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# Fast and Frobenius: Rational Isogeny Evaluation over Finite Fields

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**Abstract.** Consider the problem of efficiently evaluating isogenies  $\phi : \mathcal{E} \rightarrow \mathcal{E}/H$  of elliptic curves over a finite field  $\mathbb{F}_q$ , where the kernel  $H = \langle G \rangle$  is a cyclic group of odd (prime) order: given  $\mathcal{E}$ ,  $G$ , and a point (or several points)  $P$  on  $\mathcal{E}$ , we want to compute  $\phi(P)$ . This problem is at the heart of efficient implementations of group-action- and isogeny-based post-quantum cryptosystems such as CSIDH. Algorithms based on Vélu's formulæ give an efficient solution to this problem when the kernel generator  $G$  is defined over  $\mathbb{F}_q$ . However, for general isogenies,  $G$  is only defined over some extension  $\mathbb{F}_{q^k}$ , even though  $\langle G \rangle$  as a whole (and thus  $\phi$ ) is defined over the base field  $\mathbb{F}_q$ ; and the performance of Vélu-style algorithms degrades rapidly as  $k$  grows. In this article we revisit the isogeny-evaluation problem with a special focus on the case where  $1 \leq k \leq 12$ . We improve Vélu-style isogeny evaluation for many cases where  $k = 1$  using special addition chains, and combine this with the action of Galois to give greater improvements when  $k > 1$ .

## 1 Introduction

Faced with the rising threat of quantum computing, demand for quantum-secure, or post-quantum, cryptographic protocols is increasing. Isogenies have emerged as a useful candidate for post-quantum cryptography thanks to their generally small key sizes, and the possibility of implementing post-quantum group actions which offer many simple post-quantum analogues of classical discrete-log-based algorithms (see e.g. [27]).

A major drawback of isogeny-based cryptosystems is their relatively slow performance compared with many other post-quantum systems. In this paper, we improve evaluation times for isogenies of many prime degrees  $\ell > 3$  given a generator of the kernel; these computations are the fundamental building blocks

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\* Authors listed in alphabetical order: see <https://www.ams.org/profession/leaders/CultureStatement04.pdf>.

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of most isogeny-based cryptosystems. Specifically, we propose simple alternative differential addition chains to enumerate points of (subsets of) the kernel more efficiently. This speeds up many  $\ell$ -isogeny computations over the base field by a factor depending on  $\ell$ , and also permits a full additional factor-of- $k$  speedup for  $\ell$ -isogenies over  $\mathbb{F}_q$  whose kernel generators are defined over an extension  $\mathbb{F}_{q^k}$ .

Our techniques have constructive and destructive applications. First, accelerating basic isogeny computations can speed up isogeny-based cryptosystems. The methods in §4 apply for many  $\ell > 3$ , so they would naturally improve the performance of commutative isogeny-based schemes such as CSIDH [5], and CSI-FiSh [4] and its derivatives (such as [12] and [14]), which require computing many  $\ell$ -isogenies for various primes  $\ell$ . They may also improve the performance of other schemes like SQISign [17], which computes many  $\ell$ -isogenies in its signing process. (We discuss applications further in §6.)

In §5 we focus on rational isogenies with irrational kernels; our methods there could be used to improve the performance of Couveignes–Rostovtsev–Stolbunov key exchange (CRS) and related protocols of Stolbunov [11, 25, 28, 29], further accelerating the improvements of [16]. This is a small step forward on the road to making CRS a practical “ordinary” fallback for CSIDH in the event of new attacks making specific use of the full supersingular isogeny graph (continuing the approach of [6], for example).

Our results also have applications in cryptanalysis: the best classical and quantum attacks on commutative isogeny-based schemes involve computing massive numbers of group actions, each comprised of a large number of  $\ell$ -isogenies (see e.g. [3] and [8]). Any algorithm that reduces the number of basic operations per  $\ell$ -isogeny will improve the effectiveness of these attacks.

*Disclaimer.* In this paper, we quantify potential speedups by counting finite field operations. We make no predictions of real-world speed increases, since these depend on too many additional variables including parameter sizes; the application context; implementation choices; the runtime platform (including the specificities of the architecture, vectorization, and hardware acceleration); and the availability of optimized low-level arithmetic.

## 2 Background

We work over (extensions of) the base field  $\mathbb{F}_q$ , where  $q$  is a power of a prime  $p > 3$ . The symbol  $\ell$  always denotes a prime  $\neq p$ . In our applications,  $3 < \ell \ll p$ .

*Elliptic curves.* For simplicity, in this work every elliptic curve will be supposed to be in a general Weierstrass form  $\mathcal{E} : y^2 = f(x)$ . Our algorithms and applications are focused on<sup>4</sup> *Montgomery models*

$$\mathcal{E} : By^2 = x(x^2 + Ax + 1) \quad \text{where} \quad B(A^2 - 4) \neq 0.$$

<sup>4</sup> We will focus exclusively on Montgomery models, since these are the most common in isogeny-based cryptography, but our results extend easily to other models such as traditional short Weierstrass models (for number-theoretic applications).

The multiplication-by- $m$  map is denoted by  $[m]$ . The  $q$ -power Frobenius endomorphism is  $\pi : (x, y) \mapsto (x^q, y^q)$ .

*Field operations.* While the curve  $\mathcal{E}$  will always be defined over  $\mathbb{F}_q$ , we will often work with points defined over  $\mathbb{F}_{q^k}$  for  $k \geq 1$ . We write  $\mathbf{M}$ ,  $\mathbf{S}$ , and  $\mathbf{a}$  for the cost of multiplication, squaring, and adding (respectively) in  $\mathbb{F}_{q^k}$ . We write  $\mathbf{C}$  for the cost of multiplying an element of  $\mathbb{F}_{q^k}$  by an element of  $\mathbb{F}_q$  (typically a curve constant, or an evaluation-point coordinate). Note that  $\mathbf{C} \approx (1/k)\mathbf{M}$  (when  $k$  is not too large). Later, we will write  $\mathbf{F}$  for the cost of evaluating the Frobenius map on  $\mathbb{F}_{q^k}$ ; see §5.1 for discussion on this.

*$x$ -only arithmetic.* Montgomery models are designed to optimize  $x$ -only arithmetic (see [22] and [10]). The  $\mathbf{xADD}$  operation is

$$\mathbf{xADD} : (x(P), x(Q), x(P - Q)) \mapsto x(P + Q);$$

it can be computed at a cost of  $4\mathbf{M} + 2\mathbf{S} + 6\mathbf{a}$  using the formulæ

$$\begin{cases} X_+ = Z_- [(X_P - Z_P)(X_Q + Z_Q) + (X_P + Z_P)(X_Q - Z_Q)]^2, \\ Z_+ = X_- [(X_P - Z_P)(X_Q + Z_Q) - (X_P + Z_P)(X_Q - Z_Q)]^2 \end{cases}, \quad (1)$$

(where  $(X_P : Z_P)$ ,  $(X_Q : Z_Q)$ ,  $(X_+ : Z_+)$ , and  $(X_- : Z_-)$  are the  $x$ -coordinates  $x(P)$ ,  $x(Q)$ ,  $x(P + Q)$ , and  $x(P - Q)$ , respectively).

The  $\mathbf{xDBL}$  operation is

$$\mathbf{xDBL} : x(P) \mapsto x([2]P);$$

it can be computed at a cost of  $2\mathbf{M} + 2\mathbf{S} + \mathbf{C} + 4\mathbf{a}$  using the formulæ

$$\begin{cases} X_{[2]P} = (X_P + Z_P)^2(X_P - Z_P)^2, \\ Z_{[2]P} = (4X_P Z_P)((X_P - Z_P)^2 + ((A + 2)/4)(4X_P Z_P)). \end{cases} \quad (2)$$

*Isogenies.* Let  $\mathcal{E}_1, \mathcal{E}_2$  be elliptic curves over a finite field  $\mathbb{F}_q$ . An isogeny  $\phi : \mathcal{E}_1 \rightarrow \mathcal{E}_2$  is a non-constant morphism mapping the identity point of  $\mathcal{E}_1$  to the identity point of  $\mathcal{E}_2$ . Such a morphism is automatically a homomorphism. For more details see [26, Chapter 3, §4]. The kernel of  $\phi$  is a finite subgroup of  $\mathcal{E}_1$ , and vice versa: every finite subgroup  $\mathcal{G}$  of  $\mathcal{E}_1$  determines a separable *quotient isogeny*  $\mathcal{E}_1 \rightarrow \mathcal{E}_1/\mathcal{G}$ .

Let  $G$  be the generator of the kernel group. The kernel polynomial can be expressed as:

$$D(X) := \prod_{P \in S} (X - x(P))$$

where  $S \subset \langle G \rangle$  is any subset that satisfies the conditions:

$$S \cap -S = \emptyset \quad \text{and} \quad S \cup -S = \langle G \rangle \setminus \{0\}. \quad (3)$$

Every separable isogeny  $\phi : \mathcal{E}_1 \rightarrow \mathcal{E}_2$  defined over  $\mathbb{F}_q$  can be represented by a rational map in the form

$$\phi : (x, y) \mapsto (\phi_x(x), \phi_y(x, y)) \quad (4)$$

with

$$\phi_x(x) = \frac{N(x)}{D(x)^2} \quad \text{and} \quad \phi_y(x, y) = c \cdot y \frac{d\phi_x}{dx}(x)$$

where  $D$  is the kernel polynomial of  $\phi$ ,  $N$  is a polynomial derived from  $D$ , and  $c$  is a normalizing constant in  $\mathbb{F}_q$ .

*Vélu's formulæ.* Given a curve  $\mathcal{E}$  and a finite subgroup  $\mathcal{G} \subset \mathcal{E}$ , Vélu [30] gives explicit formulæ for the rational functions that define a separable isogeny  $\phi : \mathcal{E} \rightarrow \mathcal{E}' := \mathcal{E}/\mathcal{G}$  with kernel  $\mathcal{G}$ , as well as the resulting codomain curve  $\mathcal{E}'$ . Although the quotient curve  $\mathcal{E}'$  and the isogeny  $\phi$  are defined up to isomorphism, Vélu's formulæ construct a unique *normalized* isogeny, ensuring that if  $\omega$  and  $\omega'$  are the invariant differentials on  $\mathcal{E}$  and  $\mathcal{E}'$ , respectively, then  $\phi^*(\omega') = \omega$ .

