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# Feedback Interconnections of Passive Linear Cone Complementarity Systems

Bernard Brogliato\*      Aneel Tanwani†

June 22, 2023

## Abstract

This paper is largely concerned with generic interconnections of a class of passive nonsmooth nonlinear dynamical systems, namely linear cone complementarity systems (LCCS). Criteria which guarantee the passivity of the interconnection are given. Asymptotic stability is studied in a particular case. Examples from nonsmooth circuits and switching DAEs (Differential Algebraic Equations) illustrate the theory.

## 1 Introduction

We consider a particular class of nonlinear nonsmooth dynamical systems called *Linear Cone Complementarity Systems* (LCCS), which has found applications in circuits, mechanics, economics, *etc.* [16, 1], and are computational friendly [2]. The dynamics of such systems are described by

$$\dot{x} = Ax + B\lambda + Lu \tag{1a}$$

$$K \ni \lambda \perp w = Cx + D\lambda + Eu \in K^* \tag{1b}$$

with state  $x \in \mathbb{R}^n$  and the complementarity variables  $\lambda, w \in \mathbb{R}^m$ ,  $K \subseteq \mathbb{R}^m$  a constant closed convex nonempty cone,  $K^*$  its dual cone,  $u$  is an input. The notation  $K \ni \lambda \perp w \in K^*$  means:  $w \in K^*$ ,  $\lambda \in K$  and  $\langle \lambda, w \rangle = 0$ . When  $K = \mathbb{R}_+^m$ , the inequalities

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\*Univ. Grenoble Alpes, INRIA, CNRS, LJK, Grenoble INP, 38000 Grenoble, France; (bernard.brogliato@inria.fr)

†LAAS CNRS, 7, avenue du Colonel Roche, BP 54200, 31031 Toulouse cedex 4, France (aneel.tanwani@cnrs.fr)

are to be interpreted componentwise and (1) is named a Linear Complementarity System (LCS) [19]. The variable  $\lambda$  may be seen as a Lagrange multiplier associated with the constraint  $w \in K^*$ . The interconnections of LCCS involve coupling with both  $x$  and  $\lambda$  variables. Moreover the considered passive input/output operator associated with (1) is  $\lambda \mapsto w$ , because it is this pair of variables which plays a crucial role in the LCCS well-posedness [19, 18, 21, 17, 14, 13], and passivity of this input/output map naturally holds in applications like circuits or mechanics. These peculiar features render the interconnection analysis different from standard passivity theorems [37, 15, 36], where the interconnection variables also define the passive operators.

This article is organized as follows: in section 2 generic interconnections of two passive LCCS are analyzed: criteria such that passivity (section 2.1), strong passivity (section 2.2) and strict state passivity (section 2.3) are preserved after interconnection are given. In section 3 the link with maximal monotonicity and incremental passivity is established. In section 4 the asymptotic stability of the interconnected system is studied in several cases. Examples from circuits with ideal diodes and switching DAEs are given in section 5, conclusions are given in section 6. Some technical results are provided in the appendix (sections A, B, C and D), in particular many results taken in [7, 8] and which are used in the article are recalled in appendix D.

**Notation** The scalar product in  $\mathbb{R}^n$  is denoted  $\langle \cdot, \cdot \rangle$ , the associated norm is  $\|x\| = \sqrt{\langle x, x \rangle}$ . Let  $M \in \mathbb{R}^{n \times m}$ ,  $\sigma_i(M)$  are its singular values,  $\sigma_1(M) \geq \sigma_2(M) \geq \dots \geq \sigma_{\min\{n,m\}}(M) \geq 0$ ,  $\sigma_{\max}(M)$  is its largest singular value. Let  $m = n$ , its eigenvalues are denoted  $\lambda_i(M)$ ,  $1 \leq i \leq n$ ,  $\lambda_{\max}(M)$  is its largest eigenvalue. Positive definite matrix:  $M \succ 0$  if and only if  $x^\top M x > 0$  for all  $x \neq 0$ , positive semidefinite matrix:  $M \succcurlyeq 0$  if and only if  $x^\top M x \geq 0$  for all  $x$ . Let  $M = M^\top \succcurlyeq 0$ , then  $\lambda_i(M) \geq 0$ . The Moore-Penrose pseudo-inverse of  $M \in \mathbb{R}^{n \times m}$  is denoted  $M^\dagger$ . The range of  $M$  is denoted  $\text{Im}(M)$ , its null space as  $\text{Ker}(M)$ . The  $m \times m$  identity matrix is denoted  $I_m$ . The normal cone to a nonempty closed convex set  $\Phi \subseteq \mathbb{R}^n$  at  $x \in \mathbb{R}^n$  is denoted as  $\mathcal{N}_\Phi(x)$ . Its inverse mapping is denoted as  $\mathcal{N}_\Phi^{-1}(\cdot)$  (this is the subdifferential of the support function  $\sigma_\Phi(\cdot)$ , *i.e.*,  $\mathcal{N}_\Phi^{-1}(\cdot) = \partial\sigma_\Phi(\cdot)$ ). The dual cone of  $\Phi$  is  $\Phi^* = \{z \in \mathbb{R}^n \mid \langle z, w \rangle \geq 0, \forall w \in \Phi\}$ . Let  $x, q \in \mathbb{R}^m$ ,  $M \in \mathbb{R}^{m \times m}$ , the linear complementarity problem  $\text{LCP}(M, q)$  is defined as:  $x \geq 0$ ,  $w = Mx + q \geq 0$ ,  $w^\top x = 0$ , compactly  $0 \leq x \perp w \geq 0$ , and  $\text{SOLLCP}(M, q)$  is the set of solutions to  $\text{LCP}(M, q)$ . AC is for absolutely continuous functions. The relative interior of a set  $\Phi$  is denoted  $\text{rint}(\Phi)$ , its interior  $\text{Int}(\Phi)$ . The unit ball in  $\mathbb{R}^n$  is denoted  $\mathbb{B}_n$ .

