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STATURES AND SOBRIFICATION RANKS OF NOETHERIAN SPACES

JEAN GOUBAULT-LARRECQ AND BASTIEN LABOUREIX

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ABSTRACT. There is a rich theory of maximal order types of well-partial-orders (wpos), pioneered by de Jongh and Parikh (1977) and Schmidt (1981). Every wpo is Noetherian in its Alexandroff topology, and there are more; this prompts us to investigate an analogue of that theory in the wider context of Noetherian spaces.

The notion of maximal order type does not seem to have a direct analogue in Noetherian spaces per se, but the equivalent notion of stature, investigated by Blass and Gurevich (2008) does: we define the stature $||X||$ of a Noetherian space X as the ordinal rank of its poset of proper closed subsets. We obtain formulas for statures of sums, of products, of the space of words on a space X , of the space of finite multisets on X , in particular. They confirm previously known formulas on wpos, and extend them to Noetherian spaces.

The proofs are, by necessity, rather different from their wpo counterparts, and rely on explicit characterizations of the sobrifications of the corresponding spaces, as obtained by Finkel and the first author (2020).

We also give formulas for the statures of some natural Noetherian spaces that do not arise from wpos: spaces with the cofinite topology, Hoare powerspaces, powersets, and spaces of words on X with the so-called prefix topology.

Finally, because our proofs require it, and also because of its independent interest, we give formulas for the ordinal ranks of the sobrifications of each of those spaces, which we call their sobrification ranks.

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1. INTRODUCTION

A *well-quasi-ordering* \leq (or *wqo*) on a set P is a preordering such that every sequence $(x_n)_{n \in \mathbb{N}}$ is good, namely: there are indices $m < n$ such that $x_m \leq x_n$. Then (P, \leq) , or just P for short, is a *well-quasi-order*; we abbreviate this as wqo as well. A *well-partial-order* (*wpo*) is an antisymmetric wqo. A wpo is the same thing as a partial ordering whose linear extensions are all well-founded [Wol67]. In that case, a well-known result by de Jongh and Parikh states that the supremum of the ordinal ranks of those linear extensions is attained [JP77, Theorem 2.13]; that supremum is called the *maximal order type* $o(P)$ of the wpo P .

The study of maximal order types of wpos grew out of questions in infinite combinatorics, pioneered by de Jongh and Parikh [JP77] and Schmidt [Sch79]. Schmidt found the maximal order types of wpos of finite words, and of wpos of finite trees (a.k.a. terms of first-order logic). This has profound consequences in proof theory where, roughly speaking, the maximal order type of P coincides with the largest proof-theoretical ordinal of natural logical theories that cannot prove that P is well-partial-ordered, see [Sim85, RW93] for example. This also has implications on upper bounds of the lengths of rewrite sequences of those rewrite systems that can be shown to terminate using some familiar well-founded orderings [Wei94], and in studying the exact complexity of certain decidable, but highly complex, problems in computer science, and notably in verification [FFSS11, SS11].

The purpose of this paper is to propose an extension of the theory of maximal order types of wpos to Noetherian spaces. A *Noetherian space* is a topological space in which every open subset is compact, meaning that each of its open covers has a finite subcover; we do not require separation. (See Section 9.7 of [GL13] for more information on Noetherian spaces.) The basic premise of this work is that a preordered set (P, \leq) is wqo if and only if P is Noetherian in the Alexandroff topology of \leq [GL13, Proposition 9.7.17]. This was already used in [Gou10] in order to generalize some classic wqo-based verification algorithms [ABJ98, FS01]. With the eventual aim of understanding the complexity of those algorithms, one may think of following a similar route as in the wpo case [FFSS11, SS11]. A first step in this endeavor—and our only concern in this paper—is to find a notion that would generalize the notion of maximal order type to all Noetherian spaces X , and to study it. This will be the *stature* of X (Definition 3.1).

Explicitly, we will define the stature of X as the ordinal rank of its poset of proper closed subsets, imitating a characterization of the maximal order type due to—and taking the name from—Blass and Gurevich [BG08]. A related notion is

the ordinal rank of the poset of *irreducible* closed subsets of X , which we will call the *sobrification rank* of X . We will see that the stature and sobrification rank of a Noetherian space are related, and this will help us in proofs of formulas for the stature of certain Noetherian spaces. Additionally, this notion of sobrification rank has independent interest, even on wpos: the ideal Karp-Miller procedure of [BFG20] requires, among other conditions, that the sobrification rank of the state space of the well-structured transition system under study is strictly less than ω^2 (see Section 5 of that paper).

1.1. Outline. Section 2 provides some preliminary notions, mostly on ordinals and on topology. We explore a few ways of defining an adequate generalization of statures to Noetherian spaces in Section 3. We settle on the definition we gave above, and we show that it coincides with the maximal order type of X (Blass and Gurevich's own stature) in the case where X is a wpo. On wpos, that ordinal rank coincides with the length of a longest chain in the poset of closed sets (minus 1), but one should be warned that this coincidence fails in the realm of Noetherian spaces, except for countable Noetherian spaces, as we see in the same section.

We introduce our notion of sobrification rank in Section 4, and we show that, up to some $+1$ or $1+$ terms, the sobrification rank is smaller than the stature, and the stature is smaller than ω to the power of the sobrification rank. Section 5 is devoted to a few technical tools, which will allow us to compare the statures and sobrification ranks of spaces X and Y once we have certain maps from X to Y , and to compute the stature of a space X from the statures of a cofinal family of proper closed subsets of X .

We compute the statures and sobrification ranks of finite T_0 spaces, and of well-founded chains in various topologies, in Section 6. We do the same for spaces with a cofinite topology in Section 7, for topological sums in Section 8, for lexicographic sums in Section 9, for topological products in Section 10, for Hoare powerspaces and powersets in Section 11, for spaces of finite words with the so-called word topology in Section 12 (generalizing the case of wpos of words explored by de Jongh and Parikh [JP77] and Schmidt [Sch79]), for spaces of so-called heterogeneous words in the prefix topology in Section 13, and for spaces of finite multisets in Section 14 (generalizing the case of wpos of multisets explored by Aschenbrenner and Pong [AP04], Weiermann [Wei09] and van der Meeren, Rathjen and Weiermann [MRW15]). We have chosen this particular list of constructions since it matches roughly those introduced [GL13, Section 9.7] and further studied in

[FGL20]. Finite words are particularly useful in several applications in verification, where they are typically used to model the contents of communication queues; see [Gou10, Section 6] for example.

In all cases, we obtain exact formulae for stature and sobrification rank, except for the sobrification rank of spaces of multisets and for the stature of Hoare powerspaces, for which we obtain non-matching lower and upper bounds; but those bounds are optimal, as we will demonstrate.

The formulae that we obtain for statures of constructions that already exist in the world of wqos, namely for finite sets, sums, products, finite words, and finite multisets, are exactly the same as in wqo case, and no surprise should be expected there. This is a good thing: the notions and results we obtain smoothly extend those already known on wqos—though with pretty different proofs, as in the cases of finite words and finite multisets.

We obtain new formulae for constructions that make sense on Noetherian spaces but not on wqos. Spaces with a cofinite topology, Hoare powerspaces, powersets, and spaces of words with the prefix topology are examples of Noetherian spaces that do not arise from wpos, and therefore form proper generalizations of the theory of wpos. One should also note that our results on statures on spaces obtained from constructions that are classical on wqos, such as X^* , despite yielding the same formulae, are not consequences of the result from the wqo literature that they generalize. Finally, our study of sobrification ranks seems new as well.

We conclude in Section 15.

2. PRELIMINARIES

Given any function f from a product set $X \times Y$ to Z , the application of f to a pair (x, y) will be written as $f(x, y)$, not $f((x, y))$. The symbols \subseteq , \subsetneq , and $\not\subseteq$ stand for inclusion (or equality), strict inclusion, and the negation of inclusion respectively.

We usually write \leq for the preordering of any preordered space, and $<$ for its strict part. A function f is *monotonic* if and only if $x \leq y$ implies $f(x) \leq f(y)$, and an *order embedding* if and only if $x \leq y$ and $f(x) \leq f(y)$ are equivalent, for all points x and y .

If all the considered preorderings are orderings, then every order embedding is injective and monotonic, and every injective monotonic map is strictly monotonic; we say that f is a *strictly monotonic* between posets if and only if $x < y$ implies $f(x) < f(y)$.

2.1. Ordinals. We assume some basic familiarity with ordinals, ordinal sum $\alpha + \beta$, ordinal multiplication $\alpha\beta$, and ordinal exponentiation α^β . They are all defined by induction on β , e.g., $\alpha^0 \stackrel{\text{def}}{=} 1$, $\alpha^{\beta+1} \stackrel{\text{def}}{=} \alpha^\beta \alpha$, and $\alpha^\beta \stackrel{\text{def}}{=} \sup\{\alpha^\gamma \mid \gamma < \beta\}$ for every limit ordinal β ; in particular, one should not confuse ordinal exponentiation with cardinal exponentiation: if α and β are countable cardinals, say, then α^β is countable. Also, one should beware that ordinal addition and ordinal multiplication are not commutative. We refer to [Jec06] for more information.

Addition $\alpha + \beta$ is monotonic in α , namely $\alpha \leq \alpha'$ implies $\alpha + \beta \leq \alpha' + \beta$; it is strictly monotonic and continuous in β , namely $\beta < \beta'$ implies $\alpha + \beta < \alpha + \beta'$, and $\alpha + \sup_{i \in I} \beta_i = \sup_{i \in I} (\alpha + \beta_i)$ for every non-empty family $(\beta_i)_{i \in I}$ of ordinals. Similarly, multiplication $\alpha\beta$ is monotonic in α and β , continuous in β , and strictly monotonic in β if $\alpha \neq 0$, and exponentiation α^β is monotonic in α and β , continuous in β , and strictly monotonic in β if $\alpha \geq 2$. Additionally, addition is left-cancellative: $\alpha + \beta = \alpha + \gamma$ implies $\beta = \gamma$.

Every ordinal α can be written in a unique way as a finite sum $\omega^{\alpha_1} + \dots + \omega^{\alpha_m}$ with $\alpha \geq \alpha_1 \geq \dots \geq \alpha_m$, $m \in \mathbb{N}$. This is the so-called *Cantor normal form* of α .

The ordinals of the form ω^β are exactly the *additively indecomposable* ordinals, namely the ordinals α such that any finite sum of ordinals strictly smaller than α is still strictly smaller than α . The additively decomposable ordinals are those whose Cantor normal form is such that $m \neq 1$.

Similarly, the *multiplicatively indecomposable* ordinals, namely the ordinals α such that any product of ordinals strictly smaller than α is still strictly smaller than α , are 0, 1, 2, and those of the form ω^{ω^β} .

One can compare ordinals $\alpha \stackrel{\text{def}}{=} \omega^{\alpha_1} + \dots + \omega^{\alpha_m}$ and $\beta \stackrel{\text{def}}{=} \omega^{\beta_1} + \dots + \omega^{\beta_n}$ in Cantor normal form by: $\alpha \leq \beta$ if and only if the list $\alpha_1, \dots, \alpha_m$ is lexicographically smaller than or equal to the list β_1, \dots, β_n , namely, either the two lists are equal or there is an index i with $1 \leq i \leq \min(m, n)$ such that $\alpha_1 = \beta_1, \dots, \alpha_{i-1} = \beta_{i-1}$, and $\alpha_i < \beta_i$.

The *natural* (or Hessenberg) *sum* $\alpha \oplus \beta$ and the *natural product* $\alpha \otimes \beta$ are defined as follows, where α and β are written in Cantor normal form, respectively $\omega^{\alpha_1} + \dots + \omega^{\alpha_m}$ and $\omega^{\beta_1} + \dots + \omega^{\beta_n}$: $\alpha \oplus \beta$ is equal to $\omega^{\gamma_1} + \dots + \omega^{\gamma_{m+n}}$, where $\gamma_1 \geq \dots \geq \gamma_{m+n}$ is the list obtained by sorting the list $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n$ in decreasing order; $\alpha \otimes \beta$ is equal to $\bigoplus_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \omega^{\alpha_i \oplus \beta_j}$. Those operations are associative and commutative. Natural sum is strictly monotonic in both arguments; natural product is monotonic in each argument, and strictly monotonic in each argument provided that the other one is non-zero.

The following is an equivalent inductive definition of $\alpha \oplus \beta$ by well-founded recursion on (α, β) : $\alpha \oplus \beta$ is the smallest ordinal strictly larger than all the ordinals $\alpha' \oplus \beta$ with $\alpha' < \alpha$ and $\alpha \oplus \beta'$ with $\beta' < \beta$.

One should note that the ordinals of the form ω^α are \oplus -*indecomposable* as well, namely that any finite natural sum of ordinals strictly smaller than ω^α is still strictly smaller than ω^α . Similarly, the ordinals of the form ω^{ω^α} are \otimes -*indecomposable*, in the sense that any finite natural product of ordinals strictly smaller than ω^{ω^α} is still strictly smaller than ω^{ω^α} .

The *ordinal rank* (or *height*) $|P|$ of a well-founded poset P is the least ordinal α such that there is a strictly monotonic map from P into α , namely into the set of ordinals strictly smaller than α . This can be defined equivalently as follows [Mos09, Exercise 2G.7]. By well-founded induction on $x \in P$, we let $\text{rk}_P(x)$ be the least ordinal strictly larger than $\text{rk}_P(x')$, for every $x' < x$, namely $\text{rk}_P(x) \stackrel{\text{def}}{=} \sup_{x' < x} (\text{rk}_P(x') + 1)$. (We take the supremum of an empty family to be 0, so the rank of a minimal element is always 0.) Then $|P|$ is defined as $\sup_{x \in P} (\text{rk}_P(x) + 1)$, or equivalently as $\text{rk}_{P^\top}(\top)$, where P^\top is P with a fresh element \top added above all others.

One can show that for every ordinal $\alpha < |P|$, there is an element $x \in P$ such that $\text{rk}_P(x) = \alpha$, and that for every $x \in P$ and for every $\alpha < \text{rk}_P(x)$, there is an element $y \in P$ such that $y < x$ and $\text{rk}_P(y) = \alpha$ [BG08, Lemma 13].

We allow ourselves to write $\alpha - 1$ for the unique ordinal of which α is the successor, if α is a successor ordinal. For example, if P is a well-founded poset with a largest element, then $|P| - 1$ makes sense. We will also use the notation $\alpha - 1$ for zero and limit ordinals, in which case we agree that $\alpha - 1$ simply denotes α itself.

A *chain* D in a poset P is a non-empty, totally ordered subset. When P is well-founded, D is isomorphic to a unique non-zero ordinal, which happens to be $|D|$, and which is called the *length* of the chain D . The *chain length* $\ell(P)$ of a well-founded poset P is the supremum of the lengths of its chains. It is fairly easy to see that $\ell(P) \leq |P|$ for every well-founded poset P , but the reverse inequality fails in general. We will give a counterexample as a special case of Proposition 3.4 below, see also the paragraph before Theorem 1 in [Sch81], or the end of Section 3 in [Křif97].

2.2. Topology. We refer the reader to [GL13] for basic notions of topology. Some information on Noetherian spaces can be found in Section 9.7 of that book. Additional information can be found in [FGL20].

We write $\mathcal{O}X$ for the lattice of open subsets of a topological space X .

Every topological space X has a *specialization preordering*, which we will always write as \leq , and defined by $x \leq y$ if and only if every open neighborhood of x contains y . The closure of a point x is always equal to its downward closure $\downarrow x \stackrel{\text{def}}{=} \{x' \in X \mid x' \leq x\}$ in its specialization preordering, and therefore $x \leq y$ if and only if x belongs to the closure of $\{y\}$. X is T_0 if and only if \leq is a partial ordering. The specialization preordering of any subspace F of X is the restriction $\leq|_F$ of \leq to F .

There are several topologies on a preordered set (P, \leq) whose specialization preordering is \leq . The finest one is the *Alexandroff topology*, whose open subsets are exactly the *upwards-closed* subsets, namely the subsets U such that every element larger than an element already in U is itself in U . The coarsest one is the *upper topology*, which is the coarsest one such that the downward closure $\downarrow x$ of any point, with respect to \leq , is closed. Its closed subsets are the intersections of *finitary closed* subsets, namely of subsets of the form $\downarrow\{x_1, \dots, x_n\} \stackrel{\text{def}}{=} \bigcup_{i=1}^n \downarrow x_i$. An intermediate topology is the *Scott topology*, which is of fundamental importance in computer science and elsewhere; its open sets are the upwards-closed subsets U such that every directed family D that has a supremum in U already intersects U .

The *extended Hoare powerdomain* \mathcal{H}_0X of a topological space X is the poset of closed subsets of X , ordered by inclusion. It turns out that X is Noetherian if and only if \mathcal{H}_0X is well-founded, namely if and only if there is no strictly descending chain of non-empty closed subsets of X . This is a rephrasing of Proposition 9.7.6 of [GL13], which states it in terms of open sets.

The *Hoare powerdomain* $\mathcal{H}X$ is \mathcal{H}_0X minus the empty set, and is a familiar object in domain theory [GHK⁺03, Section IV.8]. $\mathcal{H}X$ is also well-founded if and only if X is Noetherian.

One should note that we take a rather unusual view at \mathcal{H}_0 and \mathcal{H} , and also at the upcoming construction \mathcal{S} : we consider them as mapping topological spaces to *posets*, and Noetherian spaces to well-founded posets. It is certainly possible to consider \mathcal{H}_0X and $\mathcal{H}X$ as topological spaces of their own, with suitable topologies, and we will do so in Section 11; but we will want to consider ordinal ranks of \mathcal{H}_0X and $\mathcal{H}X$, and this requires seeing the latter as well-founded posets, not spaces.

A closed subset C of X is *irreducible* if and only if it is non-empty and, given any two closed subsets F_1 and F_2 of X whose union contains C , one of them already contains C . The closures of points $\downarrow x$ are always irreducible closed, and the space X is *sober* if and only if every irreducible closed subset of X is the closure of a unique point. Given any space X , one can form its *sobrification* SX :

its points are the irreducible closed subsets of X , and its open subsets are the sets $\diamond U \stackrel{\text{def}}{=} \{C \in \mathcal{S}X \mid C \cap U \neq \emptyset\}$, $U \in \mathcal{O}X$. The specialization ordering of $\mathcal{S}X$ is inclusion.

$\mathcal{S}X$ is always a sober space, and the map $\eta_X : x \mapsto \downarrow x$ is continuous, and a topological embedding if X is T_0 . More is true: the inverse map η_X^{-1} is an order isomorphism between $\mathcal{O}X$ and $\mathcal{O}(\mathcal{S}X)$, for every topological space X ; its inverse is the map $U \mapsto \diamond U$. Since a space is Noetherian if and only if its lattice of open sets has no infinite ascending chain, it follows that X is Noetherian if and only if $\mathcal{S}X$ is.

In a Noetherian space X , every closed subset F is the union of finitely many irreducible closed subsets C_1, \dots, C_m . See Lemma 3.6 of [FGL20], for example. This is a fundamental result, which we will use often. More generally, this holds if and only if X is a *FAC space* [GL19], namely a space with no infinite discrete subspace; every Noetherian space is FAC.

By keeping only the maximal elements (with respect to inclusion) in the list C_1, \dots, C_m , we obtain that C_1, \dots, C_m are exactly the maximal irreducible closed subsets of X included in F , and this list is unique up to permutation. We call the finitely many maximal irreducible closed subsets of X included in F the *components* of F .

3. STATURE

We will define a notion of stature for Noetherian spaces which, when applied to wpos equipped with their Alexandroff topology, will coincide with the usual notion of maximal order type.

A naïve attempt to do so would be the following. The topological analogue of a poset is a T_0 space. The topological analogue of an extension of an ordering is a coarser T_0 topology, and therefore the topological analogue of a linear extension (namely, a *maximal* ordering extending the original ordering) should be a minimal T_0 topology coarser than the original T_0 topology. However, Larson [Lar69, Example 6] showed that such minimal T_0 topologies may fail to exist. This is true even in the realm of Noetherian spaces: Larson's example is \mathbb{R} with its cofinite topology, and every set is Noetherian in its cofinite topology.

Instead, we turn to the following notion.

The *stature* $\|P\|$ of a well-partial order P is defined by Blass and Gurevich [BG08, Definition 28] as the ordinal rank of the poset of non-empty bad sequences ordered by reverse extension, and coincides with the maximal order type of P [BG08, Theorem 10]. It also coincides with the ordinal rank of the poset $\mathcal{I}(P)$

of proper downwards-closed subsets of P , ordered by inclusion [BG08, Proposition 31]. (Blass and Gurevich call the downwards-closed subsets *ideals*, but we reserve this term to downwards-closed and directed subsets. A proper subset of P is any subset of P different from P , and that includes the empty set.) Those results had already appeared as Proposition 2.2 in [Kř197], but were known even before, as a combination of [BP69, Proposition 2] and of [PZ85, Theorem 5.3] (equivalently, of Theorems III-2.4 and III-2.6 in Chapter I of N. Zaguia's thesis [Nag83]).

The latter definition is the one that extends nicely to Noetherian spaces.

Definition 3.1 (Stature). The *stature* $\|X\|$ of a Noetherian space X is $|\mathcal{H}_0X| - 1$.

Note that $|\mathcal{H}_0X| - 1$ is well-defined, because \mathcal{H}_0X is well-founded, and has a largest element, namely X itself.

Equivalently, $\|X\|$ is equal to the rank $\text{rk}_{\mathcal{H}_0X}(X)$ of its largest element X , or to the ordinal rank $|\mathcal{H}_0X \setminus \{X\}|$.

Remark 3.2. Despite the fact that \mathcal{H}_0X has just one more element than $\mathcal{H}X$, it would be wrong to believe that $\|X\| = |\mathcal{H}X|$. In general, $1 + |\mathcal{H}X| = \|X\| + 1$, but ordinal addition is not commutative. For example, if X is \mathbb{N} with the Alexandroff topology of its natural ordering, we have $|\mathcal{H}X| = |\mathcal{H}_0X| = \omega + 1$, and $\|X\| = \omega$.

Let us verify that Definition 3.1 generalizes the notion of stature (a.k.a. maximal order type) of well-partial orders to Noetherian spaces. Given any poset P , considered as a topological space with its Alexandroff topology, the closed subsets of P are exactly its downwards-closed subsets. Therefore $\mathcal{H}_0P \setminus \{P\} = \mathcal{I}(P)$, and hence, when P is a well-partial order, our $\|P\|$ coincides with Blass and Gurevich's $|\mathcal{I}(P)|$. Let us state this explicitly.

Fact 3.3. *For every well-partial order P , $\|P\|$ is the maximal order type of P .*

For a well-partial order P , the stature also coincides with the chain-length of $\mathcal{I}(P)$, and there is a chain of that precise length. This is the meaning of the equality “ $\mu(\downarrow A) = \lambda(\downarrow A)$ ” in [Kř197, Proposition 2.2]: in Kř197's notation, and with $A \stackrel{\text{def}}{=} P$, $\mu(\downarrow A)$ denotes the ordinal rank of $\mathcal{I}(P)^\top$, and $\lambda(\downarrow A)$ denotes the chain length of $\mathcal{I}(P)^\top$, provided that the supremum that defines it is attained. (See also [PZ85, Theorem 5.3].) One may wonder whether a similar coincidence would hold in the realm of Noetherian spaces, namely whether $\ell(\mathcal{H}_0X) = \|X\| + 1$ for every Noetherian space X , and the answer is negative.

Proposition 3.4. *For every ordinal α , there is a non-empty, sober Noetherian space X_α whose stature is at least α , and such that all the chains in $\mathcal{H}_0 X_\alpha$ are countable. In particular, when $\alpha \geq \aleph_1$, $\ell(\mathcal{H}_0 X_\alpha) < \|X_\alpha\| + 1$.*

PROOF. We build X_α by induction. This is a slight variant of a construction mentioned by D. Schmidt [Sch81, paragraph before Theorem 1].

We build X_α as a poset and equip it with the upper topology. It turns out that the sober Noetherian spaces are exactly the posets that are well-founded and have properties T and W, in the upper topology of their ordering [GL13, Theorem 9.7.12]. Property T states that the whole set itself is finitary closed, and Property W states that any binary intersection $\downarrow x \cap \downarrow y$ is finitary closed. In a sober Noetherian space, every closed subset is finitary closed.

We let X_0 be a one element set. For every ordinal α , we let $X_{\alpha+1}$ be X_α^\top . Finally, for every limit ordinal α , we let X_α be $(\coprod_{\beta < \alpha} X_\beta)^\top$. In the disjoint sum $\coprod_{\beta < \alpha} X_\beta$, any pair of elements coming from X_β and X_γ with $\beta \neq \gamma$ is incomparable; the elements coming from the same summand X_β are compared as in X_β .

It is easy to see that X_α is a (reversed) tree for every ordinal α , namely: there is a largest element (the *root*), and the upward closure $\uparrow x \stackrel{\text{def}}{=} \{y \in X_\alpha \mid x \leq y\}$ of any element x of X_α is a finite chain. This is proved by induction on α . Then properties T and W are obvious. X_α is also clearly well-founded, by induction on α once again. Therefore X_α is sober and Noetherian in its upper topology.

Every element x of X_α is the largest element of a poset isomorphic to X_β for some unique ordinal $\beta \leq \alpha$. By induction on β , $\text{rk}_{X_\alpha}(x) = \beta$. Also, since $\downarrow x$ is a proper closed subset of $\downarrow y$ if $x < y$, a similar induction on β yields $\text{rk}_{\mathcal{H}_0 X_\alpha}(\downarrow x) \geq \beta$. Taking $\beta \stackrel{\text{def}}{=} \alpha$, we obtain that $\|X_\alpha\| = \text{rk}_{\mathcal{H}_0 X_\alpha}(X_\alpha)$ is at least α .

We now claim that all the chains in $\mathcal{H}_0 X_\alpha$ are countable. This is again by induction on α . The only interesting case is when α is a limit ordinal. Let D be a chain in $\mathcal{H}_0 X_\alpha$. D can contain at most one closed set containing the top element of X_α , since there is only one such set, which is X_α itself. Without loss of generality, we will therefore assume that D only contains closed subsets of X_α that do not contain its top element \top . All those closed subsets F are finitary, hence must be of the form $\bigcup_{i=1}^n \downarrow x_i$, where each x_i is taken from some summand X_{β_i} of the sum $\coprod_{\beta < \alpha} X_\beta$; let us call the set $\{\beta_1, \dots, \beta_n\}$ the *support* $s(F)$ of F . For any two closed subsets F, F' not containing \top , $F \subseteq F'$ implies $s(F) \subseteq s(F')$. Therefore the family $D' \stackrel{\text{def}}{=} \{s(F) \mid F \in D\}$ forms a chain of finite subsets of α . It is easy to see that there can be at most one element of D' of each given cardinality,

so that D' is countable. It follows that there is a countable subset I of α such that $s(F) \subseteq I$ for every $F \in D$.

For every $\beta \in I$, let $D_{|\beta}$ denote the chain consisting of those sets of the form $F \cap X_\beta$, where F ranges over D . This is a chain of closed subsets of X_β , which is therefore countable by induction hypothesis. We should mention a subtle point here: those are closed in the subspace topology, but we require to work with closed subsets in the *upper* topology of X_β to be able to use the induction hypothesis. They are indeed closed in the upper topology of X_β , because they are downward closures of finitely many points in the downward closed subset X_β .

Any element F of D can be written as the finite union of the sets $F \cap X_\beta$, where β ranges over $s(F)$. It follows that there are at most as many elements of D as there are finite subsets of the countable disjoint union $\bigsqcup_{\beta \in I} D_{|\beta}$, hence that D is countable.

If $\alpha \geq \aleph_1$, finally, $\ell(\mathcal{H}_0 X_\alpha) \leq \aleph_1 < \alpha + 1 \leq \|X_\alpha\| + 1$. \square

To the contrary, Proposition 3.6 below states that every Noetherian space X with a *countable* Hoare powerdomain is such that $\ell(\mathcal{H}_0 X) = \|X\| + 1$. Since having a countable Hoare powerdomain may seem like a strange condition, we first note that this is a familiar property.

Lemma 3.5. *The following properties are equivalent for a Noetherian space X :*

- (i) X is second-countable;
- (ii) $\mathcal{H}_0 X$ is countable;
- (iii) $\mathcal{S}X$ is countable.

PROOF. (i) \Rightarrow (ii). if X is second-countable, then every open subset U can be written as a union of open sets from a countable base B . Since X is Noetherian, U is compact, so U is already equal to a finite subunion. Hence X only has countably many open sets, and therefore only countably many closed sets.

(ii) \Rightarrow (iii). If $\mathcal{H}_0 X$ is countable, then its subset $\mathcal{S}X$ is countable.

(iii) \Rightarrow (i). Let us assume that $\mathcal{S}X$ is countable. Since X is Noetherian, every closed set is a finite union of elements of $\mathcal{S}X$, so X has only countably many closed subsets. \square

Proposition 3.6. *For every non-empty second-countable Noetherian space X , $\ell(\mathcal{H}_0 X) = \|X\| + 1$, and there is a chain in $\mathcal{H}_0 X$ of maximal length.*

PROOF. By Theorem III-1.1 of [Nag83, Chapter I], every distributive, well-founded lattice of countable rank α contains a chain of length α . (A even more general statement appears in [Kří97, Remark 3.4], but we will not need it.) We apply

this to the lattice \mathcal{H}_0X , using Lemma 3.5 in order to guarantee that its ordinal rank $\|X\| + 1$ is countable. \square

Example 3.7. Let us look back at the counterexample of Proposition 3.4: there, we built a space X_α , and we obtained the strict inequality $\ell(\mathcal{H}_0X_\alpha) < \|X_\alpha\| + 1$ when $\alpha \geq \aleph_1$. When $\alpha < \aleph_1$, namely when α is countable, we do have equality: $\ell(\mathcal{H}_0X_\alpha) = \|X_\alpha\| + 1$. We use Proposition 3.6 in order to show this. It suffices to observe that X_α is sober, hence isomorphic to $\mathcal{S}X_\alpha$, and countable; the latter is proved by an easy induction of the countable ordinal α .

Remark 3.8. Figure 2 of [FGL20] displays a catalogue of Noetherian spaces. (We will deal with most spaces in that list, to the exception of spectra of Noetherian rings and spaces of trees.) This starts with a few basic Noetherian spaces, such as any finite poset or \mathbb{N} in their Alexandroff topology, or spectra of Noetherian rings. The latter may fail to be second-countable, but the other basic Noetherian spaces are. The catalogue then proceeds by building new Noetherian spaces from old, using Noetherian-preserving operators F —namely, if X_1, \dots, X_m are Noetherian, then so is $F(X_1, \dots, X_m)$. For example, any finite product, any finite sum of Noetherian spaces is Noetherian, the space X^* of finite words over X with a suitable topology (see Section 12) is Noetherian, and so on. The main results of [FGL20] consist in giving explicit descriptions of $\mathcal{S}(F(X_1, \dots, X_m))$ from $\mathcal{S}X_1, \dots, \mathcal{S}X_m$. For example, $\mathcal{S}(X^*)$ consists of certain regular expressions called word products over $\mathcal{S}(X)$, which we will define and use in Section 12. It is then easy to see that if $\mathcal{S}(X)$ is countable, then so is $\mathcal{S}(X^*)$. That extends to all the operators F of [FGL20, Figure 2]. As a consequence of Lemma 3.5, and taking the proviso that we only consider spectra of Noetherian rings with countably many radical ideals, all the Noetherian spaces displayed there are second-countable. This yields an ample supply of Noetherian spaces on which Proposition 3.6 applies.

4. SOBRIFICATION RANK

Since the closed subsets of a Noetherian space X are the finite unions of elements of $\mathcal{S}X$, it will be useful to also consider the ordinal rank of $\mathcal{S}X$. Here we see $\mathcal{S}X$ not as a topological space, rather as a poset under its specialization ordering, which is inclusion. As such, it is a well-founded poset, since it is included in \mathcal{H}_0X , which is itself well-founded under inclusion.

Definition 4.1 (Sobrification Rank). The *sobrification rank* $\text{sob } X$ of a Noetherian space is $|\mathcal{S}X|$.

Lemma 4.2. *For every non-empty Noetherian space X ,*

$$\text{sob } X = \max_{i=1}^m \text{rk}_{\mathcal{S}X}(C_i) + 1,$$

where C_1, \dots, C_m are the components of X ($m \geq 1$).

PROOF. The components of X exist because X is closed in X , and $m \geq 1$ because X is non-empty. For every $C \in \mathcal{S}X$, the inclusion $C \subseteq X = C_1 \cup \dots \cup C_m$ together with the fact that C is irreducible imply that C is included in some C_i . It follows that $|\mathcal{S}X| = \max(\text{rk}_{\mathcal{S}X}(C_1) + 1, \dots, \text{rk}_{\mathcal{S}X}(C_m) + 1)$, whence the claim. \square

In particular, $|\mathcal{S}X|$ is a successor ordinal for every non-empty Noetherian space X . Hence the following definition makes sense.

Definition 4.3 (Reduced sobrification rank). The *reduced sobrification rank* of a non-empty Noetherian space X is $\text{rsob } X \stackrel{\text{def}}{=} \text{sob } X - 1$. This is equal to $\max_{i=1}^m \text{rk}_{\mathcal{S}X}(C_i)$, where C_1, \dots, C_m are the components of X ($m \geq 1$).

There is a notion of *Krull dimension* of Noetherian spaces, defined as the supremum of the lengths, minus 1, of (finite non-empty) chains of elements of $\mathcal{S}X$. The result is a natural number or infinity. One can extend this notion and consider the ordinal-valued chain length $\ell(\mathcal{S}X)$.

Remark 4.4. In contrast to Proposition 3.6, $\ell(\mathcal{S}X)$ is in general not equal to $|\mathcal{S}X| = \text{sob } X$, even when X is countable. The space X_α of Proposition 3.4 with $\alpha \stackrel{\text{def}}{=} \omega$ is sober, so $\mathcal{S}X_\omega$ is isomorphic to X_ω ; X_ω only has finite chains (of arbitrary length), so $\ell(X_\omega) = \ell(\mathcal{S}X_\omega) = \omega$, while $|X_\omega| = |\mathcal{S}X_\omega| = \omega + 1$.

Proposition 4.5. *For every Noetherian space X ,*

- (1) $1 + \text{sob } X \leq ||X|| + 1$, or equivalently $1 + \text{rsob } X \leq ||X||$;
- (2) $||X|| + 1 \leq \omega^{\text{sob } X}$;
- (3) if X is non-empty, then $||X|| \leq \omega^{\text{rsob } X} \otimes n$, where n is the number of components of X .

PROOF. (1) The map $\text{rk}_{\mathcal{H}_0X}$, once restricted to $\mathcal{S}X \cup \{\emptyset\}$, is a strictly increasing map into the set of ordinals strictly smaller than $|\mathcal{H}_0X| = ||X|| + 1$. Hence $|\mathcal{S}X \cup \{\emptyset\}| \leq ||X|| + 1$. We now observe that $|\mathcal{S}X \cup \{\emptyset\}| = 1 + \text{sob } X$, because the ordinal rank of a disjoint union of two posets A and B , where every element of A is below every element of B , is the sum of the ordinal ranks of A and B .

(2) For every $F \in \mathcal{H}_0X$, let C_1, \dots, C_m be the components of F . We recall that those are the maximal irreducible closed subsets of X included in F ; in

particular, they are pairwise incomparable. We define $\varphi(F)$ as $\bigoplus_{i=1}^m \omega^{\text{rk}_{\mathcal{S}X}(C_i)}$, and we claim that φ is a strictly increasing map.