See Kohel's Thesis [20, §2.4] for more details about explicit isogenies and a treatment of Vélu's results better-adapted to finite fields. For more information concerning isogenies and their use in cryptography we refer the reader to [13].

### 3 Evaluating isogenies

Let  $\mathcal{E}$  be an elliptic curve over  $\mathbb{F}_q$ , and let  $\langle G \rangle$  be a subgroup of prime order  $\ell$  (where  $\ell$  is not equal to the field characteristic  $p$ ). We suppose  $\langle G \rangle$  is defined over  $\mathbb{F}_q$ ; then, the quotient isogeny  $\phi : \mathcal{E} \rightarrow \mathcal{E}/\langle G \rangle$  is also defined over  $\mathbb{F}_q$ .

When we say  $\langle G \rangle$  is defined over  $\mathbb{F}_q$ , this means  $\langle G \rangle$  is *Galois stable*: that is,  $\pi(\langle G \rangle) = \langle G \rangle$  (where  $\pi$  is the  $q$ -power Frobenius endomorphism). We will mostly be concerned with algorithms taking  $x(G)$  as an input, so it is worth noting that

$$x(G) \in \mathbb{F}_{q^{k'}} \quad \text{where} \quad k' := \begin{cases} k & \text{if } k \text{ is odd,} \\ k/2 & \text{if } k \text{ is even.} \end{cases}$$

The set of projective  $x$ -coordinates of the nonzero kernel points is

$$\mathcal{X}_G := \{(X_P : Z_P) = x(P) : P \in \langle G \rangle \setminus \{0\}\} \subset \mathbb{P}^1(\mathbb{F}_{q^{k'}});$$

each  $X_P/Z_P$  corresponds to a root of the kernel polynomial  $D(X)$ , and vice versa. If  $\#\langle G \rangle$  is an odd prime  $\ell$ , then  $\#\mathcal{X}_G = (\ell - 1)/2$ .

#### 3.1 The isogeny evaluation problem

We want to evaluate the isogeny  $\phi : \mathcal{E} \rightarrow \mathcal{E}/\langle G \rangle$ . More precisely, we want efficient solutions to the problem of Definition 1:

**Definition 1 (Isogeny Evaluation).** *Given an elliptic curve  $\mathcal{E}$  over  $\mathbb{F}_q$ , a list of points  $(P_1, \dots, P_n)$  in  $\mathcal{E}(\mathbb{F}_q)$ , and a finite subgroup  $\mathcal{G}$  of  $\mathcal{E}$  corresponding to the separable isogeny  $\phi : \mathcal{E} \rightarrow \mathcal{E}/\mathcal{G}$ , compute  $(\phi(P_1), \dots, \phi(P_n))$ .*

In most cryptographic applications, the number  $n$  of evaluation points is relatively small, especially compared to the isogeny degree  $\ell$ . We do *not* assume the codomain curve  $\mathcal{E}/\mathcal{G}$  is known. If required, an equation for the codomain curve can be interpolated through the image of well-chosen evaluation points.

For each separable isogeny  $\phi$  of degree  $d$  defined over  $\mathbb{F}_q$ , there exists a sequence of primes  $(\ell_1, \dots, \ell_n)$  and a sequence of isogenies  $(\phi_1, \dots, \phi_n)$ , all defined over  $\mathbb{F}_q$ , such that  $\phi_n \circ \dots \circ \phi_1$  and

- $\phi_i = [\ell_i]$  (the non-cyclic case) or
- $\phi_i$  has cyclic kernel of order  $\ell_i$ .

The kernel of  $\phi_1$  is  $\ker \phi \cap \mathcal{E}[\ell_1]$ , and so on. The multiplication maps  $[\ell_i]$  can be computed in  $O(\log \ell_i)$   $\mathbb{F}_q$ -operations, so we reduce quickly to the case where  $\phi$  has prime degree  $\ell$ , assuming the factorization of  $d$  is known (which is always the case in our applications).

In general, the isogeny evaluation problem can be reduced to evaluating the map  $\alpha \mapsto D(\alpha)$ , where  $D$  is the kernel polynomial and  $\alpha$  is in  $\mathbb{F}_q$  or some  $\mathbb{F}_q$ -algebra (see e.g. [2, §4]). We note that the polynomial  $D$  does *not* need to be explicitly computed itself.

### 3.2 The Costello-Hisil algorithm

The Costello-Hisil algorithm [9] is the state-of-the-art for evaluating isogenies.<sup>5</sup> This algorithm is a variation of Vélu’s formulæ working entirely on the level of  $x$ -coordinates, using the fact that for an  $\ell$ -isogeny  $\phi$  of Montgomery models with kernel  $\langle G \rangle$ , the rational map on  $x$ -coordinates is

$$\phi_x(x) = x \cdot \left( \prod_{i=1}^{(\ell-1)/2} \left( \frac{x \cdot x([i]G) - 1}{x - x([i]G)} \right) \right)^2. \quad (5)$$

Moving to projective coordinates  $(U : V)$  such that  $x = U/V$  and using the fact that  $\mathcal{X}_G = \{(x([i]G) : 1) : 1 \leq i \leq (\ell-1)/2\}$ , Eq. (5) becomes

$$\phi_x((U : V)) = (U' : V')$$

where

$$\begin{cases} U' = U \left[ \prod_{(X_Q:Z_Q) \in \mathcal{X}_G} (UX_Q - VZ_Q) \right]^2, \\ V' = V \left[ \prod_{(X_Q:Z_Q) \in \mathcal{X}_G} (UZ_Q - VX_Q) \right]^2. \end{cases} \quad (6)$$

Algorithm 1 (from [9]) and Algorithm 2 (our space-efficient variant) compute  $\phi_x$  at a series of input points using an efficient evaluation of the expressions in (6). For the moment, we assume that we have subroutines

<sup>5</sup> The algorithms of [9] focus on odd-degree isogenies, treating 2 and 4-isogenies as special cases; Renes [24] extends the approach to even-degree isogenies, and provides a satisfying theoretical framework.

- **KernelPoints** (for Algorithm 1): given  $(X_G : Z_G)$ , returns  $\mathcal{X}_G$  as a list.
- **KernelRange** (for Algorithm 2): a *generator* coroutin which, given  $(X_G : Z_G)$ , constructs and yields the elements of  $\mathcal{X}_G$  to the caller one by one.
- **CrissCross** (for Algorithms 1 and 2, from [9, Algorithm 1]) takes  $(\alpha, \beta, \gamma, \delta)$  in  $\mathbb{F}_{q^k}^4$  and returns  $(\alpha\delta + \beta\gamma, \alpha\delta - \beta\gamma)$  in  $\mathbb{F}_{q^k}^2$  at a cost of  $2\mathbf{M} + 2\mathbf{a}$ .

We discuss algorithms to implement **KernelPoints** and **KernelRange** in §4

---

**Algorithm 1:** Combines Algorithms 3 and 4 from [9] to evaluate an  $\ell$ -isogeny of Montgomery models at a list of input points. The total cost is  $2n\ell\mathbf{M} + 2n\mathbf{S} + ((n+1)(\ell+1) - 2)\mathbf{a}$ , plus the cost of **KernelPoints**.

---

**Input:** The  $x$ -coordinate  $(X_G : Z_G)$  of a generator  $G$  of the kernel of an  $\ell$ -isogeny  $\phi$ , and a list of evaluation points  $((U_i : V_i) : 1 \leq i \leq n)$

**Output:** The list of images  $((U'_i : V'_i) = \phi_x((U_i : V_i)) : 1 \leq i \leq n)$

```

1  $((X_1, Z_1), \dots, (X_{(\ell-1)/2}, Z_{(\ell-1)/2})) \leftarrow \text{KernelPoints}((X_G : Z_G))$  // See §4
2 for  $1 \leq i \leq (\ell-1)/2$  do
3    $(\hat{X}_i, \hat{Z}_i) \leftarrow (X_i + Z_i, X_i - Z_i)$  // 2a
4 for  $i = 1$  to  $n$  do
5    $(\hat{U}_i, \hat{V}_i) \leftarrow (U_i + V_i, U_i - V_i)$  // 2a
6    $(U'_i, V'_i) \leftarrow (1, 1)$ 
7   for  $j = 1$  to  $(\ell-1)/2$  do
8      $(t_0, t_1) \leftarrow \text{CrissCross}(\hat{X}_j, \hat{Z}_j, \hat{U}_i, \hat{V}_i)$  // 2M + 2a
9      $(U'_i, V'_i) \leftarrow (t_0 \cdot U'_i, t_1 \cdot V'_i)$  // 2M
10     $(U'_i, V'_i) \leftarrow (U_i \cdot (U'_i)^2, V_i \cdot (V'_i)^2)$  // 2M + 2S
11 return  $((U'_1, V'_1), \dots, (U'_n, V'_n))$ 

```

---

## 4 Accelerating Vélu: faster iteration over the kernel

Let  $\mathcal{E}/\mathbb{F}_q$  be an elliptic curve, and let  $G$  be a point of prime order  $\ell$  in  $\mathcal{E}$ . For simplicity, in this section we will assume that  $G$  is defined over  $\mathbb{F}_q$ , but all of the results here apply when  $G$  is defined over an extension  $\mathbb{F}_{q^k}$ : in that case, the only change is that  $\mathbf{M}$ ,  $\mathbf{S}$ , and  $\mathbf{a}$  represent operations in the extension field  $\mathbb{F}_{q^k}$ , while  $\mathbf{C}$  represents multiplication of an element of  $\mathbb{F}_{q^k}$  by a curve constant of the subfield  $\mathbb{F}_q$  (which is roughly  $k$  times cheaper than  $\mathbf{M}$ ). We will return to the case where  $G$  is defined over an extension in §5, where we can combine results from this section with the action of Frobenius.