## 2 Generic Interconnections of Passive LCCS

In this section we consider interconnections of two LCCS. The dynamics of both systems are described by:

$$\dot{x}_i = A_i x_i + B_i \lambda_i + L_i u_i \quad (2a)$$

$$K_i \ni \lambda_i \perp w_i = C_i x_i + D_i \lambda_i + E_i u_i \in K_i^*, \quad (2b)$$

for  $i \in \{1, 2\}$ , with  $x_i \in \mathbb{R}^{n_i}$ ,  $u_1, y_2 \in \mathbb{R}^{n_{u_1}}$ ,  $u_2, y_1 \in \mathbb{R}^{n_{u_2}}$ ,  $w_i, \lambda_i \in \mathbb{R}^{m_i}$ ,  $n = n_1 + n_2$ ,  $m = m_1 + m_2$ ,  $K_i \subseteq \mathbb{R}^{m_i}$  are closed convex nonempty cones. The interconnection between both subsystems in (2) is defined as:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ y_1 \end{pmatrix} = \begin{pmatrix} H_2 x_2 + G_2 \lambda_2 \\ H_1 x_1 + G_1 \lambda_1 \end{pmatrix} \quad (3)$$

The outputs  $y_1$  and  $y_2$  can be seen as the measured outputs for each subsystem, while

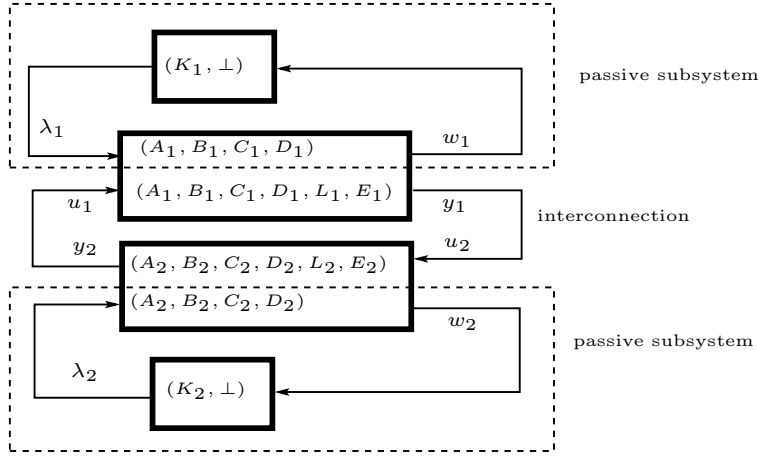


Figure 1: LCCS interconnection.

$w_1$  and  $w_2$  are internal variables (which can be considered as outputs of a passive internal subsystem as seen below). This yields the following interconnected system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} A_1 & L_1 H_2 \\ L_2 H_1 & A_2 \end{pmatrix}}_{\triangleq \tilde{A} \in \mathbb{R}^{n \times n}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\triangleq x} + \underbrace{\begin{pmatrix} B_1 & L_1 G_2 \\ L_2 G_1 & B_2 \end{pmatrix}}_{\triangleq \tilde{B} \in \mathbb{R}^{n \times m}} \underbrace{\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}}_{\triangleq \lambda} \quad (4)$$

$$K \ni \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \perp w = \underbrace{\begin{pmatrix} C_1 & E_1 H_2 \\ E_2 H_1 & C_2 \end{pmatrix}}_{\triangleq \tilde{C} \in \mathbb{R}^{m \times n}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} D_1 & E_1 G_2 \\ E_2 G_1 & D_2 \end{pmatrix}}_{\triangleq \tilde{D} \in \mathbb{R}^{m \times m}} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \in K^*,$$

with  $K \triangleq \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$ ,  $K^* \triangleq \begin{pmatrix} K_1^* \\ K_2^* \end{pmatrix}$ . The interconnection matrices are  $L_1$ ,  $L_2$ ,  $E_1$  and  $E_2$ . Notice that  $E_1 G_2 \in \mathbb{R}^{m_1 \times m_2}$  and  $E_2 G_1 \in \mathbb{R}^{m_2 \times m_1}$ .

**Assumption 1.** *The quadruples  $(A_i, B_i, C_i, D_i)$ ,  $i = 1, 2$ , define a passive system. Equivalently, there exist  $P_i = P_i^\top \succcurlyeq 0$  such that:*

$$\begin{pmatrix} -A_i^\top P_i - P_i A_i & P_i B_i - C_i^\top \\ (P_i B_i - C_i^\top)^\top & D_i + D_i^\top \end{pmatrix} \succcurlyeq 0. \quad (5)$$

This means that the passivity of each LCS in (2) with  $u_i \equiv 0$  holds with supply rates  $\lambda_i^\top w_i$ , and not  $u_i^\top y_i$  as usual in the Passivity Theorem [15] (notice that only AC solutions are considered throughout this article). This is the reason why the interconnection in (3) imposes  $u_1 = y_2$  and not  $u_1 = -y_2$  as in the usual passivity interconnections: the signum of the feedback is irrelevant in this context. The passivity is preserved under the interconnection constraint, if and only if there exists an  $n \times n$  matrix  $P = P^\top \succcurlyeq 0$  such that:

$$\tilde{Q} \triangleq \begin{pmatrix} -\tilde{A}^\top P - P \tilde{A} & P \tilde{B} - \tilde{C}^\top \\ (P \tilde{B} - \tilde{C}^\top)^\top & \tilde{D} + \tilde{D}^\top \end{pmatrix} \succcurlyeq 0. \quad (6)$$