Given any two elements F, F' of \mathcal{H}_0X , let C_1, \dots, C_m be the components of F , and C'_1, \dots, C'_n be those of F' . If $F \subseteq F'$, then each C_i is included in F' , hence in some C'_j , since C_i is irreducible. We pick one such j and call it $f(i)$. Then we split the natural sum $\varphi(F) = \bigoplus_{i=1}^m \omega^{\text{rk}_{\mathcal{S}X}(C_i)}$ as the natural sum of the quantities $\alpha_j \stackrel{\text{def}}{=} \bigoplus_{i \in f^{-1}(j)} \omega^{\text{rk}_{\mathcal{S}X}(C_i)}$, $1 \leq j \leq n$.

We note that $\alpha_j \leq \omega^{\text{rk}_{\mathcal{S}X}(C'_j)}$, with equality if and only if there is a unique element i in $f^{-1}(j)$ and $C_i = C'_j$. Indeed, if $f^{-1}(j) = \{i\}$ and $C_i = C'_j$, then $\alpha_j = \omega^{\text{rk}_{\mathcal{S}X}(C_i)} = \omega^{\text{rk}_{\mathcal{S}X}(C'_j)}$. Otherwise, for every $i \in f^{-1}(j)$, C_i is a proper subset of C'_j : if C_i were equal to C'_j for some $i \in f^{-1}(j)$, then C_i would contain $C_{i'}$ for every $i' \in f^{-1}(i)$, and since the sets C_i are pairwise incomparable, that would force $f^{-1}(j)$ to consist of exactly one element. Since $C_i \subsetneq C'_j$ for every $i \in f^{-1}(j)$, $\text{rk}_{\mathcal{S}X}(C_i) < \text{rk}_{\mathcal{S}X}(C'_j)$, so $\omega^{\text{rk}_{\mathcal{S}X}(C_i)} < \omega^{\text{rk}_{\mathcal{S}X}(C'_j)}$. Since $\omega^{\text{rk}_{\mathcal{S}X}(C'_j)}$ is \oplus -indecomposable, $\alpha_j < \omega^{\text{rk}_{\mathcal{S}X}(C'_j)}$.

Using the fact that natural sum is strictly monotonic in all of its arguments, $\varphi(F) = \bigoplus_{j=1}^n \alpha_j$ is then smaller than or equal to $\bigoplus_{j=1}^n \omega^{\text{rk}_{\mathcal{S}X}(C'_j)} = \varphi(F')$, with equality if and only if $f^{-1}(j)$ consists of a single element i and $C_i = C'_j$, for every j such that $1 \leq j \leq n$. The latter case is equivalent to the fact that f is a bijection, and that the lists C_1, \dots, C_m and C'_1, \dots, C'_n are equal up to permutation, namely to the equality $F = F'$.

It follows that φ is a strictly monotonic map from \mathcal{H}_0X to a set of ordinals. The largest value it takes is $\varphi(X) = \bigoplus_{i=1}^n \omega^{\text{rk}_{\mathcal{S}X}(C_i)}$, where C_1, \dots, C_n are the components of X . Using the additive indecomposability of $\omega^{|\mathcal{S}X|}$, together with the fact that $\text{rk}_{\mathcal{S}X}(C_i) < |\mathcal{S}X|$ (see Lemma 4.2), we obtain that $\varphi(X) < \omega^{|\mathcal{S}X|}$.

The existence of a strictly monotonic map $\varphi: \mathcal{H}_0X \rightarrow \omega^{|\mathcal{S}X|}$ in turn implies that $\|X\| + 1 = |\mathcal{H}X| \leq \omega^{|\mathcal{S}X|} = \omega^{\text{sob } X}$.

(3) We use the same map φ as in item (2). Since φ is strictly monotonic, $\varphi(F) \geq \text{rk}_{\mathcal{H}_0X}(F)$ for every closed subset F of X , by an easy well-founded induction on F . Taking $F \stackrel{\text{def}}{=} X$, we obtain that $\|X\| = \text{rk}_{\mathcal{H}_0X}(X) \leq \varphi(X) = \bigoplus_{i=1}^n \omega^{\text{rk}_{\mathcal{S}X}(C_i)}$, where C_1, \dots, C_n are the components of X . For each C_i , $\text{rk}_{\mathcal{S}X}(C_i) \leq \text{sob } X - 1 = \text{rsob } X$, by Lemma 4.2, whence the claim. \square

While we are on the subject of sobrification, we should mention the following.

Fact 4.6. *For every Noetherian space X , $\|SX\| = \|X\|$ and $\text{sob}(SX) = \text{sob } X$.*

Indeed, the lattices of open subsets of X and of $\mathcal{S}X$ are isomorphic through $U \mapsto \diamond U$. Hence X and $\mathcal{S}X$ have isomorphic lattices of closed subsets ($\mathcal{H}_0 X$ and $\mathcal{H}_0(\mathcal{S}X)$); the isomorphism also restricts to one between $\mathcal{S}X$ and $\mathcal{S}(\mathcal{S}X)$.

5. DIRECT AND INVERSE IMAGES

We collect a few tools that we will use in order to evaluate the stature of several kinds of spaces. We will discover a few others along the way. We start with a technical lemma.

Lemma 5.1. *Let $f: X \rightarrow Y$ be a continuous map.*

- (1) *The map $\mathcal{S}f: \mathcal{S}X \rightarrow \mathcal{S}Y$, where for every $C \in \mathcal{S}X$, $\mathcal{S}f(C)$ is the closure $cl(f(C))$ of the image $f(C)$ of C by f , is a well-defined, monotonic map.*
- (2) *If f is surjective, then $f^{-1}: \mathcal{H}_0(Y) \rightarrow \mathcal{H}_0(X)$ is injective.*
- (3) *If $f^{-1}: \mathcal{H}_0(Y) \rightarrow \mathcal{H}_0(X)$ is injective, then it is an order embedding.*

PROOF. (1) For every $C \in \mathcal{S}X$, we have that $\mathcal{S}f(C)$ is irreducible closed, see [GL13, Lemma 8.2.42] for example. For a short argument, if $\mathcal{S}f(C)$ is included in the union $F_1 \cup F_2$ of two closed sets, then $C \subseteq f^{-1}(F_1 \cup F_2) = f^{-1}(F_1) \cup f^{-1}(F_2)$, and the claim follows from the irreducibility of C . The monotonicity of $\mathcal{S}f$ is clear.

(2) Let F and F' be two closed subsets of Y such that $f^{-1}(F) = f^{-1}(F')$. For every $y \in F$, we can write y as $f(x)$ for some $x \in X$, and then x is in $f^{-1}(F)$. Since $f^{-1}(F) = f^{-1}(F')$, $f(x) = y$ is also in F' . We show the converse inclusion $F' \subseteq F$ similarly.

(3) We show that $f^{-1}(F) \subseteq f^{-1}(F')$ is equivalent to $F \subseteq F'$. Indeed, $f^{-1}(F) \subseteq f^{-1}(F')$ if and only if $f^{-1}(F) \cup f^{-1}(F') = f^{-1}(F')$, if and only if $f^{-1}(F \cup F') = f^{-1}(F')$, if and only if $F \cup F' = F'$, if and only if $F \subseteq F'$. \square

We will say that a continuous map $f: X \rightarrow Y$ is *Skula dense* if and only if $f^{-1}: \mathcal{H}_0(Y) \rightarrow \mathcal{H}_0(X)$ is injective. By Lemma 5.1, every surjective map is Skula dense. The name ‘‘Skula dense’’ stems from the following observation, which we present for completeness only, and is a slight relaxation of [GHK⁺03, Exercise V-5.32]. The *Skula topology* [Sku69] on Y (called the *b-topology* there, and sometimes also called the *strong topology*) has all crescents as basic open sets, where a *crescent* is a difference $U \setminus V$ of two open sets. This is a remarkable topology. For example, a topological space Y is sober Noetherian if and only if it is compact Hausdorff in its Skula topology [Hof79, Theorem 3.1], and for any space X embedded in a sober space Y , the Skula closure of X in Y is homeomorphic to $\mathcal{S}X$ [KL08, Proposition 3.4].

Lemma 5.2. *A continuous map $f: X \rightarrow Y$ between topological spaces is Skula dense, in the sense that $f^{-1}: \mathcal{H}_0(Y) \rightarrow \mathcal{H}_0(X)$ is injective, if and only if the image of f is dense in Y with the Skula topology.*

PROOF. If f^{-1} is injective, then we claim that any non-empty Skula open subset of Y intersects the image of f . Such a Skula open set must contain a basic non-empty Skula open set $U \setminus V$, where U and V are open in Y . Then $f^{-1}(U \setminus V) \neq f^{-1}(\emptyset)$, since f^{-1} is injective. Hence $f^{-1}(U \setminus V)$ is non-empty. We pick $x \in f^{-1}(U \setminus V)$, then $f(x)$ is both in $U \setminus V$ and in the image of f .

Conversely, if f has Skula dense image, let U and V be two open subsets of Y such that $f^{-1}(U) = f^{-1}(V)$. Then both $f^{-1}(U \setminus V)$ and $f^{-1}(V \setminus U)$ are empty, so neither $U \setminus V$ nor $V \setminus U$ intersects the image of f . Since the latter is Skula dense, $U \setminus V$ and $V \setminus U$ must be empty, whence $U = V$. \square

We remark that, as a consequence, the Skula dense maps between T_0 spaces are exactly the epimorphisms in the category of T_0 spaces [GHK⁺03, Exercise V-5.33].

Lemma 5.3. *Let $f: X \rightarrow Y$ be a continuous map from a Noetherian space X to a topological space Y . If f is Skula dense, then Y is Noetherian, and $\|Y\| \leq \|X\|$.*

PROOF. Let F, F' be two closed subsets of Y . If $F \subsetneq F'$, then $f^{-1}(F) \subsetneq f^{-1}(F')$, and that inclusion is strict because f^{-1} is injective. In particular, any infinite decreasing sequence in $\mathcal{H}_0 Y$ would be mapped through f^{-1} to an infinite descending sequence in $\mathcal{H}_0 X$, which is impossible since X is Noetherian. Therefore Y is Noetherian.

The map $\text{rk}_{\mathcal{H}_0 X} X \circ f^{-1}: \mathcal{H}_0 Y \rightarrow \|X\| + 1$ is a strictly increasing map, and this immediately entails that $|\mathcal{H}_0 Y| = \|Y\| + 1$ is less than or equal to $\|X\| + 1$, hence that $\|Y\| \leq \|X\|$. \square

Remark 5.4. It is not the case that, under the assumptions of Lemma 5.3, $\text{sob } Y \leq \text{sob } X$. Consider the poset $X \stackrel{\text{def}}{=} \{1, 2, 3\}$ with $1 < 2$ and 3 incomparable with both 1 and 2, and $Y \stackrel{\text{def}}{=} \{1, 2, 3\}$ with $1 < 2 < 3$. Equip both with their Alexandroff topologies, and let f be the identity map. Then $\text{sob } Y = 3 \not\leq \text{sob } X = 2$.

A dual statement is as follows. We will call *full* any map $f: X \rightarrow Y$ such that every open subset U of X can be written as $f^{-1}(V)$ for some open subset V of Y^1 . Every full map $f: X \rightarrow Y$ is order-reflecting with respect to the specialization

¹A word on naming. This notion was called ‘‘almost open’’ in [GL13]. However, ‘‘almost open’’ is improper, for several reasons. One is that it runs into a conflict with another notion, due to Klaus Keimel. Another one is that an equivalent statement is that for every closed

preorderings, namely $f(x) \leq f(y)$ implies $x \leq y^2$. A typical example is given by topological embeddings, which are full, continuous injective maps.

Lemma 5.5. *Let $f: X \rightarrow Y$ be a full map from a topological space X to a Noetherian space Y .*

- (1) X is Noetherian;
- (2) $\|X\| \leq \|Y\|$;
- (3) if f is also continuous, then $\text{sob } X \leq \text{sob } Y$;
- (4) if f is not just full and continuous, but also Skula dense, then $\|X\| = \|Y\|$ and $\text{sob } X = \text{sob } Y$.

PROOF. Let us define $f_*(F)$ as $\text{cl}(f(F))$, for every closed subset F of X . This is the same definition as $\mathcal{S}f$, except that f is no longer assumed to be continuous.

The map f_* is monotonic, and we claim that it is injective. Let us assume that $f_*(F) = f_*(F')$, where F and F' are closed in X . For every open subset U of X , we write U as $f^{-1}(V)$ for some open subset V of Y . Then U intersects F if and only if V intersects $f(F)$. An open set intersects a set A if and only if it intersects its closure $\text{cl}(A)$, so U intersects F if and only if V intersects $\text{cl}(f(F)) = f_*(F)$. Similarly, U intersects F' if and only if V intersects $f_*(F')$. Since $f_*(F) = f_*(F')$, F and F' intersect the same open subsets U of X . Therefore, they are equal.

Since f_* is monotonic and injective, it is strictly monotonic. Every infinite decreasing sequence in \mathcal{H}_0X would be mapped by f_* to an infinite decreasing sequence in \mathcal{H}_0Y , showing (1). Additionally, $\text{rk}_{\mathcal{H}_0Y} \circ f_*: \mathcal{H}_0X \rightarrow \|Y\| + 1$ is strictly monotonic, showing that $|\mathcal{H}_0X| = \|X\| + 1$ is less than or equal to $\|Y\| + 1$. This shows (2).

If additionally f is continuous, then $\mathcal{S}f$ is well-defined and monotonic by Lemma 5.1 (1), and coincides with the injective map f_* on $\mathcal{S}X$, hence is strictly monotonic. Then $\text{rk}_{\mathcal{S}Y} \circ f_*: \mathcal{S}X \rightarrow \text{sob } Y$ is strictly monotonic, showing (3).

Finally, if f is also Skula dense, then f^{-1} is an order embedding by part (3) of Lemma 5.1. The fact that f is full means that f^{-1} is surjective. Hence f^{-1} defines an order isomorphism between \mathcal{H}_0Y and \mathcal{H}_0X , which proves (4). \square

subset C of X , there is a closed subset D of Y such that $C = f^{-1}(D)$, showing that the concept has little to do with openness. In order to fix this, the concept was called ‘‘initial’’ in [GL22, Section 4], since an initial morphism in the concrete category of topological spaces is precisely a full, continuous map. However, initiality also implies continuity, which fullness does not. Fullness also evokes some form of surjectivity, and indeed f is full if and only if $f^{-1}: \mathcal{O}Y \rightarrow \mathcal{O}X$ is surjective.

²Regarding naming considerations again, we may look at preordered sets as certain categories, and then an order-reflection is a special case of a full functor, namely one that is surjective on homsets.

Remark 5.6. A full, continuous, Skula dense map $f: X \rightarrow Y$ is the same thing as a map f such that f^{-1} is an order-isomorphism of $\mathcal{H}_0 Y$ onto $\mathcal{H}_0 X$, or equivalently of $\mathcal{O}Y$ onto $\mathcal{O}X$. Such maps are called *quasihomeomorphisms* in [GHK⁺03, Definition V-5.8].

In the sequel, F and other subsets of X are given the subspace topology induced from X . The inclusion map is a topological embedding, by definition.

Corollary 5.7. *For every subset F of a Noetherian space X , $\|F\| \leq \|X\|$ and $\text{sob } F \leq \text{sob } X$.*

The following will allow us to trade ranks of closed subsets for statures of closed subspaces.

Lemma 5.8. *For every closed subset F of a Noetherian space X , we have that $\|F\| = \text{rk}_{\mathcal{H}_0 X}(F)$.*

PROOF. By well-founded induction on F , observing that the closed subsets of F , seen as a topological subspace of X , are exactly the closed subsets of X that are included in F . \square

It will often be the case that we are able to determine the stature of larger and larger proper closed subsets F of a space X . The following will allow us to determine the stature of X as a consequence.

Proposition 5.9. *Let X be a Noetherian space, and $(F_i)_{i \in I}$ be a family of closed subsets of X that is cofinal in the sense that every proper closed subset of X is included in some F_i . Then $\|X\| \leq \sup_{i \in I} (\|F_i\| + 1)$, and equality holds if all the subsets F_i are proper.*

PROOF. For every proper closed subset F of X , we have $F \subseteq F_i$ for some $i \in I$ by cofinality, hence $\|F\| \leq \|F_i\|$ by Corollary 5.7. Since $\|X\| = \text{rk}_{\mathcal{H}_0 X}(X)$ is the supremum of $\text{rk}_{\mathcal{H}_0 X}(F) + 1$ when F ranges over the proper closed subsets of X , and $\text{rk}_{\mathcal{H}_0 X}(F) = \|F\|$ by Lemma 5.8, $\|X\| \leq \sup_{i \in I} (\|F_i\| + 1)$. If every F_i is proper, then $\|F_i\| < \|X\|$ for every $i \in I$, by Corollary 5.7, so $\|F_i\| + 1 \leq \|X\|$ for every i . \square

Remark 5.10. Let X be non-empty and Noetherian, and $(F_i)_{i \in I}$ be a cofinal family of proper closed subsets of X , as in Proposition 5.9. One may form the colimit of the diagram formed by the subspaces F_i and the corresponding inclusion maps. The result is X with the topology *determined* by the subspace topologies on each F_i ; writing τ for that topology, a subset F of X is τ -closed if and only if $F \cap F_i$ is closed in F_i for every $i \in I$. The topology τ contains the topology of

X , but is in general much finer, as it contains closed sets that are included in no single F_i . In general, a colimit of Noetherian spaces will fail to be Noetherian. A typical counterexample is the collection of subsets $\{0, \dots, n\}$ of $X \stackrel{\text{def}}{=} \mathbb{N}$, each with the discrete topology, which arises from this construction by giving X the cofinite topology. In that case, the topology τ is the discrete topology on \mathbb{N} . The cofinite topology on \mathbb{N} is Noetherian, the discrete topology is not.

6. FINITE SPACES, WELL-FOUNDED CHAINS

Let X be a finite T_0 space, of cardinality n . The topology of X is necessarily the Alexandroff topology of its specialization ordering \leq , and X is automatically sober. Clearly, $0 \leq \text{sob } X \leq n$. The sobrification rank of X can be as low as 0, if X is empty. For non-empty spaces, the sobrification rank of X can be as low as 1, if \leq is the equality ordering (namely, if X is T_1), and as high as n , if X is a chain.

The stature of X , however, must be n in all cases.

Lemma 6.1. *The stature $\|X\|$ of any finite T_0 space X of cardinality $n \in \mathbb{N}$ is n .*

PROOF. This can be obtained from the fact that the maximal order type of any finite poset with cardinality n is n . A direct proof is equally easy. By well-founded induction on $F \in \mathcal{H}_0 X$, $\text{rk}_{\mathcal{H}_0 X}(F)$ is less than or equal to the cardinality of F . Therefore $\|X\| = \text{rk}_{\mathcal{H}_0 X}(X) \leq n$. In order to prove the reverse inequality, we exhibit a chain of length $n+1$ in $\mathcal{H}_0 X$. We build points x_i , $1 \leq i \leq n$, and closed subsets F_i of X , $0 \leq i \leq n$, by induction on i , such that $F_i = \{x_1, \dots, x_i\}$ for every i and $F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_n$. We do this by letting $F_0 \stackrel{\text{def}}{=} \emptyset$, and, at stage i , by picking a minimal point x_i in $X \setminus F_{i-1}$, so that $F_i \stackrel{\text{def}}{=} F_{i-1} \cup \downarrow x_i = F_{i-1} \cup \{x_i\}$. \square

One can see that $\text{sob } \mathbb{N} = |\mathbb{N}_\omega| = \omega + 1$, where \mathbb{N} is given the Alexandroff topology of its usual ordering. Here $\mathbb{N}_\omega = \mathcal{S}\mathbb{N}$ is \mathbb{N} plus a fresh element ω on top of all others, with the upper topology [FGL20, Theorem 5.4]. Every non-empty closed subset of \mathbb{N} is irreducible, so $\|\mathbb{N}\| = |\mathcal{H}_0 \mathbb{N}| - 1 = (1 + \text{sob } \mathbb{N}) - 1 = \omega$.

We generalize this to ordinals α . The upper topology on α coincides with its Scott topology, and its open subsets are the sets $\uparrow \beta$, where β ranges over the non-limit ordinals strictly smaller than α , plus the empty set. It is indeed easy to see that such sets are Scott-open, while $\uparrow \beta$ is not when β is a limit ordinal.

We recall that the notation $\alpha - 1$ makes sense even when α is a limit ordinal, in which case it denotes α itself.

Lemma 6.2. *Let α be any ordinal.*

- (1) *With its Alexandroff topology, α has stature α ; its sobrification rank is α if α is finite, $\alpha + 1$ otherwise.*
- (2) *With its upper (=Scott) topology, α has stature α if α is finite, $\alpha - 1$ otherwise; its sobrification rank is $\alpha + 1$ if α is a limit ordinal, α otherwise.*

PROOF. (1) Let us agree to write α for both the poset α and the space α with its Alexandroff topology. The stature of α is its maximal order type, since α is a wpo. That maximal order type is just the order type of α , namely α itself, see [BG08, Lemma 46], for instance.

The (downwards-)closed subsets of α are totally ordered by inclusion. Hence if C, C_1, C_2 are three closed subsets of α such that $C \subseteq C_1 \cup C_2$, then C is included in the larger of C_1 and C_2 ; hence every non-empty closed subset C of α is irreducible. It follows that $\mathcal{H}_0\alpha = \mathcal{S}\alpha \cup \{\emptyset\}$, so $|\mathcal{S}\alpha \cup \{\emptyset\}| = 1 + \text{sob } \alpha$ is equal to $|\mathcal{H}_0\alpha| = \|\alpha\| + 1 = \alpha + 1$. If α is finite, then so is $1 + \text{sob } \alpha$, and $1 + \text{sob } \alpha = \text{sob } \alpha + 1$ is equal to $\alpha + 1$, so $\text{sob } \alpha = \alpha$. If α is infinite, then so is $1 + \text{sob } \alpha$, and then $1 + \text{sob } \alpha = \text{sob } \alpha$; therefore $\text{sob } \alpha = \alpha + 1$.

(2) We now agree to write α for the space α with its upper topology. Its closed subsets are exactly the empty set, the sets $\downarrow\beta$, with $\beta < \alpha$, plus the whole set α if α is a limit ordinal. Hence, if α is a limit ordinal, then $|\mathcal{H}_0X| = 1 + \alpha + 1$, so $\|X\| = 1 + \alpha$; since α is infinite, $1 + \alpha = \alpha$, which is equal to $\alpha - 1$ by convention, since α is infinite. If α is not a limit ordinal, then $|\mathcal{H}_0X| = 1 + \alpha$ instead; then, if α is finite, then $\|X\| = \alpha$, otherwise α is a successor ordinal and $\|X\| = \alpha - 1$.

All the non-empty closed subsets are irreducible, so a similar analysis applies to $\mathcal{S}\alpha$. If α is a limit ordinal, then $|\mathcal{S}X| = \alpha + 1$. Otherwise, $|\mathcal{S}X| = \alpha$. \square

7. SPACES WITH A COFINITE TOPOLOGY

Any space obtained by equipping a set X with the cofinite topology is Noetherian, since \mathcal{H}_0X is obviously well-founded in that case. The specialization ordering of such a space is equality, which is not a wqo unless X is finite. Note that every such space is T_1 .

Theorem 7.1. *The sobrification rank of a space X with the cofinite topology is 0 if X is empty, 1 if X is finite and non-empty, and 2 if X is infinite.*

PROOF. The case of the empty space is obvious. If X is finite, then the cofinite topology is the discrete topology. In that case, X is sober, so $\mathcal{S}X$ and X are homeomorphic, and no point is strictly above any other, so the rank of every point is 0. It follows that, if X is non-empty, then $\text{sob } X = 1$.

If X is infinite, then the irreducible closed subsets of X are the sets of the form $\{x\}$ with $x \in X$, plus the whole space itself. In order to see this, it suffices to show that X is irreducible, and that no proper closed subset with at least two points is. As far as the first point is concerned, if X is included in the union of two closed subsets F_1 and F_2 , then at least one of them is infinite (since X is) and therefore equal to X , since the only infinite closed subset of X is X itself. As far as the second point is concerned, let F be a proper closed subset of X . Hence F is finite. If F contained at least two points x and y , then it would be included in the union of the two closed sets $F \setminus \{x\}$ and $\{x\}$ without being included in either.

Now the rank of each irreducible closed subset of the form $\{x\}$ is 0. Then the rank of X in $\mathcal{S}X$ is 1, and therefore $\text{sob } X = 2$. \square

Theorem 7.2. *The stature $\|X\|$ of a space X with the cofinite topology is the cardinality of X if X is finite, and ω otherwise.*

PROOF. When X is finite, this is Lemma 6.1. Let us assume X infinite. For every proper closed subset F of X , $\text{rk}_{\mathcal{H}_0 X}(F) = \|F\|$ by Lemma 5.8, and this is equal to the cardinality of F , since F is finite, by Lemma 6.1. Those values span the whole of \mathbb{N} as F varies, since X is infinite. It follows that $\|X\| = \text{rk}_{\mathcal{H}_0 X}(X)$, which is the least ordinal strictly larger than those, is equal to ω . \square

As a corollary, spaces with a cofinite topology yield examples of spaces with very low sobrification rank and stature, and with arbitrarily high cardinality.

8. SUMS

Let $P + Q$ denote the coproduct of the two posets P and Q in the category of posets and monotonic maps. This is the disjoint union of P and Q , where all elements of P are incomparable with all elements of Q , and the ordering relations inside P and inside Q are preserved.

The maximal order type $o(P + Q)$ of the sum of two wpos P and Q is equal to $o(P) \oplus o(Q)$. We have a similar result for statures, which we will prove using the following lemma, which one can find in [Fra00, Section 4.8.3], for example.

Lemma 8.1. *Let P and Q be two well-founded posets. For all $p \in P$ and $q \in Q$, $\text{rk}_{P \times Q}(p, q) = \text{rk}_P(p) \oplus \text{rk}_Q(q)$.*

Let $X + Y$ denote the topological sum of two topological spaces X and Y , namely their coproduct in the category **Top**. This is Noetherian as soon as both X and Y are.

Proposition 8.2. *For all Noetherian spaces X and Y , $\|X + Y\| = \|X\| \oplus \|Y\|$.*

PROOF. Every closed subset F of $X + Y$ can be written in a unique way as the disjoint union of $F \cap X$ and of $F \cap Y$, which are closed subsets of X and of Y , respectively. It follows that $\mathcal{H}_0(X + Y)$ and $\mathcal{H}_0 X \times \mathcal{H}_0 Y$ are order-isomorphic. Then $\|X + Y\| = \text{rk}_{\mathcal{H}_0(X+Y)}(X + Y) = \text{rk}_{\mathcal{H}_0 X \times \mathcal{H}_0 Y}(X, Y) = \text{rk}_{\mathcal{H}_0 X}(X) \oplus \text{rk}_{\mathcal{H}_0 Y}(Y) = \|X\| \oplus \|Y\|$, where the next-to-last equality is by Lemma 8.1. \square

Lemma 8.3. *Given any two well-founded posets P and Q , we have that $|P + Q| = \max(|P|, |Q|)$.*

PROOF. $|P + Q|$ is the least ordinal strictly larger than $\text{rk}_{P+Q}(p) = \text{rk}_P(p)$ for every $p \in P$ and than $\text{rk}_{P+Q}(q) = \text{rk}_Q(q)$ for every $q \in Q$. \square

Proposition 8.4. *For all Noetherian spaces X and Y , $\text{sob}(X + Y) = \max(\text{sob } X, \text{sob } Y)$ and $\text{rsob}(X + Y) = \max(\text{rsob } X, \text{rsob } Y)$.*

PROOF. By Lemma 8.3, since the sobrification of $X + Y$ is the disjoint sum of the posets $\mathcal{S}X$ and $\mathcal{S}Y$. The latter can be seen by realizing that any irreducible closed subset of $X + Y$ must be included in X or in Y . \square

9. LEXICOGRAPHIC SUMS

There is another notion of sum, which we call the *lexicographic sum* of two spaces.

Definition 9.1 (Lexicographic sum). The lexicographic sum $X +_{\text{lex}} Y$ of two topological spaces X and Y is the disjoint sum of X and Y , and its open subsets are the open subsets of Y , plus the sets of the form $U + Y$, where U is an open subset of X .

If X and Y are Noetherian, then so is $X +_{\text{lex}} Y$, since $X + Y$ is and every topology coarser than a Noetherian topology is Noetherian. The specialization preordering of $X +_{\text{lex}} Y$ is the lexicographic sum of those of X and of Y , where all elements of X are below all elements of Y , and the topology of $X +_{\text{lex}} Y$ is Alexandroff if those of X and Y are.

Proposition 9.2. *For all Noetherian spaces X and Y , $\|X +_{\text{lex}} Y\| = \|X\| + \|Y\|$.*

PROOF. The closed subsets of $X +_{\text{lex}} Y$ are the proper closed subsets F of X , plus the sets of the form $X + F'$, where F' ranges over the closed subsets of Y . By well-founded induction, we see that $\text{rk}_{X+_{\text{lex}} Y}(F) = \text{rk}_X(F)$ for subsets of the first kind, then that $\text{rk}_{X+_{\text{lex}} Y}(X + F') = \|X\| + \text{rk}_Y(F')$ for subsets of the second kind. It follows that $\|X +_{\text{lex}} Y\| = \text{rk}_{X+_{\text{lex}} Y}(X + Y) = \|X\| + \|Y\|$. \square

Lemma 9.3. *For any two topological spaces X and Y , the irreducible closed subsets of $X +_{\text{lex}} Y$ are those of X , plus sets of the form $X + D$, where D is irreducible closed in Y .*

PROOF. If C is a closed subset of X that is irreducible in X , then we claim that it is irreducible in $X +_{\text{lex}} Y$. By assumption, C is non-empty. Let us assume that C is included in the union of two closed subsets F_1 and F_2 of $X +_{\text{lex}} Y$. If one of them is of the form $X + F'$ for some closed subset F' of Y , then C is included in that one. Otherwise, both are closed subsets of X , and therefore C is included in F_1 or in F_2 .

Conversely, if C is irreducible closed in $X +_{\text{lex}} Y$, and included in X , then it is easy to see that C is irreducible closed in X .

Let us consider any set of the form $X + D$, where D is closed in Y . If D is irreducible closed in Y , then D is non-empty, hence so is $X + D$. Let us assume that $X + D$ is included in the union of two closed subsets F_1 and F_2 of $X +_{\text{lex}} Y$. It cannot be that F_1 and F_2 are both included in X , since $X + D$ is not, owing to the fact that D is non-empty. If F_1 is of the form $X + F'_1$ for some closed subset of Y , and F_2 is included in X , then $F_1 \cup F_2 = F_1$, so $X + D$ is included in F_1 . We omit the symmetrical case. If $F_1 = X + F'_1$ and $F_2 = X + F'_2$ for some closed subsets F'_1 and F'_2 of Y , finally, then $X + D \subseteq (X + F'_1) \cup (X + F'_2)$ implies that D is included in $F'_1 \cup F'_2$, hence in one of F'_1 or F'_2 , since D is irreducible. Therefore $X + D$ is included in F_1 or in F_2 .

Conversely, let us assume that $X + D$ is irreducible closed in $X +_{\text{lex}} Y$. If D is empty, then X is irreducible closed in $X +_{\text{lex}} Y$, and we have seen that X must be irreducible closed in X . Henceforth, we assume that D is non-empty. Then, given any two closed subsets F'_1 and F'_2 of Y whose union contains D , the union of $X + F'_1$ and of $X + F'_2$ contains $X + D$. Since $X + D$ is irreducible, $X + D$ is included in one of them, and therefore D is included in F'_1 or in F'_2 . \square

Proposition 9.4. *For all Noetherian spaces X and Y , $\text{sob}(X +_{\text{lex}} Y) = \text{sob } X + \text{sob } Y$.*

PROOF. We leave all references to Lemma 9.3 implicit here. By well-founded induction on $C \in \mathcal{S}X$, $\text{rk}_{\mathcal{S}(X +_{\text{lex}} Y)}(C) = \text{rk}_X(C)$. Given any minimal element D of $\mathcal{S}Y$, hence such that $\text{rk}_{\mathcal{S}Y}(D) = 0$, $\text{rk}_{\mathcal{S}(X +_{\text{lex}} Y)}(X + D)$ is the smallest ordinal strictly larger than $\text{rk}_{\mathcal{S}X}(C)$ for every $C \in \mathcal{S}X$, hence is equal to $\text{sob } X$, by definition. This is the start of an induction on $D \in \mathcal{S}Y$, showing that $\text{rk}_{\mathcal{S}(X +_{\text{lex}} Y)}(X + D) = \text{sob } X + \text{rk}_Y(D)$. By adding one and taking suprema, we obtain that $\text{sob}(X +_{\text{lex}} Y) = \text{sob } X + \text{sob } Y$. \square

The special case of *liftings* is of particular importance.

Definition 9.5 (Lifting). The *lifting* X_\perp of a topological space is the lexicographic sum of a one-point space with X , in short, $\{\perp\} +_{\text{lex}} X$.

The open subsets of X_\perp are those of X , plus X_\perp itself. X_\perp is Noetherian if and only if X is. The following is an easy consequence of Proposition 9.4 and of Proposition 9.2.

Proposition 9.6. *For every Noetherian space X , $\text{sob } X_\perp = 1 + \text{sob } X$ and $\|X_\perp\| = 1 + \|X\|$.*

10. PRODUCTS

The sobrification rank of a product is easily obtained. It suffices to observe that the irreducible closed subsets of $X \times Y$ are exactly the products $C \times D$ of an irreducible closed subset C of X and of an irreducible closed subset D of Y , and therefore $\mathcal{S}(X \times Y)$ is order-isomorphic to the poset product $\mathcal{S}X \times \mathcal{S}Y$; this is originally due to Hoffmann [Hof79, Theorem 1.4], see also [GL13, Proposition 8.4.7].

Proposition 10.1. *For all non-empty Noetherian spaces X and Y , $\text{sob}(X \times Y) = (\text{sob } X \oplus \text{sob } Y) - 1$. If one of them is empty, then $\text{sob}(X \times Y) = 0$.*

PROOF. If X is empty, then so is $X \times Y$, and therefore $\text{sob}(X \times Y) = 0$; similarly if Y is empty.

Let us assume that X and Y are both non-empty. We write X as a finite union of irreducible closed subsets C_1, \dots, C_m , and Y as a finite union of irreducible closed subsets D_1, \dots, D_n . Using Lemma 8.1, $\text{rk}_{\mathcal{S}(X \times Y)}(C_i \times D_j) = \text{rk}_{\mathcal{S}X \times \mathcal{S}Y}(C_i, D_j) = \text{rk}_{\mathcal{S}X}(C_i) \oplus \text{rk}_{\mathcal{S}Y}(D_j)$ for all i and j . Then $\text{sob}(X \times Y) = \max_{i,j} \text{rk}_{\mathcal{S}(X \times Y)}(C_i \times D_j) + 1 = \max_{i,j} \text{rk}_{\mathcal{S}X}(C_i) \oplus \text{rk}_{\mathcal{S}Y}(D_j) + 1$, while $\text{sob } X \oplus \text{sob } Y = (\max_i \text{rk}_{\mathcal{S}X}(C_i) + 1) \oplus (\max_j \text{rk}_{\mathcal{S}Y}(D_j) + 1) = (\max_i \text{rk}_{\mathcal{S}X}(C_i) \oplus 1) \oplus (\max_j \text{rk}_{\mathcal{S}Y}(D_j) \oplus 1) = \max_{i,j} \text{rk}_{\mathcal{S}X}(C_i) \oplus \text{rk}_{\mathcal{S}Y}(D_j) \oplus 2$, which is therefore equal to $\text{sob}(X \times Y) + 1$. \square

We turn to statures of products. The corresponding result on wpos is that the maximal order type $o(P \times Q)$ of the product of two wpos P and Q is equal to $o(P) \otimes o(Q)$, as shown by de Jongh and Parikh [JP77, Section 3]. We generalize this to Noetherian spaces, replacing maximal order types by statures. The general outline of the argument resembles de Jongh and Parikh's, but the details vary considerably: de Jongh and Parikh extensively build upwards-closed subsets as

upward closures of finite sets of points, and that is a technique that is not available to us in general Noetherian spaces.

