



**HAL**  
open science

# Locally Homogeneous Finite-time Stabilization of Quasi-Linear Systems

Merichel Ayamou, Andrey Polyakov, Nicolás Espitia

► **To cite this version:**

Merichel Ayamou, Andrey Polyakov, Nicolás Espitia. Locally Homogeneous Finite-time Stabilization of Quasi-Linear Systems. The 62nd IEEE Conference on Decision and Control, CDC 2023, Dec 2023, Singapore, France. hal-04139119

**HAL Id: hal-04139119**

**<https://inria.hal.science/hal-04139119v1>**

Submitted on 23 Jun 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

# Locally Homogeneous Finite-time Stabilization of Quasi-Linear Systems

M. Ayamou, A. Polyakov, N. Espitia,

**Abstract**—In this paper, an algorithm of a local finite-time control design is developed for a class of quasi-linear systems. The design procedure is essentially based on the concept of generalized homogeneity and the convex embedding technique. The adjustment of control parameters is realized by solving a system of linear matrix inequalities (LMIs). Theoretical results are supported by numerical simulations.

## I. INTRODUCTION

### A. State of the art

A symmetry with respect to a dilation is known as homogeneity [18], [7], [4]. In control theory, the homogeneity is utilized for stability/controlability analysis, controller and observer design (e.g. [5], [13], [2], [12] and references therein). The standard (Euler) homogeneity means that a function  $f(x)$  is symmetric with respect to the scaling of its argument  $f(e^s x) = e^{\nu s} f(x), \forall s, x \in \mathbb{R}$ , where the constant  $\nu$  is called the homogeneity degree. The generalized homogeneity is introduced by V.I. Zubov in 1958 [18] using the weighted dilation  $(x_1, x_2, \dots, x_n) \rightarrow (e^{r_1 s} x_1, e^{r_2 s} x_2, \dots, e^{r_n s} x_n), s \in \mathbb{R}$ , where the positive numbers  $r_1, r_2, \dots, r_n$  are the weights specifying the dilation rate of each coordinate. Nonlinear (geometric) dilations are studied in [8], [6], [14]. This paper deals with the *linear (geometric) dilation* [11] given by  $x \rightarrow e^{G_d s} x$ , where  $G_d \in \mathbb{R}^{n \times n}$  is anti-Hurwitz matrix<sup>1</sup>. Being a relaxation of linearity, the homogeneity can provide an extra degree of freedom for advanced control design. In particular, the finite/fixed-time stabilization can be guaranteed by a proper selection of the homogeneity degree [3], [17].

As shown in [9], the finite stabilization of affine-in-control systems can be realised using the Sontag's universal formula [15]. However, it requires a design of an appropriate control Lyapunov function. The convex embedding approach for homogeneous affine-in-control systems has been introduced recently [16]. It essentially uses the homogeneity (symmetry) of the vector field to define a stability condition in terms of LMIs. This paper generalizes the latter approach to a special class of locally homogeneous (quasi-linear) systems, which consideration is important for control applications in fluid mechanics. To the best of our knowledge, the homogeneous finite-time controllers have never been designed for such a class of systems.

The authors are with Univ. Lille, Inria, CNRS CRISTAL, Centrale Lille, FR-59000, France (e-mails: mericel.ayamou(andrey.polyakov)@inria.fr, nicolas.espitia-hoyos@univ-lille.fr)

<sup>1</sup>The matrix  $G_d \in \mathbb{R}^n$  is anti-Hurwitz, if  $-G_d$  is Hurwitz.

### B. Motivating Example

A quasi-linear Ordinary Differential Equation (ODE) may appear as reduced order model of mathematical physics. For example, the finite difference discretization of the viscous Burgers or Navier-Stokes equations with controlled boundary conditions provides nonlinear reduced order models (ODEs), that are affine in control. Let us consider one dimensional viscous Burgers equation

$$\partial_t v + v \partial_x v = \nu \partial_{xx} v, \quad t > 0, \quad x \in (0, L) \quad (1)$$

$\nu > 0$ , with the initial condition  $v(0, x) = f(x)$  and time-varying Dirichlet boundary conditions:

$$v(t, 0) = v_0 + \xi(t), \quad v(t, L) = v_L \quad (2)$$

where  $v_0, v_L$  are positive constants and  $\xi$  is assumed to be a positive control input. The positivity of the control inputs is a specific feature of many real control problems. For the models of fluid dynamics, the positive control may be inspired by the use of air-blowers as actuators.

Let  $N$  be strictly positives integers,  $h_x = L/N$  be the steps of spatial discretization and  $x_i = i h_x$ , with  $i \in \{0, \dots, N\}$ .

Let denote  $v_i(t)$  respectively the approximations of  $v(t, x_i)$ . The system (1) can be discretized as follows:

$$\dot{v}_i(t) + v_i(t) \frac{v_{i+1}(t) - v_{i-1}(t)}{2h_x} = \nu \frac{v_{i+1}(t) - 2v_i(t) + v_{i-1}(t)}{h_x^2}, \quad (3)$$

with  $v_i(0) = f_i$ , and  $v_0(t) = v_0 + \xi(t), v_N(t) = v_L$ . The obtained system has a rather specific structure: it is affine in control and the finite dimensional vector field has linear and bilinear/quadratic components. Notice that the above approximates of partial derivatives have the second order of approximation  $O(h_x^2)$ . The use of more accurate approximates of partial derivatives (e.g.,  $O(h_x^3), O(h_x^4)$ , etc) would lead to a system of the similar structure. In this paper, we consider such a class of control systems, which covers the above and many other examples of reduced order models of fluid mechanics.

### C. Contributions and organization

In this paper the problem of the finite-time set-point tracking is studied for a class of quasi-linear systems. The stabilizing feedback is designed using the concept of linear (geometric) homogeneity and the convex embedding approach. The control parameters tuning is based on solving of the system of LMIs, which is feasible in the case of local controlability of the system.

The paper is organized as follows. First, the problem statement and basic assumptions are given. Next, preliminaries about homogeneous systems are discussed. After

that, the finite-time control for the quasi-linear system is designed. Finally, the numerical simulation results and some conclusions are presented.