Then the interconnected system is passive with the supply rate  $w^\top \lambda$  and storage functions  $V(x) = x^\top P x$ . Using (2) (3) (4), we obtain the differential inclusion (DI):

$$\begin{aligned} \dot{x}_1 &\in A_1 x_1 - B_1 (D_1 + \mathcal{N}_{K_1^*}^{-1})^{-1} (C_1 x_1 + E_1 H_2 x_2 + E_1 G_2 \lambda_2) + L_1 (H_2 x_2 + G_2 \lambda_2) \\ \dot{x}_2 &\in A_2 x_2 - B_2 (D_2 + \mathcal{N}_{K_2^*}^{-1})^{-1} (C_2 x_2 + E_2 H_1 x_1 + E_2 G_1 \lambda_1) + L_2 (H_1 x_1 + G_1 \lambda_1) \\ \lambda &\in -(\tilde{D} + \mathcal{N}_{K^*}^{-1})^{-1} (\tilde{C} x) \end{aligned} \quad (7)$$

Let  $P \triangleq \begin{pmatrix} P_1 & P_{12} \\ P_{12}^\top & P_2 \end{pmatrix}$ ,  $P_1 \in \mathbb{R}^{n_1 \times n_1}$ ,  $P_2 \in \mathbb{R}^{n_2 \times n_2}$ ,  $P_{12} \in \mathbb{R}^{n_1 \times n_2}$ . Then:  $-\tilde{A}^\top P - P \tilde{A} = \text{diag}(Q_1, Q_2) + \bar{Q}$  with  $Q_1 \triangleq -A_1^\top P_1 - P_1 A_1$ ,  $Q_2 \triangleq -A_2^\top P_2 - P_2 A_2$ , and

$$\bar{Q} = \begin{pmatrix} P_{12} L_2 H_1 + H_1^\top L_2^\top P_{12}^\top & \underbrace{P_1 L_1 H_2 + P_{12} A_2 + A_1^\top P_{12}^\top + H_1^\top L_2^\top P_2}_{\triangleq \bar{Q}_{12}} \\ \bar{Q}_{12}^\top & P_{12}^\top L_1 H_2 + H_2^\top L_1^\top P_{12} \end{pmatrix}. \quad (8)$$

Also  $\bar{Q}_1 \triangleq Q_1 + P_{12} L_2 H_1 + H_1^\top L_2^\top P_{12}^\top$ ,  $\bar{Q}_2 \triangleq Q_2 + P_{12}^\top L_1 H_2 + H_2^\top L_1^\top P_{12}$ ,

$$\tilde{D} + \tilde{D}^\top = \begin{pmatrix} D_1 + D_1^\top & E_1 G_2 + G_1^\top E_2^\top \\ E_2 G_1 + G_2^\top E_1^\top & D_2 + D_2^\top \end{pmatrix}, \quad (9)$$

$$\begin{aligned} P \tilde{B} - \tilde{C}^\top &= \begin{pmatrix} P_1 B_1 - C_1^\top + P_{12} L_2 G_1 & P_1 L_1 G_2 + P_{12} B_2 - H_1^\top E_2^\top \\ P_{12}^\top B_1 + P_2 L_2 G_1 - H_2^\top E_1^\top & P_2 B_2 - C_2^\top + P_{12}^\top L_1 G_2 \end{pmatrix} \\ &\triangleq \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \in \mathbb{R}^{n \times n}. \end{aligned} \quad (10)$$

Finally the matrix inequality in (6) is equivalent to:

$$\begin{aligned}
\text{(a)} \quad & P\tilde{B} - \tilde{C}^\top = (P\tilde{B} - \tilde{C}^\top)(\tilde{D} + \tilde{D}^\top)^\dagger(\tilde{D} + \tilde{D}^\top) \\
\text{(b)} \quad & \tilde{D} + \tilde{D}^\top \succcurlyeq 0 \\
\text{(c)} \quad & \tilde{A}^\top P + P\tilde{A} \preccurlyeq 0 \\
\text{(d)} \quad & -\tilde{A}^\top P - P\tilde{A} \succcurlyeq (P\tilde{B} - \tilde{C}^\top)(\tilde{D} + \tilde{D}^\top)^\dagger(\tilde{B}^\top P - \tilde{C})
\end{aligned} \tag{11}$$

The conditions (11) follow from [7, Proposition 8.2.4]. As is well-known, in case  $\tilde{D} + \tilde{D}^\top = 0$ , the conditions (11) reduce to **(a)** and **(c)**. Solving the matrix equalities/inequalities in (11) with unknowns  $P_1, P_2, P_{12}, H_1, H_2, G_1, G_2$  (and possibly  $L_1, L_2, E_1, E_2$  in some cases) is difficult in general. In the following sections paths which facilitate their analysis are studied. The results in sections 2.1, 2.2, 2.3, provide conditions such that the interconnection quadruple  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  possesses required passivity properties. It is noteworthy that the matrices' structure is tightly related to LCCS structure.

## 2.1 Passivity Preservation after Interconnection

Let us start with several lemmata which state under which conditions the conditions in (11) hold. It is assumed that  $P$  is as above and  $P_i$  is a solution of the LMI (5),  $i = 1, 2$ .

**Lemma 1.** (11) **(a)** holds if and only if:

$$\begin{aligned}
& \text{Im} \begin{pmatrix} B_1^\top P_1 - C_1 + G_1^\top L_2^\top P_{12}^\top \\ G_2^\top L_1^\top P_1 + B_2^\top P_{12}^\top - E_2 H_1 \end{pmatrix} + \text{Im} \begin{pmatrix} B_1^\top P_{12} + G_1^\top L_2^\top P_2 - E_1 H_2 \\ B_2^\top P_2 - C_2 + G_2^\top L_1^\top P_{12} \end{pmatrix} \\
& \subseteq \text{Im} \begin{pmatrix} D_1 + D_1^\top \\ E_2 G_1 + G_2^\top E_1^\top \end{pmatrix} + \text{Im} \begin{pmatrix} E_1 G_2 + G_1^\top E_2^\top \\ D_2 + D_2^\top \end{pmatrix}.
\end{aligned} \tag{12}$$