Lemma 10.2. *For every Noetherian space X and every subset F of X ,*

- (1) $\|X\| \leq \|F\| \oplus \|X \setminus F\|$;
- (2) *if F is closed, then $\|F\| + \|X \setminus F\| \leq \|X\|$.*

PROOF. (1) The identity map is continuous from $F + (X \setminus F)$ to X , because every open subset U of X can be written as $(U \cap F) + (U \setminus F)$. The inequality then follows from Lemma 5.3, together with Proposition 8.2.

(2) We assume that F is closed. We claim that the identity map is continuous from X to $F +_{\text{lex}} (X \setminus F)$. In order to verify this, we consider any open set of the latter space, and we verify that it is open in X . There are two kinds of open subsets of $F +_{\text{lex}} (X \setminus F)$. The open subsets U of $X \setminus F$ are intersections $V \cap (X \setminus F)$ of an open subset V of X with $X \setminus F$; this intersection must also be open in X since F is closed. The open subsets of the form $U + (X \setminus F)$ where U is open in F , namely where $U = V \cap F$ for some open subset V of X , are equal to $V \cup (X \setminus F)$, hence are open in X , once again because F is closed.

Now (2) follows from Lemma 5.3, together with Proposition 9.2. \square

The following is the key result on which the main theorem of this section is built. The corresponding result in [JP77], in the special case of wpos, is Corollary 2.17.

Corollary 10.3. *For every Noetherian space X , whose stature is a decomposable ordinal α , written in Cantor normal form as $\omega^{\alpha_1} + \dots + \omega^{\alpha_m}$ with $\alpha \geq \alpha_1 \geq \dots \geq \alpha_m$, $m \geq 2$,*

- (1) *there is a closed subset F of X such that $\|F\| = \omega^{\alpha_1} + \dots + \omega^{\alpha_{m-1}}$;*
- (2) *for each such closed subset F , $\|X \setminus F\| = \omega^{\alpha_m}$.*

PROOF. (1) Let $\beta \stackrel{\text{def}}{=} \omega^{\alpha_1} + \dots + \omega^{\alpha_{m-1}}$. We have $\beta < \alpha \leq \|X\| + 1 = |\mathcal{H}_0 X|$, so there is an element F of $\mathcal{H}_0 X$ such that $\text{rk}_{\mathcal{H}_0 X}(F) = \beta$. Therefore $\beta = \|F\|$, by Lemma 5.8.

(2) By Lemma 10.2, $\|F\| + \|X \setminus F\| \leq \|X\| \leq \|F\| \oplus \|X \setminus F\|$, namely $\beta + \|X \setminus F\| \leq \beta + \omega^{\alpha_m} \leq \beta \oplus \|X \setminus F\|$. The first inequality implies $\|X \setminus F\| \leq \omega^{\alpha_m}$. Let us write $\|X \setminus F\|$ in Cantor normal form as $\omega^{\beta_1} + \dots + \omega^{\beta_n}$ with $\beta_1 \geq \dots \geq \beta_n$, $n \in \mathbb{N}$. The second inequality tells us that $\beta + \omega^{\alpha_m} = \omega^{\alpha_1} + \dots + \omega^{\alpha_{m-1}} + \omega^{\alpha_m}$ is less than or equal to the sum of the terms $\omega^{\alpha_1}, \dots, \omega^{\alpha_{m-1}}$ and $\omega^{\beta_1}, \dots, \omega^{\beta_n}$, sorted in decreasing order. Hence the list $\alpha_1, \dots, \alpha_m$ is lexicographically smaller than or equal to the list obtained by merging the two lists $\alpha_1, \dots, \alpha_{m-1}$ and

β_1, \dots, β_n and sorting the result in descending order. It follows that $n \geq 1$ and $\alpha_i \leq \beta_1$ for some i , $1 \leq i \leq m$. This entails $\alpha_m \leq \beta_1$, hence $\omega^{\alpha_m} \leq \|X \setminus F\|$. Therefore $\|X \setminus F\| = \omega^{\alpha_m}$. \square

Lemma 10.4. *For every finite list of closed subsets F_1, \dots, F_n of a Noetherian space X , $\|\bigcup_{i=1}^n F_i\| \leq \bigoplus_{i=1}^n \|F_i\|$.*

PROOF. The map from $F_1 + \dots + F_n$ that sends each element of F_i to itself in $\bigcup_{i=1}^n F_i$ is surjective, and continuous. By Lemma 5.3, $\|\bigcup_{i=1}^n F_i\| \leq \|F_1 + \dots + F_n\| = \|F_1\| \oplus \dots \oplus \|F_n\|$; the second equality follows from Proposition 8.2. \square

Proposition 10.5. *For all Noetherian spaces X and Y , $\|X \times Y\| \leq \|X\| \otimes \|Y\|$.*

PROOF. By induction on the pair of ordinals $\|X\|$ and $\|Y\|$, ordered lexicographically. The claim is clear if $\|X\| = 0$ or if $\|Y\| = 0$. Otherwise, let us write $\|X\|$ in Cantor normal form as $\omega^{\alpha_1} + \dots + \omega^{\alpha_m}$, with $\alpha_1 \geq \dots \geq \alpha_m$, $m \geq 1$, and $\|Y\|$ in Cantor normal form as $\omega^{\beta_1} + \dots + \omega^{\beta_n}$, with $\beta_1 \geq \dots \geq \beta_n$, $n \geq 1$.

If $m \geq 2$, then by Corollary 10.3, there is a closed subset F of X such that $\|F\| = \omega^{\alpha_1} + \dots + \omega^{\alpha_{m-1}}$, and $\|X \setminus F\| = \omega^{\alpha_m}$. Then $F \times Y$ is a closed subset of $X \times Y$, and therefore $\|X \times Y\| \leq \|F \times Y\| \oplus \|(X \setminus F) \times Y\|$ by Lemma 10.2 (1). By induction hypothesis, $\|X \times Y\| \leq (\|F\| \otimes \|Y\|) \oplus (\omega^{\alpha_m} \otimes \|Y\|) = (\|F\| \oplus \omega^{\alpha_m}) \otimes \|Y\| = \|X\| \otimes \|Y\|$.

The case where $n \geq 2$ is symmetric.

Finally, we examine the case where $m = n = 1$. Then $\|X\| = \omega^{\alpha_1}$ and $\|Y\| = \omega^{\beta_1}$. We wish to show that $\|X \times Y\| = \text{rk}_{\mathcal{H}_0(X \times Y)}(X \times Y)$ is smaller than or equal to $\|X\| \otimes \|Y\| = \omega^{\alpha_1 \oplus \beta_1}$. To this end, it suffices to show that $\text{rk}_{\mathcal{H}_0(X \times Y)}(F) < \omega^{\alpha_1 \oplus \beta_1}$ for every proper closed subset F of $X \times Y$. We write F as a finite union of irreducible closed subsets of $X \times Y$. Each one must be of the form $C_k \times D_k$, where C_k is irreducible closed in X and D_k is irreducible closed in Y , where k ranges from 1 to p , say. (The fact that they are irreducible will not matter. The important thing is that we can write F as a finite union of products of non-empty closed subsets.) Additionally, C_k is a proper subset of X or D_k is a proper subset of Y . Hence we can apply the induction hypothesis, to the effect that $\|C_k \times D_k\| \leq \|C_k\| \otimes \|D_k\|$. Moreover, $\|C_k\| \leq \|X\| = \omega^{\alpha_1}$ and $\|D_k\| \leq \|Y\| = \omega^{\beta_1}$, where at least one equality is strict. Whatever the case is, we obtain that $\|C_k \times D_k\| < \omega^{\alpha_1} \otimes \omega^{\beta_1} = \omega^{\alpha_1 \oplus \beta_1}$, since natural product is monotonic, and strictly monotonic in each of its arguments, provided the other one is not zero. Using Lemma 10.4, we have $\|F\| \leq \bigoplus_{k=1}^p \|C_k \times D_k\|$. Since $\omega^{\alpha_1 \oplus \beta_1}$ is \oplus -indecomposable, $\|F\| < \omega^{\alpha_1 \oplus \beta_1}$, and therefore, $\text{rk}_{\mathcal{H}_0(X \times Y)}(F) < \omega^{\alpha_1 \oplus \beta_1}$, using Lemma 5.8. \square

We turn to the reverse inequality.

Lemma 10.6. *For every Noetherian space X , whose stature is written in Cantor normal form as $\omega^{\alpha_1} + \dots + \omega^{\alpha_m}$, with $\alpha_1 \geq \dots \geq \alpha_m$, $m \in \mathbb{N}$, there are closed subsets $X = F_m \supseteq \dots \supseteq F_1 \supseteq F_0 = \emptyset$ such that $\|F_i\| = \omega^{\alpha_1} + \dots + \omega^{\alpha_i}$ for every i , $0 \leq i \leq m$, and $\|F_i \setminus F_{i-1}\| = \omega^{\alpha_i}$ for every i , $1 \leq i \leq m$.*

PROOF. By induction on m , using Corollary 10.3. □

Lemma 10.7. *For every Noetherian space X , and every sequence $X = F_m \supseteq \dots \supseteq F_1 \supseteq F_0 = \emptyset$ of closed subsets of X ,*

$$\|F_1 \setminus F_0\| + \|F_2 \setminus F_1\| + \dots + \|F_m \setminus F_{m-1}\| \leq \|X\| \leq \bigoplus_{i=1}^m \|F_i \setminus F_{i-1}\|.$$

PROOF. If $m = 0$, then X is empty, and this is clear. Otherwise, we use Lemma 10.2 (1), on $X = F_m, F_{m-1}, \dots, F_1$ in succession in order to obtain $\|X\| \leq \|F_{m-1}\| \oplus \|F_m \setminus F_{m-1}\|$, $\|F_{m-1}\| \leq \|F_{m-2}\| \oplus \|F_{m-1} \setminus F_{m-2}\|$, \dots , $\|F_2\| \leq \|F_1\| \oplus \|F_2 \setminus F_1\| = \|F_1 \setminus F_0\| \oplus \|F_2 \setminus F_1\|$, from which the second inequality follows. The first inequality is proved similarly, using Lemma 10.2 (2), instead. □

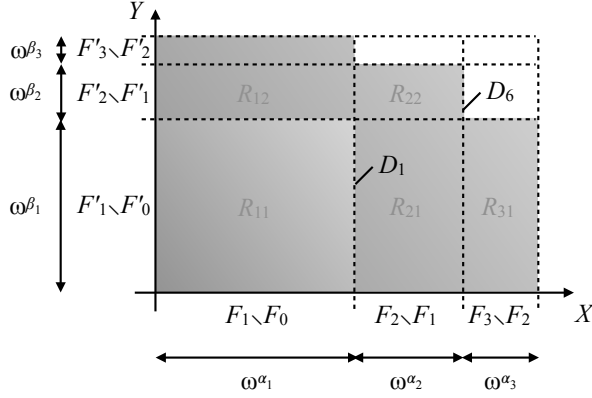
Proposition 10.8. *For all Noetherian spaces X and Y , $\|X \times Y\| \geq \|X\| \otimes \|Y\|$.*

PROOF. By induction on the pair of ordinals $\|X\|$ and $\|Y\|$, ordered lexicographically. The claim is clear if $\|X\| = 0$ or if $\|Y\| = 0$. Otherwise, let us write $\|X\|$ in Cantor normal form as $\omega^{\alpha_1} + \dots + \omega^{\alpha_m}$, with $\alpha_1 \geq \dots \geq \alpha_m$, $m \geq 1$, and $\|Y\|$ in Cantor normal form as $\omega^{\beta_1} + \dots + \omega^{\beta_n}$, with $\beta_1 \geq \dots \geq \beta_n$, $n \geq 1$.

The case $m \geq 2$ or $n \geq 2$. By Lemma 10.6, there are closed subsets $X = F_m \supseteq \dots \supseteq F_1 \supseteq F_0 = \emptyset$ such that $\|F_i\| = \omega^{\alpha_1} + \dots + \omega^{\alpha_i}$ for every i , $0 \leq i \leq m$, and $\|F_i \setminus F_{i-1}\| = \omega^{\alpha_i}$ for every i , $1 \leq i \leq m$. Similarly, there are closed subsets $Y = F'_n \supseteq \dots \supseteq F'_1 \supseteq F'_0 = \emptyset$ such that $\|F'_j\| = \omega^{\beta_1} + \dots + \omega^{\beta_j}$ for every j , $0 \leq j \leq n$, and $\|F'_j \setminus F'_{j-1}\| = \omega^{\beta_j}$ for every j , $1 \leq j \leq n$. The situation is illustrated in Figure 1, where $m = n = 3$. The sets D_1 and D_6 are instances of a sequence of closed subsets D_k that we will construct below.

There is a strict ordering \prec_0 on pairs (i, j) defined by: $(i, j) \prec_0 (i', j')$ if and only if $\alpha_i \oplus \beta_j > \alpha_{i'} \oplus \beta_{j'}$, or $\alpha_i \oplus \beta_j = \alpha_{i'} \oplus \beta_{j'}$ and $F_i \times F'_j \subsetneq F_{i'} \times F'_{j'}$. Let \prec be any linear extension of \prec_0 , and let us enumerate the pairs (i, j) as $(i_1, j_1) \prec \dots \prec (i_{mn}, j_{mn})$. (In other words, let us sort the pairs (i, j) with respect to \prec_0 .)

For every pair $(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$, we abbreviate the rectangle $(F_i \setminus F_{i-1}) \times (F'_j \setminus F'_{j-1})$ as R_{ij} . For every $k \in \{1, \dots, mn\}$, we claim that

FIGURE 1. Proving $\|X \times Y\| \geq \|X\| \otimes \|Y\|$

$D_k \stackrel{\text{def}}{=} \bigcup_{\ell=1}^k R_{i_\ell j_\ell}$ is also equal to $\bigcup_{\ell=1}^k (F_{i_\ell} \times F'_{j_\ell})$, and is therefore closed. The inclusion $D_k \subseteq \bigcup_{\ell=1}^k (F_{i_\ell} \times F'_{j_\ell})$ is clear. In order to show the reverse inclusion, it suffices to show that $F_{i_\ell} \times F'_{j_\ell}$ is included in D_k for every $\ell \in \{1, \dots, k\}$. One checks easily that every point (x, y) of $F_{i_\ell} \times F'_{j_\ell}$ lies in some rectangle R_{ij} with $i \leq i_\ell$ and $j \leq j_\ell$. If $i = i_\ell$ and $j = j_\ell$, then (x, y) is in $R_{i_\ell j_\ell}$, hence in D_k . Otherwise, $\alpha_i \geq \alpha_{i_\ell}$, $\beta_j \geq \beta_{j_\ell}$, $F_i \subseteq F_{i_\ell}$, and $F'_j \subseteq F'_{j_\ell}$, where one of the two inclusions is strict. It follows that $(i, j) \prec_0 (i_\ell, j_\ell)$. Hence (i, j) occurs as $(i_{\ell'}, j_{\ell'})$ for some ℓ' such that $1 \leq \ell' < \ell$. It follows that (x, y) is in $R_{i_{\ell'} j_{\ell'}}$, hence in D_k .

Let $D_0 \stackrel{\text{def}}{=} \emptyset$. We now claim that $D_k \setminus D_{k-1} = R_{i_k j_k}$ for every k , $1 \leq k \leq mn$. Indeed, this follows from the fact that the union $\bigcup_{\ell=1}^k R_{i_\ell j_\ell}$ defining D_k is a disjoint union, and can be reorganized as the union of $\bigcup_{\ell=1}^{k-1} R_{i_\ell j_\ell} = D_{k-1}$ and of $R_{i_k j_k}$.

We now have $D_{mn} \supseteq \dots \supseteq D_1 \supseteq D_0 = \emptyset$, and $D_{mn} = \bigcup_{\ell=1}^{mn} F_{i_\ell} \times F'_{j_\ell} = X \times Y$. This allows us to use Lemma 10.7, so $\|X \times Y\| \geq \|D_1 \setminus D_0\| + \|D_2 \setminus D_1\| + \dots + \|D_{mn} \setminus D_{mn-1}\|$. In other words, $\|X \times Y\| \geq \|R_{i_1 j_1}\| + \|R_{i_2 j_2}\| + \dots + \|R_{i_{mn} j_{mn}}\|$. By induction hypothesis, for every pair (i, j) , $\|R_{ij}\| = \|(F_i \setminus F_{i-1}) \times (F'_j \setminus F'_{j-1})\|$ is equal to $\omega^{\alpha_i} \otimes \omega^{\beta_j}$, namely to $\omega^{\alpha_i \oplus \beta_j}$. (Note that the induction hypothesis applies because the pair $(\omega^{\alpha_i}, \omega^{\beta_j})$ is lexicographically smaller than the pair $(\|X\|, \|Y\|)$, and this is so because $m \geq 2$ or $n \geq 2$.) We have obtained that $\|X \times Y\|$ is larger than or equal to $\omega^{\alpha_{i_1} \oplus \beta_{j_1}} + \dots + \omega^{\alpha_{i_{mn}} \oplus \beta_{j_{mn}}}$.

Let us observe that $\alpha_{i_k} \oplus \beta_{j_k} \geq \alpha_{i_{k+1}} \oplus \beta_{j_{k+1}}$ for every k with $1 \leq k < mn$. Indeed, otherwise we would have $\alpha_{i_{k+1}} \oplus \beta_{j_{k+1}} > \alpha_{i_k} \oplus \beta_{j_k}$, hence $(i_{k+1}, j_{k+1}) \prec_0 (i_k, j_k)$ by definition of \prec_0 , and therefore $(i_{k+1}, j_{k+1}) \prec (i_k, j_k)$, which is impossible.

It follows that the list of ordinals $\omega^{\alpha_{i_1} \oplus \beta_{j_1}}, \dots, \omega^{\alpha_{i_{mn}} \oplus \beta_{j_{mn}}}$ is sorted in decreasing order. That list enumerates all the ordinals $\omega^{\alpha_i \oplus \beta_j}$ with $1 \leq i \leq m$ and $1 \leq j \leq n$. Their sum therefore equals $(\omega^{\alpha_1} + \dots + \omega^{\alpha_m}) \otimes (\omega^{\beta_1} + \dots + \omega^{\beta_n}) = \|X\| \otimes \|Y\|$.

The case $m = n = 1$. We now assume that $\|X\| = \omega^{\alpha_1}$ and $\|Y\| = \omega^{\beta_1}$. If $\alpha_1 = 0$, then $\|X\| = 1$, so there are exactly two closed subsets of X , the empty set and X itself. The closed subsets of $X \times Y$ are then exactly the sets of the form $X \times F$, where F is closed in Y . This implies that $\|X \times Y\| = \|Y\|$, and since $\|X\| = 1$, that is equal to $\|X\| \otimes \|Y\|$. We reason similarly if $\beta_1 = 0$.

If α_1 is a successor ordinal, then $\|X\| = \omega^{\alpha_1 - 1} \times \omega$. For every $m \in \mathbb{N}$, there is a proper closed subset F_m of X such that $\text{rk}_{\mathcal{H}_0 X}(F_m) = \omega^{\alpha_1 - 1} \times m$, since $\omega^{\alpha_1 - 1} \times m < \|X\|$. By Lemma 5.8, $\|F_m\| = \omega^{\alpha_1 - 1} \times m$. By induction hypothesis, $\|F_m \times Y\| \geq (\omega^{\alpha_1 - 1} \times m) \otimes \omega^{\beta_1} = (\bigoplus_{i=1}^m \omega^{\alpha_1 - 1}) \otimes \omega^{\beta_1} = \bigoplus_{i=1}^m \omega^{(\alpha_1 - 1) \oplus \beta_1} = \omega^{(\alpha_1 - 1) \oplus \beta_1} \times m$. Since $\|X \times Y\| \geq \|F_m \times Y\|$, and taking suprema over all $m \in \mathbb{N}$, $\|X \times Y\| \geq \omega^{(\alpha_1 - 1) \oplus \beta_1} \times \omega$. Now $\omega^{(\alpha_1 - 1) \oplus \beta_1} \times \omega = \omega^{((\alpha_1 - 1) \oplus \beta_1) + 1}$, and $((\alpha_1 - 1) \oplus \beta_1) + 1 = ((\alpha_1 - 1) \oplus \beta_1) \oplus 1 = ((\alpha_1 - 1) \oplus 1) \oplus \beta_1 = \alpha_1 \oplus \beta_1$.

The argument is symmetric if β_1 is a successor ordinal.

Finally, if α_1 and β_1 are both limit ordinals, then $\alpha_1 \oplus \beta_1$ is also a limit ordinal, as one can see from its Cantor normal form. Hence $\alpha_1 \oplus \beta_1$ is the supremum of all ordinals $\gamma < \alpha_1 \oplus \beta_1$. Since $\gamma \mapsto \omega^\gamma$ is continuous, $\omega^{\alpha_1} \otimes \omega^{\beta_1} = \omega^{\alpha_1 \oplus \beta_1} = \sup_{\gamma < \alpha_1 \oplus \beta_1} \omega^\gamma$. In order to show that $\|X \times Y\| \geq \omega^{\alpha_1} \otimes \omega^{\beta_1}$, it therefore suffices to show that $\|X \times Y\| \geq \omega^\gamma$ for every $\gamma < \alpha_1 \oplus \beta_1$.

Using the inductive definition of the natural sum, $\alpha_1 \oplus \beta_1$ is the smallest ordinal strictly larger than $\alpha \oplus \beta_1$ for every $\alpha < \alpha_1$ and $\alpha_1 \oplus \beta$ for every $\beta < \beta_1$. Since $\gamma < \alpha_1 \oplus \beta_1$, we must therefore have $\gamma \leq \alpha \oplus \beta_1$ for some $\alpha < \alpha_1$, or $\gamma \leq \alpha_1 \oplus \beta$ for some $\beta < \beta_1$. In the first case, $\omega^\alpha < \omega^{\alpha_1} = \|X\|$, so there is a proper closed subset F_α of X such that $\text{rk}_{\mathcal{H}_0 X}(F_\alpha) = \omega^\alpha$; namely, $\|F_\alpha\| = \omega^\alpha$, by Lemma 5.8. By induction hypothesis, $\|F_\alpha \times Y\| \geq \omega^{\alpha \oplus \beta_1} \geq \omega^\gamma$, and therefore $\|X \times Y\| \geq \omega^\gamma$. Similarly, in the second case, $\|X \times Y\| \geq \omega^{\alpha_1 \oplus \beta} \geq \omega^\gamma$. \square

Combining Proposition 10.5 and Proposition 10.8, we finally obtain the desired result.

Theorem 10.9. *For all Noetherian spaces X and Y , $\|X \times Y\| = \|X\| \otimes \|Y\|$.*

11. HOARE POWERSPACES AND POWERSSETS

Here is another example of Noetherian spaces which do not arise from wqos in general. The *Hoare powerspace* $\mathcal{H}_V X$ of X is just its space of non-empty closed subsets $\mathcal{H}X$, with the so-called *lower Vietoris topology*. We also consider the *lifted Hoare powerspace* $\mathcal{H}_{0V} X$, which also includes the empty set. A subbase of the lower Vietoris topology is given by sets $\diamond U$, defined as the set of those closed sets F that intersect U , where U ranges over the open subsets of X .

It was observed in [Gou07] that $\mathcal{H}_V X$ and $\mathcal{H}_{0V} X$ are Noetherian for every Noetherian space X . That may seem surprising at first, considering that the specialization ordering of each one is inclusion, and that the inclusion ordering on the downwards-closed subsets of a wqo P is *not* in general a wqo [Rad54].

The trick is that the lower Vietoris topology is in general strictly coarser than the Alexandroff topology. In fact, the lower Vietoris topology coincides with the upper topology, since the complement of $\diamond U$ is equal to $\downarrow F$, where F is the complement of U . In the sequel, we will write $\square F$ instead of $\downarrow F$ for the set of closed subsets of F . This will dispel any ambiguity, since $\downarrow F$ is also accepted notation for the downward closure of F in X , not $\mathcal{H}_0 X$.

11.1. The sobrification rank and stature of $\mathcal{H}_{0V} X$. Schalk observed that every up-complete sup-semilattice, namely every poset with suprema of all non-empty families, is sober in its upper topology [Sch93, Proposition 1.7]. $\mathcal{H}X$ and $\mathcal{H}_0 X$ are up-complete sup-semilattices, and the lower Vietoris topology is the upper topology of inclusion. Hence both $\mathcal{H}_V X$ and $\mathcal{H}_{0V} X$ are sober. This makes the following a triviality.

Theorem 11.1. *For every Noetherian space X , $\text{sob } \mathcal{H}_{0V} X = \|X\| + 1$ and $\text{rsob } \mathcal{H}_{0V} X = \|X\|$.*

PROOF. Since $\mathcal{H}_{0V} X$ is sober, $\text{sob } \mathcal{H}_{0V} X$ is just the ordinal rank of $\mathcal{H}_0 X$, namely $\|X\| + 1$. \square

The stature of $\mathcal{H}_{0V} X$ is much more elusive.

Proposition 11.2. *For every Noetherian space X , $1 + \|X\| \leq \|\mathcal{H}_{0V} X\| \leq \omega^{\|X\|}$.*

PROOF. By Proposition 4.5 (1) and Theorem 11.1, $1 + \|X\| = 1 + \text{rsob } \mathcal{H}_{0V} X \leq \|\mathcal{H}_{0V} X\|$. $\mathcal{H}_{0V} X$ is not empty, and has exactly one component, which is $\square X = \mathcal{H}_{0V} X$. By Proposition 4.5 (3), $\|\mathcal{H}_{0V} X\| \leq \omega^{\text{rsob } \mathcal{H}_{0V} X} = \omega^{\|X\|}$. \square

We claim that those lower and upper bounds are tight in general. Before we give substance to this claim, we observe that several other Noetherian spaces are related to $\mathcal{H}_{0V}X$.

11.2. Powersets, and finitary variants. One is the *powerset* $\mathbb{P}X$ of X , with a topology that we will still call the lower Vietoris topology, whose subbasic open sets we will still write as $\diamond U$, and which now denote $\{A \in \mathbb{P}X \mid A \cap U \neq \emptyset\}$. Another one is $\mathcal{H}_{\text{fin}}X$, the subspace of $\mathcal{H}_{0V}X$ consisting of all the finitary closed subsets $\downarrow\{x_1, \dots, x_n\}$ of X . Finally, there is the finitary powerset $\mathbb{P}_{\text{fin}}X$, which is the subspace of $\mathbb{P}X$ consisting of its finite subsets.

The specialization ordering of $\mathcal{H}_{\text{fin}}X$, just like $\mathcal{H}_{0V}X$, is inclusion. Indeed, the closure of any point $F \in \mathcal{H}_{\text{fin}}X$ is $\square F$, which is also the downward closure of F under inclusion. The specialization preordering of $\mathbb{P}_{\text{fin}}X$, just like $\mathbb{P}X$, is inclusion of closures. This was proved for $\mathbb{P}X$ in [FGL20, Lemma 4.9], and follows from the fact that the closure of $\{A\}$ in $\mathbb{P}_{\text{fin}}X$ (resp., $\mathbb{P}X$) is $\square cl(A)$, where $cl(A)$ denotes the closure of A in X .

Remark 11.3. While $\mathcal{H}_{0V}X$ and $\mathbb{P}X$ do not arise from wqos, it so happens that $\mathcal{H}_{\text{fin}}X$ and $\mathbb{P}_{\text{fin}}X$ do, provided that X is wqo (in its Alexandroff topology). Indeed, it suffices to verify that the lower Vietoris topology on each coincides with the Alexandroff topology of their specialization preorderings. To this end, it is enough to show that the upward closure of each point F is open in the lower Vietoris topology. For $\mathcal{H}_{\text{fin}}X$, the specialization preordering is inclusion, and the upward closure of $F \stackrel{\text{def}}{=} \downarrow\{x_1, \dots, x_n\}$ is $\diamond(\uparrow x_1) \cap \dots \cap \diamond(\uparrow x_n)$. For $\mathbb{P}_{\text{fin}}X$, the specialization ordering \leq^b is given by $A \leq^b B$ if and only if the closure of A is included in the closure of B ; when X is Alexandroff, that is equivalent to the fact that the downward closure of A is included in that of B , or equivalently that every element of A is smaller than or equal to some element of B . Then the upward closure of $\{x_1, \dots, x_n\}$ is $\diamond(\uparrow x_1) \cap \dots \cap \diamond(\uparrow x_n)$, as with $\mathcal{H}_{\text{fin}}X$.

Proposition 11.4. *For every topological space X , the following maps are full, continuous, and Skula dense:*

- (1) *the functions cl that map every set to its closure in X , from $\mathbb{P}X$ to $\mathcal{H}_{0V}X$, and from $\mathbb{P}_{\text{fin}}X$ to $\mathcal{H}_{\text{fin}}X$;*
- (2) *the inclusion maps from $\mathcal{H}_{\text{fin}}X$ into $\mathcal{H}_{0V}X$ and from $\mathbb{P}_{\text{fin}}X$ into $\mathbb{P}X$.*

In particular, for every Noetherian space X ,

$$\text{sob } \mathbb{P}_{\text{fin}}X = \text{sob } \mathcal{H}_{\text{fin}}X = \text{sob } \mathbb{P}X = \text{sob } \mathcal{H}_{0V}X = ||X|| + 1,$$

and

$$\|\mathbb{P}_{\text{fin}}X\| = \|\mathcal{H}_{\text{fin}}X\| = \|\mathbb{P}X\| = \|\mathcal{H}_{0V}X\|.$$

PROOF. We start with cl . Let us reserve the notation $\diamond U$ for the subbasic open subsets of $\mathcal{H}_{0V}X$ (resp., $\mathcal{H}_{\text{fin}}X$), and let us use $\diamond_{\mathbb{P}}U$ to denote the corresponding subbasic open subset of $\mathbb{P}X$ (resp., $\mathbb{P}_{\text{fin}}X$).

For every open subset U of X , $cl^{-1}(\diamond U)$ is the set of subsets (resp., finite subsets) A of X such that $cl(A)$ intersects U . Since $cl(A)$ intersects U if and only if A does, $cl^{-1}(\diamond U) = \diamond_{\mathbb{P}}U$. This shows that cl is continuous. Also, cl is full since every open subset of $\mathbb{P}X$ (resp., $\mathbb{P}_{\text{fin}}X$) is of the form $\bigcup_{i \in I} \bigcap_{j \in J_i} \diamond_{\mathbb{P}}U_{ij}$, where each J_i is a finite set; and that is equal to $\bigcup_{i \in I} \bigcap_{j \in J_i} cl^{-1}(\diamond U_{ij}) = cl^{-1}(\bigcup_{i \in I} \bigcap_{j \in J_i} \diamond U_{ij})$. For any two open subsets U and V of $\mathcal{H}_{0V}X$ (resp., $\mathcal{H}_{\text{fin}}X$) such that $cl^{-1}(U) = cl^{-1}(V)$, every $F \in U$ is also in $cl^{-1}(U)$ since $F = cl(F)$, hence in $cl^{-1}(V)$; so $F = cl(F')$ for some $F' \in V$. But $F' \in V$ implies that F' is closed in X , so $F = F'$, and therefore F is in V . This shows that U is included in V , and the reverse inclusion is proved similarly. It follows that cl is Skula dense.

The inclusion maps from $\mathcal{H}_{\text{fin}}X$ into $\mathcal{H}_{0V}X$ and from $\mathbb{P}_{\text{fin}}X$ into $\mathbb{P}X$ are topological embeddings by definition, hence are full and continuous. Let us write i for any of those maps. In order to show that it is Skula dense, we now reserve the notation $\diamond U$ for subbasic open subsets of $\mathcal{H}_{0V}X$, resp. $\mathbb{P}X$, and write $\diamond_{\text{fin}}X$ for the corresponding sets in $\mathcal{H}_{\text{fin}}X$, resp. $\mathbb{P}_{\text{fin}}X$. For any open subset $U \stackrel{\text{def}}{=} \bigcup_{i \in I} \bigcap_{j \in J_i} \diamond U_{ij}$ of $\mathcal{H}_{0V}X$ (resp., $\mathbb{P}X$), where each set J_i is finite and each U_{ij} is open in X , $i^{-1}(U)$ is equal to $\bigcup_{i \in I} \bigcap_{j \in J_i} \diamond_{\text{fin}}U_{ij}$. Given another open subset V , such that $i^{-1}(U) = i^{-1}(V)$, we claim that $U = V$; by symmetry, it suffices to show $U \subseteq V$. Let F be any element of U . There is an index $i \in I$ such that F intersects U_{ij} for every $j \in J_i$, say at x_j . Let $F' \stackrel{\text{def}}{=} \downarrow \{x_j \mid j \in J_i\}$ (resp., $F' \stackrel{\text{def}}{=} \{x_j \mid j \in J_i\}$). Then F' is in $\bigcap_{j \in J_i} \diamond_{\text{fin}}U_{ij} \subseteq i^{-1}(U) = i^{-1}(V)$, so $i(F') = F'$ is in V . We now observe that $F' \subseteq F$ (resp., $cl(F') \subseteq cl(F)$), and that V is upwards-closed in the specialization preordering of $\mathcal{H}_{0V}X$ (resp., $\mathbb{P}X$), so F is in V .

The remaining claims follow from Lemma 5.5 (4) and Theorem 11.1. \square

11.3. The bounds on $\|\mathcal{H}_{0V}X\|$ are tight. We proceed through a series of examples.

Example 11.5. The lower bound of Proposition 11.2 is attained. Consider $X \stackrel{\text{def}}{=} \alpha$, where α is an ordinal, with its Alexandroff topology. Then $\|X\| = \alpha$ by Lemma 6.2 (1). The closed subsets of X are themselves totally ordered by inclusion, and every non-empty closed subset is irreducible. Hence $\mathcal{H}_0(\mathcal{H}_{0V}X) =$

$\mathcal{S}(\mathcal{H}_{0V}X) \cup \{\emptyset\}$, so that $\|\mathcal{H}_{0V}X\| = |\mathcal{S}(\mathcal{H}_{0V}X) \cup \{\emptyset\}| - 1 = (1 + |\mathcal{S}(\mathcal{H}_{0V}X)|) - 1$. Since $\mathcal{H}_{0V}X$ is sober, it is homeomorphic to its sobrification, so $\|\mathcal{H}_{0V}X\| = 1 + |\alpha| = 1 + \alpha = 1 + \|X\|$. This result can also be obtained by verifying that $\mathcal{H}_{0V}X$ is equal to $(1 + \alpha) + 1$ if α is a limit ordinal, to $1 + \alpha$ otherwise, and with the upper topology in both cases; then apply Lemma 6.2 (2).