#### D. Notation

$\mathbb{R}_+ = \{a \in \mathbb{R} : a > 0\}$ ;  $\bar{\mathbb{R}}_+ = \{a \in \mathbb{R} : a \geq 0\}$ ;  $|z| = \sqrt{z^\top z}$  is Euclidean norm of  $z = (z_1, \dots, z_n)^\top \in \mathbb{R}^n$ ;  $\|z\|_1 = \sum_{k=1}^n |z_k|$ ; we write  $P \succ 0$  (resp.  $\prec 0, \succeq 0, \preceq 0$ ) if the symmetric matrix  $P = P^\top \in \mathbb{R}^{n \times n}$  is positive (resp., negative, semi-positive, semi-negative) definite;  $P^{1/2}$  means that  $P = P^{1/2} P^{1/2}$  (always exists for symmetric positive definite matrices);

### II. PROBLEM STATEMENT

Let us consider the system

$$\dot{x} = (L + W(x))x + (F + Q(x))\xi + g, \quad x(0) = x_0, \quad (4)$$

where  $x \in \mathbb{R}^n$  is the system state,  $\xi \in \bar{\mathbb{R}}_+^m$  is the control input, which is assumed to be positive (due to the physical constraints),  $g \in \mathbb{R}^n$  is a known constant vector (inspired, for instance, by the boundary conditions in the above motivating example),  $L \in \mathbb{R}^{n \times n}$  and  $F \in \mathbb{R}^{n \times m}$  are known constant matrices and  $W(x) \in \mathbb{R}^{n \times n}$  are state dependent matrices, whose values can be computed for any  $x \in \mathbb{R}^n$ .

Notice that the matrix-valued functions  $W : \mathbb{R}^n \mapsto \mathbb{R}^{n \times n}$  and  $Q : \mathbb{R}^n \mapsto \mathbb{R}^{n \times m}$  are linear. The latter implies that  $\exists \tilde{W} : \mathbb{R}^n \mapsto \mathbb{R}^{n \times n}$  and  $\tilde{Q} : \mathbb{R}^m \mapsto \mathbb{R}^{n \times n}$  such that

$$W(x)y = \tilde{W}(y)x, \quad Q(x)\xi = \tilde{Q}(\xi)x, \quad \forall x, y \in \mathbb{R}^n, \quad \forall \xi \in \mathbb{R}^m. \quad (5)$$

*Assumption 1:* Given  $x^* \in \mathbb{R}^n$  let there exists  $\xi^* \in \bar{\mathbb{R}}_+^m$  such that

$$(L + W(x^*))x^* + (F + Q(x^*))\xi^* + g = 0.$$

The state  $x = x^*$  is an equilibrium (possibly unstable) of the system (4) with the control  $\xi = \xi^*$ . We consider  $\alpha^*$  as a desired set point of for tracking by the system.

Given  $r > 0$ ,  $x^* \in \mathbb{R}^n$ ,  $\xi^* \in \bar{\mathbb{R}}_+^m$ , the *control aim* is to design a finite-time stabilizing feedback  $\xi = \xi(x)$  such that for any  $x_0 \in \mathbb{R}^n : |x_0 - x^*| \leq r$  the trajectory of the closed-loop system satisfies

$$x(t) \rightarrow x^* \quad \text{as} \quad t \rightarrow T(x_0), \quad (6)$$

$$\xi(x(t)) \in \bar{\mathbb{R}}_+^m, \quad \forall t \geq 0, \quad (7)$$

where  $T : \mathbb{R}^n \mapsto \bar{\mathbb{R}}_+$  is the settling-time function.

### III. PRELIMINARIES ON HOMOGENEITY

#### A. Linear dilations

Let us recall that a *family of operators*  $\mathbf{d}(s) : \mathbb{R}^n \mapsto \mathbb{R}^n$  with  $s \in \mathbb{R}$  is a *group* if  $\mathbf{d}(0)x = x$ ,  $\mathbf{d}(s) \circ \mathbf{d}(t)x = \mathbf{d}(s+t)x$ ,  $\forall x \in \mathbb{R}^n, \forall s, t \in \mathbb{R}$ . A *group*  $\mathbf{d}$  is

- *continuous* if the mapping  $s \mapsto \mathbf{d}(s)x$  is continuous,  $\forall x \in \mathbb{R}^n$ ;
- *linear* if  $\mathbf{d}(s)$  is a linear mapping (i.e.,  $\mathbf{d}(s) \in \mathbb{R}^{n \times n}$ ),  $\forall s \in \mathbb{R}$ ;

- a *dilation* in  $\mathbb{R}^n$  if  $\liminf_{s \rightarrow +\infty} \|\mathbf{d}(s)x\| = +\infty$  and  $\limsup_{s \rightarrow -\infty} \|\mathbf{d}(s)x\| = 0, \forall x \neq \mathbf{0}$ .

A continuous linear group in  $\mathbb{R}^n$  admits the representation [10]:

$$\mathbf{d}(s) = e^{sG_{\mathbf{d}}} = \sum_{j=1}^{\infty} \frac{s^j G_{\mathbf{d}}^j}{j!}, \quad s \in \mathbb{R}. \quad (8)$$

where  $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$  is a generator of  $\mathbf{d}$ . A continuous linear group (8) is a dilation in  $\mathbb{R}^n$  if and only if  $G_{\mathbf{d}}$  is an anti-Hurwitz matrix [11]. In this paper we deal only with continuous linear dilations. A *dilation*  $\mathbf{d}$  in  $\mathbb{R}^n$  is

- *monotone* if the function  $s \mapsto \|\mathbf{d}(s)x\|$  is strictly increasing,  $\forall x \neq \mathbf{0}$ ;
- *strictly monotone* if  $\exists \beta > 0$  such that  $\|\mathbf{d}(s)x\| \leq e^{\beta s} \|x\|$ ,  $\forall s \leq 0, \forall x \in \mathbb{R}^n$ .

The following result is the straightforward consequence of the existence of the quadratic Lyapunov function for asymptotically stable LTI systems.