### 4.1 Kernel point enumeration and differential addition chains

We now turn to the problem of enumerating the set  $\mathcal{X}_G$ . This process, which we call *kernel point enumeration*, could involve constructing the entire set (as in **KernelPoints**) or constructing its elements one by one (for **KernelRange**).

---

**Algorithm 2:** A generator-based version of Algorithm 1, with much lower space requirements when  $\ell \gg n$ . The total cost is  $2n\ell\mathbf{M} + 2n\mathbf{S} + (2n + (\ell - 1)(n + 1))\mathbf{a}$ , plus the cost of a full run of `KernelRange`.

---

**Input:** The  $x$ -coordinate  $(X_G : Z_G)$  of a generator  $G$  of the kernel of an  $\ell$ -isogeny  $\phi$ , and a list of evaluation points  $((U_i : V_i) : 1 \leq i \leq n)$

**Output:** The list of images  $((U'_i : V'_i) = \phi_x((U_i : V_i)) : 1 \leq i \leq n)$

```

1 for  $1 \leq i \leq n$  do
2    $(\hat{U}_i, \hat{V}_i) \leftarrow (U_i + V_i, U_i - V_i)$  // 2a
3    $(U'_i, V'_i) \leftarrow (1, 1)$ 
4 for  $(X : Z)$  in KernelRange $((X_G : Z_G))$  do // See §4
5    $(\hat{X}, \hat{Z}) \leftarrow (X + Z, X - Z)$  // 2a
6   for  $1 \leq i \leq n$  do
7      $(t_0, t_1) \leftarrow \text{CrissCross}(\hat{X}, \hat{Z}, \hat{U}_i, \hat{V}_i)$  // 2M + 2a
8      $(U'_i, V'_i) \leftarrow (t_0 \cdot U'_i, t_1 \cdot V'_i)$  // 2M
9 for  $1 \leq i \leq n$  do
10   $(U'_i, V'_i) \leftarrow (U_i \cdot (U'_i)^2, V_i \cdot (V'_i)^2)$  // 2M + 2S
11 return  $((U'_1, V'_1), \dots, (U'_n, V'_n))$ 

```

---

For  $\ell = 2$  and  $3$ , there is nothing to be done because  $\mathcal{X}_G = \{(X_G : Z_G)\}$ ; so from now on we consider the case  $\ell > 3$ .

We allow ourselves two curve operations for kernel point enumeration: `xADD` and `xDBL`. In §5, where  $G$  is assumed to be defined over a nontrivial extension of the base field, we will also allow the Frobenius endomorphism.

Every algorithm constructing a sequence of elements of  $\mathcal{X}_G$  using a series of `xADD` and `xDBL` instructions corresponds to a *modular differential addition chain*.

**Definition 2.** A *Modular Differential Addition Chain (MDAC)* for a set  $S \subset \mathbb{Z}/\ell\mathbb{Z}$  is a sequence of integers  $(c_0, c_1, c_2, \dots, c_n)$  such that

1. every element of  $S$  is represented by some  $c_i \pmod{\ell}$ ,
2.  $c_0 = 0$  and  $c_1 = 1$ , and
3. for each  $1 < i \leq n$  there exist  $0 \leq j(i), k(i), d(i) < i$  such that  $c_i \equiv c_{j(i)} + c_{k(i)} \pmod{\ell}$  and  $c_{j(i)} - c_{k(i)} \equiv c_{d(i)} \pmod{\ell}$ .

Algorithms to enumerate  $\mathcal{X}_G$  using `xADD` and `xDBL` correspond to MDACs  $(c_0, \dots, c_n)$  for  $\{1, \dots, (\ell - 1)/2\}$ : the algorithm starts with  $x([c_0]G) = x(0) = (1 : 0)$  and  $x([c_1]G) = x(G) = (X_G : Z_G)$ , then computes each  $x([c_i]G)$  using

$$x([c_i]G) = \begin{cases} \text{xADD}(x([c_{j(i)}]G), x([c_{k(i)}]G), x([c_{d(i)}]G)) & \text{if } d(i) \neq 0, \\ \text{xDBL}([c_{j(i)}]G) & \text{if } d(i) = 0. \end{cases}$$

## 4.2 Additive kernel point enumeration

The classic approach is to compute  $\mathcal{X}_G$  using repeated `xADD`s. Algorithm 3 is Costello and Hisil's `KernelPoints` function [9, Algorithm 2]. This corresponds



to the MDAC  $(0, 1, 2, 3, \dots, (\ell - 1)/2)$  computed by repeatedly adding 1 (in the notation of Definition 2,  $(j(i), k(i), d(i)) = (i - 1, 1, i - 2)$ ), except for 2 which is computed by doubling 1. The simplicity of this MDAC means that Algorithm 3 adapts almost trivially to `KernelRange` using a relatively small internal state: in order to generate the next  $(X_{i+1} : Z_{i+1})$ , we need only keep the values of  $(X_i : Z_i)$ ,  $(X_{i-1} : Z_{i-1})$ , and  $(X_1 : Z_1)$ .

---

**Algorithm 3:** Basic kernel point enumeration by repeated addition.

Uses exactly 1 `xDBL` and  $(\ell - 5)/2$  `xADD` operations (for prime  $\ell > 3$ ).

---

**Input:** The  $x$ -coordinate  $(X_G : Z_G)$  of the generator  $G$  of a cyclic subgroup of order  $\ell$  in  $\mathcal{E}(\mathbb{F}_q)$

**Output:**  $\mathcal{X}_G$  as a list

```

1  $(X_1 : Z_1) \leftarrow (X_G : Z_G)$ 
2  $(X_2 : Z_2) \leftarrow \text{xDBL}((X_G : Z_G))$ 
3 for  $i = 3$  to  $(\ell - 1)/2$  do           // Invariant:  $(X_i : Z_i) = x([i]G)$ 
4    $(X_i : Z_i) \leftarrow \text{xADD}((X_{i-1} : Z_{i-1}), (X_G : Z_G), (X_{i-2}, Z_{i-2}))$ 
5 return  $((X_1 : Z_1), \dots, (X_{(\ell-1)/2} : Z_{(\ell-1)/2}))$ 

```

---

### 4.3 Replacing `xADDs` with `xDBLs`

Comparing  $x$ -only operations on Montgomery curves, replacing an `xADD` with an `xDBL` trades **2M** and **2a** for **1C**. We would therefore like to replace as many `xADDs` as possible in our kernel enumeration with `xDBLs`.

As a first attempt, we can replace Line 4 of Algorithm 3 with

$$(X_i : Z_i) \leftarrow \begin{cases} \text{xDBL}((X_{i/2} : Z_{i/2})) & \text{if } i \text{ is even,} \\ \text{xADD}((X_{i-1} : Z_{i-1}), (X_G : Z_G), (X_{i-2}, Z_{i-2})) & \text{if } i \text{ is odd.} \end{cases}$$

But applying this trick systematically requires storing many more intermediate values, reducing the efficiency of `KernelRange`. It also only replaces half of the `xADDs` with `xDBLs`, and it turns out that we can generally do much better.

### 4.4 Multiplicative kernel point enumeration

We can do better for a large class of  $\ell$  by considering the quotient

$$M_\ell := (\mathbb{Z}/\ell\mathbb{Z})^\times / \langle \pm 1 \rangle.$$

(We emphasize that  $M_\ell$  is a quotient of the *multiplicative* group.) For convenience, we write

$$m_\ell := \#M_\ell = (\ell - 1)/2.$$

We can now reframe the problem of enumerating  $\mathcal{X}_G$  as the problem of enumerating a complete set of representatives for  $M_\ell$ . The MDAC of Algorithm 3

computes the set of representatives  $\{1, 2, \dots, m_\ell\}$ , but for the purposes of enumerating  $\mathcal{X}_G$ , *any* set of representatives will do. Example 1 is particularly useful.

*Example 1.* Suppose 2 generates  $M_\ell$ . This is the case if 2 is a primitive element modulo  $\ell$ —that is, if 2 has order  $(\ell-1)$  modulo  $\ell$ —but also if 2 has order  $(\ell-1)/2$  modulo  $\ell$  and  $\ell \equiv 3 \pmod{4}$ . In this case

$$M_\ell = \{2^i \bmod \ell : 0 \leq i < m_\ell\},$$

so  $(0, 1, 2, 4, 8, \dots, 2^{m_\ell})$  is an MDAC for  $M_\ell$  that can be computed using *only* doubling, and *no* differential additions.

The `KernelPoints` and `KernelRange` driven by the MDAC of Example 1 replace *all* of the  $(\ell-1)/2 - 2$  `xADDs` in Algorithm 3 with cheaper `xDBLs`: we save  $\ell - 5$  `M` and  $\ell - 5$  `a` at the cost of  $(\ell - 5)/2$  `C`. The `KernelRange` based on this MDAC is particularly simple: each element depends only on its predecessor, so the internal state consists of a single  $(X_i : Z_i)$ .

So, how often does this trick apply? Theoretically, the quantitative form of Artin’s primitive root conjecture (proven by Hooley under GRH) says that  $M_\ell = \langle 2 \rangle$  for a little over half of all  $\ell$  (see [31]). Experimentally, 5609420 of the first  $10^7$  odd primes  $\ell$  satisfy  $M_\ell = \langle 2 \rangle$ .

One might try to generalize Example 1 to other generators of  $M_\ell$ : for example, if  $M_\ell = \langle 3 \rangle$ , then we could try to find an MDAC for  $\{3^i \bmod \ell : 0 \leq i < (\ell-1)/2\}$ . But this is counterproductive: *x*-only tripling (or multiplication by any scalar  $> 2$ ) is *slower* than differential addition.

#### 4.5 Stepping through cosets

What can we do when  $M_\ell \neq \langle 2 \rangle$ ? A productive generalization is to let

$$A_\ell := \langle 2 \rangle \subseteq M_\ell \quad \text{and} \quad a_\ell := \#A_\ell,$$

and to try to compute a convenient decomposition of  $M_\ell$  into cosets of  $A_\ell$ . Within each coset, we can compute elements using repeated `xDBLs` as in Example 1; then, it remains to step from one coset into another using differential additions.