Example 11.6. Let us consider $X \stackrel{\text{def}}{=} \alpha$, where α is any ordinal, with its Scott topology. As in Example 11.5, every non-empty closed subset is irreducible; the non-empty closed subsets are the subsets $\downarrow \beta$ with $\beta < \alpha$, plus α itself if α is zero or a limit ordinal. Hence $\mathcal{H}_{0V}X$ can be equated to $1 + \alpha$ if α is finite, to $\alpha + 1$ if α is a limit ordinal, and to α otherwise. The topology is the upper topology, which coincides with the Scott topology. By Lemma 6.2 (2), $\|\mathcal{H}_{0V}X\|$ is equal to $1 + \alpha$ if α is finite, to α if α is a limit ordinal, and to $\alpha - 1$ otherwise. Using Lemma 6.2 (2), we check that this is equal to $1 + \|X\|$ in all cases. This has the curious consequence that, if α is an infinite ordinal with its Scott topology, then $\|X\| = \|\mathcal{H}_{0V}X\| = \|\mathcal{H}_{0V}(\mathcal{H}_{0V}X)\| = \dots = \|\mathcal{H}_{0V}^n X\|$ for every $n \in \mathbb{N}$.

Remark 11.7. If $\|X\|$ is a *critical* ordinal ϵ , namely an ordinal such that $\epsilon = \omega^\epsilon$, then the lower and upper bounds of Proposition 11.2 match. It follows that $\|\mathcal{H}_{0V}X\| = \epsilon = \|X\|$ in this case. As in Example 11.6, this implies that $\|\mathcal{H}_{0V}^n X\| = \|X\| = \epsilon$ for every $n \in \mathbb{N}$.

Example 11.8. Here is a case where $\|\mathcal{H}_{0V}X\|$ is strictly between the lower and upper bounds of Proposition 11.2. Let X be a finite set of cardinality n , with its discrete topology. Then $\|X\| = n$ by Lemma 6.1. \mathcal{H}_0X is the powerset of X . The specialization preordering of $\mathcal{H}_{0V}X$ is inclusion, which is an ordering, so $\mathcal{H}_{0V}X$ is T_0 . By Lemma 6.1, $\|\mathcal{H}_{0V}X\| = 2^n$.

We will use the following to show that the upper bound of Proposition 11.2 is attainable.

Example 11.9. Given any space X , let $s(X)$ denote the space obtained by forming the sum of countably many disjoint copies of X , and adding a fresh top element \top . Formally, the closed subsets of $s(X)$ are the disjoint unions $F_1 + \dots + F_n$ of closed sets from finitely many copies of X , plus $s(X)$ itself. By Proposition 8.2, $\|F_1 + \dots + F_n\| = \bigoplus_{i=1}^n \|F_i\|$; with n fixed, but letting F_1, \dots, F_n vary, the largest value taken by that sum is $\|X\| \otimes n$. If $\|X\| = \omega^\alpha$, that is equal to $\omega^\alpha \times n$; taking suprema over all n , we obtain that $\|s(X)\| = \omega^{\alpha+1}$. For every $n \in \mathbb{N}$, there is an obvious embedding of $(\mathcal{H}_{0V}X)^n$ into $\mathcal{H}_{0V}(s(X))$, which maps every n -tuple (F_1, \dots, F_n) to $F_1 + \dots + F_n$, each F_i being located in the i th

copy of X . By Lemma 5.5 (3) and Theorem 10.9, $\|\mathcal{H}_{0V}(s(X))\| \geq \bigotimes_{i=1}^n \|\mathcal{H}_{0V}X\|$. If $\|\mathcal{H}_{0V}X\| \geq \omega^{\omega^\beta}$, we obtain that $\|\mathcal{H}_{0V}(s(X))\| \geq \omega^{\omega^\beta \times n}$. Since n is arbitrary, $\|\mathcal{H}_{0V}(s(X))\| \geq \omega^{\omega^{\beta+1}}$.

We iterate this construction, starting from a Noetherian space X such that $\|X\| = \omega$, for example ω itself or any infinite set with the cofinite topology. By taking $\alpha \stackrel{\text{def}}{=} 1$ and $\beta \stackrel{\text{def}}{=} 0$ (since $\|\mathcal{H}_{0V}X\| \geq 1 + \|X\| = \omega$ by Proposition 11.2), we obtain that $\|s(X)\| = \omega^2$ and $\|\mathcal{H}_{0V}(s(X))\| \geq \omega^\omega$. We can now take $\alpha \stackrel{\text{def}}{=} 2$ and $\beta \stackrel{\text{def}}{=} 1$, and obtain $\|s^2(X)\| = \omega^3$ and $\|\mathcal{H}_{0V}(s^2(X))\| \geq \omega^{\omega^2}$. In general, $\|s^k(X)\| = \omega^{k+1}$ and $\|\mathcal{H}_{0V}(s^k(X))\| \geq \omega^{\omega^k}$ for every $k \in \mathbb{N}$.

We now build a form of limit of the spaces $s^k(X)$, $k \in \mathbb{N}$. For every space Y , let $i_Y: Y \rightarrow s(Y)$ map every $y \in Y$ to y itself in copy number 0 of Y inside $s(Y)$. We note that i_Y is an embedding, and a closed map. Let $s^\omega(X)$ be the quotient of the disjoint sum $\coprod_{k \in \mathbb{N}} s^k(X)$ by the smallest equivalence relation \equiv that equates y with $i_{s^k(X)}(y)$ for every $y \in s^k(X)$. For each $k \in \mathbb{N}$ and each closed subset F of $s^k(X)$, let $[F]_k$ be the set of equivalence classes of points of $F \subseteq s^k(X)$ modulo \equiv . We note that $[F]_k = [i_{s^k(X)}(F)]_{k+1}$, so that we can always write any finite collection of sets $[F]_k$ with the same index k . It follows that the collection of such sets is closed under finite unions. It is also closed under arbitrary non-empty intersections, which in fact reduce to finite non-empty intersections in some subspace $s^k(X)$. Therefore, the collection of sets $[F]_k$, where $k \in \mathbb{N}$ and F is closed in X , plus the whole space $s^\omega(X)$ itself, form a Noetherian topology on $s^\omega(X)$. Since $s^k(X)$ embeds into $s^\omega(X)$ for every $k \in \mathbb{N}$, $\|\mathcal{H}_{0V}(s^\omega(X))\| \geq \omega^{\omega^k}$ for every $k \in \mathbb{N}$, and therefore $\|\mathcal{H}_{0V}(s^\omega(X))\| \geq \omega^{\omega^\omega}$. The family $([s^k(X)]_k)_{k \in \mathbb{N}}$ forms a cofinal family of proper closed sets, so by Proposition 5.9, $\|s^\omega(X)\| = \sup_{k \in \mathbb{N}} (\omega^{k+1} + 1) = \omega^\omega$. In particular, $\|\mathcal{H}_{0V}(s^\omega(X))\|$ is larger than or equal to, and therefore equal to the upper bound $\omega^{\|s^\omega(X)\|}$ (namely, ω^{ω^ω}) given in Proposition 11.2.

12. FINITE WORDS

12.1. The structure of $\mathcal{S}(X^*)$. Given any set X , let X^* be the set of finite words on X . We write ϵ for the empty word, and ww' for the concatenation of two words w and w' . Given two subsets A and B of X^* , we also write AB for $\{ww' \mid w \in A, w' \in B\}$.

When X is a topological space, we give X^* the *word topology*, which is defined in [GL13, Definition 9.7.26] as the topology generated by subsets of the form $X^*U_1X^*U_2X^*\dots X^*U_nX^*$, where $n \in \mathbb{N}$, and U_1, U_2, \dots, U_n range over open subsets of X .

For every Noetherian space X , X^* is Noetherian [GL13, Theorem 9.7.33]. Let us write \leq for the specialization preordering of X . The specialization preordering of X^* is the *word embedding* quasi-ordering \leq^* , defined by $w \leq^* w'$ if and only if one can obtain w' from w by increasing some of its letters and inserting arbitrarily many letters at any position [GL13, Exercise 9.7.29]. This is the familiar preordering at the heart of Higman's Lemma, which says that for every wqo \leq , \leq^* is a wqo [Hig52]. That X^* is Noetherian for every Noetherian space X is a topological generalization of this result, in the sense that if X is Alexandroff and Noetherian (i.e., wqo), then so is X^* [GL13, Exercise 9.7.30].

A *word-product* P on a space X is any expression of the form $A_1 A_2 \cdots A_n$, where $n \in \mathbb{N}$, and each A_i is an *atomic expression*, either F_i^* with F_i closed in X , or $C_i^?$ with C_i irreducible closed in X . The semantics of an atomic expression A will be written \mathbf{A} . The semantics of $C_i^?$ is the collection of words containing at most one letter, and that letter must be in C_i . That is sometimes written $C_i + \epsilon$, or $C_i \cup \{\epsilon\}$, by abuse of language. The semantics of F_i^* is the set of finite words, of arbitrary lengths, whose letters are in F_i , and the semantics of $A_1 A_2 \cdots A_n$ is $\mathbf{A}_1 \mathbf{A}_2 \cdots \mathbf{A}_n$. For example, if we take X to be $\{a, b, c\}$ with the discrete topology, then $\{a, b\}^* \{c\}^? \{a\}^? \{b\}^? \{a, c\}^*$ is the set of words $w_1 w w_2$ where w_1 is a word of arbitrary length whose letters are among $\{a, b\}$, w_2 is any word whose letters are among $\{a, c\}$, and w is cab or any of its subwords ab, cb, ca, c, a, b and ϵ .

When $n = 0$, P is abbreviated as ϵ , and denotes the one-element set $\{\epsilon\}$. In general, we write \mathbf{P} for the semantics of P . Note that \mathbf{P} always contains the empty word ϵ .

We call *word-SRE* any finite sum of word-products, where sum is interpreted as union. ‘‘SRE’’ stands for ‘‘simple regular expression’’ [ABJ98]. It turns out that, given any Noetherian space X , the closed subsets of X^* are exactly the semantics of word-SREs, and the irreducible closed subsets of X^* are exactly the semantics of word-products [FGL20, Proposition 6.14]. This extends the corresponding result on wqos, due to Kabil and Pouzet [KP92], which itself extends a previous result of Jullien on words on a finite set, ordered by equality [Jul69, chapitre VI].

The inclusion ordering on $\mathcal{S}(X^*)$ is characterized as follows. First, we observe that inclusion of atomic expressions is characterized by:

- (1) $C^? \subseteq C'^?$ if and only if $C \subseteq C'$;
- (2) $F^* \subseteq F'^*$ if and only if $F \subseteq F'$;
- (3) $C^? \subseteq F'^*$ if and only if $C \subseteq F'$;
- (4) $F^* \subseteq C'^?$ if and only if F is empty.

Then we have the following.

Lemma 12.1 (Lemmata 7.9 and 7.10, [FGL20]). *Given two word-products P and P' on a Noetherian space X , $P \subseteq P'$ if and only if P is ϵ , or $P' = \epsilon$ and P is a product of atomic expressions all equal to \emptyset^* , or P can be written as A_1Q and P' as A'_1Q' , where A_1 and A'_1 are atomic expressions, and one of the following occurs:*

- (1) $A_1 \not\subseteq A'_1$ and $P \subseteq Q'$;
- (2) or A_1 is of the form $C^?$, A'_1 is of the form $C'^?$, $C \subseteq C'$, and $Q \subseteq Q'$;
- (3) or A'_1 is of the form F'^* , $A_1 \subseteq A'_1$, and $Q \subseteq P'$;
- (4) or $A_1 = \emptyset^*$ and $Q \subseteq P'$.

Equivalently, inclusion of word-products is axiomatized in a sound and complete way by the following rules:

$$\frac{}{\epsilon \subseteq P'} \quad (0) \quad \frac{A_1Q \subseteq Q'}{A_1Q \subseteq A'_1Q'} \quad (1) \quad \frac{Q \subseteq Q'}{C^?Q \subseteq C'^?Q'} \quad (2)$$

if $A_1 \not\subseteq A'_1$ if $C \subseteq C'$

$$\frac{Q \subseteq F'^*Q'}{A_1Q \subseteq F'^*Q'} \quad (3) \quad \frac{Q \subseteq P'}{\emptyset^*Q \subseteq P'} \quad (4)$$

if $A_1 \subseteq F'^*$

A word-product $A_1A_2 \cdots A_n$ is *reduced* if and only if no A_i is equal to \emptyset^* , and for every A_i of the form F^* , F^* is not included in A_{i+1} (if $i < n$) and not included in A_{i-1} (if $i > 1$). It is easy to see that every word-product can be rewritten into a reduced word-product with the same semantics. One can also show that reduced word-products are canonical forms for irreducible closed subsets of X^* , namely that two reduced word-products denote the same irreducible closed set if and only if they are syntactically equal. The proof is identical to the corresponding result on wqos, see [GHK⁺20, Theorem 4.22]; but we will not make use of that fact.

12.2. The sobrification rank of X^* . We will need the following ordinal adjustment operation. We recall that a critical ordinal is an ordinal ϵ such that $\epsilon = \omega^\epsilon$.

Definition 12.2 (α°). For every ordinal α , the ordinal α° is defined as $\alpha + 1$ if $\alpha = \epsilon + n$ for some critical ordinal ϵ and some natural number n , as α otherwise.

Our goal in this subsection is to show that, for every non-empty Noetherian space X , $\text{sob } X^* = \omega^{\|X\|^\circ} + 1$.

Lemma 12.3. *The map $\alpha \mapsto \alpha^\circ$ is strictly increasing.*

PROOF. Let $\alpha_1 < \alpha_2$. We need to consider four cases in order to show that $\alpha_1^\circ < \alpha_2^\circ$, but only one is non-trivial, namely when $\alpha_1 = \epsilon + n$ for some critical ordinal ϵ and some $n \in \mathbb{N}$, and $\alpha_2 > \alpha_1$ is not of that form. Then $\alpha_2 \geq \epsilon + \omega$, so $\alpha_2^\circ = \alpha_2 > \epsilon + n + 1 = \alpha_1^\circ$. \square

Lemma 12.4. *For every ordinal α , $\alpha < \omega^{\alpha^\circ}$.*

PROOF. By induction on α , we have $\alpha \leq \omega^\alpha$. The inequality is strict unless α is critical, by definition. In particular, if α is not of the form $\epsilon + n$ with ϵ critical and $n \in \mathbb{N}$, then $\alpha < \omega^\alpha = \omega^{\alpha^\circ}$ (since $\alpha^\circ = \alpha$ in this case). When $\alpha = \epsilon + n$ where ϵ is critical and $n \in \mathbb{N}$, then $\omega^{\alpha^\circ} = \omega^{\epsilon+n+1} = \omega^\epsilon \times \omega^{n+1} = \epsilon \times \omega^{n+1}$. Since $\omega^{n+1} \geq \omega > 2$, this is larger than $\epsilon \times 2 = \epsilon + \epsilon > \epsilon + n = \alpha$. \square

Remark 12.5. The map $\alpha \mapsto \alpha^\circ$ is in fact the smallest strictly increasing map such that $\alpha < \omega^{\alpha^\circ}$ for every ordinal α , as one can check.

From now on, we use the notation $\|F\|$ instead of $\text{rk}_{\mathcal{H}_0X}(F)$, profiting from Lemma 5.8.

Lemma 12.6. *Let X be a Noetherian space. For all $C \in \mathcal{S}X$, $F, F' \in \mathcal{H}_0X$,*

- (1) *if $C \subseteq F'$ then $1 + \text{rk}_{\mathcal{S}X}(C) < \omega^{\|F'\|^\circ}$;*
- (2) *if $F \subsetneq F'$ then $\omega^{\|F\|^\circ} < \omega^{\|F'\|^\circ}$.*

PROOF. (1) The ordinal $1 + \text{rk}_{\mathcal{S}X}(C)$ is the rank of C in $\mathcal{S}X \cup \{\emptyset\}$. Since that is included in \mathcal{H}_0X , $1 + \text{rk}_{\mathcal{S}X}(C) \leq \text{rk}_{\mathcal{H}_0X}(C) \leq \text{rk}_{\mathcal{H}_0X}(F') = \|F'\|$. We conclude since $\|F'\| < \omega^{\|F'\|^\circ}$ by Lemma 12.4.

(2) The map $\gamma \mapsto \omega^\gamma$ is strictly monotonic, and so is the map $\alpha \mapsto \alpha^\circ$ by Lemma 12.3. \square

Lemma 12.7. *Let X be a Noetherian space, and let us define $\varphi(C^?)$ as $1 + \text{rk}_{\mathcal{S}X}(C)$ for every $C \in \mathcal{S}X$, $\varphi(F^*)$ as $\omega^{\|F\|^\circ}$ for every $F \in \mathcal{H}_0X$, and $\varphi(\mathbf{P})$ as $\bigoplus_{i=1}^n \varphi(\mathbf{A}_i)$ for every reduced word-product $\mathbf{P} \stackrel{\text{def}}{=} \mathbf{A}_1 \cdots \mathbf{A}_n$. For all reduced word-products \mathbf{P} and \mathbf{P}' , $\mathbf{P} \subseteq \mathbf{P}'$ implies $\varphi(\mathbf{P}) \leq \varphi(\mathbf{P}')$, and if the former inclusion is strict, then so is the latter inequality.*

PROOF. We proceed by induction on the sum of the lengths of \mathbf{P} and of \mathbf{P}' .

When $\mathbf{P} = \epsilon$, the claim is clear since $\varphi(\epsilon) = 0$, and $\varphi(\mathbf{P}') \neq 0$ for every reduced word-product $\mathbf{P}' \neq \epsilon$. We therefore assume that $\mathbf{P} \neq \epsilon$ in the sequel.

In particular, \mathbf{P} is of the form $\mathbf{A}_1 \cdots \mathbf{A}_n$ with $n \geq 1$. Each \mathbf{A}_i contains a non-empty word, using the fact that no \mathbf{A}_i is of the form \emptyset^* . Therefore \mathbf{P} contains a non-empty word. Since $\mathbf{P} \subseteq \mathbf{P}'$, it follows that \mathbf{P}' cannot be equal to ϵ . Let us write \mathbf{P}' as $\mathbf{A}'_1 \mathbf{Q}'$. We also write \mathbf{P} as $\mathbf{A}_1 \mathbf{Q}$.

If $\mathbf{A}_1 \not\subseteq \mathbf{A}'_1$, then only clause (1) of Lemma 12.1 can have been used to infer $\mathbf{P} \subseteq \mathbf{P}'$, so $\mathbf{P} \subseteq \mathbf{Q}'$. By induction hypothesis, $\varphi(\mathbf{P}) \leq \varphi(\mathbf{Q}')$; $\varphi(\mathbf{Q}')$ is strictly less than $\varphi(\mathbf{P}') = \varphi(\mathbf{A}'_1) \oplus \varphi(\mathbf{Q}')$, so $\varphi(\mathbf{P}) < \varphi(\mathbf{P}')$. Note that the inequality is strict because $\varphi(\mathbf{A}'_1)$ cannot be equal to 0.

Let us now assume that $\mathbf{A}_1 \subseteq \mathbf{A}'_1$.

If $\mathbf{A}_1 = C^?$ and $\mathbf{A}'_1 = C'^?$, then only clause (2) can have been used to derive $\mathbf{P} \subseteq \mathbf{P}'$. In this case, $\mathbf{Q} \subseteq \mathbf{Q}'$, so $\varphi(\mathbf{Q}) \leq \varphi(\mathbf{Q}')$ by induction hypothesis. Additionally, $C \subseteq C'$, so $\varphi(\mathbf{A}_1) = 1 + \text{rk}_{\mathcal{S}X}(C) \leq 1 + \text{rk}_{\mathcal{S}X}(C') = \varphi(\mathbf{A}'_1)$. It follows that $\varphi(\mathbf{P}) = \varphi(\mathbf{A}_1) \oplus \varphi(\mathbf{Q}) \leq \varphi(\mathbf{A}'_1) \oplus \varphi(\mathbf{Q}') = \varphi(\mathbf{P}')$. If additionally $\mathbf{P} \neq \mathbf{P}'$, then one of the inclusions $C \subseteq C'$ and $\mathbf{Q} \subseteq \mathbf{Q}'$ must be strict. In the first case, $1 + \text{rk}_{\mathcal{S}X}(C) < 1 + \text{rk}_{\mathcal{S}X}(C')$, and in the second case, $\varphi(\mathbf{Q}) < \varphi(\mathbf{Q}')$ by induction hypothesis, so that in both cases $\varphi(\mathbf{P}) < \varphi(\mathbf{P}')$.

It remains to deal with the case where $\mathbf{A}_1 \subseteq \mathbf{A}'_1$ and \mathbf{A}'_1 is of the form F'^* for some $F' \in \mathcal{H}_0X$. Since \mathbf{P} is reduced, clause (4) does not apply, so only clause (3) applies. We apply clause (3) repeatedly until we can no longer. This allows us to write \mathbf{P} as $\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_k\mathbf{R}$, for some word-product \mathbf{R} , where $k \geq 1$ is largest so that $\mathbf{A}_1, \dots, \mathbf{A}_k \subseteq F'^*$, and $\mathbf{R} \subseteq \mathbf{P}'$. By the maximality of k , the inequality $\mathbf{R} \subseteq \mathbf{P}'$ cannot be obtained by using clause (3), and clauses (2) and (4) do not apply; so it must have been obtained by using clause (1), or because \mathbf{R} is equal to ϵ , to the effect that $\mathbf{R} \subseteq \mathbf{Q}'$.

If some \mathbf{A}_i ($1 \leq i \leq k$) is equal to F'^* , then the fact that \mathbf{P} is reduced implies that k cannot be larger than or equal to 2. Hence $k = 1$. By induction hypothesis, $\varphi(\mathbf{R}) \leq \varphi(\mathbf{Q}')$, and therefore $\varphi(\mathbf{P}) = \varphi(F'^*) \oplus \varphi(\mathbf{R}) \leq \varphi(F'^*) \oplus \varphi(\mathbf{Q}') = \varphi(\mathbf{P}')$. Additionally, if $\mathbf{P} \neq \mathbf{P}'$, then \mathbf{R} must be different from \mathbf{Q}' , so $\varphi(\mathbf{R}) < \varphi(\mathbf{Q}')$, and therefore $\varphi(\mathbf{P}) = \varphi(F'^*) \oplus \varphi(\mathbf{R}) < \varphi(F'^*) \oplus \varphi(\mathbf{Q}') = \varphi(\mathbf{P}')$.

In the remaining case, every \mathbf{A}_i is different from F'^* . We claim that $\varphi(\mathbf{A}_i) < \varphi(F'^*)$. This follows from Lemma 12.6 (1) if \mathbf{A}_i is of the form $C^?$, and from Lemma 12.6 (2) if \mathbf{A}_i is of the form F^* . Since $\varphi(F'^*) = \omega^{\|F'\|^\circ}$ is \oplus -indecomposable, $\varphi(\mathbf{A}_1) \oplus \cdots \oplus \varphi(\mathbf{A}_k) < \varphi(F'^*)$, and therefore $\varphi(\mathbf{P}) = \varphi(\mathbf{A}_1) \oplus \cdots \oplus \varphi(\mathbf{A}_k) + \varphi(\mathbf{R}) < \varphi(F'^*) \oplus \varphi(\mathbf{Q}') = \varphi(\mathbf{P}')$. \square

Proposition 12.8. *For every Noetherian space X , $\text{sob } X^* \leq \omega^{\|X\|^\circ} + 1$, or equivalently, $\text{rsob } X^* \leq \omega^{\|X\|^\circ}$.*

PROOF. As a corollary of Lemma 12.7, given any two word-products \mathbf{P} and \mathbf{P}' such that $\mathbf{P} = \mathbf{P}'$, we have $\varphi(\mathbf{P}) = \varphi(\mathbf{P}')$. Hence φ defines a strictly monotonic map from irreducible closed subsets \mathbf{P} of X^* (not just word-products \mathbf{P}) to the class of ordinals. Its largest value is $\varphi(X^*)$, since X^* itself is a word-product, so the image of φ lies entirely inside the ordinal $\varphi(X^*) + 1 = \omega^{\|X\|^\circ} + 1$. \square

We turn to the matching lower bound on $\text{sob } X^*$. The specialization preordering \leq of a space X induces an equivalence relation \equiv by $x \equiv y$ if and only if $x \leq y$ and $y \leq x$. This allows us to partition X into equivalence classes. If X is T_0 , then those equivalence classes contain exactly one point, and the following lemma would simply say that F and F' differ by exactly one point.

Lemma 12.9. *Let F, F' be two closed subsets of a Noetherian space X , with $F \subseteq F'$ and $\|F'\| = \|F\| + 1$. Then $F' \setminus F$ is a single equivalence class with respect to the specialization preordering of X . Given any point x in that class, $F' = F \cup \downarrow x$.*

PROOF. Since $F \subseteq F'$ and $\|F'\| \neq \|F\|$, there is a point in $F' \setminus F$. Let us imagine that $F' \setminus F$ contains at least two non-equivalent points x and y . Without loss of generality, we may assume that $y \not\leq x$. Then F is strictly included in the closed set $F \cup \downarrow x$, which is strictly included in $F \cup \downarrow x \cup \downarrow y \subseteq F'$. It follows that $\|F'\| \geq \|F \cup \downarrow x \cup \downarrow y\| \geq \|F \cup \downarrow x\| + 1 \geq \|F\| + 2$, which is impossible. Finally, given any $x \in F' \setminus F$, the points of F' consist of those of F , plus those that are equivalent to x , hence are in $\downarrow x$. Conversely, x is in F' , so $\downarrow x$ is included in F' , and therefore $F \cup \downarrow x \subseteq F'$. \square

In the situation of Lemma 12.9, F' is obtained by adding an irreducible closed set C to F , namely $C \stackrel{\text{def}}{=} \downarrow x$. Using such sets F and C , we build word-products of the form $(F^*C^?)^m \mathbf{P}$, where $m \in \mathbb{N}$ and \mathbf{P} is a word-product built from subsets of F (equivalently, such that $\mathbf{P} \subseteq F^*$).

Lemma 12.10. *Let F be a closed subset of a Noetherian space X , C be an irreducible closed subset of X that is not included in F . Let also \mathbf{P} and \mathbf{Q} be two word-products such that $\mathbf{P}, \mathbf{Q} \subseteq F^*$. For all $m, n \in \mathbb{N}$, $(F^*C^?)^m \mathbf{P} \subseteq (F^*C^?)^n \mathbf{Q}$ if and only if (m, \mathbf{P}) is lexicographically smaller than or equal to (n, \mathbf{Q}) , namely if and only if $m < n$, or $m = n$ and $\mathbf{P} \subseteq \mathbf{Q}$.*

PROOF. Let us assume $(F^*C^?)^m \mathbf{P} \subseteq (F^*C^?)^n \mathbf{Q}$. We pick an element x from $C \setminus F$. The word x^m consisting of m copies of x is in $(F^*C^?)^m \mathbf{P}$ (noting that the empty word is in the semantics of every word-product, in particular in \mathbf{P}), hence in $(F^*C^?)^n \mathbf{Q}$. Since $\mathbf{Q} \subseteq F^*$ and $x \notin F$, x^m must be in $(F^*C^?)^n$, and that in turn implies that $n \geq m$. If $m < n$, (m, \mathbf{P}) is lexicographically smaller than (n, \mathbf{Q}) , so we are left to show that $(F^*C^?)^n \mathbf{P} \subseteq (F^*C^?)^n \mathbf{Q}$ implies $\mathbf{P} \subseteq \mathbf{Q}$, for every $n \in \mathbb{N}$. This is by induction on n . This is clear if $n = 0$. Otherwise, only clause (3) or (4) of Lemma 12.1 can have led to this inclusion, and this can only occur if $C^?(F^*C^?)^{n-1} \mathbf{P} \subseteq (F^*C^?)^n \mathbf{Q}$. Since $C^? \not\subseteq F^*$, only clause (1) can have

led to the latter, so $C^?(F^*C^?)^{n-1}\mathbf{P} \subseteq C^?(F^*C^?)^{n-1}\mathbf{Q}$. This can only have been obtained through clause (2), from the inclusion $(F^*C^?)^{n-1}\mathbf{P} \subseteq (F^*C^?)^{n-1}\mathbf{Q}$, and therefore $\mathbf{P} \subseteq \mathbf{Q}$ by the induction hypothesis.

In the converse direction, we first claim that $m < n$ implies $(F^*C^?)^m\mathbf{P} \subseteq (F^*C^?)^n\mathbf{Q}$. Indeed, the words of $(F^*C^?)^m\mathbf{P}$ are the words w_1w_2 where $w_1 \in (F^*C^?)^m$ and $w_2 \in \mathbf{P}$. Every such w_2 is in F^* . This shows that $(F^*C^?)^m\mathbf{P} \subseteq (F^*C^?)^mF^*$. Every word w in $(F^*C^?)^mF^*$ is also in $(F^*C^?)^n\mathbf{Q}$, as the concatenation of $w \in (F^*C^?)^mF^*$ and of $\epsilon \in C^?(F^*C^?)^{n-m-1}\mathbf{Q}$.

It remains to show that if $m = n$ and $\mathbf{P} \subseteq \mathbf{Q}$, then $(F^*C^?)^m\mathbf{P} \subseteq (F^*C^?)^n\mathbf{Q}$, and that is obvious. \square

We will also need the following simpler construction.

Lemma 12.11. *Let X be a Noetherian space. For all $C, C', C'' \in \mathcal{S}X$ such that $C', C'' \subseteq C$, for all $m, n \in \mathbb{N}$, $(C^?)^m C'^? \subseteq (C^?)^n C''?$ if and only if (m, C') is lexicographically smaller than or equal to (n, C'') .*

PROOF. Let us assume $(C^?)^m C'^? \subseteq (C^?)^n C''?$. Let x be any point in C' , hence also in C . Then x^{m+1} is in the left-hand side, hence also in the right-hand side, and this implies that $n \geq m$. If additionally $n = m$, then $(C^?)^n C'^? \subseteq (C^?)^n C''?$ can only be derived by n applications of clause (2) of Lemma 12.1, implying that $C'^? \subseteq C''?$, namely $C' \subseteq C''$.

Conversely, if $m < n$, then we have that $(C^?)^m C'^? \subseteq (C^?)^n C''?$, since every word in $(C^?)^m C'^?$ consists of at most $m+1$ letters, all from C (the last one possibly being in C' , hence in C). If $m = n$ and $C' \subseteq C''$, then $(C^?)^m C'^? \subseteq (C^?)^n C''?$, too, as one sees easily. \square

We will use the latter lemmata in conjunction with the well-known fact that, given two well-founded posets P and Q , and points $p \in P$ and $q \in Q$,

$$\mathrm{rk}_{P \times_{\mathrm{lex}} Q}(p, q) = |Q| \times \mathrm{rk}_P(p) + \mathrm{rk}_Q(q),$$

where $P \times_{\mathrm{lex}} Q$ denotes the lexicographic product of P and Q . As a consequence, $|P \times_{\mathrm{lex}} Q| = |Q| \times |P|$. It also follows that every well-founded poset R that admits a strictly monotonic map from $P \times_{\mathrm{lex}} Q$ to R must have rank at least $|Q| \times |P|$.

Proposition 12.12. *For every non-empty Noetherian space X , we have that $\mathrm{sob} X^* \geq \omega^{\|X\|^\circ} + 1$.*

PROOF. By well-founded induction on non-empty closed subsets F of X , we show that $\mathrm{rk}_{\mathcal{S}(X^*)}(F^*) \geq \omega^{\|F\|^\circ}$. The claim will follow since $\mathrm{sob} X^* = \mathrm{rk}_{\mathcal{S}(X^*)}(X^*) + 1$, as X^* is the largest element of $\mathcal{S}(X^*)$, and using Lemma 4.2.

Since F is non-empty, $\|F\| \geq \|\emptyset\| + 1 = 1$.

If $\|F\| = 1$, then using Lemma 12.9 on the strict inclusion $\emptyset \subsetneq F$, F itself is an irreducible closed subset of the form $\downarrow x$, and this contains no proper closed subset except the empty set. The elements of $\mathcal{S}(X^*)$ included in F^* are $(F^?)^n$, $n \in \mathbb{N}$, plus F^* ; the rank of $(F^?)^n$ is n , so $\text{rk}_{\mathcal{S}(X^*)}(F^*) = \omega$.

If $\|F\|$ is a successor ordinal $\alpha + 1$ different from 1, then by the inductive definition of rank (and Lemma 5.8), there is a closed subset F' of F such that $\|F'\| = \alpha$. By induction hypothesis, $\text{rk}_{\mathcal{S}(X^*)}(F'^*) \geq \omega^{\alpha^\circ}$. Hence there are irreducible closed subsets \mathbf{P} of F'^* of arbitrary rank between 0 and ω^{α° in $\mathcal{S}(X^*)$. By Lemma 12.9, F is equal to $F' \cup C$, where C is an irreducible closed subset of the form $\downarrow x$ with $x \notin F'$. Using Lemma 12.10, there is a strictly monotonic map from $\mathbb{N} \times_{\text{lex}} Q$ into the poset of proper irreducible closed subsets of F^* , where Q is the poset of irreducible closed subsets of F'^* of rank strictly less than ω^{α° , through $(n, \mathbf{P}) \mapsto (F'^* C^?)^n \mathbf{P}$. It follows that $\text{rk}_{\mathcal{S}(X^*)}(F^*) \geq \omega^{\alpha^\circ} \times \omega = \omega^{\alpha^\circ + 1}$. Now $\alpha^\circ + 1 = (\alpha + 1)^\circ$, in both cases of the definition of α° .

If $\|F\|$ is a limit ordinal α , then there are closed subsets F' of F of arbitrarily high rank $\beta < \alpha$, and by induction $\text{rk}_{\mathcal{S}(X^*)}(F^*) \geq \text{rk}_{\mathcal{S}(X^*)}(F'^*) \geq \omega^{\beta^\circ}$. When α is not critical (and since α is a limit ordinal), we realize that $\alpha^\circ = \alpha$, and that the latter inequality implies $\text{rk}_{\mathcal{S}(X^*)}(F^*) \geq \omega^\beta$ for every $\beta < \alpha$, and therefore $\text{rk}_{\mathcal{S}(X^*)}(F^*) \geq \omega^\alpha = \omega^{\alpha^\circ}$, by taking suprema over $\beta < \alpha$.

When α is a critical ordinal, we require another argument. We first observe that $\text{sob } F \geq \alpha$. Indeed, if $\text{sob } F < \alpha$, then using Proposition 4.5 (2) and the fact that α is critical, $\|F\| + 1 \leq \omega^\alpha = \alpha$, which is impossible since $\|F\| = \alpha$. Using Lemma 4.2, there is an irreducible closed subset C of F such that $\text{rk}_{\mathcal{S}X}(C) + 1 \geq \alpha$. As a consequence, $\text{rk}_{\mathcal{S}X}(C) \geq \alpha$.

Using Lemma 12.11, there is a strictly monotonic map from $\mathbb{N} \times_{\text{lex}} Q$ into the poset of proper irreducible closed subsets of F^* , where Q is the poset of irreducible closed subsets of C of rank strictly less than α , through $(n, C') \mapsto (C^?)^n C'^?$. Therefore $\text{rk}_{\mathcal{S}(X^*)}(F^*) \geq \alpha \times \omega$. Now $\alpha \times \omega = \omega^\alpha \times \omega = \omega^{\alpha+1} = \omega^{\alpha^\circ}$. \square

We finally combine Proposition 12.8 and Proposition 12.12, and we include the case of the empty Noetherian space in the following theorem.

Theorem 12.13. *For every Noetherian space X , $\text{sob } X^* = \omega^{\|X\|^\circ} + 1$ (equivalently, $\text{rsob } X^* = \omega^{\|X\|^\circ}$) if X is non-empty, 1 otherwise.*

A special case of this result appears as Proposition 5.5 of [BFG20], where X was assumed to be non-empty and finite.