*Corollary 1:* A linear continuous dilation in  $\mathbb{R}^n$  is strictly monotone with respect to the weighted Euclidean norm  $\|x\| = \sqrt{x^\top P x}$ ,  $0 \prec P = P^\top \in \mathbb{R}^{n \times n}$  if and only if

$$P G_{\mathbf{d}} + G_{\mathbf{d}}^\top P \succ 0, \quad P \succ 0. \quad (9)$$

Any dilation in  $\mathbb{R}^n$  defines an alternative topology (balls, spheres, cones, etc) in  $\mathbb{R}^n$ .

#### B. Canonical homogeneous norm

Any linear continuous and monotone dilation in a normed vector space introduces also an alternative norm topology defined by the so-called canonical homogeneous norm [11].

*Definition 1 (Canonical homogeneous norm):* Let a linear dilation  $\mathbf{d}$  in  $\mathbb{R}^n$  be continuous and monotone with respect to a norm  $\|\cdot\|$ . A function  $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \mapsto [0, +\infty)$  defined as follows:  $\|\mathbf{0}\|_{\mathbf{d}} = 0$  and

$$\|x\|_{\mathbf{d}} = e^{s_x}, \quad \text{where } s_x \in \mathbb{R} : \|\mathbf{d}(-s_x)x\| = 1, \quad x \neq \mathbf{0} \quad (10)$$

is said to be a canonical  $\mathbf{d}$ -homogeneous norm in  $\mathbb{R}^n$ . For standard dilation  $\mathbf{d}_1(s) = e^s I_n$  we have  $\|x\|_{\mathbf{d}_1} = \|x\|$ . In other cases,  $\|x\|_{\mathbf{d}}$  with  $x \neq \mathbf{0}$  is implicitly defined by a nonlinear algebraic equation, which always have a unique solution due to monotonicity of the dilation. In some particular cases [12], this implicit equation has explicit solution even for non-standard dilations. Since  $\|\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x\| = 1$  for  $x \neq \mathbf{0}$ , then the operator

$$\pi_{\mathbf{d}}(x) := \begin{cases} \mathbf{d}(-\ln \|x\|_{\mathbf{d}})x & \text{if } x \neq \mathbf{0}, \\ \mathbf{0} & \text{if } x = \mathbf{0}, \end{cases} \quad (11)$$

is a projector of a nonzero vector  $x$  on the unit sphere  $\|x\| = 1$ . For the standard dilation, such a projector is defined as  $\pi_{\mathbf{d}_1}(x) = \mathbf{d}_1(-\ln \|x\|_{\mathbf{d}_1})x = \frac{x}{\|x\|}$  for  $x \neq \mathbf{0}$ . We have  $\pi_{\mathbf{d}}(x) = \text{sign}(x)$  in the scalar case  $n = 1$ .

*Lemma 1:* [11] If a linear continuous dilation  $\mathbf{d}$  in  $\mathbb{R}^n$  is strictly monotone with respect to a norm  $\|x\| = \sqrt{x^\top P x}$  then

- 1)  $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \mapsto \mathbb{R}_+$  is single-valued, continuous on  $\mathbb{R}^n$  and continuously differentiable on  $\mathbb{R}^n \setminus \{\mathbf{0}\}$ :

$$\frac{\partial \|x\|_{\mathbf{d}}}{\partial x} = \frac{\|x\|_{\mathbf{d}} x^{\top} \mathbf{d}^{\top} (-\ln \|x\|_{\mathbf{d}}) P \mathbf{d} (-\ln \|x\|_{\mathbf{d}})}{x^{\top} \mathbf{d}^{\top} (-\ln \|x\|_{\mathbf{d}}) P G_{\mathbf{d}} \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x}, \quad x \neq \mathbf{0}. \quad (12)$$

- 2) for  $\alpha = \frac{\lambda_{\max}(P^{1/2} G_{\mathbf{d}} P^{-1/2} + P^{-1/2} G_{\mathbf{d}}^{\top} P^{1/2})}{\lambda_{\min}(P^{1/2} G_{\mathbf{d}} P^{-1/2} + P^{-1/2} G_{\mathbf{d}}^{\top} P^{1/2})}$  and  $\beta = \frac{2}{2}$  it holds

$$\|x\|_{\mathbf{d}}^{\alpha} \leq \|x\| \leq \|x\|_{\mathbf{d}}^{\beta}, \quad \forall x \in \mathbb{R}^n : \|x\| \leq 1; \quad (13)$$

$$\|x\|_{\mathbf{d}}^{\beta} \leq \|x\| \leq \|x\|_{\mathbf{d}}^{\alpha}, \quad \forall x \in \mathbb{R}^n : \|x\| \geq 1. \quad (14)$$

### C. Homogeneous functions and vector fields

Below we study systems which are symmetric on homogeneous cones with respect to a linear dilation  $\mathbf{d}$ . The dilation symmetry introduced by the following definition is known as a generalized homogeneity [18], [7], [13], [3].

*Definition 2:* [7] Let  $\mathbf{d}$  be dilation in  $\mathbb{R}^n$ . A function  $h : \mathbb{R}^n \mapsto \mathbb{R}$  (resp., a vector field  $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ ) is  $\mathbf{d}$ -homogeneous of degree  $\nu \in \mathbb{R}$  if

$$h(\mathbf{d}(s)x) = e^{\nu s} h(x), \quad \forall x \in \mathbb{R}^n, \quad \forall s \in \mathbb{R}.$$

(resp., if  $f(\mathbf{d}(s)x) = e^{\mu s} \mathbf{d}(s) f(x)$ ,  $\forall s \in \mathbb{R}$ ,  $\forall x \in \mathbb{R}^n$ .) Formally, to avoid a collision in the above definition for  $n = 1$ , a vector field  $g$  should be defined as  $g : \Xi \mapsto T\Xi$ , where  $T\Xi$  is the tangent space for  $\Xi$ . Since the tangent space of  $\mathbb{R}^n$  can be associated with  $\mathbb{R}^n$ , we simply write  $g : \mathbb{R}^n \mapsto \mathbb{R}^n$ .