This can be done in a particularly simple way for the primes  $\ell$  such that

$$M_\ell \text{ is generated by 2 and 3.} \tag{*}$$

If  $(*)$  holds, then

$$M_\ell = \bigsqcup_{i=0}^{m_\ell/a_\ell-1} 3^i A_\ell.$$

We can move from the  $i$ -th to the  $(i+1)$ -th coset using the elementary relations

$$\begin{cases} c \cdot 2^{j+1} + c \cdot 2^j = 3c \cdot 2^j \\ c \cdot 2^{j+1} - c \cdot 2^j = c \cdot 2^j \end{cases} \quad \text{for all integers } c \text{ and } j \geq 0. \tag{7}$$

In particular, if we have enumerated some coset  $3^i A_\ell$  by repeated doubling, then we can compute an element of  $3^{i+1} A_\ell$  by applying a differential addition to any two consecutive elements of  $3^i A_\ell$  (and the difference is the first of them). Algorithm 4 minimises storage overhead by using the last two elements of the previous coset to generate the first element of the next one. The `KernelRange` of Algorithm 4 therefore has an internal state of only two  $x$ -coordinates—so not only is it faster than the `KernelRange` of Algorithm 3 where it applies, but it also has a smaller memory footprint.

---

**Algorithm 4:** Kernel enumeration for  $\ell > 3$  satisfying (\*). Cost:  $(1 - 1/a_\ell) \cdot m_\ell$  `xDBLs` and  $m_\ell/a_\ell - 1$  `xADDs`.

---

**Input:** Projective  $x$ -coordinate  $(X_G : Z_G)$  of the generator  $G$  of a cyclic subgroup of order  $\ell$  in  $\mathcal{E}(\mathbb{F}_q)$ , where  $\ell$  satisfies (\*).  
**Output:**  $\mathcal{X}_G$  as a list

```

1  $(a, b) \leftarrow (a_\ell, m_\ell/a_\ell)$ 
2 for  $i = 0$  to  $b - 1$  do // Invariant:  $(X_{ai+j} : Z_{ai+j}) = x([3^i 2^{i(a-2)+(j-1)}]G)$ 
3   if  $i = 0$  then
4      $(X_1 : Z_1) \leftarrow (X_G : Z_G)$ 
5   else // Compute new coset representative
6      $(X_{ai+1} : Z_{ai+1}) \leftarrow \mathbf{xADD}((X_{ai} : Z_{ai}), (X_{ai-1} : Z_{ai-1}), (X_{ai-1} : Z_{ai-1}))$ 
7   for  $j = 2$  to  $a$  do // Exhaust coset by doubling
8      $(X_{ai+j} : Z_{ai+j}) \leftarrow \mathbf{xDBL}((X_{ai+j-1} : Z_{ai+j-1}))$ 
9 return  $((X_1 : Z_1), \dots, (X_{(\ell-1)/2} : Z_{(\ell-1)/2}))$ 

```

---

Algorithm 4 performs better the closer  $a_\ell$  is to  $m_\ell$ . In particular, when  $A_\ell = M_\ell$ , it uses  $m_\ell - 1$  `xDBLs` and no `xADDs` at all. The worst case for Algorithm 4 is when the order of 2 in  $M_\ell$  is as small as possible: that is, when  $\ell = 2^k - 1$ . In this case  $a_\ell = k$ , and compared with Algorithm 3 we still reduce the number of `xADDs` to be done by a factor of  $k$ .

#### 4.6 The remaining primes

While 1878 of the 2261 odd primes  $\ell \leq 20000$  satisfy (\*), there are still 383 primes that do not. We can, to some extent, adapt Algorithm 4 to handle these primes, but on a case-by-case basis and with somewhat less satisfactory results.

For example, the CSIDH-512 parameter set specifies 74 isogeny-degree primes

$$\ell = 3, 5, 7, 11, 13, \dots, 367, 373, \text{ and } 587.$$

Of these 74 primes, all but seven satisfy (\*): the exceptions are  $\ell = 73, 97, 193, 241, 313, \text{ and } 337$ . Table 1 lists these primes and a candidate decomposition of  $M_\ell$  into cosets of  $A_\ell$ . In each case, we need to produce either an element of  $5A_\ell$  or  $7A_\ell$ . This can certainly be done using previously-computed elements, but it

requires careful tracking of those elements, which implies a larger internal state and a more complicated execution pattern, ultimately depending on the value of  $\ell$ .

**Table 1.** Primes  $\ell$  in the CSIDH-512 parameter set that do not satisfy (\*).

Prime $\ell$	$a_\ell$	$[M_\ell : \langle 2, 3 \rangle]$	Coset decomposition of $M_\ell$	Notes
73	9	2	$M_{73} = A_{73} \sqcup 3A_{73} \sqcup 5A_{73} \sqcup 5 \cdot 3A_{73}$	
97	24	2	$M_{97} = A_{97} \sqcup 5A_{97}$	3 is in $A_{97}$
193	48	2	$M_{193} = A_{193} \sqcup 5A_{193}$	3 is in $A_{193}$
241	12	2	$M_{241} = (\bigsqcup_{i=0}^4 3^i A_{241}) \sqcup (\bigsqcup_{i=0}^4 7 \cdot 3^i A_{241})$	
307	51	3	$M_{307} = A_{307} \sqcup 5A_{307} \sqcup 7A_{307}$	3 is in $A_{313}$
313	78	2	$M_{313} = A_{313} \sqcup 5A_{313}$	3 is in $A_{193}$
337	21	2	$M_{337} = (\bigsqcup_{i=0}^3 3^i A_{337}) \sqcup (\bigsqcup_{i=0}^3 5 \cdot 3^i A_{337})$	

*Example 2.* Consider  $\ell = 97$ . In this case, 3 is in  $A_{97}$  (in fact  $3 \equiv 2^{19} \pmod{97}$ ), and we find that  $M_{97} = A_{97} \sqcup 5A_{97}$ .

To adapt Algorithm 4 to this case, we can still enumerate  $A_{97}$  using repeated doubling. Then, we need to construct an element of  $5A_{97}$  from elements of  $A_{97}$ , for which we can use a differential addition like  $5 \cdot 2^i = 2^{i+2} + 2^i$  (with difference  $3 \cdot 2^i$ ) or  $5 \cdot 2^i = 2^{i+1} + 3 \cdot 2^i$  (with difference  $2^i$ ). Each involves near powers of 2 (modulo 97), but also  $3 \cdot 2^i$ —which we know is in  $A_{97}$ , so it need not be recomputed, but we need to know that  $3 \cdot 2^i \equiv 2^{i+19} \pmod{97}$  so that we can identify and store the  $x$ -coordinate corresponding to  $3^i$  (for a chosen  $i$ ) while enumerating  $A_{97}$ . The end result is an algorithm that uses one **xADD** and 48 **xDBLs**, just like Algorithm 4, but the internal state is slightly larger and the more complicated execution pattern specific to  $\ell = 97$ .

Alternatively, after (or while) enumerating  $A_{97}$ , we could just recompute  $3 = 1 + 2$  (difference 1) to get 5 as  $1 + 4$  (difference 3) or  $2 + 3$  (difference 1), but this recomputation of 3 is redundant.

Ultimately, there does not seem to be any “one size fits all” generalization of Algorithm 4 for enumerating  $\mathcal{X}_G$  without either a more complicated state or redundant recomputations. The obvious approach of finding an MDAC to enumerate a set of representatives for  $M_\ell / \langle 2, 3 \rangle$  and then using Algorithm 4 to exhaust the coset containing each representative can give reasonable results for many  $\ell$  not satisfying (\*), but the savings are generally not optimal.

#### 4.7 (In)Compatibility with Vélusqrt

One natural question is whether these techniques can be used to further accelerate the Vélusqrt algorithm of [2], which can evaluate isogenies of large prime

degree  $\ell$  in  $\tilde{O}(\sqrt{\ell})$  (with  $O(\sqrt{\ell})$  space). Vélusqrt never explicitly computes all of  $\mathcal{X}_G$ . Instead, it relies on the existence of a decomposition

$$S := \{1, 3, 5, \dots, \ell - 2\} = (I + J) \sqcup (I - J) \sqcup K \quad (8)$$

where  $I$ ,  $J$ , and  $K$  are sets of integers of size  $O(\sqrt{\ell})$  such that the maps  $(i, j) \rightarrow i + j$  and  $(i, j) \rightarrow i - j$  are injective with disjoint images. In [2], these sets are

$$\begin{aligned} I &:= \{2b(2i + 1) : 0 \leq i < b'\}, \\ J &:= \{2j + 10 \leq j < b\}, \\ K &:= \{4bb' + 1, \dots, \ell - 4, \ell - 2\} \end{aligned}$$

where  $b := \lfloor \sqrt{\ell - 1}/2 \rfloor$  and  $b' := \lfloor (\ell - 1)/4b \rfloor$ . (Note that  $I$  contains “giant steps”,  $J$  contains “baby steps”, and  $K$  contains the rest of  $S$ ).

The key thing to note here is that this decomposition is essentially additive, and the elements of  $I$ ,  $J$ , and  $K$  form arithmetic progressions. Algorithm 4, however, is essentially multiplicative: it works with subsets in geometric progression, not arithmetic progression. We cannot exclude the existence of subsets  $I$ ,  $J$ , and  $K$  of size  $O(\sqrt{\ell})$  satisfying Equation (8) and which are amenable to enumeration by 2-powering or a variation of Algorithm 4 for some, or even many  $\ell$ , but it seems difficult to construct nontrivial and useful examples.

## 5 Irrational kernel points: exploiting Frobenius

Now suppose  $G$  is defined over a nontrivial extension  $\mathbb{F}_{q^k}$  of  $\mathbb{F}_q$ , but  $\langle G \rangle$  is defined over the subfield  $\mathbb{F}_q$ : that is, it is Galois-stable. In particular, the  $q$ -power Frobenius endomorphism  $\pi$  of  $\mathcal{E}$ , which maps points in  $\mathcal{E}(\mathbb{F}_{q^k})$  to their conjugates under  $\text{Gal}(\mathbb{F}_{q^k}/\mathbb{F}_q)$ , maps  $\langle G \rangle$  into  $\langle G \rangle$ . In Appendix A we show how we can find the point  $G$ .