12.3. The stature of X^* . We start with an easy upper bound.

Lemma 12.14. *For every non-empty Noetherian space X , $\|X^*\| \leq \omega^{\omega^{\alpha^\circ}}$, where $\alpha \stackrel{\text{def}}{=} \|X\|$.*

PROOF. X^* is irreducible closed, so the space X^* has exactly one component. By Proposition 4.5 (3), $\|X^*\| \leq \omega^{\text{rsob } X^*}$, and $\text{rsob } X^* = \omega^{\alpha^\circ}$ by Theorem 12.13. \square

One can improve upon that upper bound when $\|X\|$ is finite. We need the following simple observation first.

Lemma 12.15. *For every Noetherian space X such that $\alpha \stackrel{\text{def}}{=} \|X\|$ is finite, X has only finitely many open subsets. Additionally, X has exactly α equivalence classes with respect to its specialization preordering.*

In particular, every T_0 , finite Noetherian space X contains exactly $\|X\|$ points. This is a form of converse to Lemma 6.1.

PROOF. We prove the second claim by induction on α . If $\alpha = 0$, then X is empty, and the claim is clear. Otherwise, there is a closed subset F of X such that $\text{rk}_{\mathcal{H}_0 X}(F) = \alpha - 1$, while $\text{rk}_{\mathcal{H}_0 X}(X) = \alpha$ by definition. By Lemma 12.9 with $F' \stackrel{\text{def}}{=} X$, there is a point x such that $X = F \cup \downarrow x$, and $X \setminus F$ is a single equivalence class with respect to the specialization preordering \leq of X . By Lemma 5.8, $\|F\| = \alpha - 1$, so the subspace F has exactly $\alpha - 1$ equivalence classes with respect to $\leq|_F$, hence X has exactly α equivalence classes.

The first claim follows, since every open subset of X is upwards-closed with respect to \leq , hence is closed under the associated equivalence relation, and therefore is a union of equivalence classes. \square

Lemma 12.16. *For every Noetherian space X such that $\alpha \stackrel{\text{def}}{=} \|X\|$ is finite, we have:*

- (1) if $\alpha = 0$, then $\|X^*\| = 1$;
- (2) if $\alpha = 1$, then $\|X^*\| = \omega$;
- (3) if $\alpha \neq 0$, then $\|X^*\| \leq \omega^{\omega^{\alpha-1}}$.

PROOF. (1) If $\alpha = 0$, then X is empty, so $X^* = \{\epsilon\}$, and therefore $\|X^*\| = 1$.

(2) When $\alpha = 1$, all the proper closed subsets of X have rank equal to 0, hence are empty. In other words, the topology of X is the indiscrete topology, whose sole closed sets are X and the empty set. Then X is the sole element of SX . The only irreducible closed subsets of X^* are $(X^?)^n$, $n \in \mathbb{N}$, plus X^* . They form a chain, so any non-empty finite union of word-products reduces to a single word-product. This entails that $\mathcal{H}_0(X^*)$ is the same chain, with the empty set added as a new bottom element. In particular, $\|X^*\| = \omega$.

(3) We prove this by induction on $\alpha \geq 1$. Item (2) is the base case. Let $\alpha \geq 2$. By Lemma 12.15, X has only finitely many closed subsets. We enumerate its *proper* closed subsets as F_1, \dots, F_m , and its components as C_1, \dots, C_p . For every i with $1 \leq i \leq m$, $\|F_i\| = \text{rk}_{\mathcal{H}_0 X}(F_i) < \text{rk}_{\mathcal{H}_0 X}(X) = \|X\| = \alpha$ (using Lemma 5.8), so $\|F_i^*\| \leq \omega^{\alpha-2}$ for every i , by induction hypothesis.

We build word-products P_n , $n \in \mathbb{N}$, by induction on n , by letting $P_0 \stackrel{\text{def}}{=} \epsilon$ and $P_{n+1} \stackrel{\text{def}}{=} F_1^* \cdots F_m^* C_1^? \cdots C_p^? P_n$. The point of this construction is that, for every word-product P such that $P \neq X^*$, the inclusion $P \subseteq P_n$ holds for n large enough, namely for every n larger than or equal to the length $|P|$ of P . This is shown by induction on $|P|$. If $P = \epsilon$, then $P \subseteq P_n$ for every n , by Lemma 12.1 (or rule (0)). If P is of the form $A_1 Q$, then $|P| = 1 + |Q|$. Let us fix $n \geq |P|$. We observe that $Q \neq X^*$, otherwise P would contain every word on X and therefore be equal to X^* . Hence we can apply the induction hypothesis, so that $Q \subseteq P_{n-1}$. We claim that $P \subseteq P_n$. If A_1 is of the form $C^?$, then the words of P are those of Q , which are in P_{n-1} , hence also in P_n , plus those of the form yw with $y \in C$ and $w \in Q \subseteq P_{n-1}$; then C is included in some C_j , so yw is in P_n . If A_1 is of the form F^* , then F cannot be equal to the whole of X , since in that case P would contain every word on X and therefore be equal to X^* . Hence $F = F_i$ for some i , and then the inclusion $P = F_i^* Q \subseteq P_n = X^? F_1^* \cdots F_m^* P_{n-1}$ is immediate.

In particular, every proper closed subset A of X^* is included in P_n for n large enough. This follows from the previous claim by writing A as a finite union of word-products, all different from X^* . If the first one is included in P_{n_1} , the second one is included in P_{n_2}, \dots , and the last one is included in P_{n_k} , then their union A is included in P_n , where $n \stackrel{\text{def}}{=} \max(n_1, n_2, \dots, n_k)$.

We now claim that $\|P_n\| < \omega^{\alpha-1}$ for every $n \in \mathbb{N}$. We prove this by induction on n . The base case reduces to $\|P_0\| = 1 < \omega^{\alpha-1}$. In the inductive case, we know that $\|P_n\| < \omega^{\alpha-1}$, and we aim to prove that $\|P_{n+1}\| < \omega^{\alpha-1}$. The function $j: X_\perp \rightarrow X^*$ that maps every $x \in X$ to the one-letter word x , and \perp to ϵ is continuous: for all open subsets U_1, \dots, U_k of X , $j^{-1}(X^* U_1 X^* \cdots X^* U_k X^*)$ is equal to U_1 if $k = 1$, to X_\perp if $k = 0$, and is empty if $k \geq 2$; in any case, this is open. Let $f: F_1^* \times \cdots \times F_m^* \times X_\perp^p \times P_n \rightarrow P_{n+1}$ map $(w_1, \dots, w_m, x_1, \dots, x_p, w)$ to $w_1 \cdots w_m j(x_1) \cdots j(x_p) w$. Using the fact that the concatenation map from $X^* \times X^*$ to X^* is continuous [FGL20, Lemma B.1], f is continuous. It is also clearly surjective, hence Skula dense by Lemma 5.1 (2). By Lemma 5.3, $\|P_{n+1}\|$ is less than or equal to $\|F_1^* \times \cdots \times F_m^* \times X_\perp^p \times P_n\|$. By Theorem 10.9, the latter is equal to the natural product of $\|F_1^*\|, \dots, \|F_m^*\|, \|X_\perp\|$ p times, and $\|P_n\|$. We have $\|X_\perp\| = 1 + \|X\| = 1 + \alpha$ by Proposition 9.6, we recall that

$\|F_i^*\| \leq \omega^{\omega^{\alpha-2}}$ for every i , and that $\|\mathbf{P}_n\| < \omega^{\omega^{\alpha-1}}$. In particular, all the terms in the natural product are strictly smaller than $\omega^{\omega^{\alpha-1}}$. (For the first one, we use the fact that $\omega^{\omega^{\alpha-1}} \geq \omega$, while $1 + \alpha$ is finite.) Since $\omega^{\omega^{\alpha-1}}$ is \otimes -indecomposable, $\|\mathbf{P}_{n+1}\| < \omega^{\omega^{\alpha-1}}$.

We put everything together. The family $(\mathbf{P}_n)_{n \in \mathbb{N}}$ is a cofinal family of closed subsets of X^* . By Proposition 5.9, $\|X^*\| \leq \sup_{n \in \mathbb{N}} (\|\mathbf{P}_n\| + 1) \leq \omega^{\omega^{\alpha-1}}$. \square

In the search for a lower bound of $\|X^*\|$, we will need the following trick. Given a well-founded poset (P, \leq) , let $<$ be the strict part of \leq , and let us call *step* of P any pair (p, p^+) , where $p, p^+ \in P$ and $p < p^+$. A step (p, p^+) is an *increment* if and only $\text{rk}_P(p^+) = \text{rk}_P(p) + 1$. We write $\text{Step}(P)$ for the set of steps of P , and $\text{Inc}(P)$ for the subset of increments of P . Both steps and increments are ordered strictly by $(p, p^+) < (q, q^+)$ if and only if $p^+ \leq q$. We reuse the same notation $<$; no confusion should arise. We also write $(p, p^+) \leq (q, q^+)$ if and only if $(p, p^+) < (q, q^+)$ or $(p, p^+) = (q, q^+)$.

Proposition 12.17. *For every well-founded poset P , $|\text{Step}(P)| = |\text{Inc}(P)| = |P| - 1$.*

We recall that, when α is not a successor ordinal, $\alpha - 1 = \alpha$ by convention.

PROOF. When $|P| = 0$, the sets P , $\text{Step}(P)$ and $\text{Inc}(P)$ are empty, so the claim is clear. We now assume that $|P|$ is either a successor or a limit ordinal. We note that $|\text{Inc}(P)| \leq |\text{Step}(P)|$, since $\text{Inc}(P) \subseteq \text{Step}(P)$.

The strictly monotonic map $(p, p^+) \mapsto \text{rk}_P(p)$ shows that $|\text{Step}(P)| \leq |P|$. When $|P|$ is a successor ordinal $\alpha + 1$, this map takes its values in α , since for every $(p, p^+) \in \text{Step}(P)$, $\text{rk}_P(p) < \text{rk}_P(p^+) < \alpha + 1$; in that case, we can improve the former inequality to $|\text{Step}(P)| \leq \alpha$, namely $|\text{Step}(P)| \leq |P| - 1$.

With an eye toward showing the converse, we prove that for every ordinal α ,

- (i) for every $q^+ \in P^\top$ such that $\text{rk}_{P^\top}(q^+) = \alpha + 1$, there is an element $q < q^+$ such that $\text{rk}_{P^\top}(q) = \alpha$ and $\text{rk}_{\text{Inc}(P^\top)}(q, q^+) \geq \alpha$;
- (ii) for every $q \in P^\top$ such that $\text{rk}_{P^\top}(q) = \alpha$ and α is a limit ordinal, $\sup\{\text{rk}_{\text{Inc}(P^\top)}(p, p^+) \mid (p, p^+) \in \text{Inc}(P^\top), p^+ < q\} \geq \alpha$.

This is by mutual induction on α . We start with (i). In this case, there is an element $q < q^+$ such that $\text{rk}_{P^\top}(q) = \alpha$, and therefore (q, q^+) is a increment of P^\top . If α is itself a successor ordinal $\beta + 1$, then there is an element $p < q$ such that $\text{rk}_{P^\top}(p) = \beta$ and $\text{rk}_{\text{Inc}(P^\top)}(p, q) \geq \beta$, by induction hypothesis. We note that $(p, q) < (q, q^+)$, so $\text{rk}_{\text{Inc}(P^\top)}(q, q^+) \geq \beta + 1 = \alpha$. If $\alpha = 0$, then the claim that $\text{rk}_{\text{Inc}(P^\top)}(q, q^+) \geq \alpha$ is obvious. If α is a limit ordinal, then for every $\beta < \alpha$, there

is an increment (p, p^+) such that $p^+ < q$ such that $\text{rk}_{\text{Inc}(P^\top)}(p, p^+) \geq \beta$ by the induction hypothesis, part (ii). Since $(p, p^+) < (q, q^+)$, $\text{rk}_{\text{Inc}(P^\top)}(q, q^+) \geq \beta + 1$. Taking suprema over $\beta < \alpha$, $\text{rk}_{\text{Inc}(P^\top)}(q, q^+) \geq \alpha$. We turn to (ii). For every ordinal $\beta < \alpha$, $\beta + 1$ is still strictly smaller than α . Then, there is an element $p^+ < q$ such that $\text{rk}_{P^\top}(p^+) = \beta + 1$. By induction hypothesis, part (i), there is an element $p < p^+$ such that $\text{rk}_{P^\top}(p) = \beta$ and $\text{rk}_{\text{Inc}(P^\top)}(p, p^+) \geq \beta$. Hence $\sup\{\text{rk}_{\text{Inc}(P^\top)}(p, p^+) \mid (p, p^+) \in \text{Inc}(P^\top), p^+ < q\} \geq \sup_{\beta < \alpha} \beta = \alpha$.

We now fix $\alpha \stackrel{\text{def}}{=} \text{rk}_{P^\top}(\top)$, namely $\alpha \stackrel{\text{def}}{=} |P|$.

If α is a successor ordinal, by (i) there is an increment (q, \top) such that $\text{rk}_{\text{Inc}(P^\top)}(q, \top) \geq \alpha - 1$. Therefore $|\text{Inc}(P^\top)| \geq \alpha$. The strictly monotonic map which maps every element (p, p^+) of $\text{Inc}(P)$ to $\text{rk}_{\text{Inc}(P)}(p, p^+)$ and the remaining elements (p, \top) of $\text{Inc}(P^\top)$ to $|\text{Inc}(P)|$ shows that $|\text{Inc}(P^\top)| \leq |\text{Inc}(P)| + 1$, so $\alpha \leq |\text{Inc}(P)| + 1$. Therefore $|\text{Inc}(P)| \geq |P| - 1$.

If α is a limit ordinal, by (ii) the supremum of the ranks $\text{rk}_{\text{Inc}(P^\top)}(p, p^+)$ over all elements (p, p^+) of $\text{Inc}(P^\top)$ such that $p^+ < \top$, namely over all elements of $\text{Inc}(P)$, is at least α . The same therefore holds of the supremum of the ordinals $\text{rk}_{\text{Inc}(P^\top)}(p, p^+) + 1$, so $|\text{Inc}(P)| \geq \alpha = |P| = |P| - 1$. \square

We will only require steps here, but increments will be needed in our study of multisets, in Section 14.

In the following, we will build closed subsets of X^* by concatenation. For this, we need to observe that, for any two closed subsets \mathbf{A}, \mathbf{B} of X^* , where X is Noetherian, \mathbf{AB} is also closed. In order to see this, we write \mathbf{A} as a finite union $\bigcup_{i=1}^m \mathbf{P}_i$ of word-products, and similarly \mathbf{B} as a finite union $\bigcup_{j=1}^n \mathbf{Q}_j$ of word-products. We can do this because the closed sets are exactly the semantics of word-SREs, namely the finite unions of (semantics of) word-products. Then we note that $\mathbf{AB} = \bigcup_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \mathbf{P}_i \mathbf{Q}_j$, a finite union of word-products. This rests on the easily checked fact that concatenation distributes over union.

Our main gadget consists of closed sets of the form $(F^*C^?)^{n+1}\mathbf{B} \cup \mathbf{AC}^?\mathbf{B}^+ \cup (F^*C^?)^n F^*$, where F and C are as in Lemma 12.10, \mathbf{A} is a closed subset of X^* and $(\mathbf{B}, \mathbf{B}^+)$ is a step of $\mathcal{H}_0(X^*)$. We will need to compare them with respect to inclusion, and this will boil down to comparing pairs of sets of one of the two forms $(F^*C^?)^{n+1}\mathbf{B}$ or $\mathbf{AC}^?\mathbf{B}^+$. This leads us to examine four different cases, which are explored in the following lemma.

Lemma 12.18. *Let F be a closed subset of a Noetherian space X , C be an irreducible closed subset of X that is not included in F , and $n \in \mathbb{N}$. For all closed subsets $\mathbf{A}, \mathbf{B}, \mathbf{B}', \mathbf{P}, \mathbf{Q}$ of X^* :*

- (1) $(F^*C^?)^n \mathbf{B}' \subseteq (F^*C^?)^n \mathbf{B}$ if and only if $\mathbf{B}' \subseteq \mathbf{B}$;
- (2) if $(F^*C^?)^{n+1} \mathbf{Q} \subseteq \mathbf{AC}^? \mathbf{B}'$, $\mathbf{Q} \neq \emptyset$, and $\mathbf{B}' \subseteq F^*$, then $(F^*C^?)^n F^* \subseteq \mathbf{A}$;
- (3) if \mathbf{P} is non-empty, the conditions $\mathbf{PC}^? \mathbf{B}' \subseteq (F^*C^?)^{n+1} \mathbf{B}$, $\mathbf{B} \subseteq F^*$, and [if $n \geq 1$ then $\mathbf{P} \not\subseteq (F^*C^?)^{n-1} F^*$], entail $\mathbf{B}' \subseteq \mathbf{B}$;
- (4) if $\mathbf{PC}^? \mathbf{B}' \subseteq \mathbf{AC}^? \mathbf{B}'$, $\mathbf{B}' \neq \emptyset$, and $\mathbf{B}' \subseteq F^*$, then $\mathbf{P} \subseteq \mathbf{A}$.

PROOF. Let $x \in C \setminus F$.

(1) The if direction is obvious. Conversely, let us assume that $(F^*C^?)^n \mathbf{B}' \subseteq (F^*C^?)^n \mathbf{B}$. For every $w \in \mathbf{B}'$, $x^n w$ is in $(F^*C^?)^n \mathbf{B}'$, hence in $(F^*C^?)^n \mathbf{B}$. Since x belongs to C but not to F , $x^m w$ must be in \mathbf{B} for some $m \leq n$. Now $w \leq^* x^m w$, and we recall that \leq^* is the specialization ordering of X^* . Every closed set is downwards-closed with respect to the specialization ordering, so w is in \mathbf{B} .

(2) For every $w \in (F^*C^?)^n F^*$, we can write w as $w_0 c_1 w_1 \cdots w_{n-1} c_n w_n$, where each w_i is in F^* and each c_i is either in C or is the empty word. For each i , let c'_i be x if $c_i = \epsilon$, c_i otherwise. Let us form the word $w' \stackrel{\text{def}}{=} w_0 c'_1 w_1 \cdots w_{n-1} c'_n w_n$. Since \mathbf{Q} is non-empty and (downwards-)closed, ϵ is in \mathbf{Q} . Therefore $w' x \epsilon = w' x$ is in $(F^*C^?)^{n+1} \mathbf{Q}$. The assumption implies that it is in $\mathbf{AC}^? \mathbf{B}'$. Since $\mathbf{B}' \subseteq F^*$, and therefore the final x of $w' x$ can only be in $C^?$, not in \mathbf{B}' , w' is in \mathbf{A} or $w' x$ is in \mathbf{A} . Now $w \leq^* w'$ and $w \leq^* w' x$, and \mathbf{A} is (downwards-)closed, so w is in \mathbf{A} .

(3) Under the given assumptions, there is a word w in \mathbf{P} , and if $n \geq 1$, we may assume that $w \notin (F^*C^?)^{n-1} F^*$. We reason by contradiction and we assume that there is also a word w' in $\mathbf{B}' \setminus \mathbf{B}$. Then $w x w'$ is in $\mathbf{PC}^? \mathbf{B}'$, hence in $(F^*C^?)^{n+1} \mathbf{B}$. Since x is in $C \setminus F$, and since $\mathbf{B} \subseteq F^*$, w must be in $(F^*C^?)^i F^*$ and w' must be in $(F^*C^?)^{n-i} \mathbf{B}$ for some i , $0 \leq i \leq n$. If $n = i$, then the latter would imply $w' \in \mathbf{B}$, which is impossible. Therefore $n - i \geq 1$. In particular, $n \geq 1$, and $i \leq n - 1$. Since w is in $(F^*C^?)^i F^*$, and since $i \leq n - 1$, w is in the larger set $(F^*C^?)^{n-1} F^*$, which is impossible.

(4) Let us fix a word $w' \in \mathbf{B}'$. For every $w \in \mathbf{P}$, $w x w'$ is in $\mathbf{PC}^? \mathbf{B}'$, hence in $\mathbf{AC}^? \mathbf{B}'$. Therefore we can write $w x w'$ as $w_1 w_2$ where $w_1 \in \mathbf{AC}^?$ and $w_2 \in \mathbf{B}'$. Since $\mathbf{B}' \subseteq F^*$, w_2 cannot contain x . Therefore $w x$ is a prefix of w_1 , in particular $w x \leq^* w_1$, so that $w x$ is in $\mathbf{AC}^?$. Then w is in \mathbf{A} or $w x$ is in \mathbf{A} , and in any case w is in \mathbf{A} . \square

Lemma 12.19. *Let F be a closed subset of a Noetherian space X , and C be an irreducible closed subset of X that is not included in F . Let $\mathbf{C}_{n+1} \stackrel{\text{def}}{=} (F^*C^?)^n F^*$ for every $n \in \mathbb{N}$, and $\mathbf{C}_0 \stackrel{\text{def}}{=} \emptyset$. Let also \mathbf{A}_n be the set of all closed subsets of X^* containing \mathbf{C}_n and strictly included in \mathbf{C}_{n+1} .*

For every $n \in \mathbb{N}$, the map $((\mathbf{B}, \mathbf{B}^+), \mathbf{A}) \mapsto (F^*C^?)^{n+1}\mathbf{B} \cup AC^?\mathbf{B}^+ \cup \mathbf{C}_{n+1}$ is a strictly monotonic map from $\text{Step}(\mathcal{H}_0(F^*)) \times_{\text{lex}} \mathcal{A}_n$ to \mathcal{A}_{n+1} .

PROOF. We first check that for all $(\mathbf{B}, \mathbf{B}^+) \in \text{Step}(\mathcal{H}_0(F^*))$ and $\mathbf{A} \in \mathcal{A}_n$, $(F^*C^?)^{n+1}\mathbf{B} \cup AC^?\mathbf{B}^+ \cup \mathbf{C}_{n+1}$ is in \mathcal{A}_{n+1} . That set is clearly closed and contains \mathbf{C}_{n+1} . Since $\mathbf{B}, \mathbf{B}^+ \subseteq F^*$ and $\mathbf{A} \subseteq \mathbf{C}_{n+1}$, it follows that $(F^*C^?)^{n+1}\mathbf{B} \cup AC^?\mathbf{B}^+ \cup \mathbf{C}_{n+1}$ is included in $(F^*C^?)^{n+1}F^* \cup \mathbf{C}_{n+1}C^?F^* \cup \mathbf{C}_{n+1} = \mathbf{C}_{n+2} \cup \mathbf{C}_{n+2} \cup \mathbf{C}_{n+1} = \mathbf{C}_{n+2}$. If it were the whole of \mathbf{C}_{n+2} , then $\mathbf{C}_{n+2} = (F^*C^?)^{n+1}F^*$ would be included in $(F^*C^?)^{n+1}\mathbf{B} \cup AC^?\mathbf{B}^+ \cup \mathbf{C}_{n+1}$, hence in $(F^*C^?)^{n+1}\mathbf{B}$ or in $AC^?\mathbf{B}^+$ or in \mathbf{C}_{n+1} , since \mathbf{C}_{n+2} is irreducible (a word-product). If $(F^*C^?)^{n+1}F^* \subseteq (F^*C^?)^{n+1}\mathbf{B}$, then $F^* \subseteq \mathbf{B}$ by Lemma 12.18 (1); that is impossible, since $\mathbf{B} \subsetneq \mathbf{B}^+ \subseteq F^*$. If $(F^*C^?)^{n+1}F^* \subseteq AC^?\mathbf{B}^+$, then $(F^*C^?)^nF^* \subseteq \mathbf{A}$ by Lemma 12.18 (2); that is impossible because \mathbf{A} is strictly included in \mathbf{C}_{n+1} . Finally, the inclusion $(F^*C^?)^{n+1}F^* \subseteq \mathbf{C}_{n+1} = (F^*C^?)^nF^*$ is also impossible, since, given any $x \in C \setminus F$, x^{n+1} is in the former but not in the latter.

Let $(\mathbf{B}, \mathbf{B}^+), (\mathbf{B}', \mathbf{B}'^+) \in \text{Step}(\mathcal{H}_0(F^*))$, $\mathbf{A}, \mathbf{A}' \in \mathcal{A}_n$, and let us assume that $((\mathbf{B}, \mathbf{B}^+), \mathbf{A})$ is lexicographically smaller than $((\mathbf{B}', \mathbf{B}'^+), \mathbf{A}')$. We verify that $(F^*C^?)^{n+1}\mathbf{B} \cup AC^?\mathbf{B}^+ \cup \mathbf{C}_{n+1}$ is a proper subset of $(F^*C^?)^{n+1}\mathbf{B}' \cup A'C^?\mathbf{B}'^+ \cup \mathbf{C}_{n+1}$.

If $(\mathbf{B}, \mathbf{B}^+) < (\mathbf{B}', \mathbf{B}'^+)$, namely if $\mathbf{B}^+ \subseteq \mathbf{B}'$, then both $(F^*C^?)^{n+1}\mathbf{B}$ and $AC^?\mathbf{B}^+$ are included in $(F^*C^?)^{n+1}\mathbf{B}'$. This is obvious for the first one, once we note that $\mathbf{B} \subseteq \mathbf{B}'$. For the second one, we use the inclusions $\mathbf{A} \subseteq \mathbf{C}_{n+1} = (F^*C^?)^nF^*$ and $\mathbf{B}^+ \subseteq \mathbf{B}'$ to deduce

$$AC^?\mathbf{B}^+ \subseteq (F^*C^?)^nF^*C^?\mathbf{B}' = (F^*C^?)^{n+1}\mathbf{B}'.$$

Hence $(F^*C^?)^{n+1}\mathbf{B} \cup AC^?\mathbf{B}^+ \cup \mathbf{C}_{n+1} \subseteq (F^*C^?)^{n+1}\mathbf{B}' \cup A'C^?\mathbf{B}'^+ \cup \mathbf{C}_{n+1}$. If that inclusion were an equality, then $(F^*C^?)^{n+1}\mathbf{B}'$ would be included in $(F^*C^?)^{n+1}\mathbf{B} \cup AC^?\mathbf{B}^+ \cup \mathbf{C}_{n+1}$. We write \mathbf{B}' as a finite union $\bigcup_{k=1}^p \mathbf{Q}_k$ of irreducible closed subsets of X^* . For every k , $(F^*C^?)^{n+1}\mathbf{Q}_k$ is included in $(F^*C^?)^{n+1}\mathbf{B}$ or in $AC^?\mathbf{B}^+$ or in \mathbf{C}_{n+1} , by irreducibility. If $(F^*C^?)^{n+1}\mathbf{Q}_k \subseteq AC^?\mathbf{B}^+$, then $(F^*C^?)^nF^* \subseteq \mathbf{A}$ by Lemma 12.18 (2); that is impossible since $\mathbf{A} \subsetneq \mathbf{C}_{n+1} = (F^*C^?)^nF^*$. The case $(F^*C^?)^{n+1}\mathbf{Q}_k \subseteq \mathbf{C}_{n+1} = (F^*C^?)^nF^*$ is also impossible since, given any $x \in C \setminus F$, x^{n+1} is in $(F^*C^?)^{n+1}\mathbf{Q}_k$ but not in $(F^*C^?)^nF^*$. Therefore, for every k , $(F^*C^?)^{n+1}\mathbf{Q}_k$ is included in $(F^*C^?)^{n+1}\mathbf{B}$; by Lemma 12.18 (1) once again, \mathbf{Q}_k is included in \mathbf{B} . Since that holds for every k , \mathbf{B}' is included in \mathbf{B} , which is impossible since $\mathbf{B} \subsetneq \mathbf{B}^+ \subseteq \mathbf{B}'$.

If $(\mathbf{B}, \mathbf{B}^+) = (\mathbf{B}', \mathbf{B}'^+)$ and $\mathbf{A} \subsetneq \mathbf{A}'$, we need to show that $(F^*C^?)^{n+1}\mathbf{B} \cup AC^?\mathbf{B}^+ \cup \mathbf{C}_{n+1}$ is a proper subset of $(F^*C^?)^{n+1}\mathbf{B} \cup A'C^?\mathbf{B}^+ \cup \mathbf{C}_{n+1}$. It is clearly a subset. If the two sets were equal, then $A'C^?\mathbf{B}^+$ would be included

in $(F^*C^?)^{n+1}\mathbf{B} \cup \mathbf{A}C^?\mathbf{B}^+ \cup \mathbf{C}_{n+1}$. Let us write \mathbf{A}' as a finite union of irreducible closed sets. Amongst the latter, let $\mathbf{P}_1, \dots, \mathbf{P}_m$ be those that are not included in \mathbf{C}_n . Since \mathbf{A}' contains \mathbf{C}_n , \mathbf{A}' is equal to $\mathbf{C}_n \cup \bigcup_{i=1}^m \mathbf{P}_i$. For each i , $\mathbf{P}_iC^?\mathbf{B}^+$ is included in $(F^*C^?)^{n+1}\mathbf{B}$ or in $\mathbf{A}C^?\mathbf{B}^+$ or in \mathbf{C}_{n+1} , by irreducibility. If $\mathbf{P}_iC^?\mathbf{B}^+ \subseteq (F^*C^?)^{n+1}\mathbf{B}$, then we recall that $\mathbf{B} \subseteq F^*$, that \mathbf{P}_i is not included in $\mathbf{C}_n = (F^*C^?)^{n-1}F^*$ (if $n \geq 1$), and that \mathbf{B}^+ is not included in \mathbf{B} ; but those statements are contradictory, by Lemma 12.18 (3). The assumption $\mathbf{P}_iC^?\mathbf{B}^+ \subseteq \mathbf{C}_{n+1} = (F^*C^?)^nF^*$ is contradictory, too. Indeed, given any $x \in C \setminus F$, that would imply that for every $w \in \mathbf{P}_i$, wx is in $(F^*C^?)^nF^*$, hence contains at most n occurrences of x ; then $n \geq 1$ and w contains at most $n-1$ occurrences of x , and since w is arbitrary, \mathbf{P}_i would be included in $(F^*C^?)^{n-1}F^* = \mathbf{C}_n$. Hence, for each i , only the last assumption remains, namely that $\mathbf{P}_iC^?\mathbf{B}^+$ is included in $\mathbf{A}C^?\mathbf{B}^+$. By Lemma 12.18 (4), which applies since \mathbf{B}^+ is non-empty, being a proper superset of \mathbf{B} , \mathbf{P}_i is included in \mathbf{A} . Since that holds for every i , and since \mathbf{C}_n is also included in \mathbf{A} , $\mathbf{A}' = \mathbf{C}_n \cup \bigcup_{i=1}^m \mathbf{P}_i$ is included in \mathbf{A} . This contradicts our assumption that $\mathbf{A} \subsetneq \mathbf{A}'$. \square

Corollary 12.20. *Let F be a closed subset of a Noetherian space X , and C be an irreducible closed subset of X that is not included in F . Let α be any ordinal such that $\|F^*\| \geq \omega^{\omega^\alpha}$. Then $\|(F \cup C)^*\| \geq \omega^{\omega^{\alpha+1}}$.*

PROOF. We have $|\text{Step}(\mathcal{H}_0(F^*))| = |\mathcal{H}_0(F^*)| - 1 = \|F^*\|$ by Proposition 12.17 and the definition of stature.

Using the notations of Lemma 12.19, $|\mathcal{A}_0| = |\mathcal{C}_1| = \|F^*\|$. Then by part (3) of the Lemma, $|\mathcal{A}_1| \geq |\mathcal{A}_0| \times |\text{Step}(\mathcal{H}_0(F^*))| = \|F^*\|^2$. By an easy induction on n , $|\mathcal{A}_n| \geq \|F^*\|^{n+1}$. Since $\|F^*\| \geq \omega^{\omega^\alpha}$, $|\mathcal{A}_n| \geq \omega^{\omega^\alpha \times (n+1)}$.

Now every element of \mathcal{A}_n is clearly included in $(F \cup C)^*$. It follows that $\|(F \cup C)^*\| \geq \omega^{\omega^\alpha \times (n+1)}$ for every $n \in \mathbb{N}$. The supremum of the right-hand side as n varies is $\omega^{\omega^\alpha \times \omega} = \omega^{\omega^{\alpha+1}}$, which allows us to conclude. \square

Using a notation of Schmidt's [Sch79, Theorem 9], we define α' as $\alpha - 1$ if α is finite and non-zero, and as α° otherwise.

Proposition 12.21. *For every non-empty Noetherian space X , $\|X^*\| \geq \omega^{\omega^{\alpha'}}$, where $\alpha \stackrel{\text{def}}{=} \|X\|$.*

PROOF. By induction on the non-zero ordinal α . When $\alpha = 1$, $\|X^*\| = \omega = \omega^{\omega^{\alpha-1}}$ by Lemma 12.16 (2).

When α is a successor ordinal, there is a closed subset F of X such that $\text{rk}_{\mathcal{H}_0X}(F) = \alpha - 1$. By Lemma 5.8, $\|F\| = \alpha - 1$. We use the induction hypothesis,

and we obtain that $\|F^*\| \geq \omega^{\omega^{(\alpha-1)'}}$. By Lemma 12.9, there is a point x such that $X = F \cup \downarrow x$. Let $C \stackrel{\text{def}}{=} \downarrow x$. Applying Corollary 12.20, we obtain that $\|X^*\| = \|(F \cup C)^*\| \geq \omega^{\omega^{(\alpha-1)'+1}}$. A simple case analysis on the definition of α' shows that $(\alpha-1)'+1 = \alpha'$.

When α is a limit ordinal, for every $\beta < \alpha$, there is a closed subset F of X such that $\|F\| = \text{rk}_{\mathcal{H}_0 X}(F) = \beta$, and by induction hypothesis $\|F^*\| \geq \omega^{\omega^{\beta'}}$. It follows that $\|X^*\| \geq \sup_{\beta < \alpha} \omega^{\omega^{\beta'}} = \omega^{\omega^{\sup_{\beta < \alpha} \beta'}}$. Since β and β' differ by at most 1, and α is a limit ordinal, $\sup_{\beta < \alpha} \beta' = \alpha$. This shows that $\|X^*\| \geq \omega^{\omega^\alpha}$. In particular, $\|X^*\| \geq \omega^{\omega^{\alpha'}}$, unless α is a critical ordinal. (We recall that α is a limit ordinal, and is therefore of the form $\epsilon + n$ with ϵ critical and $n \in \mathbb{N}$ only if $n = 0$.)

We finally deal with the case where α is a critical ordinal. Let $\mathcal{S}'X$ be the subset of those elements of $\mathcal{S}X$ of non-maximal rank, namely of rank different from $\text{rsob } X = \text{sob } X - 1$ (see Lemma 4.2).

For every $n \in \mathbb{N}$, we consider n -elementary word-products, which we define as those of the form $C_0^? C_1^? \cdots C_n^?$, where each C_i is an irreducible closed subset of X , and not all are of maximal rank $\text{rsob } X$. Let \mathcal{E}_n be the collection of closed subsets of X^* that one obtains as finite unions of n -elementary word-products.

We note that every element \mathbf{A} of \mathcal{E}_n is a subset of the set \mathbf{B}_n of words of length at most $n+1$, and we claim that this inclusion is proper. Indeed, otherwise \mathbf{B}_n would be included in \mathbf{A} . By Lemma 4.2, one of the components C of X has maximal rank $\text{rsob } X$. Then $(C^?)^{n+1}$ would be included in \mathbf{A} . Since $(C^?)^{n+1}$ is a word-product hence irreducible, it would be included in one of the n -elementary word-products $C_0^? C_1^? \cdots C_n^?$ whose union comprises \mathbf{A} . In turn, using clauses (1) and (2) of Lemma 12.1, this would force $C \subseteq C_i$ for each i , contradicting the fact that at least one C_i is of non-maximal rank.