The homogeneity of a mapping is inherited by other mathematical objects induced by this mapping. In particular, solutions of  $\mathbf{d}$ -homogeneous system<sup>2</sup>

$$\dot{x} = g(x), \quad t > 0, \quad x(0) = x_0 \in \mathbb{R}^n \quad (15)$$

are symmetric with respect to the dilation  $\mathbf{d}$  in the following sense [18], [7], [3]:

$$x(t, \mathbf{d}(s)x_0) = \mathbf{d}(s)x(e^{\mu s}t, x_0), \quad (16)$$

where  $x(\cdot, x_0)$  denotes a solution of (15) with  $x(0) = x_0 \in \mathbb{R}^n$  and  $\mu \in \mathbb{R}$  is the homogeneity degree of  $g$ . The solution symmetry (16) implies that any asymptotically stable homogeneous systems of negative degree are finite-time stable [3]. The latter property holds for locally homogeneous systems as well [2]. We follow the idea of local homogeneity for the finite-time stabilizing feedback design and we adapt the control design procedure, developed originally for linear plants in [17], to the quasi-linear system (4).

## IV. CONTROL DESIGN

Let us derive first the error equation and denote

$$\xi = \xi^* + u, \quad z = x - x^*, \quad (17)$$

<sup>2</sup>A system is homogeneous if its is governed by a  $\mathbf{d}$ -homogeneous vector field

where  $u \in \mathbb{R}^m$  is a new (virtual) control such that  $\xi^* + u \in \mathbb{R}_+^m$ . We derive

$$\begin{aligned} \dot{z} &= (L + W(x))x + F\xi^* + Fu + Q(x)\xi^* + Q(x)u + g \\ &= Lx + W(x)x - Lx^* - W(x^*)x^* + Fu \\ &\quad - Q(x^*)\xi^* + Q(x)\xi^* + Q(x)u \\ &= Lz + W(x - x^*)x + W(x^*)z + (F + Q(x))u \\ &\quad + Q(x - x^*)\xi^*. \end{aligned} \quad (18)$$

Therefore, from (18) and using (5) we obtain the following bilinear system:

$$\dot{z} = Az + Bu + D(z, u)z, \quad (19)$$

with

$$A = L + \tilde{W}(x^*) + W(x^*) + \tilde{Q}(\xi^*), \quad (20)$$

$$B = F + Q(x^*), \quad (21)$$

and

$$D(z, u) = W(z) + \tilde{Q}(u). \quad (22)$$

A finite-time stabilizing feedback for a linear plant can be designed if and only if the pair  $\{A, B\}$  is controllable. The homogeneous controllers for this case can be found in [17], where, in particular, it is shown that for any controllable  $\{A, B\}$ , there exists a solution  $Y_0 \in \mathbb{R}^{m \times n}$ ,  $G_0 \in \mathbb{R}^{n \times n}$  of the linear algebraic equation

$$AG_0 - G_0A + BY_0 = A, \quad G_0B = \mathbf{0} \quad (23)$$

such that the matrix  $G_{\mathbf{d}} := I_n + \mu G_0$  is anti-Hurwitz for any  $\mu \in [-1, 1/\tilde{n}]$ , where  $\tilde{n}$  is a minimal natural number such that  $\text{rank}[B, AB, \dots, A^{\tilde{n}-1}B] = n$ , and the matrix  $A_0 = A + BK_0$  with  $K_0 := Y_0(G_0 - I_n)^{-1}$  satisfies the identity  $A_0G_{\mathbf{d}} = (G_{\mathbf{d}} + \mu I_n)A_0$ ,  $G_{\mathbf{d}}B = B$ . (24)

The above property of the matrix  $A_0$  guarantees that the vector field  $z \mapsto A_0z$  is  $\mathbf{d}$ -homogeneous of the degree  $\mu$ , so the feedback  $K_0z$  ‘‘homogenize’’ (in the generalized sense) the linear part of the system [17].

*Theorem 1:* Let the pair  $\{A, B\}$  be controllable, the matrices  $K_0, G_{\mathbf{d}}$  be defined by solving the equation (23) and  $\mu \in [-1, 0)$ . If

- 1)  $G_{\mathbf{d}}Q(z) = Q(z)$  for all  $s \in \mathbb{R}$  and all  $z \in \mathbb{R}^n$ , where  $\mathbf{d}(s) = e^{sG_{\mathbf{d}}}$  is the linear dilation.
- 2) if for some  $\rho, \beta, \omega_1, \omega_2 \in \mathbb{R}_+$  the matrices  $X = X^{\top} \in \mathbb{R}^{n \times n}$  and  $Y \in \mathbb{R}^{m \times n}$  satisfy the LMIs

$$A_0X + XA_0^{\top} + BY + Y^{\top}B^{\top} + \rho(G_{\mathbf{d}}X + XG_{\mathbf{d}}) + (\omega_1 + \omega_2)X \preceq 0, \quad (25)$$

$$\begin{pmatrix} \omega_1 X & rYQ(h_k)^{\top} \\ rQ(h_k)Y & \omega_1 X \end{pmatrix} \succeq 0, \quad k = \overline{1, n} \quad (26)$$

$$\begin{pmatrix} \omega_2 X & rX(W(h_k) + Q(h_k)K_0)^{\top} \\ r(W(h_k) + Q(h_k)K_0)X & \omega_2 X \end{pmatrix} \succeq 0, \quad (27)$$

$$\begin{pmatrix} 0.5\xi_j^* X & e_j^{\top} K_0 X \\ X K_0^{\top} e_j & 0.5\xi_j^* \end{pmatrix} \succeq 0, \quad \begin{pmatrix} 0.5\xi_j^* X & e_j^{\top} Y \\ Y^{\top} e_j & 0.5\xi_j^* \end{pmatrix} \succeq 0, \quad j = \overline{1, m}, \quad (28)$$

$$2(2\beta - \mu)X \succ G_{\mathbf{d}}X + XG_{\mathbf{d}}^{\top} \succeq 2\beta X, \quad (29)$$

$$\frac{r^2}{n} I_n \succeq X \succ 0, \quad (30)$$

where  $\lambda_j \geq 0$ ,  $h_k = (0, \dots, 1, \dots, 0)^\top$  is the unit Euclidean vector in  $\mathbb{R}^n$ ,  $e_j = (0, \dots, 1, \dots, 0)^\top$  is the unit Euclidean vector in  $\mathbb{R}^m$  and  $\xi_j^*$  is the  $j$ -th element of the vector  $\xi^* \in \mathbb{R}_+^m$ , then the system (19) with the feedback control