Since  $\pi$  maps  $\langle G \rangle$  into  $\langle G \rangle$ , it restricts to an endomorphism of  $\langle G \rangle$ —and since the endomorphisms of  $\langle G \rangle$  are  $\mathbb{Z}/\ell\mathbb{Z}$ , and Frobenius has no kernel (so  $\pi$  is not 0 on  $\langle G \rangle$ ), it must act as multiplication by an eigenvalue  $\lambda \neq 0$  on  $\langle G \rangle$ . The precise value of  $\lambda$  is not important here, but we will use the fact that  $\lambda$  has order  $k$  in  $(\mathbb{Z}/\ell\mathbb{Z})^\times$  and order  $k'$  in  $(\mathbb{Z}/\ell\mathbb{Z})^\times / \langle \pm 1 \rangle$ .

Now let

$$F_\ell := \langle \lambda \rangle \subseteq M_\ell \quad \text{and} \quad c_F := [M_\ell : F_\ell] = \frac{m_\ell}{k'}.$$

Let  $R_0$  be a set of representatives for the cosets of  $F_\ell$  in  $M_\ell$ , and let  $S_0 = \{[r]G : r \in R_0\}$ . Note that

$$\#S_0 = (\ell - 1)/k'.$$

### 5.1 The cost of Frobenius

In this section, we seek to use the Galois action to replace (many)  $\mathbf{M}$  and  $\mathbf{S}$  with a few  $\mathbf{F}$ . For this to be worthwhile,  $\mathbf{F}$  must be cheap: and they are, even

if this is not obvious given the definition of the Frobenius map on  $\mathbb{F}_{q^k}$  as  $q$ -th powering. It is important to note that we do not compute Frobenius by powering. Instead, we use the fact that Frobenius is an  $\mathbb{F}_q$ -linear map on  $\mathbb{F}_{q^k}$  viewed as an  $\mathbb{F}_q$ -vector space: that is, Frobenius acts as a  $k \times k$  matrix (with entries in  $\mathbb{F}_q$ ) on the coefficient vectors of elements in  $\mathbb{F}_{q^k}$ .

The form of the Frobenius matrix, and the cost of applying it, depends on the basis of  $\mathbb{F}_{q^k}/\mathbb{F}_q$ . For example:

1. If  $k = 2$  and  $\mathbb{F}_{q^2} = \mathbb{F}_q(\sqrt{\Delta})$ , then Frobenius simply negates  $\sqrt{\Delta}$  and the matrix is  $\text{diag}(1, -1)$ , so  $\mathbf{F} \approx 0$ .
2. If  $\mathbb{F}_{q^k}/\mathbb{F}_q$  is represented with a normal basis, then the matrix represents a cyclic permutation, and again  $\mathbf{F} \approx 0$ .

Even in the worst case where the basis of  $\mathbb{F}_{q^k}/\mathbb{F}_q$  has no special Galois structure,  $\mathbf{F}$  is just the cost of multiplying a  $k$ -vector by a  $k \times k$  matrix over  $\mathbb{F}_q$ : that is,  $k^2$   $\mathbb{F}_q$ -multiplications and  $k(k-1)$   $\mathbb{F}_q$ -additions. This is close to the cost of a single  $\mathbb{F}_{q^k}$ -multiplication using the ‘‘schoolbook’’ method; so when  $k \leq 12$ , we have  $\mathbf{F} \approx \mathbf{M}$  in the worst case.

## 5.2 Galois orbits

Each point  $P \in \mathcal{E}(\mathbb{F}_{q^k})$  is contained in a *Galois orbit* containing all the conjugates of  $P$ . The kernel subgroup  $\langle G \rangle$  breaks up (as a set) into *Galois orbits*: if we write

$$\mathcal{O}_P := \{P, \pi(P), \dots, \pi^{k-1}(P)\} \quad \text{for } P \in \mathcal{E}(\mathbb{F}_{q^k}),$$

then

$$\langle G \rangle = \{0\} \sqcup \begin{cases} \bigsqcup_{P \in S_0} \mathcal{O}_P & \text{if } k \text{ is even,} \\ \left( \bigsqcup_{P \in S_0} \mathcal{O}_P \right) \sqcup \left( \bigsqcup_{P \in S_0} \mathcal{O}_{-P} \right) & \text{if } k \text{ is odd.} \end{cases} \quad (9)$$

To get a picture of where we are going, recall from §3 that in general, isogeny evaluation can be reduced to evaluations of the kernel polynomial

$$D(X) := \prod_{P \in S} (X - x(P)),$$

where  $S \subset \langle G \rangle$  is any subset such that  $S \cap -S = \emptyset$  and  $S \cup -S = \langle G \rangle \setminus \{0\}$ . The decomposition of (9) can be seen in the factorization of  $D(X)$  over  $\mathbb{F}_{q^k}$ :

$$\begin{aligned} D(X) &= \prod_{P \in S} (X - x(P)) = \prod_{P \in S_0} \prod_{i=0}^{k'-1} (X - x(\pi^i(P))) \\ &= \prod_{P \in S_0} \prod_{i=0}^{k'-1} (X - x(P)^{q^i}), \end{aligned}$$

and the factors corresponding to each  $P$  in  $S_0$  are the irreducible factors of  $D$  over  $\mathbb{F}_q$ . Transposing the order of the products, if we let

$$D_0(X) := \prod_{P \in S_0} (X - x(P))$$

then for  $\alpha$  in the base field  $\mathbb{F}_q$ , we can compute  $D(\alpha)$  by computing  $D_0(\alpha)$  and taking the norm:

$$D(\alpha) = \text{Norm}(D_0(\alpha)) \quad \text{for all } \alpha \in \mathbb{F}_q.$$

where

$$\text{Norm}(x) := \prod_{i=0}^{k'-1} x^{q^i} = x(x(\cdots(x(x)^q)\cdots)^q)^q,$$

which can be computed for the cost of  $(k-1)\mathbf{F} + (k-1)\mathbf{M}$  (some multiplications can be saved with more storage, but for small  $k$  this may not be worthwhile).

Similarly, we can rewrite the rational map  $\phi_x$  from (5) as

$$\phi_x(x) = x \cdot \left( \prod_{P \in S} \left( \frac{x \cdot x(P) - 1}{x - x(P)} \right) \right)^2 = x \cdot \left( \prod_{P \in S_0} \prod_{i=0}^{k'-1} \left( \frac{x \cdot x(P)^{q^i} - 1}{x - x(P)^{q^i}} \right) \right)^2.$$

Evaluating  $\phi_x$  at  $\alpha$  in  $\mathbb{F}_q$ , rearranging the products gives

$$\phi_x(\alpha) = \alpha \cdot \left( \prod_{P \in S_0} \prod_{i=0}^{k'-1} \left( \frac{\alpha \cdot x(P)^{q^i} - 1}{\alpha - x(P)^{q^i}} \right) \right)^2 = \alpha \cdot \text{Norm}(\bar{\phi}_x(\alpha))^2,$$

where

$$\bar{\phi}_x(X) := \prod_{P \in S_0} \frac{X \cdot x(\pi^i(P)) - 1}{X - x(\pi^i(P))}$$

Projectively, from (6) we get  $\phi_x : (U : V) \mapsto (U' : V')$  where

$$U' = U \cdot \left[ \prod_{i=0}^{k'-1} \prod_{P \in S_0} (UX_P^{q^i} - Z_P^{q^i}V) \right]^2,$$

$$V' = V \cdot \left[ \prod_{i=0}^{k'-1} \prod_{P \in S_0} (UZ_P^{q^i} - X_P^{q^i}V) \right]^2,$$

so if we set

$$F(U, V) := \prod_{P \in S_0} (U \cdot X_P - Z_P \cdot V) \quad \text{and} \quad G(U, V) := \prod_{P \in S_0} (U \cdot Z_P - X_P \cdot V),$$

then for  $\alpha$  and  $\beta$  in  $\mathbb{F}_q$  we get

$$\phi_x((\alpha : \beta)) = (\alpha' : \beta') := \left( \alpha \cdot \text{Norm}(F(\alpha, \beta))^2 : \beta \cdot \text{Norm}(G(\alpha, \beta))^2 \right).$$

### 5.3 Enumerating representatives for the Galois orbits.

We now need to enumerate a set  $S_0$  of representatives for the Galois orbits modulo  $\pm 1$  or, equivalently, a set of representatives  $R_0$  for the cosets of  $F_\ell$  in  $M_\ell$ . We therefore want good MDACs for  $M_\ell/F_\ell$ .

Given an MDAC driving enumeration of the coset representatives, there are obvious adaptations of Algorithms 1 and 2 to this extension field case. Rather than iterating over all of the kernel  $x$ -coordinates, we just iterate over a subset representing the cosets of  $\mathbb{F}_\ell$ , and then compose with the norm.

Concretely, in Algorithm 2, we should

1. Replace `KernelRange` in Line 4 with a generator driven by an efficient MDAC for  $M_\ell/F_\ell$ ;
2. Replace Line 10 with  $(U'_i, V'_i) \leftarrow (U_i \cdot \text{Norm}(U'_i)^2, V_i \cdot \text{Norm}(V'_i)^2)$ .

First, we can consider Algorithm 3: that is, enumerating  $M_\ell/F_\ell$  by repeated addition. Unfortunately, we do not have a nice bound on the length of this MDAC: the coset representatives are not necessarily conveniently distributed over  $M_\ell$ , so we could end up computing a lot of redundant points.

*Example 3.* Take  $(\ell, k) = (89, 11)$ . We see that  $M_\ell \neq \langle \lambda, 2 \rangle$ . So we compute the minimal element in each Galois orbit (up to negation), and choose it to be our orbit representative. Using arithmetic modulo 89 only, we found a choice for  $R_0$  to be  $R_0 = \{1, 3, 5, 13\}$ . Now we compute an optimal MDAC, namely  $(0, 1, 2, 3, 5, 8, 13)$ . This chain computes the orbit generator  $x$ -coordinates using one `xDBL` operation and six `xADD` operations, albeit involving the computation of two intermediate points that will not be utilized in the final result.