The set \mathbf{B}_n is closed, because it is equal to the finite union of the word-products $C_0^? C_1^? \cdots C_n^?$, where C_0, C_1, \dots, C_n range over the components of X . Let $f_n: \text{Step}(\mathcal{S}X) \times \mathcal{E}_n \rightarrow \mathcal{E}_{n+1}$ map $((C, C^+), \mathbf{A})$ to $\mathbf{B}_n C^? \cup \mathbf{A} C^{+?}$. Writing \mathbf{B}_n as a finite union of word-products \mathbf{P}_i , $\mathbf{B}_n C^?$ is the union of the word-products $\mathbf{P}_i C^?$, in which C has non-maximal rank, being a proper subset of C^+ . Using this, it is easy to check that $f_n((C, C^+), \mathbf{A})$ is indeed in \mathcal{E}_{n+1} .

We claim that f_n is strictly monotonic from $\text{Step}(\mathcal{S}X) \times_{\text{lex}} \mathcal{E}_n$ to \mathcal{E}_{n+1} .

If $(C, C^+) < (C', C'^+)$, namely if $C^+ \subseteq C'$, then both $\mathbf{B}_n C^?$ and $\mathbf{A} C^{+?}$ are included in $\mathbf{B}_n C'^?$, for every $\mathbf{A} \in \mathcal{E}_n$. Let $w \in \mathbf{B}_n \setminus \mathbf{A}$ and $x \in C^+ \setminus C$. By concatenating w with enough copies of x , we can assume that w has length exactly $n+1$; the result is still not in \mathbf{A} , since \mathbf{A} is downwards-closed with respect to \leq^* .

Then wx is in $\mathbf{B}_n C'^?$, but neither in $\mathbf{B}_n C^?$ (since $x \notin C$) nor in $\mathbf{A}C'^?$ (since $w \notin \mathbf{A}$). Therefore the inclusion is strict.

If $(C, C^+) = (C', C'^+)$ and $\mathbf{A} \subsetneq \mathbf{A}'$, then $\mathbf{A}C'^? \subseteq \mathbf{A}'C'^?$, so $f_n((C, C^+), \mathbf{A}) \subseteq f_n((C', C'^+), \mathbf{A}')$. Let $w \in \mathbf{A}' \setminus \mathbf{A}$ and $x \in C^+ \setminus C$. Since w is in an n -elementary word-product $C_0^? C_1^? \cdots C_n^?$ included in \mathbf{A}' , it is of the form $w_0 w_1 \cdots w_n$ where each w_i is in $C_i^?$; for those indices i such that $w_i = \epsilon$, we may replace w_i by some letter from C_i , and we will obtain a word that is still in $C_0^? C_1^? \cdots C_n^?$, hence in \mathbf{A}' , but not in \mathbf{A} since \mathbf{A} is downwards-closed with respect to \leq^* . Hence, without loss of generality, we may assume that w has length exactly $n+1$. Then wx is in $f_n((C', C'^+), \mathbf{A}')$ and neither in $\mathbf{B}_n C^?$ (since $x \notin C$ and w is of length $n+1$) nor in $\mathbf{A}C'^?$ (since $w \notin \mathbf{A}$, w is of length $n+1$, and \mathbf{A} contains words of lengths at most $n+1$ only), hence not in $f_n((C, C^+), \mathbf{A})$. Hence the inclusion is strict.

It follows that $|\mathcal{E}_{n+1}| \geq |\mathcal{E}_n| \times |\text{Step}(\mathcal{S}X)|$, for every $n \in \mathbb{N}$. Now $\text{sob } X^* = \omega^{\|X\|+1} + 1$ by Theorem 12.13, namely, $|\mathcal{S}X| = \omega^{\alpha+1} + 1$. Using Proposition 12.17, $|\text{Step}(\mathcal{S}X)| = \omega^{\alpha+1}$. Hence $|\mathcal{E}_{n+1}| \geq |\mathcal{E}_n| \times \omega^{\alpha+1}$, for every $n \in \mathbb{N}$. Since \mathcal{E}_0 is non-empty, $|\mathcal{E}_0| \geq 1$. By induction on n , then, $|\mathcal{E}_n| \geq (\omega^{\alpha+1})^n$. In particular, $|\mathcal{E}_n| \geq (\omega^\alpha)^n = \alpha^n$, since α is critical. Since each element of \mathcal{E}_n is a proper closed subset of X^* , it follows that $\|X^*\| \geq \sup_{n \in \mathbb{N}} \alpha^n = \alpha^\omega$. But $\alpha^\omega = (\omega^\alpha)^\omega = \omega^{\alpha \times \omega}$ and $\alpha \times \omega = \omega^\alpha \times \omega = \omega^{\alpha+1}$, so $\|X^*\| \geq \omega^{\omega^{\alpha+1}} = \omega^{\omega^{\alpha'}}$. \square

Now we put together the results of Proposition 12.21, Lemma 12.14, and Lemma 12.16.

Theorem 12.22. *For every Noetherian space X , letting $\alpha \stackrel{\text{def}}{=} \|X\|$, the stature $\|X^*\|$ is equal to 1 if $\alpha = 0$, and to $\omega^{\omega^{\alpha'}}$ otherwise, where $\alpha' = \alpha - 1$ if α is finite and non-zero, $\alpha' = \alpha + 1$ if $\alpha = \epsilon + n$ for some critical ordinal ϵ and some natural number n , and $\alpha' = \alpha$ otherwise.*

13. HETEROGENEOUS WORDS AND THE PREFIX TOPOLOGY

The prefix topology is another interesting topology on X^* (see [GL13, Exercise 9.7.36]), which has no equivalent in the wqo world. Its specialization quasi-ordering is a form of the prefix ordering, which is almost never a wpo. We would like to mention that this is the topology needed to decide reachability of sets defined by forbidden patterns in the so-called oblivious k -stack system model of [Gou10, Section 5].

In general, the prefix topology makes sense not just on X^* , but on spaces of sequences of elements taken from possibly different spaces, see [FGL20, Section 9].

Henceforth, let $X_1, X_2, \dots, X_n, \dots$ be countably many topological spaces. A *heterogeneous word* over these spaces is any tuple (x_1, x_2, \dots, x_m) in $X_1 \times X_2 \times \dots \times X_m$, $m \in \mathbb{N}$. We write it as $x_1 x_2 \dots x_m$, and we call $m = |w|$ the *length* of the word $w = x_1 x_2 \dots x_m$.

We write $\triangleright_{n=1}^{+\infty} X_n$ for the space of all such heterogeneous words, with the so-called *prefix topology*. The latter is defined as follows. A *telescope* on $(X_n)_{n \geq 1}$ is a sequence $\mathcal{U} \stackrel{\text{def}}{=} U_0, U_1, \dots, U_n, \dots$ of open sets, where U_n is open in $\prod_{i=1}^n X_i$ for each $n \in \mathbb{N}$, and such that $U_n X_{n+1} \subseteq U_{n+1}$ for every $n \in \mathbb{N}$. (We write UX instead of $U \times X$, for convenience, and ϵ for the empty tuple. When $n = 0$, $\prod_{i=1}^n X_i = \{\epsilon\}$, and U_0 can only be the empty set or $\{\epsilon\}$.) A *wide telescope* is a telescope such that $U_n = \prod_{i=1}^n X_i$ for some $n \in \mathbb{N}$, or equivalently for all n large enough. The open sets of the prefix topology are the empty set and all the sets $[\mathcal{U}] \stackrel{\text{def}}{=} \{w \in \triangleright_{n=1}^{+\infty} X_n \mid w \in U_{|w|}\}$, where \mathcal{U} is a wide telescope.

The specialization preordering of $\triangleright_{n=1}^{+\infty} X_n$ is the *prefix preordering* \leq^{\triangleright} , defined by $x_1 \dots x_m \leq^{\triangleright} y_1 \dots y_n$ if and only if $m \leq n$ and $x_i \leq y_i$ for every i , $1 \leq i \leq m$. Moreover, when all the spaces X_n are Noetherian, so is $\triangleright_{n=1}^{+\infty} X_n$ [FGL20, Proposition 9.1].

The irreducible closed subsets of $\triangleright_{n=1}^{+\infty} X_n$ are exactly the sets of the form $[C_1 \dots C_n] \stackrel{\text{def}}{=} \{x_1 \dots x_m \mid m \leq n, x_1 \in C_1, \dots, x_m \in C_m\}$, where $n \in \mathbb{N}$ and each C_i is irreducible closed in X_i for each i , plus $\triangleright_{n=1}^{+\infty} X_n$ itself when all the spaces X_n are non-empty [FGL20, Lemma 9.6]. The set $\triangleright_{n=1}^{+\infty} X_n$ contains all the other irreducible subsets, and the inclusion relation between the sets of the former kind is given by $[C_1 \dots C_m] \subseteq [C'_1 \dots C'_n]$ if and only if $m \leq n$ and $C_i \subseteq C'_i$ for every i , $1 \leq i \leq m$, namely if and only if $C_1 \dots C_m \leq^{\triangleright} C'_1 \dots C'_n$; in general, if no X_n is empty, the map $i: (\triangleright_{n=1}^{+\infty} \mathcal{S}X_n)^{\top} \rightarrow \mathcal{S}(\triangleright_{n=1}^{+\infty} X_n)$ that sends \top to $\triangleright_{n=1}^{+\infty} X_n$ and $C_1 \dots C_m$ to $[C_1 \dots C_m]$ is an order-isomorphism [FGL20, Proposition 9.7].

Computing the sobrification rank of $\triangleright_{n=1}^{+\infty} X_n$ therefore boils down to computing the ordinal rank of a product of well-founded posets in the prefix ordering.

For posets P_1, \dots, P_n, \dots , we define $\triangleright_{n=1}^{+\infty} P_n$ as the set of words $p_1 \dots p_n$ where each p_i is in P_i , ordered by the prefix ordering \leq^{\triangleright} . This is also the poset underlying the space $\triangleright_{n=1}^{+\infty} P_n$, where each P_n is given the Alexandroff topology of its ordering. We also let $\triangleright_{i=1}^n P_i$ be the subposet of words of length at most n .

In the following, let \leq_P denote the ordering on a poset P .

Lemma 13.1. *For any two posets P, Q , let $P \triangleright Q$ be the set of elements that are either elements of P or pairs in $P \times Q$, ordered by: $(p, q) \leq_{P \triangleright Q} (p', q')$ if and only if $p \leq_P p'$ and $q \leq_Q q'$; $p \leq_{P \triangleright Q} p'$ if and only if $p \leq_P p'$; $p \leq_{P \triangleright Q} (p', q')$ if and only if $p \leq_P p'$; $(p, q) \not\leq_{P \triangleright Q} p'$ for any p, p', q .*

If P and Q are well-founded and non-empty, then for all $p \in P$ and $q \in Q$,

- (1) $\text{rk}_{P \triangleright Q}(p) = \text{rk}_P(p)$;
- (2) $\text{rk}_{P \triangleright Q}(p, q) = \text{rk}_P(p) \oplus (1 + \text{rk}_Q(q))$.

PROOF. Let Q_\perp be Q with a fresh element \perp added below all others. For every $q \in Q$, $\text{rk}_{Q_\perp}(q) = 1 + \text{rk}_Q(q)$, by an easy induction on $\text{rk}_Q(q)$. The map $f: P \triangleright Q \rightarrow P \times_{\text{lex}} Q_\perp$ defined by $f(p, q) \stackrel{\text{def}}{=} (p, q)$, $f(p) \stackrel{\text{def}}{=} (p, \perp)$ is an order-isomorphism. The claim then follows from Lemma 8.1. \square

Lemma 13.2. *Let P_1, \dots, P_n, \dots be non-empty well-founded posets. For every word $w \stackrel{\text{def}}{=} p_1 \cdots p_m$ in $P \stackrel{\text{def}}{=} \triangleright_{n=1}^{+\infty} P_n$ (resp., in $P \stackrel{\text{def}}{=} \triangleright_{i=1}^n P_i$, for any $n \geq m$), $\text{rk}_P(w) = 1 + (\text{rk}_{P_1}(p_1) \oplus (1 + (\text{rk}_{P_2}(p_2) \oplus (1 + \cdots \oplus (1 + \text{rk}_{P_m}(p_m))))))$ if $m \neq 0$, 0 otherwise.*

PROOF. The rank of w in $\triangleright_{n=1}^{+\infty} P_n$ or in $\triangleright_{i=1}^n P_i$, for any $n \geq m$, is the same, because the words smaller than w in any of those spaces are the same. Hence it suffices to prove that, given $P = \triangleright_{i=1}^n P_i$, for some fixed, but arbitrary natural number n , the rank of any word $w \stackrel{\text{def}}{=} p_1 \cdots p_m$ with $m \leq n$ in P is as indicated. We realize that P is order-isomorphic to $\{\epsilon\} \triangleright (P_1 \triangleright (P_2 \triangleright \cdots \triangleright P_n))$, where w is mapped to $(\epsilon, (p_1, \dots, (p_{m-1}, p_m)))$. The result then follows from Lemma 13.1, by showing that the rank of $(p_i, (p_{i+1}, \dots, (p_{m-1}, p_m)))$ in the appropriate space is $\text{rk}_{P_i}(p_i) \oplus (1 + (\text{rk}_{P_{i+1}}(p_{i+1}) \oplus \cdots \oplus (1 + (\text{rk}_{P_{m-1}}(p_{m-1}) \oplus (1 + \text{rk}_{P_m}(p_m))))))$, by induction on $m - i$. \square

The baroque formula of Lemma 13.2 simplifies as follows.

Lemma 13.3. *In the situation of Lemma 13.2, $\text{rk}_P(w) = \bigoplus_{i=1}^m \text{rk}_{P_i}(p_i) + (m - k)$, where k is the largest number between 1 and m such that $\text{rk}_{P_k}(p_k)$ is infinite, or 0 if there is none.*

PROOF. The claim is obvious if $m = 0$. Let us therefore assume $m \neq 0$. Let α_i be defined by $\alpha_m \stackrel{\text{def}}{=} 1 + \text{rk}_{P_m}(p_m)$, $\alpha_i \stackrel{\text{def}}{=} 1 + (\text{rk}_{P_i}(p_i) \oplus \alpha_{i+1})$ for every i with $1 \leq i < m$. Lemma 13.2 states that $\text{rk}_P(w) = \alpha_1$.

Since $\text{rk}_{P_{k+1}}(p_{k+1}), \dots, \text{rk}_{P_m}(p_m)$ are all finite, we have $\alpha_m = \text{rk}_{P_m}(p_m) + 1$, $\alpha_{m-1} = \text{rk}_{P_{m-1}}(p_{m-1}) + \text{rk}_{P_m}(p_m) + 2, \dots, \alpha_{k+1} = \sum_{i=k+1}^m \text{rk}_{P_i}(p_i) + (m - k)$. We may rewrite the latter as $\alpha_{k+1} = \bigoplus_{i=k+1}^m \text{rk}_{P_i}(p_i) + (m - k)$. If $k = 0$, namely if every number $\text{rk}_{P_i}(p_i)$ is finite, this proves the claim directly.

Otherwise, $\alpha_k = 1 + (\text{rk}_{P_k}(p_k) \oplus \bigoplus_{i=k+1}^m \text{rk}_{P_i}(p_i) + (m - k))$. Since $\text{rk}_{P_k}(p_k)$ is infinite, this is also equal to $\text{rk}_{P_k}(p_k) \oplus \bigoplus_{i=k+1}^m \text{rk}_{P_i}(p_i) + (m - k) = \bigoplus_{i=k}^m \text{rk}_{P_i}(p_i) + (m - k)$. Also, α_k is itself infinite, so $\alpha_{k-1} = 1 + (\text{rk}_{P_{k-1}}(p_{k-1}) \oplus \alpha_k)$ is equal to

$1 + \bigoplus_{i=k-1}^m \text{rk}_{P_i}(p_i) + (m - k) = \bigoplus_{i=k-1}^m \text{rk}_{P_i}(p_i) + (m - k)$, and is also infinite. Similarly, we show that $\alpha_j = \bigoplus_{i=j}^m \text{rk}_{P_i}(p_i) + (m - k)$ for every j with $1 \leq j \leq k$, by descending induction on j . The claim follows by taking $j \stackrel{\text{def}}{=} 1$. \square

We define the infinite natural sum $\bigoplus_{n=1}^{+\infty} \alpha_n$ as the least upper bound of the increasing sequence of ordinals $\bigoplus_{i=1}^n \alpha_i$, $n \in \mathbb{N}$. The following is more easily stated using reduced sobrification ranks.

Theorem 13.4. *Given countably many non-empty Noetherian spaces $X_1, X_2, \dots, X_n, \dots$, $\text{rsob}(\bigtriangleright_{n=1}^{+\infty} X_n)$ is equal to:*

- (1) $\bigoplus_{n=1}^k \text{rsob } X_n + \omega$ if $\text{rsob } X_n$ is finite for n large enough, where k is the largest index such that $\text{rsob } X_k$ is infinite, or 0 if there is none;
- (2) $\bigoplus_{n=1}^{+\infty} \text{rsob } X_n$ otherwise.

PROOF. First, for all $C_1 \in \mathcal{S}X_1, \dots, C_m \in \mathcal{S}X_m$, the rank of $\lceil C_1 \cdots C_m \rceil$ in $\mathcal{S}(\bigtriangleright_{n=1}^{+\infty} X_n)$ is the same as the rank of the word $C_1 \cdots C_m$ in the isomorphic poset in $(\bigtriangleright_{n=1}^{+\infty} \mathcal{S}X_n)^\top$, hence also in the latter minus its top element \top , which is $\bigtriangleright_{n=1}^{+\infty} \mathcal{S}X_n$. By Lemma 13.3, this is $\bigoplus_{i=1}^m \text{rk}_{\mathcal{S}X_i}(C_i) + (m - k)$, where k is the largest number between 1 and m such that $\text{rk}_{\mathcal{S}X_i}(C_i)$ is infinite, or 0 if there is none.

When C_1, \dots, C_m vary (with m fixed), the latter reaches a maximum at $\alpha_m \stackrel{\text{def}}{=} \bigoplus_{i=1}^m \text{rsob } X_i + (m - k)$, using Definition 4.3, where k is the largest number between 1 and m such that $\text{sob } X_i$ is infinite, 0 otherwise. Then $\text{rsob}(\bigtriangleright_{n=1}^{+\infty} X_n)$ is the rank of the additional top element of $(\bigtriangleright_{n=1}^{+\infty} \mathcal{S}X_n)^\top$, which is the supremum of those values as m varies.

Case 1. If there are infinitely many indices m such that $\text{sob } X_m$ is infinite, then the rank of the top element is also the supremum of the cofinal subfamily of the values α_m , where m ranges over the indices such that $\text{sob } X_m$ is infinite; those values are equal to $\bigoplus_{i=1}^m \text{rk}_{\mathcal{S}X_i}(C_i)$, by definition and the condition on $\text{sob } X_m$, so the rank of the top element is $\bigoplus_{n=1}^{+\infty} \text{rsob } X_n$, and this is the desired reduced sobrification rank of $\bigtriangleright_{n=1}^{+\infty} X_n$.

Case 2. Otherwise, there is a number m_0 such that $\text{rsob } X_m$ is finite for every $m \geq m_0$. Let m_0 be the least one, and $k \stackrel{\text{def}}{=} m_0 - 1$. For every $m \geq m_0$, $\alpha_m = \bigoplus_{i=1}^m \text{rsob } X_i + (m - k)$. We write the latter as $\bigoplus_{i=1}^k \text{rsob } X_i + a_m$, where a_m is the natural number $\sum_{i=m_0}^m \text{rsob } X_i + (m - k)$. The first summand $\bigoplus_{i=1}^k \text{rsob } X_i$ is independent of m , and the supremum of the numbers a_m , $m \geq m_0$, is equal to ω . Therefore the rank of the top element is $\bigoplus_{n=1}^k \text{rsob } X_n + \omega$. \square

One may simplify the formulae of Theorem 13.4 as follows.

Corollary 13.5. *Given countably many non-empty Noetherian spaces $X_1, X_2, \dots, X_n, \dots$, such that $\text{rsob } X_n \geq 1$ for infinitely many indices $n \geq 1$, we have $\text{rsob } (\triangleright_{n=1}^{+\infty} X_n) = \bigoplus_{n=1}^{+\infty} \text{rsob } X_n$.*

PROOF. We only have to prove this in case 1 of Theorem 13.4. Then we have that $\bigoplus_{n=k+1}^{+\infty} \text{rsob } X_n$ is an infinite sum of natural numbers, infinitely of which are non-zero, and is therefore equal to ω . It follows that $\bigoplus_{n=1}^k \text{rsob } X_n + \omega = \bigoplus_{n=1}^{+\infty} \text{rsob } X_n$. \square

Remark 13.6. When some X_n is empty, let $n \in \mathbb{N}$ be smallest such that $X_{n+1} = \emptyset$; then, the space $\triangleright_{i=1}^{+\infty} X_i$ still makes sense, and coincides with $\triangleright_{i=1}^n X_i$. The same argument as in Theorem 13.4 then shows that $\text{rsob } (\triangleright_{n=1}^{+\infty} X_n) = \text{rsob } (\triangleright_{i=1}^n X_i) = \bigoplus_{i=1}^n \text{rsob } X_i + (n - k)$, where k is the largest number between 1 and n such that $\text{rsob } X_k$ is infinite, or 0 if there is none. Indeed, in that case $\mathcal{S}(\triangleright_{i=1}^n X_i)$ is isomorphic to $\triangleright_{i=1}^n \mathcal{S}X_i$, without any additional top element \top [FGL20, Proposition 9.9]; so the largest rank of an element of $\triangleright_{i=1}^n X_i$ is $\bigoplus_{i=1}^n \text{rsob } X_i + (n - k)$, by Lemma 13.3.

When all the space X_n are equal to the same space X , we write X^\triangleright for $\triangleright_{n=1}^{+\infty} X_n$. This is the space of finite words over X , with the prefix topology.

Corollary 13.7. *For every non-empty Noetherian space X , $\text{rsob } X^\triangleright = \omega^{\alpha_1+1}$, where $\text{rsob } X$ is written in Cantor normal form as $\omega^{\alpha_1} + \dots + \omega^{\alpha_m}$, $\alpha_1 \geq \dots \geq \alpha_m$. When X is empty, $X^\triangleright = \{\epsilon\}$ and $\text{rsob } X^\triangleright = 0$.*

PROOF. The second claim is obvious. For the first claim, either $\text{rsob } X = 0$, in which case, by Theorem 13.4, item 1, $\text{rsob } X^\triangleright = \omega$, which shows the claim; or $\alpha \stackrel{\text{def}}{=} \text{rsob } X \geq 1$, in which case $\text{rsob } X^\triangleright$ is the infinite sum $\bigoplus_{n=1}^{+\infty} \alpha = \sup_{n \in \mathbb{N}} \bigoplus_{i=1}^n \alpha$ of the same ordinal α , by Corollary 13.5. We write α in Cantor normal form as $\omega^{\alpha_1} + \dots + \omega^{\alpha_m}$. Then $\bigoplus_{i=1}^n \alpha = \omega^{\alpha_1} \times n + \dots + \omega^{\alpha_m} \times n$. In order to compute the supremum of the latter values as n varies, we note that this supremum is larger than or equal to $\sup_{n \in \mathbb{N}} \omega^{\alpha_1} \times n = \omega^{\alpha_1} \times \omega = \omega^{\alpha_1+1}$. It is equal to it, because ω^{α_1+1} is already an upper bound of the family $\{\omega^{\alpha_1} \times n + \dots + \omega^{\alpha_m} \times n \mid n \in \mathbb{N}\}$. \square

As far as stature is concerned, we define the infinite natural product $\bigotimes_{n=1}^{+\infty} \alpha_n$ of non-zero ordinals α_n as the least upper bound of the monotonic sequence of ordinals $\bigotimes_{i=1}^n \alpha_i$, $n \in \mathbb{N}$.

Theorem 13.8. *Given countably many non-empty Noetherian spaces $X_1, X_2, \dots, X_n, \dots$,*

- (1) If $\|X_n\|$ is infinite for infinitely many values of $n \in \mathbb{N}$, then $\|\triangleright_{n=1}^{+\infty} X_n\| = \bigotimes_{n=1}^{+\infty} \|X_n\|$.
- (2) If $\|X_n\|$ is finite for n large enough, then letting k be the largest index such that $\text{sob } X_k$ is infinite, or 0 if there is none, then $\|\triangleright_{n=1}^{+\infty} X_n\|$ is equal to $\bigotimes_{m=1}^k \|X_m\| \times \omega$.

The formula in (2) is not a special case of (1), unless $\|X_n\| \geq 2$ for infinitely many values of n . Indeed, if $\|X_n\| = 1$ for all $n \geq \ell + 1$ (where $\ell \geq k$), then $\bigotimes_{n=1}^{+\infty} \|X_n\| = \bigotimes_{n=1}^{\ell} \|X_n\| = \bigotimes_{n=1}^k \|X_n\| \times p$, where p is the natural number $\prod_{n=k+1}^{\ell} \|X_n\|$; that is different from $\bigotimes_{m=1}^k \|X_m\| \times \omega$.

PROOF. For all $i, j \in \mathbb{N}$ with $i \leq j$, let \mathbf{A}_{ij} be the subset of $\triangleright_{n=i+1}^{+\infty} X_n$ of those words of length at most $j - i$. This is the complement of $[\mathcal{U}]$ where $\mathcal{U} \stackrel{\text{def}}{=} U_i, U_{i+1}, \dots, U_n, \dots$ is the wide telescope defined by $U_i \stackrel{\text{def}}{=} \dots \stackrel{\text{def}}{=} U_j \stackrel{\text{def}}{=} \emptyset$, and $U_k \stackrel{\text{def}}{=} \prod_{n=i+1}^k X_n$ for every $k \geq j + 1$. Therefore \mathbf{A}_{ij} is a closed subset of $\triangleright_{n=i+1}^{+\infty} X_n$.

When $i = j$, $\mathbf{A}_{jj} = \{\epsilon\}$, so $\|\mathbf{A}_{jj}\| = 1$.

If $i < j$, then we claim that the map f defined by $f(\perp) \stackrel{\text{def}}{=} \epsilon$, $f(x_i, w) \stackrel{\text{def}}{=} x_i w$ is a homeomorphism from $(X_{i+1} \times \mathbf{A}_{(i+1)j})_{\perp}$ onto \mathbf{A}_{ij} . In order to see that f is continuous, we show that the inverse image of any closed set by f is closed. Since \mathbf{A}_{ij} is a proper closed subset of $\triangleright_{n=i+1}^{+\infty} X_n$, every closed subset of \mathbf{A}_{ij} is a proper closed subset \mathbf{B} of $\triangleright_{n=i+1}^{+\infty} X_n$, which is included in \mathbf{A}_{ij} . Since $\triangleright_{n=i+1}^{+\infty} X_n$ is Noetherian, \mathbf{B} is a finite union of (proper) irreducible closed sets. Each one is of the form $[C_{i+1} \cdots C_n]$, where $n \geq i$ and C_{i+1}, \dots, C_n are irreducible closed in X_{i+1}, \dots, X_n respectively. We may also assume that $n \leq j$, since otherwise $[C_{i+1} \cdots C_n] \cap \mathbf{A}_{ij} = [C_{i+1} \cdots C_j]$. Now $f^{-1}([C_{i+1} \cdots C_n]) = \{\perp\} \cup (C_{i+1} \times [C_{i+2} \cdots C_n])$ if $n \geq i + 1$, or $\{\perp\}$ if $n = i$, and those sets are closed. (Note that $\{\perp\}$ is the downwards-closure, hence the closure, of \perp .) This shows that f is continuous. In order to see that its inverse is continuous, too, it suffices to show that every irreducible closed subset of $(X_{i+1} \times \mathbf{A}_{(i+1)j})_{\perp}$ is of one of the above forms. The irreducible closed subsets of $(X_{i+1} \times \mathbf{A}_{(i+1)j})_{\perp} = \{\perp\} +_{\text{lex}} (X_{i+1} \times \mathbf{A}_{(i+1)j})$ are $\{\perp\}$ itself, and the sets of the form $\{\perp\} \cup (C_{i+1} \times [C_{i+2} \cdots C_n])$, where C_{i+1}, \dots, C_n are irreducible closed and $n \geq i + 1$, using Lemma 9.3, and the fact that the irreducible closed subsets of a product are the products of irreducible closed subsets (see Section 10).

Using Proposition 9.6 and Theorem 10.9, we obtain that $\|\mathbf{A}_{ij}\| = 1 + (\|X_{i+1}\| \otimes \|\mathbf{A}_{(i+1)j}\|)$ for all $i < j$. By an easy induction on $j - i$, which proceeds along similar principles as the proof of Lemma 13.3, we obtain that for all $i \leq j$,

$\|\mathbf{A}_{ij}\| = \bigotimes_{m=i+1}^k \|X_m\| \times (1 + \|X_{k+1}\| + \|X_{k+1}\| \times \|X_{k+2}\| + \cdots + \prod_{p=k+1}^j \|X_p\|)$, where k is the largest number between $i+1$ and j such that $\|X_k\|$ is infinite, and i if there is no such number. (We take products over an empty family to be equal to 1.)

In particular, the stature of the closed subspace $\mathbf{A}_n \stackrel{\text{def}}{=} \mathbf{A}_{0n}$ of the space $\triangleright_{i=1}^{+\infty} X_i$ is $\bigotimes_{m=1}^k \|X_m\| \times (1 + \|X_{k+1}\| + \|X_{k+1}\| \times \|X_{k+2}\| + \cdots + \prod_{p=k+1}^n \|X_p\|)$, where k is the largest number between 1 and n such that $\|X_k\|$ is infinite, and 0 if there is no such number.

Clearly, \mathbf{A}_n is a proper subset of $\triangleright_{i=1}^{+\infty} X_i$. Every proper irreducible closed subset of $\triangleright_{i=1}^{+\infty} X_i$ is of the form $\lceil C_1 \cdots C_n \rceil$, where $n \in \mathbb{N}$ and each C_i is irreducible closed; then, $\lceil C_1 \cdots C_n \rceil$ is included in \mathbf{A}_n . Every proper closed subset \mathbf{B} of $\triangleright_{i=1}^{+\infty} X_i$, which one can write as a finite union of proper irreducible closed sets, is therefore also included in some \mathbf{A}_n . Proposition 5.9 then implies that $\|\triangleright_{m=1}^{+\infty} X_m\| = \sup_{n \in \mathbb{N}} \|\mathbf{A}_n\|$.

We now make two cases, as in the proof of Theorem 13.4.

(1) If there are infinitely many indices m such that $\|X_m\|$ is infinite, then the supremum on the right-hand side can equivalently be taken over those $n \in \mathbb{N}$ such that $\|X_n\|$ is infinite. For each of those, $\|\mathbf{A}_n\| = \bigotimes_{m=1}^n \|X_m\|$, so $\|\triangleright_{m=1}^{+\infty} X_m\| = \bigotimes_{m=1}^{+\infty} \|X_m\|$.

(2) Otherwise, let k be the largest number such that $\|X_k\|$ is infinite, or 0 if there is no such number. For every $n \geq k$, we have $\|\mathbf{A}_n\| = \bigotimes_{m=1}^k \|X_m\| \times (1 + \|X_{k+1}\| + \|X_{k+1}\| \times \|X_{k+2}\| + \cdots + \prod_{p=k+1}^n \|X_p\|)$. Since $\|X_p\| \geq 1$ for every p , the least upper bound of the latter values is $\bigotimes_{m=1}^k \|X_m\| \times \omega$. \square

Corollary 13.9. *For every Noetherian space X ,*

- (1) *If $\|X\| = 0$, then $\|X^\triangleright\| = 1$;*
- (2) *If $\|X\|$ is finite and non-zero, then $\|X^\triangleright\| = \omega$;*
- (3) *Otherwise, $\|X^\triangleright\|$ is equal to $\omega^{\omega^{\beta_1+1}}$, where $\|X\|$ is written in Cantor normal form as $\omega^{\alpha_1} + \cdots + \omega^{\alpha_m}$ ($\alpha_1 \geq \cdots \geq \alpha_m$), and α_1 is written in Cantor normal form as $\omega^{\beta_1} + \cdots + \omega^{\beta_n}$ ($\beta_1 \geq \cdots \geq \beta_n$).*

PROOF. (1) If $\|X\| = 0$, then X is empty, so $X^\triangleright = \{\epsilon\}$.

(2) follows from Theorem 13.8 (2).

(3) By Theorem 13.8 (1), $\|X^\triangleright\| = \bigotimes_{n=1}^{+\infty} \|X\|$. We write $\|X\|$ in Cantor normal form as $\omega^{\alpha_1} + \cdots + \omega^{\alpha_m}$ ($\alpha_1 \geq \cdots \geq \alpha_m$). Necessarily, $\alpha_1 \geq 1$, since $\|X\|$ is infinite. For every $k \in \mathbb{N}$, $\bigotimes_{n=1}^k \|X\|$ is a (natural) sum of terms of the form ω^α , where each α is equal to $\alpha_{i_1} \oplus \cdots \oplus \alpha_{i_k}$ for some tuple $(i_1, \dots, i_k) \in \{1, \dots, m\}^k$. Every such α is smaller than or equal to $\alpha_1 \otimes k$. We write α_1 in Cantor normal

form as $\omega^{\beta_1} + \dots + \omega^{\beta_n}$ ($\beta_1 \geq \dots \geq \beta_n$). Then $\alpha_1 \otimes k = \omega^{\beta_1} \times k + \dots + \omega^{\beta_n} \times k$ is strictly smaller than ω^{β_1+1} . Hence every summand ω^α of $\otimes_{n=1}^k \|X\|$ is strictly smaller than ω^{β_1+1} . It follows that $\otimes_{n=1}^k \|X\| < \omega^{\beta_1+1}$. Taking suprema over $k \in \mathbb{N}$, $\|X^\triangleright\| \leq \omega^{\beta_1+1}$.

As far as the reverse inequality is concerned, for every $k \in \mathbb{N}$, $\omega^{\alpha_1 \otimes k}$ is a summand (the largest one) in $\otimes_{n=1}^k \|X\|$, so $\|X^\triangleright\| \geq \omega^{\alpha_1 \otimes k}$ for every $k \in \mathbb{N}$. In turn, $\alpha_1 \otimes k \geq \omega^{\beta_1} \times k$. Taking suprema over $k \in \mathbb{N}$, $\|X^\triangleright\| \geq \omega^{\beta_1+1}$. \square

14. FINITE MULTISSETS

A (finite) *multiset* over a set X is a map m from X to \mathbb{N} such that $m(x) = 0$ for all but finitely many elements of X . We write X^\circledast for the set of all (finite) multisets over X . The *Parikh mapping* $\Psi: X^* \rightarrow X^\circledast$ maps every word w to the multiset m such that $m(x)$ is the number of occurrences of x in w [Par66]. We write $\{x_1, \dots, x_n\}$ for $\Psi(x_1 \dots x_n)$, and \emptyset for $\Psi(\epsilon)$. Multiset union $m \uplus m'$ maps every element x to $m(x) + m'(x)$. Clearly, $\Psi(w w') = \Psi(w' w) = \Psi(w) \uplus \Psi(w')$ for all $w, w' \in X^*$.

If X is equipped with a preordering \leq , the *sub-multiset* preordering \leq^\circledast is defined by $\{x_1, \dots, x_m\} \leq^\circledast \{y_1, \dots, y_n\}$ if and only if there is an injective map $r: \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ such that $x_i \leq y_{r(i)}$ for every i , $1 \leq i \leq m$. If X is wqo under \leq , then X^\circledast is wqo under \leq^\circledast . We need to mention the folklore lemma.