$$u = K_0 z + \|z\|_{\mathbf{d}}^{1+\mu} K \mathbf{d}(-\ln \|z\|_{\mathbf{d}}) z, \quad (31)$$

$$K_0 := Y_0 G_{\mathbf{d}}^{-1}, \quad K := Y X^{-1},$$

is locally finite-time stable with the invariant ellipsoid

$$\mathcal{E}(X) := \{z \in \mathbb{R}^n : z^\top X^{-1} z \leq 1\}; \quad (32)$$

belonging to the attraction domain. Moreover, for  $|z(0)| \leq r$ , we have

$$0 \leq \xi^* + u(z(t)), \quad \forall t \geq 0,$$

$$z(t) \rightarrow \mathbf{0} \quad \text{as} \quad t \rightarrow T(z(0)),$$

where  $T(z(0)) \leq \frac{\|z(0)\|_{\mathbf{d}}^{-\mu}}{(-\mu\rho)}$  with the canonical homogeneous norm  $\|\cdot\|_{\mathbf{d}}$  induced by the weighted Euclidean norm  $\|z\| = \sqrt{z^\top X^{-1} z}$ .

**Proof.** Using (12), for  $V = \|z\|_{\mathbf{d}}$  we derive

$$\dot{V} = \|z\|_{\mathbf{d}} \frac{z^\top \mathbf{d}^\top (-\ln \|z\|_{\mathbf{d}}) X^{-1} \mathbf{d} (-\ln \|z\|_{\mathbf{d}}) (Az + Bu + D(z, u)z)}{z^\top \mathbf{d}^\top (-\ln \|z\|_{\mathbf{d}}) X^{-1} G_{\mathbf{d}} \mathbf{d} (-\ln \|z\|_{\mathbf{d}}) z}. \quad (33)$$

Using the homogeneity (namely,  $A_0 \mathbf{d}(s) = e^{\mu s} \mathbf{d}(s) A_0$  and  $\mathbf{d}(s) B = e^s B$ ) we derive (see, [17] for mode details)

$$z^\top \mathbf{d}^\top (-\ln \|z\|_{\mathbf{d}}) X^{-1} \mathbf{d} (-\ln \|z\|_{\mathbf{d}}) (Az + Bu)$$

$$= \|z\|_{\mathbf{d}}^\mu z^\top \mathbf{d}^\top (-\ln \|z\|_{\mathbf{d}}) X^{-1} (A_0 + BK) \mathbf{d} (-\ln \|z\|_{\mathbf{d}}) z, \quad (34)$$

which in conjunction with condition (25), we obtain from (33) the following estimate:

$$\dot{V} \leq -\rho V^{1+\mu} + \frac{-(\omega_1 + \omega_2) \|z\|_{\mathbf{d}}^\mu + z^\top \mathbf{d}^\top (-\ln \|z\|_{\mathbf{d}}) X^{-1} \mathbf{d} (-\ln \|z\|_{\mathbf{d}}) D(z, u) z}{z^\top \mathbf{d}^\top (-\ln \|z\|_{\mathbf{d}}) X^{-1} G_{\mathbf{d}} \mathbf{d} (-\ln \|z\|_{\mathbf{d}}) z}. \quad (35)$$

Notice that the term  $D(z, u)z$  can be rewritten as follows:

$$D(z, u)z = W(z)z + Q(z)u$$

$$= W(z)z + Q(z)K_0 z$$

$$+ \|z\|_{\mathbf{d}}^{1+\mu} Q(z)K \mathbf{d}(-\ln \|z\|_{\mathbf{d}}) z. \quad (36)$$

Hence, using  $\mathbf{d}(s)Q(z) = e^s Q(z)$  we obtain

$$z^\top \mathbf{d}^\top (-\ln \|z\|_{\mathbf{d}}) X^{-1} \mathbf{d} (-\ln \|z\|_{\mathbf{d}}) D(z, u)z$$

$$= z^\top \mathbf{d}^\top (-\ln \|z\|_{\mathbf{d}}) X^{-1} \mathbf{d} (-\ln \|z\|_{\mathbf{d}}) (W(z) + Q(z)K_0)z$$

$$+ \|z\|_{\mathbf{d}}^\mu z^\top \mathbf{d}^\top (-\ln \|z\|_{\mathbf{d}}) X^{-1} Q(z)K \mathbf{d}(-\ln \|z\|_{\mathbf{d}}) z. \quad (37)$$

Taking into account  $z^\top \mathbf{d}^\top (-\ln \|z\|_{\mathbf{d}}) X^{-1} \mathbf{d} (-\ln \|z\|_{\mathbf{d}}) z = 1$ , and using Cauchy-Swartz inequality we derive

$$z^\top \mathbf{d}^\top (-\ln \|z\|_{\mathbf{d}}) X^{-1} Q(z)K \mathbf{d}(-\ln \|z\|_{\mathbf{d}}) z \leq \omega_1 \quad (38)$$

provided that  $K^\top Q^\top(z)X^{-1}Q(z)K \leq \omega_1^2 X^{-1}$ .