But when we say that the coset representatives are not conveniently distributed over  $M_\ell$ , we mean that with respect to addition. If we look at  $M_\ell$  multiplicatively, then the path to efficient MDACs is clearer.

The nicest case is when  $M_\ell = \langle 2, \lambda \rangle$ : then, we can take  $R_0 = \{2^i : 0 \leq i < c_F\}$ , which brings us to the 2-powering MDAC of Example 1—except that we stop after  $c_F - 1$  `xDBL`s. We thus reduce the number of `xDBL`s by a factor of  $\approx k'$ , at the expense of two norm computations.

This MDAC actually applies to more primes  $\ell$  here than it did in §4, because we no longer need 2 to generate all of  $M_\ell$ ; we have  $\lambda$  to help. In fact, the suitability of this MDAC no longer depends on  $\ell$ , but also on  $k$ .

We can go further if we assume

$$M_\ell = \langle 2, 3, \lambda \rangle. \tag{**}$$

To simplify notation, we define

$$a_{\ell,k} := [\langle 2, \lambda \rangle : F_\ell], \quad b_{\ell,k} := [\langle 2, 3, \lambda \rangle : \langle 2, \lambda \rangle] = c_F/a_{\ell,k}.$$

Algorithm 5 is a truncated version of Algorithm 4 for computing  $S_0$  instead of  $\mathcal{X}_G$  when **(\*\*)** holds. Algorithm 6 is the corresponding modification of Algorithm 1, evaluating an  $\ell$ -isogeny over  $\mathbb{F}_q$  with kernel  $\langle G \rangle$  at  $n$  points of  $\mathcal{E}(\mathbb{F}_q)$ , where  $x(G)$  is in  $\mathbb{F}_{q^{k'}}$  with  $k' > 1$ .

Table 2 compares the total cost of Algorithms 6 and 5 with that of Algorithms 1 and 3. In both algorithms, we can take advantage of the fact that many of the multiplications have one operand in the smaller field  $\mathbb{F}_q$ : notably, the



multiplications involving coordinates of the evaluation points. In the context of isogeny-based cryptography (where curve constants look like random elements of  $\mathbb{F}_q$ ), this means that in Algorithm 1, we can replace the  $2\mathbf{M} + 2\mathbf{a}$  in Line 8 and the  $2\mathbf{M} + 2\mathbf{S}$  in Line 10 with  $2\mathbf{C} + 2\mathbf{a}$  and  $2\mathbf{C} + 2\mathbf{S}$ , respectively.

**Table 2.** Comparison of  $\ell$ -isogeny evaluation algorithms for kernels  $\langle G \rangle$  defined over  $\mathbb{F}_q$  but with  $x(G) \in \mathbb{F}_{q^{k'}}$ . In this table,  $\mathbf{C}$  denotes multiplications of elements of  $\mathbb{F}_{q^{k'}}$  by elements of  $\mathbb{F}_q$  (including, but not limited to, curve constants).

	Costello–Hisil (Algorithms 1 and 3)	This work (Algorithm 6)
$\mathbf{M}$	$(\ell - 1)n + 2\ell - 8$	$2(c_F + k' - 1)n + 2c_F + 2b_{\ell,k} + 4$
$\mathbf{S}$	$2n + \ell - 3$	$2n + 2c_F - 2$
$\mathbf{C}$	$(\ell + 1)n + 1$	$2(c_F + 1)n + c_F - b_{\ell,k}$
$\mathbf{a}$	$(n + 1)(\ell + 1) + 3\ell + 17$	$2c_F(n + 1) + 4c_F + 4b_{\ell,k} - 6$
$\mathbf{F}$	0	$2(k' - 1)n$

---

**Algorithm 5:** Compute  $S_0$  when  $(**)$  holds. Cost:  $b_{\ell,k} - 1$   $\mathbf{xADD}$ s and  $(c_F - b_{\ell,k})$   $\mathbf{xDBL}$ s, or  $(2c_F + 2b_{\ell,k} + 4)\mathbf{M} + (2c_F - 2)\mathbf{S} + (c_F - b_{\ell,k})\mathbf{C} + (4c_F + 4b_{\ell,k} - 6)\mathbf{a}$

---

**Input:** Projective  $x$ -coordinate  $(X_G : Z_G)$  of the generator  $G$  of a cyclic subgroup of order  $\ell$  in  $\mathcal{E}(\mathbb{F}_{q^k})$ , where  $\ell$  satisfies  $M_\ell = \langle 2, 3, \lambda \rangle$ .

**Output:**  $S_0$  as a list

```

1 Function SZeroPoints( $(X_G : Z_G)$ )
2    $(a, b) \leftarrow (a_{\ell,k}, b_{\ell,k})$ 
3   for  $i = 0$  to  $b - 1$  do           // Invariant:  $x_{ai+j} = x([3^i 2^{i(a-2)+(j-1)}]G)$ 
4     if  $i = 0$  then
5        $x_1 \leftarrow (X_G : Z_G)$ 
6     else                               // Compute new coset representative
7        $x_{ai+1} \leftarrow \mathbf{xADD}(x_{ai}, x_{ai-1}, x_{ai-1})$ 
8     for  $j = 2$  to  $a$  do               // Exhaust coset by doubling
9        $x_{ai+j} \leftarrow \mathbf{xDBL}(x_{ai+j-1})$ 
10  return  $(x_1, \dots, x_{c_F})$ 

```

---

If the parameter choice for  $(\ell, k)$  does not satisfy any of these criteria, then we have to compute the coset representatives using some ad-hoc MDAC. We can do some precomputations here to determine an optimal, or near optimal, approach to computing  $R_0$ .

---

**Algorithm 6:** Isogeny evaluation using `SZeroPoints` and Frobenius.

Cost:  $2(c_F + k - 1)n\mathbf{M} + 2n\mathbf{S} + 2(c_F + 1)n\mathbf{C} + 2c_F(n + 1)\mathbf{a} + 2(k - 1)n\mathbf{F}$   
*plus the cost of `SZeroPoints`.*

---

**Input:** The  $x$ -coordinate  $(X_G : Z_G)$  of a generator  $G$  of the kernel of an  $\ell$ -isogeny  $\phi$ , and a list of evaluation points  $((U_i : V_i) : 1 \leq i \leq n)$

**Output:** The list of images  $((U'_i : V'_i) = \phi_x((U_i : V_i)) : 1 \leq i \leq n)$

```

1  $((X_1, Z_1), \dots, (X_{c_F}, Z_{c_F})) \leftarrow \text{SZeroPoints}((X_G : Z_G))$  // Algorithm 5
2 for  $1 \leq i \leq c_F$  do
3    $(\hat{X}_i, \hat{Z}_i) \leftarrow (X_i + Z_i, X_i - Z_i)$  // 2a
4 for  $i = 1$  to  $n$  do
5    $(\hat{U}_i, \hat{V}_i) \leftarrow (U_i + V_i, U_i - V_i)$  // 2a
6    $(U'_i, V'_i) \leftarrow (1, 1)$ 
7   for  $j = 1$  to  $c_F$  do
8      $(t_0, t_1) \leftarrow \text{CrissCross}(\hat{X}_j, \hat{Z}_j, \hat{U}_i, \hat{V}_i)$  // 2C + 2a
9      $(U'_i, V'_i) \leftarrow (t_0 \cdot U'_i, t_1 \cdot V'_i)$  // 2M
10     $(U'_i, V'_i) \leftarrow (\text{Norm}(U'_i), \text{Norm}(V'_i))$  //  $2(k' - 1)\mathbf{M} + 2(k' - 1)\mathbf{F}$ 
11     $(U'_i, V'_i) \leftarrow (U_i \cdot (U'_i)^2, V_i \cdot (V'_i)^2)$  // 2C + 2S
12 return  $((U'_1, V'_1), \dots, (U'_n, V'_n))$ 

```

---

## 5.4 Experimental results

We provide proof-of-concept implementations of our algorithms in SageMath.<sup>6</sup> Our implementations include operation-counting code to verify the counts claimed in this article. We provide the number of operations for a given  $\ell$ -isogeny and the extension field  $k$ . Table 3 displays the costs for our algorithm, highlighted in light gray, compared with the basic Costello–Hisil algorithm (Algorithms 1 and 3). As depicted in Table 3, our approach consistently employs fewer operations across all values of  $\ell$  and extension fields. For  $k$  of this size, it is reasonable to use the approximation  $\mathbf{F} \approx \mathbf{M}$  (see §5.1).

Finally, Table 4 shows our success rate (over all primes  $\ell < 10^4$ ) at finding optimal MDACs for  $k \leq 12$ . These rates are computed by choosing the minimal representative of each Galois orbit to be in  $S_0$ , and checking whether the set  $S_0$  can be computed without any intermediary additions (that would not be otherwise used). Note, this computation checks only one approach for computing the MDAC, hence the percentages in Table 4 represent a lower bound on the number of  $\ell$  that have an optimal MDAC.

## 6 Applications

Our algorithms have potential applications in any isogeny-based cryptosystem involving isogenies of prime degree  $\ell > 3$ , including key exchanges like CSIDH [5]

<sup>6</sup> Sage scripts available from <https://github.com/vgilchri/k-velu>.

**Table 3.** Cost of evaluating an  $\ell$ -isogeny at a single point over  $\mathbb{F}_q$ , using a kernel generator with  $x$ -coordinate in  $\mathbb{F}_{q^{k'}}$ .

$\ell$	$k'$	M	S	C	a	F	Algorithm
13	any	30	12	15	54	0	Costello–Hisil (Algorithm 1 with 3)
	1	22	12	19	46	0	This work (Algorithm 6)
	3	10	4	7	14	4	This work (Algorithm 6)
19	any	48	18	21	84	0	Costello–Hisil (Algorithm 1 with 3)
	1	34	18	28	70	0	This work (Algorithm 6)
	3	14	6	10	22	4	This work (Algorithm 6)
	9	18	2	4	6	16	This work (Algorithm 6)
23	any	60	22	25	104	0	Costello–Hisil (Algorithm 1 with 3)
	1	42	22	34	86	0	This work (Algorithm 6)
	11	22	2	4	6	20	This work (Algorithm 6)

**Table 4.** Percentage of primes,  $3 \leq \ell < 10^4$  for which an optimal MDAC definitely exists (using the naive choice of  $S_0$ ).

k	1	2	3	4	5	6	7	8	9	10	11	12
%	100	100	100	100	84	86	76	67	60	56	45	42

and signature schemes such as SQISign [18, 19], SeaSign [15], and CSI-FiSh [4]. We focus on key exchange here, but similar discussion applies for other schemes.