Lemma 14.1. *If \leq is a partial ordering on a set P , then \leq^\circledast is a partial ordering on P^\circledast .*

PROOF. Reflexivity and transitivity are obvious. Let $\{p_1, \dots, p_m\}$ and $\{q_1, \dots, q_n\}$ be two multisets that are less than or equal to each other with respect to \leq^\circledast . There is an injective map $r: \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ such that $p_i \leq q_{r(i)}$ for every i , $1 \leq i \leq m$, and there is an injective map $s: \{1, \dots, n\} \rightarrow \{1, \dots, m\}$ such that $q_j \leq p_{s(j)}$ for every j , $1 \leq j \leq n$. In particular, $m \leq n$ and $n \leq m$, so $m = n$ and both r and s are permutations. Let $f \stackrel{\text{def}}{=} s \circ r$. This is also a permutation, and $p_{f(i)} \geq p_i$ for every i , $1 \leq i \leq n$. By a classic trick of finite group theory, for each i , there is a number $k \geq 1$ such that $f^k(i) = i$. Then, $p_i = p_{f^k(i)} \geq p_{f^{k-1}(i)} \geq \dots \geq p_{f(i)} \geq p_i$, so that all those elements are equal. It follows that $p_{f(i)} = p_i$ for every i . Hence, for every i , $1 \leq i \leq m$, $p_i = p_{f(i)} = p_{s(r(i))} = q_{r(i)}$. Since r is a permutation, we have just shown that p_1, \dots, p_m and q_1, \dots, q_n are the same list up to permutation, namely, that $\{p_1, \dots, p_m\} = \{q_1, \dots, q_n\}$. \square

It turns out that, if X is wpo, and α is its maximal order type, written in Cantor normal form as $\omega^{\alpha_1} + \dots + \omega^{\alpha_n}$, then the maximal order type of X^\circledast is $\omega^{\omega^{\alpha_1^\circ} + \dots + \omega^{\alpha_n^\circ}}$, where α° was defined in Definition 12.2. This was proved by van der Meeren, Rathjen and Weiermann in [MRW15, Theorem 5], correcting a previous statement [Wei09, Theorem 2].

Definition 14.2 ($\widehat{\alpha}$). For every ordinal α , written in Cantor normal form as $\omega^{\alpha_1} + \dots + \omega^{\alpha_m}$ with $\alpha \geq \alpha_1 \geq \dots \geq \alpha_m$, $m \in \mathbb{N}$, the ordinal $\widehat{\alpha}$ is defined as $\omega^{\alpha_1^\circ} + \dots + \omega^{\alpha_m^\circ}$.

Hence van der Meeren, Rathjen and Weiermann's result is that $o(X^\circledast) = \omega^{\widehat{o(X)}}$.

As with other constructions, we wish to extend this result to Noetherian spaces, computing the stature $\|X^\circledast\|$, but also the sobrification rank $\text{sob } X^\circledast$.

For every topological space X , and given that X^* has the word topology, we topologize X^\circledast with the quotient topology induced by the Parikh mapping Ψ [GL13, Exercise 9.7.35]. The following can be found in [FGL20, Proposition 8.2]. If X is Noetherian, then so is X^\circledast , a base of the topology on X^\circledast is given by the sets $\langle U_1, \dots, U_n \rangle \stackrel{\text{def}}{=} \{m \uplus \{x_1, \dots, x_n\} \mid m \in X^\circledast, x_1 \in U_1, \dots, x_n \in U_n\}$, where U_1, \dots, U_n range over the open subsets of X and $n \in \mathbb{N}$. The specialization preordering of X^\circledast is \leq^\circledast , where \leq is the specialization preordering of X . Moreover, if X has the Alexandroff topology of \leq , then X^\circledast has the Alexandroff topology of \leq^\circledast .

Still assuming X Noetherian, the irreducible closed subsets of X^\circledast are the m -products $F \mid C_1, \dots, C_n \stackrel{\text{def}}{=} \Psi(F^* C_1^? \dots C_n^?)$, where F is a closed subset of X , $n \in \mathbb{N}$, and C_1, \dots, C_n are irreducible closed subsets of X , [FGL20, Proposition 8.4]; $F \mid C_1, \dots, C_n$ is the set of multisets that contain as many elements from F as one wishes, plus at most one from each C_i , $1 \leq i \leq n$. When $n = 0$, this can be written $F \mid$; alternatively, this is just F^\circledast .

We will later use the notation $F \mid C_1, \dots, C_n$ even when C_1, \dots, C_n are closed but not necessarily irreducible, to denote the set of multisets containing an arbitrary number of elements of F , plus at most one from each C_i (e.g., in the proof of Proposition 14.16, and again later). In that case, $F \mid C_1, \dots, C_n$ is not necessarily irreducible.

Inclusion between m -products is characterized as follows [FGL20, Lemma 8.6]: $F \mid C_1, \dots, C_m$ is included in $F' \mid C'_1, \dots, C'_n$ if and only if $F \subseteq F'$ and $\{C_{i_1}, \dots, C_{i_k}\} \subseteq^\circledast \{C'_1, \dots, C'_n\}$, where C_{i_1}, \dots, C_{i_k} enumerates those irreducible closed sets C_i that are not included in F' .

It is easy to see that every m -product can be written in *reduced* form as $F \mid C_1, \dots, C_m$, where no C_i is included in F . Indeed, if $C \subseteq F$, then we can remove

C from the notation $F \mid C, C_1, \dots, C_m$, obtaining $F \mid C_1, \dots, C_m$, and that denotes the same set of multisets. Reduced forms are canonical forms, as we now see.

Lemma 14.3. *If $F \mid C_1, \dots, C_m$ and $F' \mid C'_1, \dots, C'_n$ are two equal m -products in reduced form, then $F = F'$, $m = n$, and the list C_1, \dots, C_m is equal to C'_1, \dots, C'_n up to permutation.*

PROOF. Since each one is included in the other, we have $F \subseteq F'$ and $F' \subseteq F$, so $F = F'$. Additionally, we have $\{C_{i_1}, \dots, C_{i_k}\} \subseteq^{\otimes} \{C'_1, \dots, C'_n\}$, where C_{i_1}, \dots, C_{i_k} enumerates those irreducible closed sets C_i that are not included in F' . Since $F = F'$ and the first m -product is in reduced form, that simply means that $\{C_1, \dots, C_m\} \subseteq^{\otimes} \{C'_1, \dots, C'_n\}$. We obtain the reverse inequality in a similar fashion, and then we conclude since \subseteq^{\otimes} is antisymmetric, by Lemma 14.1. \square

Lemma 14.4. *Let X be a Noetherian space, and \rightsquigarrow be the smallest relation such that, for every m -product $F \mid C_1, \dots, C_m$ in reduced form (up to permutation of C_1, \dots, C_m):*

- (1) $(F \mid C_1, \dots, C_m) \rightsquigarrow (F \mid C_1, \dots, C_m, C)$, for every irreducible closed subset C of X not included in F ;
- (2) $(F \mid C_1, \dots, C_i, \dots, C_m) \rightsquigarrow (F \mid C_1, \dots, C'_i, \dots, C_m)$ for every irreducible closed subset C'_i that contains C_i strictly, for every i with $1 \leq i \leq m$;
- (3) $(F \mid C_1, \dots, C_m) \rightsquigarrow (F \cup C \mid C_1, \dots, C_k)$ for every k with $0 \leq k < m$, provided that $C_{k+1} = \dots = C_m = C$ and C_1, \dots, C_k are not included in $F \cup C$.

Let also \rightsquigarrow^+ be the transitive closure of \rightsquigarrow . For all m -products P and P' in reduced form, $P \subsetneq P'$ if and only if $P \rightsquigarrow^+ P'$.

PROOF. We note that if P is in reduced form and if $P \rightsquigarrow P'$, then P' is in reduced form as well, and $P \subseteq P'$. Additionally, $P \neq P'$, using Lemma 14.3.

Conversely, let $P \stackrel{\text{def}}{=} (F \mid C_1, \dots, C_m)$ and $P' \stackrel{\text{def}}{=} (F' \mid C'_1, \dots, C'_n)$ be in reduced form, and let us assume that $P \subseteq P'$. In particular, $F \subseteq F'$. We show that $P \rightsquigarrow^* P'$ by induction on p , where \rightsquigarrow^* is the reflexive transitive closure of \rightsquigarrow , and where p is the number of components of F' that are not included in F . This will prove our claim, since if $P \neq P'$, then the number of \rightsquigarrow steps from P to P' cannot be 0.

If $p = 0$, then F' is included in F , so $F' = F$. There is an injective map $r: \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ such that $C_i \subseteq C'_{r(i)}$ for every $i \in \{1, \dots, m\}$, using

the fact that P is in reduced form. We then have:

$$\begin{aligned}
P &= (F \mid C_1, \dots, C_m) \\
&\rightsquigarrow^* (F \mid C'_{r(1)}, \dots, C'_{r(m)}) && \text{by (2), used at most } m \text{ times} \\
&\rightsquigarrow^* (F' \mid C'_1, \dots, C'_{r(1)}, \dots, C'_{r(m)}, \dots, C'_n) && \text{by (1), used } n - m \text{ times} \\
&= P'.
\end{aligned}$$

If $p \geq 1$, then let us pick one component C of F' that is not included in F . Up to permutation, we may assume that the indices $i \in \{1, \dots, m\}$ such that $C_i \subseteq C$ are exactly those between $k+1$ and m , where $0 \leq k \leq m$. Then:

$$\begin{aligned}
P &= (F \mid C_1, \dots, C_k, C_{k+1}, \dots, C_m) \\
&\rightsquigarrow^* (F \mid C_1, \dots, C_k, \underbrace{C, \dots, C}_{m-k \text{ copies}}) && \text{using (2) at most } m - k \text{ times} \\
&\rightsquigarrow (F \mid C_1, \dots, C_k, \underbrace{C, \dots, C}_{m-k+1 \text{ copies}}) && \text{using (1)} \\
&\rightsquigarrow (F \cup C \mid C_1, \dots, C_k) && \text{by (3) (see below)} \\
&\rightsquigarrow^* P' && \text{by induction hypothesis.}
\end{aligned}$$

The use of (3) is justified by the fact that C_1, \dots, C_k are not included in C , and are not included in F since P is in reduced form; so they are not included in $F \cup C$ either, by irreducibility. The use of (1) in the previous step guarantees that the number of copies of C that we will move from the right-hand side to the left-hand side of \mid in the last step is at least 1, as is also required in order to apply (3). \square

14.1. Bounds on the sobrification rank of X^\otimes . We recall that, for every ordinal α , written in Cantor normal form as $\omega^{\alpha_1} + \dots + \omega^{\alpha_m}$, the ordinal $\alpha \otimes \omega$ is equal to $\omega^{\alpha_1+1} + \dots + \omega^{\alpha_m+1}$.

Lemma 14.5. *For all ordinals α, β and natural numbers k , if $\beta \neq 0$ then $(\alpha \otimes \omega) \oplus (\beta \otimes k) < (\alpha + \beta) \otimes \omega$.*

PROOF. We write α in Cantor normal form as $\omega^{\alpha_1} + \dots + \omega^{\alpha_m}$, with $\alpha \geq \alpha_1 \geq \dots \geq \alpha_m$, $m \in \mathbb{N}$, and similarly β as $\omega^{\beta_1} + \dots + \omega^{\beta_n}$, with $\beta \geq \beta_1 \geq \dots \geq \beta_n$. Since $\beta \neq 0$, it follows that $n \geq 1$. Let i be the largest index in $\{1, \dots, m\}$ such that $\alpha_i \geq \beta_1$, or 0 if there is none. Then $\omega^{\alpha_m} + \omega^{\beta_1}, \dots, \omega^{\alpha_{i+1}} + \omega^{\beta_1}$ are all equal to ω^{β_1} , so $\alpha + \beta = \omega^{\alpha_1} + \dots + \omega^{\alpha_i} + \omega^{\beta_1} + \dots + \omega^{\beta_n}$, and therefore $(\alpha + \beta) \otimes \omega = \omega^{\alpha_1+1} + \dots + \omega^{\alpha_i+1} + \omega^{\beta_1+1} + \dots + \omega^{\beta_n+1}$. We compare this to $(\alpha \otimes \omega) \oplus (\beta \otimes k)$, which is equal to $\omega^{\alpha_1+1} + \dots + \omega^{\alpha_i+1} + \gamma$, where $\gamma \stackrel{\text{def}}{=} (\omega^{\alpha_{i+1}+1} +$

$\dots \omega^{\alpha_m+1}) \oplus (\omega^{\beta_1} \times k + \dots + \omega^{\beta_n} \times k)$. The latter is a sum of terms of the form ω^δ with $\delta < \beta_1 + 1$, so $\gamma < \omega^{\beta_1+1} \leq \omega^{\beta_1+1} + \dots + \omega^{\beta_n+1}$. The claim follows. \square

Lemma 14.6. *Let X be a topological space, F be a closed subset of X , and C, C' be two irreducible closed subsets of X . Then:*

- (1) $C \setminus F$ is empty or irreducible closed in $X \setminus F$;
- (2) $C \setminus F \subseteq C' \setminus F$ if and only if $C \subseteq F$ or $C \subseteq C'$;
- (3) If X is Noetherian, then

$$\mathrm{rk}_{\mathcal{S}X \cup \{\emptyset\}}(C) \leq \mathrm{rk}_{\mathcal{S}X \cup \{\emptyset\}}(C') + \mathrm{rk}_{\mathcal{S}(X \setminus C') \cup \{\emptyset\}}(C \setminus C').$$

PROOF. (1) $C \setminus F$ is closed in $X \setminus F$. Let us assume that it is non-empty, namely that C is not included in F . We verify that $C \setminus F$ is irreducible. If $C \setminus F$ is included in the union of two closed subsets $F_1 \setminus F$ and $F_2 \setminus F$ of $X \setminus F$ (where F_1 and F_2 are closed in X), then C is included in $F \cup F_1 \cup F_2$. Since C is irreducible in X and not included in F , it is included in F_1 or in F_2 . Then, $C \setminus F$ is also included in $F_1 \setminus F$, or in $F_2 \setminus F$.

(2) The if direction is clear. Conversely, if $C \setminus F \subseteq C' \setminus F$, then $C \subseteq F \cup C'$, and the claim follows from the irreducibility of C .

(3) We fix C' , and we prove the inequality by well-founded induction on $C \setminus C'$. If $C \setminus C'$ is empty, then $\mathrm{rk}_{\mathcal{S}(X \setminus C') \cup \{\emptyset\}}(C \setminus C') = 0$, and $C \subseteq C'$, so $\mathrm{rk}_{\mathcal{S}X \cup \{\emptyset\}}(C) \leq \mathrm{rk}_{\mathcal{S}X \cup \{\emptyset\}}(C')$. Otherwise, it suffices to show the strict inequality $\alpha < \mathrm{rk}_{\mathcal{S}X \cup \{\emptyset\}}(C') + \mathrm{rk}_{\mathcal{S}(X \setminus C') \cup \{\emptyset\}}(C \setminus C')$ for every ordinal $\alpha < \mathrm{rk}_{\mathcal{S}X \cup \{\emptyset\}}(C)$. By the inductive definition of rank, there is a $C'' \in \mathcal{S}X \cup \{\emptyset\}$ such that $C'' \subsetneq C$ and $\mathrm{rk}_{\mathcal{S}X \cup \{\emptyset\}}(C'') = \alpha$. By induction hypothesis, $\alpha = \mathrm{rk}_{\mathcal{S}X \cup \{\emptyset\}}(C'') \leq \mathrm{rk}_{\mathcal{S}X \cup \{\emptyset\}}(C') + \mathrm{rk}_{\mathcal{S}(X \setminus C') \cup \{\emptyset\}}(C'' \setminus C')$. Now $C'' \setminus C'$ is a subset of $C \setminus C'$, and a proper one: otherwise $C \setminus C' \subseteq C'' \setminus C'$, and since $C \not\subseteq C'$, this would imply $C \subseteq C''$ by (2). Hence, using the fact that $+$ is strictly monotonic in its second argument, $\alpha < \mathrm{rk}_{\mathcal{S}X \cup \{\emptyset\}}(C') + \mathrm{rk}_{\mathcal{S}(X \setminus C') \cup \{\emptyset\}}(C \setminus C')$, as claimed. \square

Lemma 14.7. *Let X be a Noetherian space. For every m -product $P \stackrel{\mathrm{def}}{=} F \mid C_1, \dots, C_m$ in reduced form, let $\varphi(P) \stackrel{\mathrm{def}}{=} (\|F\| \otimes \omega) \oplus \bigoplus_{i=1}^m \mathrm{rk}_{\mathcal{S}(X \setminus F) \cup \{\emptyset\}}(C_i \setminus F)$. Then φ is a strictly monotonic ordinal-valued map with domain $\mathcal{S}(X^{\otimes})$.*

PROOF. We first note that $\varphi(P)$ is well defined, as $C_i \setminus F$ is an element of $\mathcal{S}(X \setminus F) \cup \{\emptyset\}$, by Lemma 14.6 (1).

Using Lemma 14.4, it suffices to show that, for all m -products P and P' in reduced form such that $P \rightsquigarrow P'$, $\varphi(P) < \varphi(P')$. This is obvious for steps of the form (1) or (2); note that in the case of form (1), $C \setminus F \neq \emptyset$ by assumption, so that $\mathrm{rk}_{\mathcal{S}(X \setminus F) \cup \{\emptyset\}}(C \setminus F) \neq 0$.

For steps of the form (3), P is of the form $F \mid C_1, \dots, C_m$, $P' = (F \cup C \mid C_1, \dots, C_k)$, $0 \leq k < m$, and $C_{k+1} = \dots = C_m = C$. (Note also that the condition that C_1, \dots, C_k are not included in $F \cup C$ implies that P' is written in reduced form.) Then, letting $\beta \stackrel{\text{def}}{=} \text{rk}_{\mathcal{S}(X \setminus F) \cup \{\emptyset\}}(C \setminus F)$, we have:

$$\varphi(P) = (\|F\| \otimes \omega) \oplus (\beta \otimes (m - k)) \oplus \bigoplus_{i=1}^k \text{rk}_{\mathcal{S}(X \setminus F) \cup \{\emptyset\}}(C_i \setminus F).$$

Using Lemma 14.6 (3), and noticing that $(C_i \setminus F) \setminus (C \setminus F) = C_i \setminus (F \cup C)$, the term $\text{rk}_{\mathcal{S}(X \setminus F) \cup \{\emptyset\}}(C_i \setminus F)$ is smaller than or equal to $\beta + \text{rk}_{\mathcal{S}(X \setminus (F \cup C)) \cup \{\emptyset\}}(C_i \setminus (F \cup C))$. The latter is smaller than or equal to $\beta \oplus \text{rk}_{\mathcal{S}(X \setminus (F \cup C)) \cup \{\emptyset\}}(C_i \setminus (F \cup C))$, since the inequality $\beta + \gamma \leq \beta \oplus \gamma$ holds for all ordinals γ . Using the associativity and the commutativity of \oplus , we obtain:

$$\varphi(P) \leq (\|F\| \otimes \omega) \oplus (\beta \otimes m) \oplus \bigoplus_{i=1}^k \text{rk}_{\mathcal{S}(X \setminus (F \cup C)) \cup \{\emptyset\}}(C_i \setminus (F \cup C)).$$

We note that, since P is reduced, $C = C_m$ is not included in F , so $\beta = \text{rk}_{\mathcal{S}(X \setminus F) \cup \{\emptyset\}}(C \setminus F)$ is non-zero. Therefore Lemma 14.5 applies, to the effect that $(\|F\| \otimes \omega) \oplus (\beta \otimes m) < (\|F\| + \beta) \otimes \omega$. It is easy to see that $\beta \leq \|C \setminus F\|$, so $\|F\| + \beta \leq \|F\| + \|C \setminus F\| = \|F\| + \|(F \cup C) \setminus F\| \leq \|F \cup C\|$, using Lemma 10.2 (2). It follows that:

$$\varphi(P) < (\|F \cup C\| \otimes \omega) \oplus \bigoplus_{i=1}^k \text{rk}_{\mathcal{S}(X \setminus (F \cup C)) \cup \{\emptyset\}}(C_i \setminus (F \cup C)) = \varphi(P').$$

This concludes the proof. \square

Proposition 14.8. *For every Noetherian space X , $\text{sob } X^{\otimes} \leq (\|X\| \otimes \omega) + 1$, or equivalently, $\text{rsob } X^{\otimes} \leq \|X\| \otimes \omega$.*

PROOF. We use Lemma 14.7, and we observe that the largest value taken by φ is obtained as $\varphi(X \mid) = \|X\| \otimes \omega$. Therefore φ takes its values in (the set of ordinals strictly below) $(\|X\| \otimes \omega) + 1$. \square

Given an ordinal α , written in Cantor normal form as $\omega^{\alpha_1} + \dots + \omega^{\alpha_m}$ with $\alpha \geq \alpha_1 \geq \dots \geq \alpha_m$, $m \in \mathbb{N}$, the ordinal $\omega \times \alpha$ is equal to $\omega^{1+\alpha_1} + \dots + \omega^{1+\alpha_m}$. This is in general smaller than or equal to $\alpha \otimes \omega = \omega^{\alpha_1+1} + \dots + \omega^{\alpha_m+1}$, and equal to it if and only if every α_i is finite, if and only if $\alpha < \omega^\omega$.

Proposition 14.9. *For every Noetherian space X , $\text{sob } X^{\otimes} \geq (\omega \times \|X\|) + 1$; equivalently, $\text{rsob } X^{\otimes} \geq \omega \times \|X\|$.*

PROOF. This is clear if $\|X\| = 0$, namely if X is empty, since then X^\otimes has exactly one element, the empty multiset $\{\emptyset\}$. Henceforth, we assume that X is non-empty, and we prove the claim by induction on $\|X\|$.

For every increment $(F, F') \in \text{Inc}(\mathcal{H}_0 X)$, $F' \setminus F$ is the equivalence class of some point x with respect to the specialization preordering of X , and $F' = F \cup C$, where $C \stackrel{\text{def}}{=} \downarrow x$, by Lemma 12.9. We will simply write such increments as $(F, F \cup C)$ in the sequel, without restating the associated requirements on F and C . Given such an increment, for every $n \in \mathbb{N}$, we form the m-product $F \mid C^n$, meaning $F \mid C, \dots, C$ with n copies of C . We claim that the map $f: \text{Inc}(\mathcal{H}_0 X) \times_{\text{lex}} \mathbb{N} \rightarrow \mathcal{S}(X^\otimes)$ defined by $f((F, F \cup C), n) \stackrel{\text{def}}{=} (F \mid C^n)$ is strictly monotonic.

Given two increments $(F, F \cup C) < (F', F' \cup C')$, we have $f((F, F \cup C), n) \subsetneq f((F', F' \cup C'), n')$ for all $n, n' \in \mathbb{N}$. Inclusion follows from the fact that both F and C are included in F' , so $(F \mid C^n) \subseteq (F \cup C)^\otimes \subseteq F'^\otimes \subseteq (F' \mid C'^{n'})$. The inclusion is strict: given $x \in C \setminus F$, the multiset that contains $n + 1$ copies of x is in $F' \mid C'^{n'}$ since in $(F \cup C)^\otimes \subseteq F'^\otimes$, but not in $F \mid C^n$.

Next, we show that $f((F, F \cup C), n) = F \mid C^n$ is strictly included in $F \mid C^{n'}$ for all $n < n'$. The inclusion is obvious. Given $x \in C \setminus F$, the multiset containing exactly n' copies of x is in $F \mid C^{n'}$ but not in $F \mid C^n$.

Since f is strictly monotonic, $\text{rk}_{\mathcal{H}_0(X^\otimes)}(F \mid C^n) \geq \text{rk}_{\text{Inc}(\mathcal{H}_0 X) \times_{\text{lex}} \mathbb{N}}((F, F \cup C), n) = \omega \times \text{rk}_{\text{Inc}(\mathcal{H}_0 X)}(F, F \cup C) + n$. When $(F, F \cup C)$ varies and $n = 0$, the supremum of those quantities is $\omega \times \|X\|$, using Lemma 12.17 and the fact that $|\mathcal{H}_0 X| = \|X\| + 1$. Therefore $\text{sob } X^\otimes \geq (\omega \times \|X\|) + 1$, proving the claim. \square

As we said, the lower bound and the upper bound only match provided that $\|X\| < \omega^\omega$. We state the following for the record.

Theorem 14.10. *Let X be a Noetherian space such that $\|X\| < \omega^\omega$, namely $\|X\|$ is of the form $\omega^{n_1} + \dots + \omega^{n_m}$, where $n_1 \geq \dots \geq n_m$ are natural numbers. Then $\text{sob } X^\otimes = \omega^{n_1+1} + \dots + \omega^{n_m+1} + 1$.*

14.2. The bounds on $\text{sob } X^\otimes$ are tight. The upper and lower bounds of Proposition 14.8 and Proposition 14.9 do not match, unless $\|X\| < \omega^\omega$. This cannot be improved upon, as we now see, by realizing that both the lower bound and the upper bound can be attained.

Proposition 14.11. *For every Noetherian space Z , let $X \stackrel{\text{def}}{=} Z^*$ and $\alpha \stackrel{\text{def}}{=} \|Z\|$. If $\alpha \geq 2$, then the lower bound on $\text{sob } X^\otimes$ given in Lemma 14.9 is attained: $\text{sob } X^\otimes = \omega^{\omega^{\alpha'}} + 1 = (\omega \times \|X\|) + 1$.*

PROOF. For every proper irreducible closed subset P of X^\otimes , P is an m-product $F \mid C_1, \dots, C_m$, which we write in reduced form. Since $P \neq X^\otimes$, F is a proper closed subset of $X = Z^*$, so $\|F\| < \|Z^*\|$. By Theorem 12.22, $\|Z^*\| = \omega^{\omega^{\alpha'}}$. Hence $\|F\| < \omega^{\omega^{\alpha'}}$. We write $\|F\|$ in Cantor normal form as $\omega^{\beta_1} + \dots + \omega^{\beta_n}$, with $\omega^{\alpha'} > \beta_1 \geq \dots \geq \beta_n$. Since $\omega^{\alpha'}$ is additively indecomposable and $\omega^{\alpha'} > 1$ (we recall that $\alpha \geq 2$, so $\alpha' \geq 1$), we also have $\omega^{\alpha'} > \beta_1 + 1 \geq \dots \geq \beta_n + 1$. Therefore $\|F\| \otimes \omega < \omega^{\omega^{\alpha'}}$.

By Theorem 12.13, $\text{sob } X = \omega^{\alpha^\circ} + 1$. By Corollary 5.7, $\text{sob } (X \setminus F) \leq \text{sob } X$, so, for every $i \in \{1, \dots, m\}$, $\text{rk}_{\mathcal{S}(X \setminus F)}(C_i \setminus F) \leq \text{sob } (X \setminus F) - 1 \leq \omega^{\alpha^\circ}$. Since $\alpha \geq 2$, $\alpha^\circ \geq 1$, so ω^{α° is infinite, and therefore $1 + \omega^{\alpha^\circ} = \omega^{\alpha^\circ}$. It follows that $\text{rk}_{\mathcal{S}(X \setminus F) \cup \{\emptyset\}}(C_i \setminus F) = 1 + \text{rk}_{\mathcal{S}(X \setminus F)}(C_i \setminus F) \leq \omega^{\alpha^\circ}$. We note that $\alpha^\circ < \omega^{\alpha'}$. (If $\alpha \geq 2$ is finite, then $\alpha^\circ = \alpha$ and $\alpha' = \alpha - 1$, and $\alpha < \omega^{\alpha-1}$ because the right-hand side is infinite. If α is infinite, then $\alpha^\circ = \alpha'$, and this is never a critical ordinal. We have $\alpha' \leq \omega^{\alpha'}$, and the inequality is strict, since α' is not critical.) It follows that $\omega^{\alpha^\circ} < \omega^{\omega^{\alpha'}}$, so $\text{rk}_{\mathcal{S}(X \setminus F) \cup \{\emptyset\}}(C_i \setminus F) < \omega^{\omega^{\alpha'}}$.

We use the map φ of Lemma 14.7: $\varphi(P)$ is the natural sum of $\|F\| \otimes \omega < \omega^{\omega^{\alpha'}}$, and of finitely many terms $\text{rk}_{\mathcal{S}(X \setminus F) \cup \{\emptyset\}}(C_i \setminus F)$, which are all strictly less than $\omega^{\omega^{\alpha'}}$. Since $\omega^{\omega^{\alpha'}}$ is \oplus -indecomposable, $\varphi(P) < \omega^{\omega^{\alpha'}}$. Adding one and taking suprema over all proper irreducible closed subsets P of X^\otimes , we obtain that the rank of X^\otimes in $\mathcal{S}(X^\otimes)$ is less than or equal to $\omega^{\omega^{\alpha'}}$. Therefore $\text{sob } X^\otimes \leq \omega^{\omega^{\alpha'}} + 1$.

By Proposition 14.9, we have $\text{sob } X^\otimes \geq (\omega \times \|X\|) + 1$. We note that $\omega \times \|X\| = \omega^{1+\omega^{\alpha'}} = \omega^{\omega^{\alpha'}}$, because $\omega^{\alpha'}$ is infinite, and the result follows. \square

We now give a class of examples where the upper bound, instead of the lower bound, is attained.

Proposition 14.12. *For every ordinal α , with its Alexandroff topology, the upper bound on $\text{sob } \alpha^\otimes$ given in Lemma 14.8 is reached: $\text{sob } \alpha^\otimes = (\|\alpha\| \otimes \omega) + 1 = (\alpha \otimes \omega) + 1$.*

PROOF. The key is that every ordinal $\beta < \alpha$ is a closed subset, and is irreducible provided that $\beta \neq 0$. This will allow us to form m-products of the form $\beta \mid \gamma^n$ for various ordinals β and $\gamma, \gamma \neq 0$.

We recall that $\|\alpha\| = \alpha$, by Lemma 6.2 (1). This means that $(\alpha \otimes \omega) + 1$ is indeed the upper bound of Proposition 14.8.

Let us write α in Cantor normal form as $\omega^{\alpha_1} + \dots + \omega^{\alpha_m}$ with $\alpha \geq \alpha_1 \geq \dots \geq \alpha_m$. For each $i \in \{1, \dots, m+1\}$, let $\bar{\alpha}_i \stackrel{\text{def}}{=} \omega^{\alpha_1} + \dots + \omega^{\alpha_{i-1}}$. We note that $\bar{\alpha}_1 = 0$ and that $\bar{\alpha}_{m+1} = \alpha$.

For the time being, let us fix $i \in \{1, \dots, m\}$. For every $n \in \mathbb{N}$ and every ordinal $\beta \leq \omega^{\alpha_i}$, we form the m-product $P_{i,n,\beta} \stackrel{\text{def}}{=} \bar{\alpha}_i \mid \underbrace{\bar{\alpha}_{i+1}, \dots, \bar{\alpha}_{i+1}}_n, \bar{\alpha}_i + \beta$. Let also $P_{i,n} \stackrel{\text{def}}{=} \bar{\alpha}_i \mid \underbrace{\bar{\alpha}_{i+1}, \dots, \bar{\alpha}_{i+1}}_n$; this is simply $P_{i,n,0}$, written in reduced form.

For each fixed n , we claim that $\text{rk}_{S(\alpha^{\otimes})}(P_{i,n,\beta}) \geq \text{rk}_{S(\alpha^{\otimes})}(P_{i,n}) + \beta$. It suffices to observe that the map $\beta \mapsto P_{i,n,\beta}$ is strictly monotonic, and to induct on β . It is clear that $\beta \leq \beta'$ implies $P_{i,n,\beta} \subseteq P_{i,n,\beta'}$. If $\beta < \beta'$, we reason by contradiction, and we assume that $P_{i,n,\beta} = P'_{i,n,\beta'}$. Since $\beta' \neq 0$, $P_{i,n,\beta'}$ is already in reduced form. If $\beta = 0$, then the reduced form of $P_{i,n,\beta}$ is the expression we gave for $P_{i,n}$, which does not have the same number of terms to the right of the vertical bar. This would contradict the equality $P_{i,n,\beta} = P'_{i,n,\beta'}$, by Lemma 14.3. Hence $\beta \neq 0$. Lemma 14.3 then implies that the lists $\underbrace{\bar{\alpha}_{i+1}, \dots, \bar{\alpha}_{i+1}}_n, \bar{\alpha}_i + \beta$ and $\underbrace{\bar{\alpha}_{i+1}, \dots, \bar{\alpha}_{i+1}}_n, \bar{\alpha}_i + \beta'$ are equal up to permutation. It is easy to see that this implies $\bar{\alpha}_i + \beta = \bar{\alpha}_i + \beta'$. Since ordinal addition is left-cancellative, $\beta = \beta'$, which is impossible.

We observe that $P_{i,n+1} = P_{i,n,\omega^{\alpha_i}}$, because $\bar{\alpha}_i + \omega^{\alpha_i} = \bar{\alpha}_{i+1}$. It follows that $\text{rk}_{S(\alpha^{\otimes})}(P_{i,n+1}) \geq \text{rk}_{S(\alpha^{\otimes})}(P_{i,n}) + \omega^{\alpha_i}$. By induction on $n \in \mathbb{N}$, and observing that $P_{i,0} = \bar{\alpha}_i^{\otimes}$, we have that $\text{rk}_{S(\alpha^{\otimes})}(P_{i,n}) \geq \text{rk}_{S(\alpha^{\otimes})}(\bar{\alpha}_i^{\otimes}) + \omega^{\alpha_i} \times n$. Since $P_{i,n}$ is included in $\bar{\alpha}_{i+1}^{\otimes}$ for every $n \in \mathbb{N}$, it follows that $\text{rk}_{S(\alpha^{\otimes})}(\bar{\alpha}_{i+1}^{\otimes}) \geq \text{rk}_{S(\alpha^{\otimes})}(\bar{\alpha}_i^{\otimes}) + \sup_{n \in \mathbb{N}}(\omega^{\alpha_i} \times n) = \text{rk}_{S(\alpha^{\otimes})}(\bar{\alpha}_i^{\otimes}) + \omega^{\alpha_i+1}$.

We now induct on $i \in \{1, \dots, m+1\}$, and we obtain that $\text{rk}_{S(\alpha^{\otimes})}(\bar{\alpha}_i^{\otimes}) \geq \omega^{\alpha_1+1} + \dots + \omega^{\alpha_{i-1}+1}$. When $i = m+1$, this implies that $\text{rk}_{S(\alpha^{\otimes})}(\alpha^{\otimes}) \geq \omega^{\alpha_1+1} + \dots + \omega^{\alpha_m+1} = \alpha \otimes \omega$. Hence $\text{sob } \alpha^{\otimes} \geq (\alpha \otimes \omega) + 1$. The reverse inequality is by Proposition 14.8. \square

14.3. The stature of X^{\otimes} . Working as for Lemma 12.14, we obtain the following, non-optimal upper bound.

Lemma 14.13. *For every non-empty Noetherian space X , $\|X^{\otimes}\| \leq \omega^{\alpha \otimes \omega}$, where $\alpha \stackrel{\text{def}}{=} \|X\|$.*

PROOF. X^{\otimes} is irreducible closed, so the space X^{\otimes} has exactly one component. By Proposition 4.5 (3), $\|X^{\otimes}\| \leq \omega^{\text{sob } X^{\otimes}-1}$, and $\text{sob } X^{\otimes} - 1 \leq \|X\| \otimes \omega$ by Proposition 14.8. \square

In order to obtain a better upper bound, we observe the following.