In addition, (12) holds if:

$$\begin{aligned}
& \text{Im}(G_1^\top L_2^\top P_{12}^\top) \subseteq \text{Im}(B_1^\top P_1 - C_1) \\
& \text{Im}(B_1^\top P_{12} + G_1^\top L_2^\top P_2 - E_1 H_2) \subseteq \text{Im}(B_1^\top P_1 - C_1) \\
& \text{Im}(G_2^\top L_1^\top P_{12}) \subseteq \text{Im}(B_2^\top P_2 - C_2) \\
& \text{Im}(G_2^\top L_1^\top P_1 + B_2^\top P_{12}^\top - E_2 H_1) \subseteq \text{Im}(B_2^\top P_2 - C_2) \\
& \text{Im}(E_1 G_2 + G_1^\top E_2^\top) \subseteq \text{Im}(D_1 + D_1^\top) \\
& \text{Im}(E_2 G_1 + G_2^\top E_1^\top) \subseteq \text{Im}(D_2 + D_2^\top).
\end{aligned} \tag{13}$$

*Proof.* Using [8, Fact 8.4.3], (11) **(a)** holds true if and only if  $\text{Im}(\tilde{B}^\top P - \tilde{C}) \subseteq \text{Im}(\tilde{D} + \tilde{D}^\top)$ , equivalently (12) holds, from [8, Fact 3.14.8] (*i.e.*,  $\text{Im}(A) + \text{Im}(B) = \text{Im}(A \ B)$  for any  $A$  and  $B$  with same number of rows). Let us now prove (13). For two matrices  $A$  and  $B$ , let  $\text{Im}(B) \subseteq \text{Im}(A)$ , then  $\text{Im}(A + B) \subseteq \text{Im}(A)$ , and  $\text{Im}(A) + \text{Im}(B) = \text{Im}(A)$ . Also

$\text{Im} \begin{pmatrix} A \\ B \end{pmatrix} \subseteq \begin{pmatrix} \text{Im}(A) \\ \text{Im}(B) \end{pmatrix}$ . Using these inclusions it follows that (13) implies (12), using  $\text{Im}(B_1^\top P_1 + C_1) \subseteq \text{Im}(D_1 + D_1^\top)$  and  $\text{Im}(B_2^\top P_2 + C_2) \subseteq \text{Im}(D_2 + D_2^\top)$  from Assumption 1.  $\square$

**Corollary 1.** *Assume that  $P_{12} = 0$ , then (12) holds if:*

$$\begin{aligned} P_1 L_1 G_2 - H_1^\top E_2^\top &= 0 \\ P_2 L_2 G_1 - H_2^\top E_1^\top &= 0 \end{aligned} \quad (14)$$

*Proof.* Follows from Assumption 1, since  $\text{Im}(B_1^\top P_1 + C_1) \subseteq \text{Im}(D_1 + D_1^\top)$  and  $\text{Im}(B_2^\top P_2 + C_2) \subseteq \text{Im}(D_2 + D_2^\top)$  (see Appendix B).  $\square$

**Lemma 2.** *Inequality (11) (b) holds if:  $\text{Im}(E_1 G_2 + G_1^\top E_2^\top) \subseteq \text{Im}(D_1 + D_1^\top)$ ,  $\sigma_{\max}^2(E_1 G_2 + G_1^\top E_2^\top) < \sigma_{r_1}(D_1 + D_1^\top) \lambda_{\max}(D_2 + D_2^\top)$  when  $D_2 + D_2^\top \succ 0$ ,  $\text{rank}(D_1 + D_1^\top) = r_1 \leq m_1$ ,  $D_1 + D_1^\top \neq 0$ . Or if:  $\text{Im}(E_2 G_1 + G_2^\top E_1^\top) \subseteq \text{Im}(D_2 + D_2^\top)$ ,  $\sigma_{\max}^2(E_1 G_2 + G_1^\top E_2^\top) < \sigma_{r_2}(D_2 + D_2^\top) \lambda_{\max}(D_1 + D_1^\top)$  when  $D_1 + D_1^\top \succ 0$ ,  $\text{rank}(D_2 + D_2^\top) = r_2 \leq m_1$ ,  $D_2 + D_2^\top \neq 0$ . Or if:  $(D_1 + D_1^\top = 0$  or  $D_2 + D_2^\top = 0)$  and  $E_1 G_2 + G_1^\top E_2^\top = 0$ .*

*Proof.* Consider (9). From Assumption 1,  $D_1 + D_1^\top \succcurlyeq 0$  and  $D_2 + D_2^\top \succcurlyeq 0$ . It follows that  $\tilde{D} + \tilde{D}^\top \succcurlyeq 0$  if and only if [7, Proposition 8.2.4]:

$$\begin{aligned} (\alpha) \quad D_2 + D_2^\top &\succcurlyeq (E_2 G_1 + G_2^\top E_1^\top)(D_1 + D_1^\top)^\dagger (E_1 G_2 + G_1^\top E_2^\top) \\ (\beta) \quad E_1 G_2 + G_1^\top E_2^\top &= (D_1 + D_1^\top)(D_1 + D_1^\top)^\dagger (E_1 G_2 + G_1^\top E_2^\top), \end{aligned} \quad (15)$$

equivalently:

$$\begin{aligned} (\alpha') \quad D_1 + D_1^\top &\succcurlyeq (E_1 G_2 + G_1^\top E_2^\top)(D_2 + D_2^\top)^\dagger (E_2 G_1 + G_2^\top E_1^\top) \\ (\beta') \quad E_1 G_2 + G_1^\top E_2^\top &= (E_1 G_2 + G_1^\top E_2^\top)(D_2 + D_2^\top)^\dagger (D_2 + D_2^\top), \end{aligned} \quad (16)$$

Let us examine the above conditions:

( $\alpha$ ) Assume that  $D_2 + D_2^\top \succ 0$ ,  $\text{rank}(D_1 + D_1^\top) = r_1 \leq m_1$ ,  $D_1 + D_1^\top \neq 0$ . Using Corollary 8 and [7, Corollary 9.6.5], the inequality ( $\alpha$ ) holds if  $\sigma_{\max}(D_2 + D_2^\top) > \sigma_{\max}((E_2 G_1 + G_2^\top E_1^\top)(D_1 + D_1^\top)^\dagger (E_1 G_2 + G_1^\top E_2^\top))$ . From [7, Corollary 9.6.5, Fact 6.3.28], it follows that  $\sigma_{\max}((E_2 G_1 + G_2^\top E_1^\top)(D_1 + D_1^\top)^\dagger (E_1 G_2 + G_1^\top E_2^\top)) \leq \sigma_{\max}^2(E_1 G_2 + G_1^\top E_2^\top) \sigma_{\max}((D_1 + D_1^\top)^\dagger) = \sigma_{\max}^2(E_1 G_2 + G_1^\top E_2^\top) \sigma_{r_1}^{-1}(D_1 + D_1^\top)$  and  $\sigma_{\max}(D_2 + D_2^\top) = \lambda_{\max}(D_2 + D_2^\top)$ .

( $\alpha'$ ) Assume that  $D_1 + D_1^\top \succ 0$ ,  $\text{rank}(D_2 + D_2^\top) = r_2 \leq m_1$ ,  $D_2 + D_2^\top \neq 0$ . Then for same reasons, the inequality ( $\alpha$ ) holds if  $\sigma_{\max}^2(E_1 G_2 + G_1^\top E_2^\top) \sigma_{r_2}^{-1}(D_2 + D_2^\top) < \lambda_{\max}(D_1 + D_1^\top)$ .

( $\beta$ ) Using [8, Fact 8.4.3], this is equivalent to  $\text{Im}(E_1 G_2 + G_1^\top E_2^\top) \subseteq \text{Im}(D_1 + D_1^\top)$ .

( $\beta'$ ) Similarly, the equality is equivalent to  $\text{Im}(E_2G_1 + G_2^\top E_1^\top) \subseteq \text{Im}(D_2 + D_2^\top)$ . Therefore the Lemma's condition imply that (15) and (16) are satisfied. Notice that if  $D_1 + D_1^\top = 0$  or if  $D_2 + D_2^\top = 0$  then  $[(\alpha)$  and  $(\beta)]$ , or  $[(\alpha')$  and  $(\beta')]$   $\Rightarrow E_1G_2 = -G_1^\top E_2^\top$ . Also  $E_1G_2 = -G_1^\top E_2^\top$  guarantees that  $(\alpha)$ ,  $(\beta)$ ,  $(\alpha')$  and  $(\beta')$  are satisfied. Thus if  $D_1 + D_1^\top = 0$  or  $D_2 + D_2^\top = 0$ , we have  $\tilde{D} + \tilde{D}^\top \succcurlyeq 0 \Leftrightarrow E_1G_2 = -G_1^\top E_2^\top$ .  $\square$

**Lemma 3.** (11) (c) holds if:  $\bar{Q}_1 \succcurlyeq 0$ ,  $\bar{Q}_2 \succcurlyeq 0$ , and:  $\text{Im}(\bar{Q}_{12}) \subseteq \text{Im}(P_{12}L_2H_1 + H_1^\top L_2^\top P_{12}^\top)$ ,  $\sigma_{\max}^2(\bar{Q}_{12}) < \sigma_{q_1}(\bar{Q}_1)\sigma_{\max}(\bar{Q}_2)$  when  $\bar{Q}_2 \succ 0$ ,  $\text{rank}(\bar{Q}_1) = q_1$ ,  $\bar{Q}_1 \neq 0$ , or  $\text{Im}(\bar{Q}_{12}^\top) \subseteq \text{Im}(P_{12}^\top L_1H_2 + H_2^\top L_1^\top P_{12})$ ,  $\sigma_{\max}^2(\bar{Q}_{12}) < \sigma_{q_2}(\bar{Q}_2)\sigma_{\max}(\bar{Q}_1)$  when  $\bar{Q}_1 \succ 0$ ,  $\text{rank}(\bar{Q}_2) = q_2$ ,  $\bar{Q}_2 \neq 0$ .

*Proof.* The conditions which guarantee that  $\tilde{A}^\top P + P\tilde{A} \preccurlyeq 0$  are studied from the fact that  $-\tilde{A}^\top P - P\tilde{A} = \text{diag}(Q_1, Q_2) + \bar{Q}$  with  $\bar{Q}$  in (8). We have  $\text{diag}(Q_1, Q_2) + \bar{Q} \succcurlyeq 0$  if and only if:  $\bar{Q}_1 = Q_1 + P_{12}L_2H_1 + H_1^\top L_2^\top P_{12}^\top \succcurlyeq 0$ ,  $\bar{Q}_2 = Q_2 + P_{12}^\top L_1H_2 + H_2^\top L_1^\top P_{12} \succcurlyeq 0$ ,  $\bar{Q}_{12} = (P_{12}L_2H_1 + H_1^\top L_2^\top P_{12}^\top)(P_{12}L_2H_1 + H_1^\top L_2^\top P_{12}^\top)^\dagger \bar{Q}_{12} \iff \text{Im}(\bar{Q}_{12}) \subseteq \text{Im}(P_{12}L_2H_1 + H_1^\top L_2^\top P_{12}^\top)$ , and  $\bar{Q}_2 \succcurlyeq \bar{Q}_{12}^\top \bar{Q}_1^\dagger \bar{Q}_{12}$ , where we used [7, Proposition 8.2.4] [8, Fact 8.4.3]. The last inequality can be treated similarly as we did in case (b) ( $\alpha$ ) and ( $\alpha'$ ). Assume that  $\bar{Q}_2 \succ 0$ ,  $\text{rank}(\bar{Q}_1) = q_1$ ,  $\bar{Q}_1 \neq 0$ . Then the last inequality holds if  $\sigma_{\max}^2(\bar{Q}_{12}) < \sigma_{q_1}(\bar{Q}_1)\sigma_{\max}(\bar{Q}_2)$ . We can use also the equivalent conditions  $\bar{Q}_{12} = \bar{Q}_{12}(P_{12}^\top L_1H_2 + H_2^\top L_1^\top P_{12})^\dagger (P_{12}^\top L_1H_2 + H_2^\top L_1^\top P_{12}) \iff \text{Im}(\bar{Q}_{12}^\top) \subseteq \text{Im}(P_{12}^\top L_1H_2 + H_2^\top L_1^\top P_{12})$ , and  $\bar{Q}_1 \succcurlyeq \bar{Q}_{12}^\top \bar{Q}_2^\dagger \bar{Q}_{12}^\top$ . Assume that  $\bar{Q}_1 \succ 0$ ,  $\text{rank}(\bar{Q}_2) = q_2$ ,  $\bar{Q}_2 \neq 0$ . Then the last inequality holds if  $\sigma_{\max}^2(\bar{Q}_{12}) < \sigma_{q_2}(\bar{Q}_2)\sigma_{\max}(\bar{Q}_1)$ . Thus the lemma is proved.  $\square$