Let us show that for  $\|z\|_1 \leq r$ , one has

$$K^\top Q^\top(z)X^{-1}Q(z)K \leq \omega_1^2 X^{-1}. \quad (39)$$

Indeed, using the Schur complement, inequality (39) can be rewritten as follows:

$$\begin{pmatrix} \omega_1 X & rYQ(z/r)^\top \\ rQ(z/r)Y & \omega_1 X \end{pmatrix} \succeq 0, \quad (40)$$

By Carathéodory lemma on convex hull, for any  $z$  such that  $\|z\|_1 \leq 1$ , there exist scalars  $\alpha_k^\pm \geq 0$  such that  $\sum_{k=1}^n \alpha_k^+ + \alpha_k^- = 1$  and

$$z = \sum_{k=1}^n (\alpha_k^+ - \alpha_k^-) h_k. \quad (41)$$

where  $h_k = (0, \dots, 1, \dots, 0)^\top \in \mathbb{R}^n$  is the unit Euclidean vector. Hence, for  $\|z\|_1 \leq r$  we derive

$$\begin{pmatrix} \omega_1 X & rYQ(z/r)^\top \\ rQ(z/r)Y & \omega_1 X \end{pmatrix} = \sum_{k=1}^n \alpha_k^+ \begin{pmatrix} \omega_1 X & rYQ(h_k)^\top \\ rQ(h_k)Y & \omega_1 X \end{pmatrix}$$

$$+ \alpha_k^- \begin{pmatrix} \omega_1 X & rYQ(-h_k)^\top \\ rQ(-h_k)Y & \omega_1 X \end{pmatrix}, \quad (42)$$

Taking into account  $Q(-z) = -Q(z)$ ,  $\forall z \in \mathbb{R}^n$  we derive that (26) implies (38) for any  $z : \|z\|_1 \leq r$ .

Similarly, we derive

$$\frac{z^\top \mathbf{d}^\top (-\ln \|z\|_{\mathbf{d}}) X^{-1} \mathbf{d} (-\ln \|z\|_{\mathbf{d}}) (W(z) + Q(z)K_0)z}{\omega_2 \|z\|_{\mathbf{d}}^\mu} \leq 1 \quad (43)$$

provided that

$$\left\| \frac{1}{\omega_2 \|z\|_{\mathbf{d}}^\mu} \mathbf{d}(-\ln \|z\|_{\mathbf{d}}) (W(z) + Q(z)K_0)z \right\| \leq 1.$$

The inequality  $2(2\beta - \mu)X \succ G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top$  guarantees that the matrix  $G_{\tilde{\mathbf{d}}} = (2\beta - \mu)I_n - G_{\mathbf{d}}$  is anti-Hurwitz and the dilation  $\tilde{\mathbf{d}}(s) = e^{sG_{\tilde{\mathbf{d}}}}$  is monotone with respect to the norm  $\|\cdot\|$ . Hence, we have

$$\left\| \|z\|_{\mathbf{d}}^{2\beta - \mu} \mathbf{d}(-\ln \|z\|_{\mathbf{d}}) \right\| = \|\tilde{\mathbf{d}}(\ln \|z\|_{\mathbf{d}})\| \leq 1 \quad (44)$$

for all  $z : \|z\|_{\mathbf{d}} \leq 1$  (which is equivalent to  $\|z\| \leq 1$ ). Thus, the inequality

$$\|\omega_2^{-1} (W(z)/\|z\|_{\mathbf{d}}^\beta + Q(z)/\|z\|_{\mathbf{d}}^\beta) K_0 z / \|z\|_{\mathbf{d}}^\beta\| \leq 1 \quad (45)$$

implies (43). The LMI  $G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top \succeq 2\beta X$  implies  $\|z\| \leq \|z\|_{\mathbf{d}}^\beta$  (see Lemma 1) provided that  $\|z\|_{\mathbf{d}} \leq 1$ . In this case, for  $\|z\| \leq 1$  the inequality (45) follows from  $\|\omega_2^{-1} (W(z) + Q(z)K_0)z\| \leq \|z\|^2 \Leftarrow$

$$(W(z) + Q(z)K_0)^\top (W(z) + Q(z)K_0) \leq \omega_2 X^{-1}. \quad (46)$$

Repeating the above consideration, we derive that LMI (26) implies (43) provided that  $\|z\| \leq 1$  and  $\|z\|_1 \leq r$ . Therefore,

$$\dot{V} \leq -\rho V^{1+\mu}, \quad (47)$$

where  $\|z\|_1 \leq r$  and  $\|z\| \leq 1$ . Since  $r^2 n^{-1} I_n \succeq X$ , then

$$\|z\|_1 r^{-1} \leq \frac{\sqrt{n}}{r} |z| = \sqrt{r^{-2} n z^\top z} \leq \sqrt{z^\top X^{-1} z} \leq 1,$$

for  $\|z\| \leq 1$ . Therefore, the closed-loop system is locally finite-time stable and the ellipsoid  $\mathcal{E}(X) = \{z : \|z\| \leq 1\}$  belongs to the domain of attraction. Moreover, since  $|z| \leq \|z\|_1 \leq r$  for  $\|z\| \leq 1$  then  $|z(0)| \leq r \Rightarrow z(t) \rightarrow 0$  within the finite time  $T(z(0)) \leq (-\mu\rho)^{-1} V(z(0))^{-\mu}$ .

The LMI (28) and the inequality  $z^\top X^{-1}z \leq 1$  guarantees

$$z^\top K_0^\top e_j e_j^\top K_0 z \leq (\xi_j^*)^2/4$$

or, equivalently,  $|e_j^\top K_0 z(t)| \leq \xi_j^*/2$  for all  $t \geq 0$ . Similarly, taking into account  $z^\top \mathbf{d}^\top (-\ln \|z\|_{\mathbf{d}}) X^{-1} \mathbf{d} (-\ln \|z\|_{\mathbf{d}}) z = 1$ , we derive

$$z^\top \mathbf{d}^\top (-\ln \|z\|_{\mathbf{d}}) X^{-1} Y^\top e_j e_j^\top Y X^{-1} \mathbf{d} (-\ln \|z\|_{\mathbf{d}}) z \leq \frac{(\xi_j^*)^2}{4}$$

or, equivalently,  $|e_j^\top K \mathbf{d} (-\ln \|z(t)\|_{\mathbf{d}}) z(t)| \leq \xi_j^*/2$  for all  $t \geq 0$ . Since  $\|z\|_{\mathbf{d}} \leq 1$  then  $\xi_j^* + e_j^\top K_0 z + \|z\|_{\mathbf{d}} e_j^\top K \mathbf{d} (-\ln \|z(t)\|_{\mathbf{d}}) z(t) \geq 0$  for all  $t \geq 0$ . The proof is complete.

The feasibility of the system (25)-(29) can be studied similarly to [17].