Lemma 14.14. *Let X be a Noetherian space, and F be a subset of X . The map $\uplus: F^{\otimes} \times (X \setminus F)^{\otimes} \rightarrow X^{\otimes}$, which maps (m_1, m_2) to $m_1 \uplus m_2$, is continuous and bijective.*

PROOF. Bijectivity is obvious. The inverse function maps every $m \in X^{\otimes}$ to the pair $(m \cap F, m \setminus F)$, where $m \cap F$ denotes the multiset of those elements of m that are in F (i.e., recalling that a multiset is a function with values in \mathbb{N} , this is the function that maps every $x \in F$ to $m(x)$, and all other elements to 0), and $m \setminus F$ is the multiset of those elements of m that are not in F .

We recall that a base of the topology on X^{\otimes} is given by the sets $\langle U_1, \dots, U_n \rangle$, where U_1, \dots, U_n range over the open subsets of X and $n \in \mathbb{N}$. Let (m_1, m_2) be such that $m_1 \uplus m_2 \in \langle U_1, \dots, U_n \rangle$. We can write $m_1 \uplus m_2$ as $m \cup \{x_1, \dots, x_n\}$ where each x_i is in U_i . We look at the indices i such that $x_i \in F$. Up to permutation, we assume that those are the indices $1, \dots, k$, where $0 \leq k \leq n$. The open set $\langle U_1 \cap F, \dots, U_k \cap F \rangle \times \langle U_{k+1} \setminus F, \dots, U_n \setminus F \rangle$ is then an open neighborhood of (m_1, m_2) whose image by \uplus is included in $\langle U_1, \dots, U_n \rangle$. \square

Remark 14.15. The inverse of the bijection \uplus of Lemma 14.14 is not continuous in general, even when F is closed. The problem is that the map $m \mapsto m \cap F$ is not even monotonic, and every continuous map is necessarily monotonic with respect to the underlying specialization preorderings. For example, let $X \stackrel{\text{def}}{=} \{1, 2\}$ in its Alexandroff topology, with $1 < 2$, and let $F \stackrel{\text{def}}{=} \{1\}$. Let $m \stackrel{\text{def}}{=} \{1\}$ and $m' \stackrel{\text{def}}{=} \{2\}$. Then $m \leq^{\otimes} m'$, but $m \cap F = \{1\} \not\leq^{\otimes} m' \cap F = \emptyset$.

Proposition 14.16. *For every non-empty Noetherian space X , $\|X^{\otimes}\| \leq \omega^{\hat{\alpha}}$, where $\alpha \stackrel{\text{def}}{=} \|X\|$.*

PROOF. We prove this by induction on α . We first note that there is a continuous map $f: X^{\otimes} \times X_{\perp} \rightarrow X^{\otimes}$, which maps (m, x) to $m \uplus \{x\}$ if $x \neq \perp$, and to m if $x = \perp$. In order to see that f is continuous, we note that for all open subsets U_1, \dots, U_n of X , $f^{-1}(\langle U_1, \dots, U_n \rangle)$ is equal to the union of $\langle U_1, \dots, U_n \rangle \times X_{\perp}$ and of the sets $\langle U_1, \dots, U_{i-1}, U_{i+1}, \dots, U_n \rangle \times U_i$, $1 \leq i \leq n$, which are all open.

We first deal with the case where $\alpha = \omega^{\beta}$ for some ordinal β . If $\beta = 0$, namely if $\alpha = 1$, then there is just one non-empty closed subset of X , and that is X itself. As such, it is irreducible, and the m-products are just $\emptyset \mid X^n$, $n \in \mathbb{N}$, plus $X \mid$. They form a chain, so any finite union of m-products is equal to just one of them. In other words, the sets $\emptyset \mid X^n$, $n \in \mathbb{N}$, and $X \mid$, exhaust all the non-empty closed subsets of X . Then $\|X^{\otimes}\| = \omega$, which is indeed equal to $\omega^{\hat{\alpha}}$, since $\hat{\alpha} = \omega^{\beta^{\circ}} = 1$.

Henceforth, we assume that $\alpha = \omega^{\beta}$ and that $\beta \geq 1$.

For every proper closed subset F of X , $\|F\| < \alpha = \omega^\beta$. Writing $\|F\|$ in Cantor normal form as $\omega^{\beta_1} + \dots + \omega^{\beta_n}$, this entails that $\beta_i < \beta$ for every i . By Lemma 12.3, $\beta_i^\circ < \beta^\circ$ for every i , so $\widehat{\|F\|} < \omega^{\beta^\circ}$. Using the induction hypothesis, we obtain that $\|F^\circ\| \leq \omega^{\widehat{\|F\|}} = \omega^{\omega^{\beta_1^\circ} + \dots + \omega^{\beta_n^\circ}} < \omega^{\omega^{\beta^\circ}}$.

For every $n \in \mathbb{N}$, $F \mid X^n$ is a closed subset of X° , although not necessarily an m-product. Indeed, letting C_1, \dots, C_m be the components of X , $F \mid X^n$ is equal to the finite union of the m-products $F \mid C_{i_1}, \dots, C_{i_n}$, where $1 \leq i_1, \dots, i_n \leq m$. Then f restricts to a continuous map from $(F \mid X^n) \times X_\perp$ to $F \mid X^{n+1}$, and that restriction is surjective. By Lemma 5.3, Theorem 10.9, and Proposition 9.6, $\|F \mid X^{n+1}\| \leq \|(F \mid X^n) \times X_\perp\| = \|F \mid X^n\| \otimes \|X_\perp\| = \|F \mid X^n\| \otimes (1 + \alpha)$. Since $\beta \geq 1$, $\alpha = \omega^\beta$ is infinite, so $1 + \alpha = \omega^\beta$, and we obtain that $\|F \mid X^{n+1}\| \leq \|F \mid X^n\| \otimes \omega^\beta$. By induction on n , $\|F \mid X^n\| \leq \|F^\circ\| \otimes \omega^{\beta \otimes n}$. We recall that $\|F^\circ\| \leq \omega^{\omega^{\beta_1^\circ} + \dots + \omega^{\beta_n^\circ}}$. For each i , $\beta_i^\circ < \beta^\circ$, so $\omega^{\beta_i^\circ} < \omega^{\beta^\circ}$. Also, $\beta \otimes n$ is the natural sum of n copies of β , and $\beta < \omega^{\beta^\circ}$ by Lemma 12.4. Since ω^{β° is additively indecomposable and \oplus -indecomposable, $\omega^{\beta_1^\circ} + \dots + \omega^{\beta_n^\circ} \oplus (\beta \otimes n) < \omega^{\beta^\circ}$. It follows that $\|F \mid X^n\| < \omega^{\omega^{\beta^\circ}} = \omega^{\widehat{\alpha}}$, for every proper closed subset F of X and for every $n \in \mathbb{N}$.

For every finite family of proper closed subsets $\mathbf{A}_1, \dots, \mathbf{A}_k$ of X° , there is a surjective continuous map $g: \mathbf{A}_1 + \dots + \mathbf{A}_k \rightarrow \bigcup_{i=1}^k \mathbf{A}_i$, which maps every $m \in \mathbf{A}_i$ to itself. By Lemma 5.3, $\|\bigcup_{i=1}^k \mathbf{A}_i\| \leq \|\mathbf{A}_1 + \dots + \mathbf{A}_k\| = \bigoplus_{i=1}^k \|\mathbf{A}_i\|$.

Every proper closed subset \mathbf{A} of X° is a finite union of m-products $\mathbf{A}_i \stackrel{\text{def}}{=} F_i \mid C_{i_1}, \dots, C_{i_{n_i}} \subseteq F_i \mid X^{n_i}$. Each of them is proper, so F_i is a proper closed subset of X . It follows that $\|\mathbf{A}\| \leq \bigoplus_{i=1}^k \|F_i \mid X^{n_i}\|$. Since each term $\|F_i \mid X^{n_i}\|$ is strictly less than $\omega^{\widehat{\alpha}}$, and since the latter is \oplus -indecomposable, $\|\mathbf{A}\| < \omega^{\widehat{\alpha}}$. By Proposition 5.9, $\|X^\circ\| \leq \omega^{\widehat{\alpha}}$.

Finally, we deal with the general case, where α is not necessarily of the form ω^β . We write α in Cantor normal form as $\omega^{\alpha_1} + \dots + \omega^{\alpha_m}$, with $\alpha_1 \geq \dots \geq \alpha_m$, $m \in \mathbb{N}$. By Lemma 10.6, there are closed subsets $X = F_m \supseteq \dots \supseteq F_1 \supseteq F_0 = \emptyset$ such that $\|F_i \setminus F_{i-1}\| = \omega^{\alpha_i}$ for every i , $1 \leq i \leq m$. Using Lemma 14.14, the multiset union map from $(F_1 \setminus F_0)^\circ \times (F_2 \setminus F_1)^\circ \times \dots \times (F_m \setminus F_{m-1})^\circ$ to X° is continuous and surjective. By Lemma 5.3 and Theorem 10.9, $\|X^\circ\| \leq \bigotimes_{i=1}^m \|(F_i \setminus F_{i-1})^\circ\|$. Since $\|F_i \setminus F_{i-1}\| = \omega^{\alpha_i}$, we have already shown that $\|(F_i \setminus F_{i-1})^\circ\| \leq \omega^{\widehat{\omega^{\alpha_i}}}$, so $\|X^\circ\| \leq \omega^{\bigoplus_{i=1}^m \widehat{\omega^{\alpha_i}}}$. Now the $\widehat{}$ operation commutes with finite natural sums, as one can check by oneself, or by referring to [MRW15, Notation 1]; so $\|X^\circ\| \leq \omega^{\widehat{\alpha}}$. \square

For every m-product $P \stackrel{\text{def}}{=} F | C_1, \dots, C_m$, for every $n \in \mathbb{N}$ and every irreducible closed subset C of X , we write $P \cdot C^n$ for the m-product $F | C_1, \dots, C_m, C^n$. For every closed subset \mathbf{A} of X^\otimes , which one can write as a finite union of m-products P_1, \dots, P_k , we write $\mathbf{A} \cdot C^n$ for the union of the m-products $P_1 \cdot C^n, \dots, P_k \cdot C^n$. Clearly, $\mathbf{A} \cdot C^n$ is closed. This is the set of multisets that one can write as $m \uplus m'$, where $m \in \mathbf{A}$ and m' is a multiset containing at most n elements, all taken from C .

Lemma 14.17. *Let F be a closed subset of a Noetherian space X , and C be an irreducible closed subset of X that is not included in F . The map $(n, \mathbf{A}) \mapsto (F | C^n) \cup (\mathbf{A} \cdot C^{n+1})$ is a strictly monotonic map from $\mathbb{N} \times_{\text{lex}} (\mathcal{H}_0(F^\otimes) \setminus \{F^\otimes\})$ to $\mathcal{H}_0((F \cup C)^\otimes) \setminus \{(F \cup C)^\otimes\}$.*

PROOF. Let f denote this map. Let us also fix a point x in $C \setminus F$.

Let $n \in \mathbb{N}$ and \mathbf{A} be a proper closed subset of F^\otimes . It is clear that $f(n, \mathbf{A})$ is a closed subset of X^\otimes included in $(F \cup C)^\otimes$, hence a closed subset of $(F \cup C)^\otimes$. There is a multiset m in $F^\otimes \setminus \mathbf{A}$, and this allows us to form the multiset $x^{n+1} \uplus m$ obtained by adding $n+1$ copies of x to m . It is clear that $x^{n+1} \uplus m$ is not in $F | C^n$, since any element of $F | C^n$ can contain at most n elements of $C \setminus F$. We claim that $x^{n+1} \uplus m$ is not in $\mathbf{A} \cdot C^{n+1}$ either. If it were, $x^{n+1} \uplus m$ would be equal to the union $m_1 \uplus m_2$ of a multiset m_1 in \mathbf{A} , hence consisting of elements of F , plus a multiset m_2 of at most $n+1$ elements of C . The $n+1$ copies of x in $x^{n+1} \uplus m$ cannot be in m_1 , hence would comprise the totality of m_2 ; this entails that $m_1 = m$, which is impossible since m is not in \mathbf{A} . It follows that $x^{n+1} \uplus m$ is not in $f(n, \mathbf{A}) = (F | C^n) \cup (\mathbf{A} \cdot C^{n+1})$, showing that $f(n, \mathbf{A})$ is a proper closed subset of $(F \cup C)^\otimes$.

If $n < n'$, then $(F | C^n) \subseteq (F | C^{n'})$, and $(\mathbf{A} \cdot C^{n+1}) \subseteq (F | C^{n'})$, since $\mathbf{A} \subseteq F^\otimes$ and $n+1 \leq n'$. Therefore $f(n, \mathbf{A}) \subseteq f(n', \mathbf{A})$. The inclusion is strict: given any $m \in F^\otimes \setminus \mathbf{A}$, $x^{n+1} \uplus m$ is in $f(n', \mathbf{A})$ but not in $f(n, \mathbf{A})$.

It remains to verify that $\mathbf{A} \subsetneq \mathbf{B}$ implies $f(n, \mathbf{A}) \subsetneq f(n, \mathbf{B})$. The inclusion is clear. We pick $m \in \mathbf{B} \setminus \mathbf{A}$, and we note that $x^{n+1} \uplus m$ is in $\mathbf{B} \cdot C^{n+1} \subseteq f(n, \mathbf{B})$ but not in $f(n, \mathbf{A})$. \square

Corollary 14.18. *Let F be a closed subset of a Noetherian space X , and C be an irreducible closed subset of X that is not included in F . Then $\|(F \cup C)^\otimes\| \geq \|F^\otimes\| \times \omega$.*

PROOF. Let us call f the map of Lemma 14.17. For every $n \in \mathbb{N}$ and every proper closed subset \mathbf{A} of F^\otimes , $\text{rk}_{\mathcal{H}_0((F \cup C)^\otimes)}(f(n, \mathbf{A})) \geq |\mathcal{H}_0(F^\otimes) \setminus \{F^\otimes\}| \times n +$

$\text{rk}_{\mathcal{H}_0(F^\otimes)}(\mathbf{A})$. We note that $|\mathcal{H}_0(F^\otimes) \setminus \{F^\otimes\}| = \|F^\otimes\|$, and that the least ordinal strictly larger than all those ranks is $\|F^\otimes\| \times \omega$. \square

A very similar argument shows the following.

Lemma 14.19. *Let F be a closed subset of a Noetherian space X , such that $\|X \setminus F\|$ is a critical ordinal ϵ . Then $\|X^\otimes\| \geq \|F^\otimes\| \times \epsilon \times \omega$.*

PROOF. For every closed subset C of X , for every $n \in \mathbb{N}$, let C^n denote the set of multisets of at most n elements, all in C . For every irreducible closed subset \mathbf{A} of F^\otimes , let us write $\mathbf{A} \cdot C^n$ for the set of multisets of the form $m \uplus m'$ where $m \in \mathbf{A}$ and $m' \in C^n$. Writing \mathbf{A} as a finite union of m-products $F_i \mid C_{i1}, \dots, C_{in_i}$, $\mathbf{A} \cdot C^n$ is the union of the m-products $F_i \mid C_{i1}, \dots, C_{in_i}, C, \dots, C$, with n copies of C ; so $\mathbf{A} \cdot C^n$ is closed.

For every irreducible closed subset C of $X \setminus F$, we write $cl(C)$ for its closure in X . Then $cl(C)$ is irreducible closed in X and $cl(C) \cap (X \setminus F) = C$ [GL13, Lemma 8.4.10]. For every $n \in \mathbb{N}$, we form the set $F \mid X^n, cl(C)$ of all multisets $m \uplus m'$ where $m \in F^\otimes$, and m' contains at most n elements, or exactly $n+1$ of which one is in $cl(C)$. Write X as a finite union of components C_1, \dots, C_m , $F \mid X^n, cl(C)$ is equal to the union of the m-products $F \mid C_{i1}, \dots, C_{in}, cl(C)$, where i_1, \dots, i_n range over $\{1, \dots, m\}$. Hence $F \mid X^n, cl(C)$ is closed.

Let $Y \stackrel{\text{def}}{=} \text{Step}(\mathcal{S}(X \setminus F))$, and Z be the poset $\mathcal{H}_0(F^\otimes) \setminus \{F^\otimes\}$ of all proper closed subsets of F^\otimes . For every $n \in \mathbb{N}$, for every $(C, C^+) \in Y$, for every $\mathbf{A} \in Z$, we let $f(n, (C, C^+), \mathbf{A}) \stackrel{\text{def}}{=} (\mathbf{A} \cdot cl(C^+)^{n+1}) \cup (F \mid X^n, cl(C))$. We claim that f is strictly monotonic from $\mathbb{N} \times_{\text{lex}} Y \times_{\text{lex}} Z$ to $\mathcal{H}_0(X^\otimes)$. We will write x^n for the multiset containing exactly n copies of x .

If $n < n'$, then for all (C, C^+) and (C', C'^+) in Y , for all $\mathbf{A}, \mathbf{A}' \in Z$, $\mathbf{A} \cdot cl(C^+)^{n+1}$ is included in $F^\otimes \cdot X^{n+1} = F \mid X^{n+1}$, hence in $F \mid X^{n'}, cl(C')$; also, $F \mid X^n, cl(C)$ is included in $F \mid X^{n'}$, hence in $F \mid X^{n'}, cl(C')$. It follows that $f(n, (C, C^+), \mathbf{A}) \subseteq f(n', (C', C'^+), \mathbf{A}')$. With the aim of showing that this inclusion is strict, we observe that C is a proper subset of C^+ , so we can pick an element x in $C^+ \setminus C$. In particular, x is in C^+ , hence not in F . Then x^{n+2} is in $\mathbf{A}' \cdot cl(C'^+)^{n'+1}$, and therefore in $f(n', (C', C'^+), \mathbf{A}')$. We claim that it is not in $f(n, (C, C^+), \mathbf{A})$. It is not in $\mathbf{A} \cdot cl(C^+)^{n+1}$ since all the multisets in that set are in $F^\otimes \cdot cl(C^+)^{n+1}$, and can therefore only contain at most $n+1$ elements of C^+ . If x^{n+2} were in $F \mid X^n, cl(C)$, then one of the $n+2$ copies of x would have to be in F , which is impossible.

Let us now fix $n \in \mathbb{N}$, let (C, C^+) and (C', C'^+) be two elements of Y such that $(C, C^+) < (C', C'^+)$, namely such that $C^+ \subseteq C'$, and let $\mathbf{A}, \mathbf{A}' \in Z$. It

is clear that $F \mid X^n, cl(C)$ is included in $F \mid X^n, cl(C')$, since $C \subsetneq C^+ \subseteq C'$. Since $\mathbf{A} \subseteq F^\otimes$ and $C^+ \subseteq C'$, $\mathbf{A} \cdot cl(C^+)^{n+1}$ is also included in $F \mid X^n, cl(C')$, so $f(n, (C, C^+), \mathbf{A}) \subseteq f(n, (C', C'^+), \mathbf{A}')$. In order to show that the inclusion is strict, let us pick an element x of $C^+ \setminus C$, and let us note that x is in C' (and hence, not in F). Since \mathbf{A} is in Z , it is strictly included in F^\otimes , so there is a multiset m' in $F^\otimes \setminus \mathbf{A}$. We see that $x^{n+1} \uplus m'$ is in $F \mid X^n, cl(C')$, hence in $f(n, (C', C'^+), \mathbf{A}')$, and we claim that it is not in $f(n, (C, C^+), \mathbf{A})$. We first assume that $x^{n+1} \uplus m'$ is in $\mathbf{A} \cdot cl(C^+)^{n+1}$, namely that it splits as $m_1 \uplus m_2$ where $m_1 \in \mathbf{A}$ and m_2 consists of at most $n+1$ elements, all from $cl(C^+)$. Since $\mathbf{A} \subseteq F^\otimes$ and x is not in F , the $n+1$ copies of x cannot be part of m_1 ; so $m_2 = x^{n+1}$, and therefore $m_1 = m'$. This entails that m' is in \mathbf{A} , which is impossible. Second, we assume that $x^{n+1} \uplus m'$ is in $F \mid X^n, cl(C)$. Among the $n+1$ copies of x , none is in F , so that at least one is in $cl(C)$. Remembering that x is not in F , x would be in $cl(C) \cap (X \setminus F)$, which, as we have said earlier, is equal to C . This is impossible, since we have chosen x in $C^+ \setminus C$.

Finally, let $n \in \mathbb{N}$, $(C, C^+) \in Y$, and let $\mathbf{A}, \mathbf{A}' \in Z$ be such that $\mathbf{A} \subsetneq \mathbf{A}'$. It is clear that $f(n, (C, C^+), \mathbf{A}) \subseteq f(n, (C, C^+), \mathbf{A}')$. In order to show that the inclusion is strict, let $m' \in \mathbf{A}' \setminus \mathbf{A}$. Let also $x \in C^+ \setminus C$ (hence $x \in X \setminus F$, as before). Then $x^{n+1} \uplus m'$ is in $\mathbf{A}' \cdot cl(C^+)^{n+1}$, hence in $f(n, (C, C^+), \mathbf{A}')$. If it were in $F \mid X^n, cl(C)$, then among the $n+1$ copies of x , since none is in F , one would be in $cl(C)$, hence in $cl(C) \cap (X \setminus F) = C$, which is impossible. If $x^{n+1} \uplus m'$ were in $\mathbf{A} \cdot cl(C^+)^{n+1}$, it would split as $m_1 \uplus m_2$ with $m_1 \in \mathbf{A}$ and $m_2 \in cl(C^+)^{n+1}$. Since $\mathbf{A} \subseteq F$, the $n+1$ copies of x cannot be in m_1 , and are therefore in m_2 . It follows that $m_2 = x^{n+1}$, so that $m_1 = m'$; but m' is not in \mathbf{A} , while m_1 is. In any case, we conclude that $x^{n+1} \uplus m'$ is not in $f(n, (C, C^+), \mathbf{A})$.

This finishes to prove that f is strictly monotonic.

We claim that $\text{rsob}(X \setminus F) \geq \epsilon$. If not then we would have that $\text{rsob}(X \setminus F) < \epsilon$, so $\omega^{\text{rsob}(X \setminus F)} < \omega^\epsilon$, and since ω^ϵ is \oplus -indecomposable, $\omega^{\text{rsob}(X \setminus F)} \otimes n < \omega^\epsilon$ for every $n \in \mathbb{N}$. By Proposition 4.5 (3), and letting n be the number of components of $X \setminus F$, $\epsilon = \|X \setminus F\| \leq \omega^{\text{rsob}(X \setminus F)} \otimes n < \omega^\epsilon = \epsilon$, which is impossible.

By Proposition 12.17, $|Y| = |\mathcal{S}(X \setminus F)| - 1 = \text{rsob}(X \setminus F)$, so $|Y| \geq \epsilon$. Since f is strictly monotonic, for all $n \in \mathbb{N}$, $(C, C^+) \in Y$, and $\mathbf{A} \in Z$, we have $\text{rk}_{\mathcal{H}_0(X^\otimes)}(f(n, (C, C^+), \mathbf{A})) \geq |Z| \times (|Y| \times n + \text{rk}_Y(C, C^+)) + \text{rk}_Z(\mathbf{A})$. Since $|Y| \geq \epsilon$ and $|Z| = \|F^\otimes\|$, we obtain that $\text{rk}_{\mathcal{H}_0(X^\otimes)}(f(n, (C, C^+), \mathbf{A})) \geq \|F^\otimes\| \times (\epsilon \times n + \text{rk}_Y(C, C^+)) + \text{rk}_Z(\mathbf{A})$. When n, C and \mathbf{A} vary, the smallest ordinal strictly larger than the latter is at least $\|F^\otimes\| \times \epsilon \times \omega$, so $\|X^\otimes\| \geq \|F^\otimes\| \times \epsilon \times \omega$. \square

Theorem 14.20. *For every Noetherian space X , $\|X^\otimes\| = \omega^{\hat{\alpha}}$, where $\hat{\alpha} \stackrel{\text{def}}{=} \|X\|$.*

PROOF. Considering Proposition 14.16, it suffices to show that $\|X^\circledast\| \geq \omega^{\widehat{\alpha}}$. We do this by induction on α . When $\alpha = 0$, X is empty, and there is exactly one element of X^\circledast , so $\|X^\circledast\| = 1 = \omega^0 = \omega^{\widehat{\alpha}}$.

Otherwise, let us write α in Cantor normal form as $\omega^{\alpha_1} + \cdots + \omega^{\alpha_m}$ with $\alpha \geq \alpha_1 \geq \cdots \geq \alpha_m$, $m \geq 1$. Let $\beta \stackrel{\text{def}}{=} \omega^{\alpha_1} + \cdots + \omega^{\alpha_{m-1}}$.

If $\alpha_m = 0$, then $\|X\| = \beta + 1$. There is a proper closed subset F of X such that $\text{rk}_{\mathcal{H}_0 X}(F) = \beta$, namely such that $\|F\| = \beta$, by Lemma 5.8. By Lemma 12.9, we can write X as $F \cup C$, where $C \stackrel{\text{def}}{=} \downarrow x$, and x is a point of $X \setminus F$. By Corollary 14.18, $\|X^\circledast\|$ is then larger than or equal to $\|F^\circledast\| \times \omega$, hence to $\omega^{\widehat{\beta}+1}$, using the induction hypothesis. We now note that $\widehat{\beta} + 1 = \widehat{\alpha}$.

If $\alpha_m \geq 1$, then ω^{α_m} is a limit ordinal. For every $\gamma < \omega^{\alpha_m}$, $\alpha' \stackrel{\text{def}}{=} \beta + \gamma$ is strictly smaller than α , so there is a proper closed subset F of X such that $\text{rk}_{\mathcal{H}_0 X}(F) = \alpha'$. By induction hypothesis, $\|F^\circledast\| \geq \omega^{\widehat{\alpha}'}$. Since $\gamma < \omega^{\alpha_m}$, the Cantor normal form of α' is of the form $\omega^{\alpha_1} + \cdots + \omega^{\alpha_{m-1}} + \omega^{\gamma_1} + \cdots + \omega^{\gamma_k}$, with $\alpha_m > \gamma_1 \geq \cdots \geq \gamma_k$. Using Lemma 12.3, the Cantor normal form of $\widehat{\alpha}'$ is equal to $\omega^{\alpha_1^\circ} + \cdots + \omega^{\alpha_{m-1}^\circ} + \omega^{\gamma_1^\circ} + \cdots + \omega^{\gamma_k^\circ}$, which is larger than or equal to $\widehat{\beta} + \gamma$. In particular, $\|X^\circledast\| \geq \|F^\circledast\| \geq \omega^{\widehat{\beta} + \gamma}$. By letting γ vary among the ordinals strictly smaller than ω^{α_m} and taking suprema, $\|X^\circledast\| \geq \omega^{\widehat{\beta} + \omega^{\alpha_m}}$.

When α_m is not of the form $\epsilon + n$ with ϵ a critical ordinal and $n \in \mathbb{N}$, $\alpha_m = \alpha_m^\circ$, and therefore we have obtained the desired lower bound $\|X^\circledast\| \geq \omega^{\widehat{\alpha}}$.

If $\alpha_m = \epsilon$ for some critical ordinal ϵ , then we use Lemma 14.19 instead. There is a proper closed subset F of X such that $\text{rk}_{\mathcal{H}_0 X}(F) = \beta$, by Lemma 10.3, and $\|X \setminus F\| = \omega^\epsilon = \epsilon$. By induction hypothesis, $\|F^\circledast\| \geq \omega^{\widehat{\beta}}$. By Lemma 14.19, $\|X^\circledast\| \geq \|F^\circledast\| \times \epsilon \times \omega \geq \omega^{\widehat{\beta}} \times \epsilon \times \omega$. Now $\epsilon \times \omega = \omega^\epsilon \times \omega = \omega^{\epsilon+1} = \omega^{\alpha_m^\circ}$, so $\|X^\circledast\| \geq \omega^{\widehat{\beta}} \times \omega^{\alpha_m^\circ} = \omega^{\widehat{\alpha}}$.

If $\alpha_m = \epsilon + n$ for some critical ordinal ϵ and some $n \in \mathbb{N}$ such that $n \geq 1$, $\omega^{\alpha_m} = \epsilon \times \omega^n$ is the supremum of ordinals of the form $\epsilon \times \omega^{n-1} \times k$, $k \in \mathbb{N}$. Given $\gamma \stackrel{\text{def}}{=} \epsilon \times \omega^{n-1} \times k = \omega^{\epsilon+n-1} \times k$, $\alpha' \stackrel{\text{def}}{=} \beta + \gamma$ is strictly smaller than α , so there is a proper closed subset F of X such that $\text{rk}_{\mathcal{H}_0 X}(F) = \alpha'$. By induction hypothesis, $\|F^\circledast\| \geq \omega^{\widehat{\alpha}'} = \omega^{\widehat{\beta} + \omega^{\epsilon+n} \times k}$. By taking suprema over k , $\|X^\circledast\| \geq \omega^{\widehat{\beta} + \omega^{\epsilon+n+1}} = \omega^{\widehat{\beta} + \omega^{\alpha_m^\circ}} = \omega^{\widehat{\alpha}}$. \square

15. CONCLUSION AND OPEN PROBLEMS

We have developed a theory of statures of Noetherian spaces that generalizes and extends the theory of maximal order types of wpos. In the process, we have also studied the related notion of sobrification rank. We have also given an

X	$\text{sob } X$		$\ X\ $	
Finite T_0	$\leq \text{card } X$		$\text{card } X$	Lem. 6.1
Ordinal α (Alex.)	$\alpha / \alpha + 1$	Lem. 6.2	α	Lem. 6.2
Ordinal α (Scott)	$\alpha / \alpha + 1$	Lem. 6.2	$\alpha / \alpha - 1$	Lem. 6.2
Cofinite topology	$1 / 2$	Thm. 7.1	$\min(\text{card } X, \omega)$	Thm. 7.2
$X + Y$	$\max(\text{sob } X, \text{sob } Y)$	Prop. 8.4	$\ X\ \oplus \ Y\ $	Prop. 8.2
$X +_{\text{lex}} Y$	$\text{sob } X + \text{sob } Y$	Prop. 9.4	$\ X\ + \ Y\ $	Prop. 9.2
X_{\perp}	$1 + \text{sob } X$	Prop. 9.6	$1 + \ X\ $	Prop. 9.6
$X \times Y$	$(\text{sob } X \oplus \text{sob } Y) - 1$	Prop. 10.1	$\ X\ \otimes \ Y\ $	Thm. 10.9
$\mathcal{H}_{0V}X, \mathcal{H}_{\text{fin}}X,$ $\mathbb{P}X, \mathbb{P}_{\text{fin}}X$	$\ X\ + 1$	Thm. 11.1	$\geq 1 + \ X\ ,$ $\leq \omega^{\ X\ }$	Prop. 11.2
X^*	$\omega^{\ X\ ^{\circ}} + 1$ $(\alpha^{\circ} \stackrel{\text{def}}{=} \alpha + 1 \text{ if } \alpha = \epsilon + n, \epsilon \text{ critical, } n \in \mathbb{N},$ $\alpha^{\circ} \text{ otherwise})$	Thm. 12.13	$\omega^{\omega^{\ X\ ^{\circ}}}$	Thm. 12.22
$\triangleright_{n=1}^{+\infty} X_n$	$\bigoplus_{n=1}^{+\infty} \text{rsob } X_n + 1 /$ $\bigoplus_{n=1}^k \text{rsob } X_n + \omega + 1$	Thm. 13.4	$\bigotimes_{n=1}^{+\infty} \ X_n\ /$ $\bigotimes_{m=1}^k \ X_m\ \times \omega$	Thm. 13.8
X^{\triangleright}	$\omega^{\alpha_1+1} + 1$ where $\text{sob } X - 1 =_{\text{CNF}} \omega^{\alpha_1} + \dots$	Cor. 13.7	$\omega^{\omega^{\beta_1+1}} / \omega$ where $\ X\ =_{\text{CNF}} \omega^{\alpha_1} + \dots,$ $\alpha_1 =_{\text{CNF}} \omega^{\beta_1} + \dots$	Cor. 13.9
X^{\otimes}	$\geq (\omega \times \ X\) + 1,$ $\leq (\ X\ \otimes \omega) + 1$	Prop. 14.8, Prop. 14.9	$\omega^{\hat{\alpha}}$ $(\hat{\alpha} \stackrel{\text{def}}{=} \omega^{\alpha_1^{\circ}} + \dots + \omega^{\alpha_m^{\circ}}$ if $\alpha =_{\text{CNF}} \omega^{\alpha_1} + \dots + \omega^{\alpha_m}$)	Thm. 14.20

TABLE 1. Statures and sobrification ranks of Noetherian constructions (all spaces assumed non-empty)

extensive list of explicit formulae for sobrification ranks and statures of several families of Noetherian spaces, arising or not from wqos (see Table 1, where only the cases of non-empty spaces X, Y, X_n are shown, in order to avoid a proliferation of cases). Among the questions that remain, let us cite the following.

- (1) We have $1 + \|X\| \leq \|\mathcal{H}_{0V}X\| \leq \omega^{\|X\|}$ (Proposition 11.2), and both the lower bounds and upper bounds are attained. What is the exact set of ordinals between those bounds that one can obtain as $\|\mathcal{H}_{0V}X\|$? Can we reach the upper bound $\omega^{\|X\|}$ for all infinite values that $\|X\|$ may take?
- (2) Similarly, the sobrification rank of X^{\otimes} lies between $(\omega \times \|X\|) + 1$ (Proposition 14.9) and $(\|X\| \otimes \omega) + 1$ (Proposition 14.8), and those bounds are attained. What other values can $\text{sob } X^{\otimes}$ evaluate to?
- (3) There are many other examples of Noetherian spaces we have not considered. An outstanding one is the space of finite trees with function symbols taken from X [FGL20, Section 10], for which we expect the sobrification

rank and stature to be equal to Schmidt's formula $f^+(\frac{||X||}{\omega})$, as in the wqo case [Sch79, Chapter II].

- (4) For a Noetherian space X , the spaces of infinite words X^ω , and of finite-or-infinite words $X^{<\omega}$ are Noetherian as well [GL22]. (This would fail with wqos instead of Noetherian spaces, and is similar to a well-known result on bqos due to Nash-Williams [NW68].) This is more generally true of the spaces $X^{<\alpha}$ of transfinite words over X of ordinal length $< \alpha$, where α is an arbitrarily large ordinal [GLHL22]. What are the sobrification ranks and statures of those spaces? An upper bound is given [GLHL22, Proposition 33].
- (5) What about Noetherian spaces obtained as spectra of Noetherian rings? There is an abundant literature on Krull dimension of Noetherian rings and variations, see [Bas71, Gul73, Rho74] for example. It is not yet clear to us what the precise relationship to our notion of sobrification rank is, and whether one can derive corresponding results on statures in general. For the Noetherian ring of polynomials on m variables over a field K , the reduced sobrification rank of its spectrum is exactly its Krull dimension, namely m . The ordinal rank of the family of all ideals in that ring is $\omega^m + 1$ [AP04, Lemma 3.18]; when K is algebraically closed, this is not far from the stature of the spectrum, which is the ordinal rank of the subfamily of *radical* ideals, minus 1, by Hilbert's Nullstellensatz.
- (6) An application of the theory of maximal order types in computer science consists in evaluating the precise complexity of reachability and related questions on well-structured transition systems, see [FFSS11, SS11] for example. The present theory should find similar complexity-theoretic applications in relation with the topological well-structured transition systems of [Gou10], and this remains to be developed.

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