As mentioned in §1, we also have cryptanalytic applications: state-of-the-art attacks on group-action cryptosystems like CSIDH involve computing a massive number of  $\ell$ -isogenies (in order to do a Pollard-style random walk, for example, or a baby-step giant-step algorithm as in [8]). In this context, even minor savings in individual  $\ell$ -isogenies quickly add up to substantial overall savings.

## 6.1 CSIDH and constant-time considerations

CSIDH is a post-quantum non-interactive key exchange based on the action of the class group of the imaginary quadratic order  $\mathbb{Z}[\sqrt{-p}]$  on the set of supersingular elliptic curves  $\mathcal{E}/\mathbb{F}_p$  with  $\text{End}_{\mathbb{F}_p}(\mathcal{E}) \cong \mathbb{Z}[\sqrt{-p}]$ . The action is computed via compositions of  $\ell_i$ -isogenies for a range of small primes  $(\ell_1, \dots, \ell_m)$ .

CSIDH works over prime fields  $\mathbb{F}_p$ , so the methods of §5 do not apply; but Algorithm 4 may speed up implementations at least for the  $\ell_i$  satisfying (\*). (We saw in §4.6 that 67 of the 74 primes  $\ell_i$  in the CSIDH-512 parameter set met (\*)).

The extent of any speedup depends on two factors. The first is the number of evaluation points. Costello and Hisil evaluate at a 2-torsion point other than  $(0, 0)$  in order to interpolate the image curve. The constant-time CSIDH of [21] evaluates at one more point (from which subsequent kernels are derived)—that is,  $n = 2$ ; We find  $n = 3$  in [23], and [7] discusses  $n > 3$ . For larger  $n$ , the cost

of Algorithm 1 overwhelms kernel enumeration, but our results may still make a simple and interesting improvement when  $n$  is relatively small.

The second factor is the organisation of primes into batches for constant-time CSIDH implementations. CTIDH [1] makes critical use of the so-called *Matryoshka* property of isogeny computations to hide the degree  $\ell$ : using Algorithms 1 and 3,  $\ell_i$ -isogeny evaluation is a sub-computation of  $\ell_j$ -isogeny computation whenever  $\ell_i < \ell_j$ . Organising primes into similar-sized batches, we can add dummy operations to disguise smaller-degree isogenies as isogenies of the largest degree in their batch.

Our Algorithm 4 has a limited Matryoshka property:  $\ell_i$ -isogenies are sub-computations of  $\ell_j$ -isogenies if  $a_{\ell_i} \leq a_{\ell_k}$  and  $m_{\ell_i}/a_{\ell_i} \leq m_{\ell_j}/a_{\ell_j}$ . For constant-time implementations, it would make more sense to make all primes in a batch satisfying (\*) a sub-computation of an algorithm using the maximum  $a_\ell$  and maximum  $m_\ell/a_\ell$  over  $\ell$  in the batch. Redistributing batches is a delicate matter with an important impact on efficiency; therefore, while our work improves the running time for a fixed  $\ell$ , its impact on batched computations remains uncertain, and ultimately depends on specific parameter choices.

## 6.2 CRS key exchange

The historical predecessors of CSIDH, due to Couveignes [11] and Rostovtsev and Stolbunov [25, 28, 29] are collectively known as CRS. Here the group is the class group of a quadratic imaginary order, acting on an isogeny (sub)class of elliptic curves with that order as their endomorphism ring; the action is computed using a composition of  $\ell_i$ -isogenies for a range of small primes  $(\ell_1, \dots, \ell_m)$ .

In [11, 25, 28, 29], isogenies are computed by finding roots of modular polynomials; this makes key exchange extremely slow at reasonable security levels. Performance was greatly improved in [16] using Vélú-style isogeny evaluation, but this requires finding ordinary isogeny classes over  $\mathbb{F}_p$  with rational  $\ell_i$ -torsion points over  $\mathbb{F}_{q^{k_i}}$  with  $k_i$  as small as possible for as many  $\ell_i$  as possible.

One such isogeny class over a 512-bit prime field is proposed in [16, §4]: the starting curve is  $\mathcal{E}/\mathbb{F}_p : y^2 = x(x^2 + Ax + 1)$  where  $p := 7 \prod_{\ell} \ell - 1$  where the product is over all primes  $2 \leq \ell < 380$ ; and

$$A = \frac{108613385046492803838599501407729470077036464083728319343246605668887327977789}{32142488253565145603672591944602210571423767689240032829444439469242521864171}.$$

This curve has rational  $\ell$ -isogenies with rational kernel generators for  $\ell = 3, 5, 7, 11, 13, 17, 103, 523$ , and 821, and irrational generators over  $\mathbb{F}_{q^k}$  for  $\ell = 19, 29, 31, 37, 61, 71, 547, 661, 881, 1013, 1181, 1321$ , and 1693; these “irrational”  $\ell$  are an interesting basis of comparison for our algorithms: all but 1321 satisfy (\*\*).

Table 5 compares operation counts for Algorithms 1 and 3 against Algorithm 6, which encapsulates the improvements in §5, for  $\ell$ -isogeny evaluation with kernel generators over  $\mathbb{F}_{q^k}$  (and arbitrary  $n$ ), for all of the “irrational”  $\ell$  above *except* 1321. We see that there are substantial savings to be had for all  $n$ .

**Table 5.** Comparison of Costello–Hisil (Algorithms 1 and 3, in white) with our approach (Algorithm 6, in grey) for the CRS parameters with  $k > 1$  proposed in [16]. The prime  $\ell = 1321$  with  $k = 5$  is omitted, since in this case  $M_\ell \neq \langle 2, 3, \lambda \rangle$ . In each row, **M**, **S**, **a**, and **F** refer to operations on elements of  $\mathbb{F}_{q^{k'}}$ , while **C** denotes multiplications of elements of  $\mathbb{F}_{q^{k'}}$  by elements of  $\mathbb{F}_q$  (including, but not limited to, curve constants).

$k$	Parameters			Operations				
	$\ell$	$a_{\ell,k}$	$b_{\ell,k}$	<b>M</b>	<b>S</b>	<b>C</b>	<b>a</b>	<b>F</b>
3	19	3	1	18n + 30	2n + 16	20n + 1	20n + 64	0
				10n + 4	2n + 4	8n + 2	8n + 14	4n
	661	110	1	660n + 1314	2n + 658	662n + 1	662n + 2632	0
				224n + 218	2n + 218	222n + 109	222n + 656	4n
4	1013	23	11	1012n + 2018	2n + 1010	1014n + 1	1014n + 4040	0
				48n + 524	2n + 504	48n + 242	48n + 1074	2n
	1181	59	5	1180n + 2354	2n + 1178	1182n + 1	1182n + 4712	0
				120n + 596	2n + 588	120n + 290	120n + 1302	2n
5	31	1	3	30n + 54	2n + 28	32n + 1	32n + 112	0
				10n + 8	2n + 4	4n	4n + 14	8n
	61	6	1	60n + 114	2n + 58	62n + 1	62n + 232	0
				20n + 10	2n + 10	14n + 5	14n + 32	8n
7	29	2	1	28n + 50	2n + 26	30n + 1	30n + 104	0
				16n + 2	2n + 2	6n + 1	6n + 8	12n
	71	5	1	70n + 134	2n + 68	72n + 1	72n + 272	0
				22n + 8	2n + 8	12n + 4	12n + 26	12n
547	39	1	546n + 1086	2n + 544	548n + 1	548n + 2176	0	
			90n + 76	2n + 76	80n + 38	80n + 230	12n	
8	881	55	2	880n + 1754	2n + 878	882n + 1	882n + 3512	0
				116n + 220	2n + 218	112n + 108	112n + 548	6n
9	37	2	1	36n + 66	2n + 34	38n + 1	38n + 136	0
				20n + 2	2n + 2	6n + 1	6n + 8	16n
	1693	94	1	1692n + 3378	2n + 1690	1694n + 1	1694n + 6760	0
				204n + 186	2n + 186	190n + 93	190n + 560	16n

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## A Computing a kernel generator

One task that poses a challenge is to find a point  $G \in \mathcal{E}(\mathbb{F}_{q^k})$ . In this section, we will illustrate an efficient method for computing a point with the necessary properties for use in the isogeny evaluation.

### A.1 The subgroup $H_k$ .

To compute a rational isogeny, our first step will be to sample a random point  $P \in \mathcal{E}(\mathbb{F}_{q^k})$  of order  $\ell$ . For this, letting  $N_k := \#\mathcal{E}(\mathbb{F}_{q^k})$ , one could sample a random point  $P$ , and compute  $P_\ell = [N_k/\ell]P$ . Then  $P_\ell$  is either 0 or a point of order  $\ell$ . If the order of  $P_\ell$  is not  $\ell$ , one tries again with a new choice of  $P$ .

*Remark 1.* In the special case that  $\ell^2$  divides  $\#\mathcal{E}(\mathbb{F}_{q^k})$ , we instead choose  $N_k = \exp(\mathbb{F}_{q^k})$ , the exponent of the group order. We do this to avoid having a cofactor,  $N_k/\ell$ , that “kills” certain torsion points.

In our context, we are assured that the  $P_\ell$  we are looking for is *not* in  $\mathcal{E}(\mathbb{F}_q)$ , or indeed in any  $\mathcal{E}(\mathbb{F}_{q^i})$ , for any proper divisor  $i$  of  $k$ . We can therefore save some effort by sampling  $P_\ell$  from the genuinely “new” subgroup of  $\mathcal{E}(\mathbb{F}_{q^k})$ .