*Remark 1:* Given  $r > 0$ , the system of LMIs (25)-(29) is feasible for  $\mu$  sufficiently close to 0 provided that  $|rW(h_i)|$  and  $|rQ(h_i)|$  are small enough but  $\xi_j^* \in \mathbb{R}_+^n$  is large enough. Indeed, taking  $\beta = -3/4$  and any  $X \succ 0$  once can see that the LMI (29) is fulfilled for  $\mu$  sufficiently close to 0. The controllability of  $\{A, B\}$  implies the feasibility of (25) (see, [17] for more details). For any fixed  $\omega_1, \omega_2 \in \mathbb{R}_+$  and  $X \succ 0$ , the LMIs (26), (27) are feasible if  $|rW(h_i)|$  and  $|rQ(h_i)|$  are small enough. For any fixed  $X \succ 0$ , the LMI (28) is feasible if  $\xi_j^* \in \mathbb{R}_+$  is large enough.

Therefore, the only critical restriction in the above theorem is  $G_{\mathbf{d}}Q(z) = Q(z)$ , but it may be fulfilled in some practical cases, for example, for a discretization of the one-dimensional viscous Burgers equation controlled on the boundary.

*Remark 2:* The closed-loop system (19), (31) is locally homogeneous and locally asymptotically stable so it is locally ISS in the view of results [2] with respect to a rather large class of perturbations (in particular with respect to measurement noises and additive disturbances).

## V. NUMERICAL EXAMPLE

Let us consider the error system (19) with the parameters

$$A = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad Q(z) = \varepsilon \begin{pmatrix} -0.5z_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$W(z) = \varepsilon \cdot \text{diag} \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} z \right\},$$

where  $z = (z_1, z_2, z_3, z_4, z_5)^\top \in \mathbb{R}^5$  and  $\text{diag}\{v\}$  denotes the diagonal matrix with the components of the vector  $v \in \mathbb{R}^n$  on the main diagonal.

Solving the algebraic equation (23) we derive

$$K_0 = (20 \quad -88 \quad 220 \quad -330 \quad 264),$$

$$G_0 = \begin{pmatrix} 0 & -8 & 6 & -4 & 2 \\ 0 & -1 & -6 & 4 & -2 \\ 0 & 0 & -2 & -4 & 2 \\ 0 & 0 & 0 & -3 & -2 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix}.$$

The dilation is given by  $G_{\mathbf{d}} = I_5 + \mu G_0$  where  $\mu \in [-1, 0)$ . Obviously,  $G_{\mathbf{d}}Q(z) = Q(z)$  so the first condition of the main

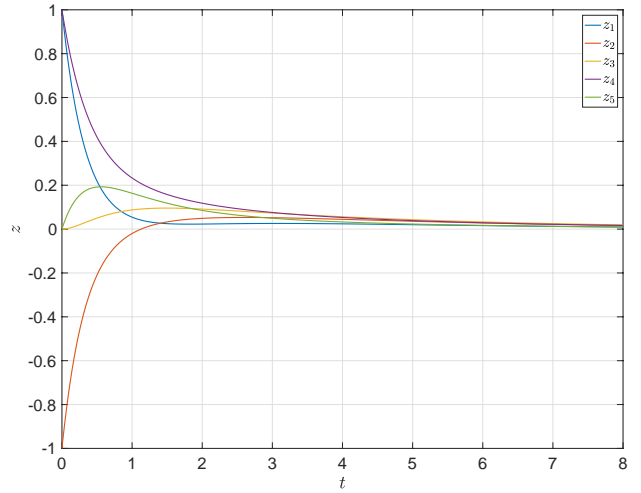


Fig. 1: Evolution of the states of the open-loop system ( $u = 0$ )

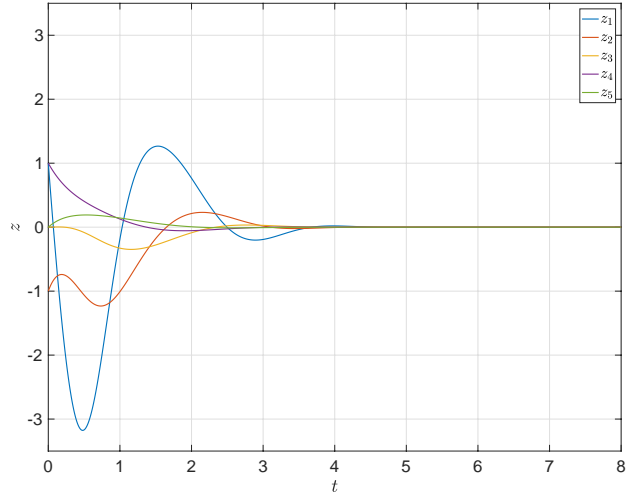


Fig. 2: Evolution of the states of the closed-loop system

theorem is fulfilled. The pair  $\{A, B\}$  is controllable, so the LMIs (25)-(27),(29),(30) feasible for small enough  $\varepsilon > 0$  (in this example, the control restrictions are omitted). Solving (25),(29),(30) for  $\mu = -0.1$  we derive

$$X = \begin{pmatrix} 148.1977 & 5.4546 & -28.9205 & -2.0518 & 9.0251 \\ 5.4546 & 25.7547 & 0.2926 & -4.9821 & 0.0481 \\ -28.9205 & 0.2926 & 6.3759 & 0.1808 & -2.3111 \\ -2.0518 & -4.9821 & 0.1808 & 1.0161 & -0.1000 \\ 9.0251 & 0.0481 & -2.3111 & -0.1000 & 1.0000 \end{pmatrix},$$

$$K = (-18.0000 \quad 65.0431 \quad -210.9065 \quad 272.6945 \quad -257.9888).$$

The simulations results for the open-loop ( $u = 0$ ) and the closed-loop systems for  $\varepsilon = 0.1$  are shown on the Figure 1 and 2. The simulation results are obtained using the explicit Euler method with the sampling period  $10^{-3}$ . Application of the homogeneous control (31) requires a computation of the canonical homogeneous norm  $\|\cdot\|_{\mathbf{d}}$  by means of the special numerical procedure [17]. Homogeneous Control Systems (HCS) Toolbox for MATLAB [1] is utilized for this purpose.