Using [8, Fact 3.13.15, equation (3.5.7)],  $\text{Im}(\bar{Q}_{12}) \subseteq P_1\text{Im}(L_1H_2) + \text{Im}(P_{12}A_2) + \text{Im}(A_1^\top P_{12}^\top) + \text{Im}(H_1^\top L_2^\top P_2)$ , and similalry for  $\text{Im}(\bar{Q}_{12}^\top)$ . This can be used as a sufficient condition in the lemma.

**Corollary 2.** When  $P_{12} = 0$ , (11) (c) holds provided that

$$P_1L_1H_2 = -(P_2L_2H_1)^\top, \quad (17)$$

*Proof.* The conditions guarantee that  $\bar{Q}_{12} = 0$ , hence  $\bar{Q} = 0$ , hence  $-\tilde{A}^\top P - P\tilde{A} = \text{diag}(Q_1, Q_2)$ .  $\square$

Let us now provide conditions for (11) (d) to be satisfied.

**Lemma 4.** Let the conditions in Lemmata 2 and 3 hold (hence (11) (b) and (c) are satisfied). The following statements hold:

1. Let  $N \triangleq \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \left( \begin{pmatrix} D_1 + D_1^\top & E_1G_2 + G_1^\top E_2^\top \\ E_2G_1 + G_2^\top E_1^\top & D_2 + D_2^\top \end{pmatrix}^\dagger \right)^{\frac{1}{2}}$ . Then (11) (d) holds true only if:



$$\text{Im}(\text{diag}(Q_1, Q_2) + \bar{Q}) \supseteq \text{Im}(N) \quad (18)$$

$$\lambda_i(\text{diag}(Q_1, Q_2) + \bar{Q}) \geq \lambda_i(NN^\top) \geq 0, \text{ for all } i \in \{1, \dots, m\}, \quad (19)$$

$$\text{tr}(\text{diag}(Q_1, Q_2) + \bar{Q}) \geq \text{tr}(NN^\top) \geq 0. \quad (20)$$

2. Let  $E_1G_2 + G_1^\top E_2^\top = 0$ ,  $P_{12} = (L_2H_1)^\top = L_1H_2$ . Let us denote  $D_{s,1} \triangleq D_1 + D_1^\top$ ,  $D_{s,2} \triangleq D_2 + D_2^\top$ . Then (11) **(d)** holds true only if:

$$\begin{aligned} \text{(i)} \quad & \text{Im}(\bar{Q}_1) \supseteq \text{Im}(p_{11}D_{s,1}^\dagger p_{11}^\top + p_{12}D_{s,2}^\dagger p_{12}^\top) \\ \text{(ii)} \quad & \text{Im}(\bar{Q}_{12} - p_{11}D_{s,1}^\dagger p_{21}^\top - p_{12}D_{s,2}^\dagger p_{22}^\top) \subseteq \text{Im}(\bar{Q}_1 - p_{11}D_{s,1}^\dagger p_{11}^\top - p_{12}D_{s,2}^\dagger p_{12}^\top) \\ \text{(iii)} \quad & \text{Im}(\bar{Q}_2 - p_{21}D_{s,1}^\dagger p_{21}^\top - p_{22}D_{s,2}^\dagger p_{22}^\top) \supseteq \text{Im}((\bar{Q}_{12}^\top - p_{21}D_{s,1}^\dagger p_{11}^\top - p_{22}D_{s,2}^\dagger p_{12}^\top) \times \\ & (\bar{Q}_1 - p_{11}D_{s,1}^\dagger p_{11}^\top - p_{12}D_{s,2}^\dagger p_{12}^\top)^\dagger (\bar{Q}_{12} - p_{11}D_{s,1}^\dagger p_{21}^\top - p_{12}D_{s,2}^\dagger p_{22}^\top)) \end{aligned} \quad (21)$$

3. Then (11) **(d)** holds true if:  $P_{12} = (L_2H_1)^\top = L_1H_2$ ,  $\text{rank}(P_{12}) = n_1$ ,  $\sigma_{\max}^2(\bar{Q}_{12}) < \lambda_{\min}(\bar{Q}_1)\lambda_{\min}(\bar{Q}_2)$ ,  $\text{rank}(\tilde{D} + \tilde{D}^\top) = d$ ,  $\tilde{D} + \tilde{D}^\top \neq 0$ , and

$$\frac{\sigma_{\max}^2(p_{11}) + \sigma_{\max}^2(p_{12}) + \sigma_{\max}^2(p_{22}) + \sigma_{\max}^2(p_{21}) + \sqrt{\Delta}}{2} < \sigma_d(\tilde{D} + \tilde{D}^\top)\lambda_{\max} \begin{pmatrix} \bar{Q}_1 & \bar{Q}_{12} \\ \bar{Q}_{12}^\top & \bar{Q}_2 \end{pmatrix} \quad (22)$$

with  $\Delta = (\sigma_{\max}^2(p_{11}) + \sigma_{\max}^2(p_{12}) - \sigma_{\max}^2(p_{22}) - \sigma_{\max}^2(p_{21})) + 4(\sigma_{\max}(p_{11})\sigma_{\max}(p_{21}) + \sigma_{\max}(p_{12})\sigma_{\max}(p_{22}))$ .