Recall that  $\mathcal{E}(\mathbb{F}_{q^i}) = \ker(\pi^i - [1])$  for each  $i > 0$ . For each  $k > 0$ , then we define an endomorphism

$$\eta_k := \Phi_k(\pi) \in \text{End}(\mathcal{E})$$

where  $\Phi_k(X)$  is the  $k$ -th cyclotomic polynomial (that is, the minimal polynomial over  $\mathbb{Z}$  of the primitive  $k$ -th roots of unity in  $\overline{\mathbb{F}}$ ). The subgroup

$$H_k := \ker(\eta_k) \subset \mathcal{E}(\mathbb{F}_{q^k})$$

satisfies

$$\mathcal{E}(\mathbb{F}_{q^k}) = H_k \oplus \sum_{i|k, i \neq k} \mathcal{E}(\mathbb{F}_{q^i}).$$

The key fact is that in our situation,  $\mathcal{E}[\ell](\mathbb{F}_{q^k}) \subset H_k$ .

*Generating elements of  $\mathcal{E}[\ell](\mathbb{F}_{q^k})$ .* We always have  $\Phi_k(X) \mid X^k - 1$ , so for each  $k > 0$  there is an endomorphism

$$\delta_k := (\pi^k - [1])/\eta_k \in \text{End}(\mathcal{E}),$$

and  $\delta_k(\mathcal{E}(\mathbb{F}_{q^k})) \subset H_k$ . We can therefore sample a point  $P_\ell$  in  $\mathcal{E}[\ell](\mathbb{F}_{q^k})$  by computing

$$P_\ell = [h_k/\ell]\delta_k(P) \quad \text{where} \quad h_k := \#H_k$$

and  $P$  is randomly sampled from  $\mathcal{E}(\mathbb{F}_{q^k})$ .

Table 6 lists the first few values of  $h_k$  and  $\delta_k$ . We see that evaluating  $\delta_k$  amounts to a few Frobenius operations (which are almost free, depending on the field representation) and a few applications of the group law, so this approach saves us a factor of at least  $1/k$  in the loop length of the scalar multiplication (compared with computing  $P_\ell$  as  $[N_k/\ell]P$ ), but for highly composite  $k$  we save much more.

The value  $\varphi(k)$  of the Euler totient function plays an important role. We have  $h_k = q^{\varphi(k)} + o(q^{\varphi(k)})$ , so computing  $[h_k/\ell]$  instead of  $[N_k/\ell]$  allows us to reduce the loop length of basic scalar multiplication algorithms from  $k \log_2 q$  to  $\varphi(k) \log_2 q$ , which is particularly advantageous when  $k$  is highly composite.

*The action of Frobenius on  $H_k$ .* The Frobenius endomorphism  $\pi$  commutes with  $\eta_k$ , and therefore restricts to an endomorphism of  $H_k$ . If  $G \subset H_k$  is a subgroup of prime order  $\ell$  and fixed by  $\pi$ , then  $\pi$  will act on  $G$  as multiplication by an integer eigenvalue  $\lambda$  (defined modulo  $\ell$ ). Since  $\eta_k = \Phi_k(\pi) = 0$  on  $H_k$  by definition, we know that  $\lambda$  is a  $k$ -th root of unity in  $\mathbb{Z}/\ell\mathbb{Z}$ .

*Scalar multiplication with Frobenius.* Now,  $H_k \cong \mathbb{Z}/d_k\mathbb{Z} \times \mathbb{Z}/e_k\mathbb{Z}$ , where  $d_k \mid e_k$  and (by the rationality of the Weil pairing)  $d_k \mid q^k - 1$ . Typically,  $d_k$  is very small compared with  $e_k$ . If  $\ell \nmid d_k$ , then we can replace  $H_k$  with the cyclic subgroup  $H'_k := [d_k]H_k$ , and  $h_k$  with  $h'_k := e_k/d_k$ . Now,  $\pi$  induces an endomorphism of  $H'_k$ , and therefore acts as multiplication by an eigenvalue  $\lambda$  defined modulo  $h'_k$ .

We want to compute  $[c_k]P$  for  $P$  in  $H'_k$ , where  $c_k := h'_k/\ell$ . Since  $\Phi_k(\pi) = 0$ , the eigenvalue  $\lambda$  is a root of  $\Phi_k$  (i.e., a primitive  $k$ -th root of unity) modulo  $h'_k$ . We can compute  $a_0, \dots, a_{k-1}$  such that

$$c_k \equiv \sum_{i=0}^{k-1} a_i \lambda^i \pmod{h'_k},$$

with each coefficient  $a_i \approx (h'_k)^{1/\varphi(k)}$  in  $O(q)$ , and then

$$[c_k]P = \sum_{i=0}^{k-1} [a_i] \pi^i(P).$$

If we precompute the various sums of conjugates of  $P$ , then we can compute  $[c_k]$  using a multiscalar multiplication algorithm with a loop of length only  $\log_2 q$ . This might be particularly interesting in the cases where  $\varphi(k) = 2$  (which corresponds to GLV multiplication) or 4.

**Table 6.** The first few values of  $h_k$  and  $\delta_k$ .

$k$	$h_k$	$\delta_k$	$\varphi(k)$
1	1	[1]	
2	$N_2/N_1 = q + O(\sqrt{q})$	$\pi - [1]$	1
3	$N_3/N_1 = q^2 + O(q^{3/2})$	$\pi - [1]$	2
4	$N_4/N_2 = q^2 + O(q)$	$\pi^2 - [1]$	2
5	$N_5/N_1 = q^4 + O(q^{7/2})$	$\pi - [1]$	4
6	$(N_6 N_1)/(N_2 N_3) = q^2 + O(q^{3/2})$	$(\pi + [1])(\pi^3 - [1])$	2
7	$N_7/N_1 = q^6 + O(q^{7/2})$	$\pi - [1]$	6
8	$N_8/N_4 = q^4 + O(q^2)$	$\pi^4 - [1]$	4
9	$N_9/N_3 = q^6 + O(q^{4/2})$	$\pi^3 - [1]$	6
10	$(N_{10} N_1)/(N_2 N_5) = q^4 + O(q^{7/2})$	$(\pi + [1])(\pi^5 - [1])$	4
11	$N_{11}/N_1 = q^{10} + O(q^{19/2})$	$\pi - [1]$	10
12	$(N_{12} N_2)/(N_4 N_6) = q^4 + O(q^3)$	$(\pi^2 + [1])(\pi^6 - [1])$	4

*Example 4.* Consider  $k = 3$ : we have  $\mathcal{E}(\mathbb{F}_{q^3}) \cong \mathcal{E}(\mathbb{F}_q) \oplus H_3$ , and  $\#H_3 = N_3/N_1$ .

We first note that  $\pi^3$  fixes the points in  $\mathcal{E}(\mathbb{F}_{q^3})$ , so  $\pi^3 - [1] = [0]$  on  $\mathcal{E}(\mathbb{F}_{q^3})$ . By similar logic, the regular Frobenius map will fix the points in  $\mathcal{E}(\mathbb{F}_q)$ , meaning  $\pi - [1] = [0]$  holds only for points contained entirely in the  $\mathcal{E}(\mathbb{F}_q)$  portion of  $\mathcal{E}(\mathbb{F}_{q^3})$ . Therefore, by computing  $P_H = (\pi - 1)P$ , we are “killing” the  $\mathcal{E}(\mathbb{F}_q)$  part of  $P$ , leaving only the part lying in the subgroup  $H_3$ . This computation is easy enough to do, and so now we need only compute  $P_\ell = [N_3/N_1/\ell]P_H$ , thereby saving us about a third of the multiplications.

*The “twist trick”.* When  $k$  is even, if we use  $x$ -only scalar multiplication, then the following lemma allows us to work over  $\mathbb{F}_{q^{k/2}}$  instead of  $\mathbb{F}_{q^k}$ . In the case  $k = 2$ , this is known as the “twist trick”.

**Lemma 1.** *If  $k$  is even, then every point  $P$  in  $H_k$  has  $x(P)$  in  $\mathbb{F}_{q^{k/2}}$ .*

*Proof.* If  $k$  is even, then  $\eta_k$  divides  $\pi^{k/2} + 1$ , so  $\pi^{k/2}$  acts as  $-1$  on  $H_k = \ker(\eta_k)$ : that is, if  $P$  is in  $H_k$ , then  $\pi^{k/2}(P) = -P$ , so  $x(P)$  is in  $\mathbb{F}_{q^{k/2}}$ .

*Example 5.* Consider  $k = 6$ . We take a random point  $R$  in  $E(\mathbb{F}_{q^6})$ , and compute  $R' := \pi^3(R) - R$ , then  $P := \pi(R') + R'$ ; now  $P = \delta_6(R)$  is in  $H_6$ , and  $x(P)$  is in  $\mathbb{F}_{q^3}$ . We have  $h_6 = N_1^2 - (q+1)N_1 + q^2 - q + 1 \approx q^2$ , and we need to compute  $x([c_{\ell,6}]P)$  where  $c_{\ell,6} := h_6/\ell$ . Since  $x(P)$  is in  $\mathbb{F}_{q^3}$ , we can do this using  $x$ -only arithmetic and the Montgomery ladder working entirely over  $\mathbb{F}_{q^3}$ .

The improvements outlined in this section are summarized in Algorithm 7.

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**Algorithm 7:** Computation of Kernel Generator.

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**Input:**  $\mathcal{E}$  an elliptic curve defined over  $\mathbb{F}_q$ ,  $\ell$  an integer,  $k$  such that  $\mathbb{F}_{q^k}$  contains an  $\ell$ -torsion point.

**Output:**  $P$ , a point on  $\mathcal{E}(\mathbb{F}_{q^k})$  of order  $\ell$ .

```

1  $P_\ell \leftarrow (0 : 1 : 0)$ 
2 repeat
3    $P \leftarrow \text{RandomPoint}(\mathcal{E}(\mathbb{F}_{q^k}))$ 
4    $c \leftarrow h_k/\ell$  //  $h_k$  taken from Table 6.
5    $P' \leftarrow \delta_k(P)$  //  $\delta_k$  taken from Table 6.
6    $P_\ell \leftarrow [c]P'$ 
7 until  $P_\ell = (0 : 1 : 0)$ 
8 return  $P_\ell$ 

```

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