The comparison of the decay rates of the open-loop and the closed-loop systems is given on the Figure 3. The

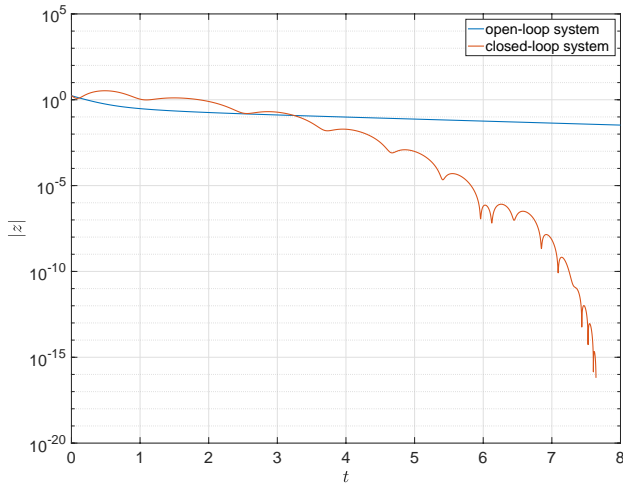


Fig. 3: Comparison of the decay rates of the open-loop and closed-loop systems in the logarithmic scale

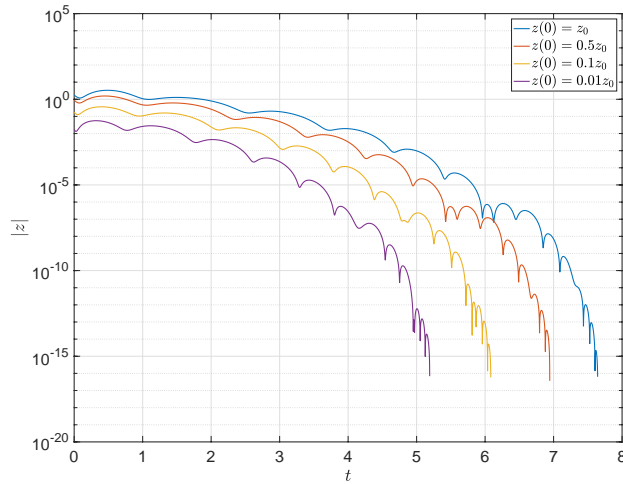


Fig. 4: Dependence of the settling time on the variation of the norm of the initial state

system with the homogeneous controller demonstrates the behavior typical for the finite-time stable system. According to simulation results the settling time for the chosen initial state  $z(0) = z_0 := (1, -1, 0, 1, 0)^T$  is about 8 units of time. The dependence of the settling time of the initial condition can be observed from the simulation results depicted on Figure 4.

## VI. CONCLUSIONS

In this paper, an algorithm for the finite-time stabilizing control design is developed for the class of quasi-linear systems. The design procedure uses the linear approximation as the reference mode to design a homogeneous controller. The parameters of the controller are tuned in such a way that allow the stabilization of the original (quasi-linear) system as well. The convex embedding technique is utilized in order to derive the control parameters from linear matrix inequalities. The obtained LMIs are shown to be feasible under conditions of controllability of the linear approximation. However,

they are still rather conservative. Further relaxation of the restriction to the quasi-linear system and derivation of less conservative LMIs are interesting problems for the future research.

## REFERENCES

- [1] <https://gitlab.inria.fr/polyakov/hcs-toolbox-for-matlab>.
- [2] V. Andrieu, L. Praly, and A. Astolfi. Homogeneous Approximation, Recursive Observer Design, and Output Feedback. *SIAM Journal of Control and Optimization*, 47(4):1814–1850, 2008.
- [3] S. P. Bhat and D. S. Bernstein. Geometric homogeneity with applications to finite-time stability. *Mathematics of Control, Signals and Systems*, 17:101–127, 2005.
- [4] L. Grüne. Homogeneous state feedback stabilization of homogeneous systems. *SIAM Journal of Control and Optimization*, 38(4):1288–1308, 2000.
- [5] H. Hermes. Nilpotent approximations of control systems and distributions. *SIAM Journal of Control and Optimization*, 24(4):731, 1986.
- [6] L.S. Husch. Topological Characterization of The Dilation and The Translation in Frechet Spaces. *Mathematical Annals*, 190:1–5, 1970.
- [7] M. Kawski. Families of dilations and asymptotic stability. *Analysis of Controlled Dynamical Systems*, pages 285–294, 1991.
- [8] V. V. Khomenuk. On systems of ordinary differential equations with generalized homogenous right-hand sides. *Izvestia vuzov. Matematika (in Russian)*, 3(22):157–164, 1961.
- [9] E. Moulay and W. Perruquetti. Finite time stability and stabilization of a class of continuous systems. *Journal of Mathematical Analysis and Applications*, 323:1430–1443, 2006.
- [10] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer, 1983.
- [11] A. Polyakov. Sliding mode control design using canonical homogeneous norm. *International Journal of Robust and Nonlinear Control*, 29(3):682–701, 2019.
- [12] A. Polyakov and M. Krstic. Finite- and fixed-time nonovershooting stabilizers and safety filters by homogeneous feedback. *IEEE Transaction on Automatic Control*, 2023.
- [13] L. Rosier. Homogeneous Lyapunov function for homogeneous continuous vector field. *Systems & Control Letters*, 19:467–473, 1992.
- [14] L. Rosier. Etude de quelques problèmes de stabilisation. *PhD Thesis, Ecole Normale Supérieure de Cachan (France)*, 1993.
- [15] E.D. Sontag. A 'universal' construction of artstein's theorem on nonlinear stabilization. *Systems & Control Letters*, 13:117–123, 1989.
- [16] K. Zimenko, A. Polyakov, and D. Efimov. Homogeneous systems stabilization based on convex embedding. *Automatica*, (Accepted, hal-04037561), 2023.
- [17] K. Zimenko, A. Polyakov, D. Efimov, and W. Perruquetti. Robust feedback stabilization of linear mimo systems using generalized homogenization. *IEEE Transactions on Automatic Control*, 2020.
- [18] V.I. Zubov. On systems of ordinary differential equations with generalized homogeneous right-hand sides. *Izvestia vuzov. Matematika (in Russian)*, 1:80–88, 1958.