4. Let both sides of (11) **(d)** be decomposed as  $MM^\top$  and  $NN^\top$ , respectively, so that (11)

**(d)** is rewritten as  $MM^\top \succcurlyeq NN^\top$ . Let  $M = U_1 \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} U_2$ , where  $U_1$  and  $U_2$  are unitary matrices,  $T = \text{diag}(\sigma_1(M), \dots, \sigma_r(M))$ ,  $r = \text{rank}(M)$ , and  $U_1^\top N = \begin{pmatrix} N_1 \\ 0 \end{pmatrix}$ . Assume that:

$$\sigma_{\max} \begin{pmatrix} N_1 \\ 0 \end{pmatrix} \leq |\lambda_r(\text{diag}(Q_1, Q_2) + \bar{Q})|, \quad (23)$$

then (11) **(d)** is satisfied.

*Proof.* Let us examine (11) **(d)**. We have  $-\tilde{A}^\top P - P\tilde{A} = \text{diag}(Q_1, Q_2) + \bar{Q}$ , with  $\bar{Q}$  in (8), and  $P\tilde{B} - \tilde{C}^\top$  in (10),  $\tilde{D} + \tilde{D}^\top$  in (9). Obviously the conditions in **(c)** are implied by the conditions in this lemma.

1. If the conditions in Lemmata 2 and 3 hold, then (11) **(b)** and **(c)** are satisfied. Then applying [8, Fact 7.17.24], both sides of (11) **(d)** are positive semidefinite and symmetric, thus they can be decomposed as  $MM^\top$  and  $NN^\top$  for some matrices  $M$  and  $N$ , respectively. Hence (11) **(d)** is equivalently rewritten as  $MM^\top \succcurlyeq NN^\top$ . From

[7, Theorems 8.6.2, 2.4.3] this implies  $\text{Im}(N) \subseteq \text{Im}(M) = \text{Im}(MM^\top)$ , which is (18), a necessary condition for (11) **(d)**. The necessary conditions (19) and (20) follow from [7, Theorem 8.4.9, Corollary 8.4.10] since both matrices are symmetric positive semidefinite.

2. Let us assume that  $E_1G_2 + G_1^\top E_2^\top = 0$ . Then  $(\tilde{D} + \tilde{D}^\top)^\dagger = \text{diag}(D_{s,1}^\dagger, D_{s,2}^\dagger)$  [7, Fact 6.5.21]. Thus the inequality in (11) **(d)** is rewritten as:

$$\begin{pmatrix} \bar{Q}_1 - p_{11}D_{s,1}^\dagger p_{11}^\top - p_{12}D_{s,2}^\dagger p_{12}^\top & \bar{Q}_{12} - p_{11}D_{s,1}^\dagger p_{21}^\top - p_{12}D_{s,2}^\dagger p_{22}^\top \\ \bar{Q}_{12}^\top - p_{21}D_{s,1}^\dagger p_{11}^\top - p_{22}D_{s,2}^\dagger p_{12}^\top & \bar{Q}_2 - p_{21}D_{s,1}^\dagger p_{21}^\top - p_{22}D_{s,2}^\dagger p_{22}^\top \end{pmatrix} \succcurlyeq 0. \quad (24)$$

Let us apply [7, Proposition 8.2.4] [8, Fact 8.4.3] to infer that (24) is equivalent to:

$$\begin{aligned} \text{(i)} \quad & \bar{Q}_1 - p_{11}D_{s,1}^\dagger p_{11}^\top - p_{12}D_{s,2}^\dagger p_{12}^\top \succcurlyeq 0 \\ \text{(ii)} \quad & \text{Im}(\bar{Q}_{12} - p_{11}D_{s,1}^\dagger p_{21}^\top - p_{12}D_{s,2}^\dagger p_{22}^\top) \subseteq \text{Im}(\bar{Q}_1 - p_{11}D_{s,1}^\dagger p_{11}^\top - p_{12}D_{s,2}^\dagger p_{12}^\top) \\ \text{(iii)} \quad & \bar{Q}_2 - p_{21}D_{s,1}^\dagger p_{21}^\top - p_{22}D_{s,2}^\dagger p_{22}^\top \succcurlyeq (\bar{Q}_{12}^\top - p_{21}D_{s,1}^\dagger p_{11}^\top - p_{22}D_{s,2}^\dagger p_{12}^\top) \times \\ & \quad \times (\bar{Q}_1 - p_{11}D_{s,1}^\dagger p_{11}^\top - p_{12}D_{s,2}^\dagger p_{12}^\top)^\dagger (\bar{Q}_{12} - p_{11}D_{s,1}^\dagger p_{21}^\top - p_{12}D_{s,2}^\dagger p_{22}^\top) \end{aligned} \quad (25)$$

Recall that  $\bar{Q}_1 = \bar{Q}_1^\top = Q_1 + P_{12}L_2H_1 + H_1^\top L_2^\top P_{12}^\top$ ,  $\bar{Q}_2 = \bar{Q}_2^\top = Q_2 + P_{12}^\top L_1H_2 + H_2^\top L_1^\top P_{12}$ . If condition  $P_{12} = (L_2H_1)^\top = L_1H_2$  holds, then  $\bar{Q}_1 \succcurlyeq 0$  and  $\bar{Q}_2 \succcurlyeq 0$ . Also  $p_{11}D_{s,1}^\dagger p_{11}^\top + p_{12}D_{s,2}^\dagger p_{12}^\top \succcurlyeq 0$  [8, Proposition 8.1.7]. Using [8, Fact 7.17.24], there exists  $M_1$  and  $N_1$  such that the decompositions  $\bar{Q}_1 = M_1M_1^\top$  and  $p_{11}D_{s,1}^\dagger p_{11}^\top + p_{12}D_{s,2}^\dagger p_{12}^\top = N_1N_1^\top$  hold. The inequality **(i)** is  $M_1M_1^\top \succcurlyeq N_1N_1^\top \implies \text{Im}(N_1) \subseteq \text{Im}(M_1)$  [8, Theorem 10.6.2]. Therefore using [7, Theorem 2.4.3], it follows that condition **(i)** in (21) is necessary for (25) **(i)**. Similarly, condition **(iii)** in (21) is necessary for in (25) **(iii)**.

