. 533–556, 2017.
- [4] P. Pepe, "A nonlinear version of Halanay's inequality for the uniform convergence to the origin," *Mathematical Control and Related Fields*, vol. 12, no. 3, pp. 789–811, 2022.
- [5] A. Halanay, *Differential Equations: Stability, Oscillations, Time Lags*. New York, NY: Academic Press, 1966.
- [6] E. Fridman, *Introduction to Time-Delay Systems: Analysis and Control*. Boston, MA: Springer, 2014.
- [7] E. Liz, A. Ivanov, and J. Ferreiro, "Discrete Halanay-type inequalities and applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 55, no. 6, pp. 669–678, 2003.
- [8] M. Grifa and P. Pepe, "On stability analysis of discrete-time systems with constrained time-delays via nonlinear Halanay-type inequality," *IEEE Control Systems Letters*, vol. 5, no. 3, pp. 869–874, 2021.
- [9] R. Lozano, P. Castillo, P. Garcia, and A. Dzul, "Robust prediction-based control for unstable delay systems: Application to the yaw control of a mini-helicopter," *Automatica*, vol. 40, no. 4, pp. 603–612, 2004.
- [10] F. Mazenc, V. Andrieu, and M. Malisoff, "Design of continuous-discrete observers for time-varying nonlinear systems," *Automatica*, vol. 57, pp. 135–144, 2015.
- [11] F. Mazenc, M. Malisoff, and Z.-P. Jiang, "Reduced-order fast converging observers for systems with discrete measurements and measurement error," *Systems & Control Letters*, vol. 150, article 104892, 2021.
- [12] F. Sauvage, M. Guay, and D. Dochain, "Design of a nonlinear finite-time converging observer for a class of nonlinear systems," *Journal of Control Science and Engineering*, vol. 2007, article 036954, 2007.
- [13] D. Efimov, W. Perruquetti, T. Raissi, and A. Zolghadri, "Interval observers for time-varying discrete-time systems," *IEEE Transactions on Automatic Control*, vol. 58, no. 12, pp. 3218–3224, 2013.
- [14] F. Mazenc, T. Dinh, and S.-I. Niculescu, "Interval observers for discrete-time systems," *International Journal of Robust and Nonlinear Control*, vol. 24, no. 17, pp. 2867–2890, 2014.
- [15] Z.-P. Jiang and Y. Wang, "Input-to-state stability for discrete-time nonlinear systems," *Automatica*, vol. 37, no. 6, pp. 857–869, 2001.
- [16] F. Mazenc and M. Malisoff, "Continuous discrete sequential observers for time-varying systems under sampling and input delays," *IEEE Transactions on Automatic Control*, vol. 65, no. 4, pp. 1704–1709, 2019.
- [17] E. Sontag, *Mathematical Control Theory, Second Edition*. New York, NY: Springer, 1998.