3. Let  $\bar{Q}_2 \succ 0$ ,  $\bar{Q}_1 \succ \bar{Q}_{12}\bar{Q}_2^{-1}\bar{Q}_{12}^\top$ , or  $\bar{Q}_1 \succ 0$ ,  $\bar{Q}_2 \succ \bar{Q}_{12}^\top\bar{Q}_1^{-1}\bar{Q}_{12}$ . If  $Q_1 \succcurlyeq 0$  and  $Q_2 \succcurlyeq 0$  (both subsystems are just passive), then  $\bar{Q}_1 \succ 0$  and  $\bar{Q}_2 \succ 0$  hold if  $P_{12} = (L_2H_1)^\top = L_1H_2$ , and  $\text{rank}(P_{12}) = n_1$ . The other two inequalities hold if  $\sigma_{\max}^2(\bar{Q}_{12}) < \lambda_{\min}(\bar{Q}_2)\lambda_{\min}(\bar{Q}_1)$  [7, Fact 8.19.2]. Then applying Corollary 8, the inequality (11) **(d)** holds if  $\sigma_{\max}^2(P\tilde{B} - \tilde{C}^\top)\sigma_{\max}((\tilde{D} + \tilde{D}^\top)^\dagger) < \sigma_{\max}(\text{diag}(Q_1, Q_2) + \bar{Q})$ . Let  $\text{rank}(\tilde{D} + \tilde{D}^\top) = d$ ,  $\tilde{D} + \tilde{D}^\top \neq 0$ , then the last inequality holds if

$$\sigma_{\max}^2 \begin{pmatrix} \sigma_{\max}(p_{11}) & \sigma_{\max}(p_{12}) \\ \sigma_{\max}(p_{21}) & \sigma_{\max}(p_{22}) \end{pmatrix} < \sigma_d(\tilde{D} + \tilde{D}^\top)\lambda_{\max} \begin{pmatrix} \bar{Q}_1 & \bar{Q}_{12} \\ \bar{Q}_{12}^\top & \bar{Q}_2 \end{pmatrix}, \quad (26)$$

where we used [7, Facts 6.3.28, 9.14.12]. The left-hand side in (26) is easily calculated explicitly to get (22). Hence item 4.3 is proved.

4. The proof is based on the proof of [7, Theorem 8.6.2]. The factorization of  $M$  follows from [7, Theorem 5.6.3], where  $0 < \sigma_1(M) = \sigma_{\max}(M)$ ,  $\sigma_r(M) > 0$  [7, section 5.6]. Let  $L \triangleq U_2^\top \begin{pmatrix} T^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} N_1 \\ 0 \end{pmatrix}$ . Using the reasoning in the proof of [7, Theorem 8.6.2], it is

inferred that  $N = ML$ , and  $NN^\top \leq \lambda_{\max}(LL^\top)MM^\top$ . Using [7, Corollary 9.6.5], we have:

$$\begin{aligned} \lambda_{\max}(LL^\top) &= \sigma_{\max}^2(L) \\ &\leq \sigma_{\max}^2(U_2^\top)\sigma_{\max}^2(U_2)\sigma_{\max}^2\begin{pmatrix} N_1 \\ 0 \end{pmatrix}\sigma_{\max}^2(N_1^\top \ 0)\sigma_{\max}^4\begin{pmatrix} T^{-1} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \sigma_{\max}^4(U_2)\sigma_{\max}^4\begin{pmatrix} N_1 \\ 0 \end{pmatrix}\sigma_{\max}^4\begin{pmatrix} T^{-1} & 0 \\ 0 & 0 \end{pmatrix} = \sigma_{\max}^4\begin{pmatrix} N_1 \\ 0 \end{pmatrix}\sigma_{\max}^4\begin{pmatrix} T^{-1} & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \quad (27)$$

since  $U_2$  is unitary. Notice that since  $T = \text{diag}(\sigma_1(M), \dots, \sigma_r(M))$ , with  $\sigma_1(M) = \sigma_{\max}(M) \geq \sigma_2(M) \geq \dots \geq \sigma_r(M) > 0$ , we have  $\sigma_{\max}(T^{-1}) = \sigma_r^{-1}(M)$ . Therefore,  $\sigma_{\max}\begin{pmatrix} N_1 \\ 0 \end{pmatrix}\sigma_r^{-1}(M) \leq 1$  implies that  $\lambda_{\max}(LL^\top) \leq 1$ , so that  $MM^\top \geq NN^\top$ . The equality  $\sigma_r(M) = \lambda_r^{\frac{1}{2}}(MM^\top) = |\lambda_r(\text{diag}(Q_1, Q_2) + \bar{Q})|$  yields (23).  $\square$

Condition (19) involves the singular values of  $N$ . Sufficient conditions such that it holds can be derived, using [8, Corollary 11.6.8] to consider both matrices in  $N$  separately. It is also worth recalling that most of these criteria simplify when  $D_i = 0$  ( $\Rightarrow P_i B_i - C_i^\top = 0$ ) which occurs often in practice. For instance  $(\tilde{D} + \tilde{D}^\top)^\dagger$  can be calculated using [8, Fact 8.9.31], simplifying the expression for  $N$ .

**Lemma 5.** *Let  $P_{12} = 0$ , assume that (14), (17) and  $E_1 G_2 = -(E_2 G_1)^\top$  are satisfied. Then (11) (d) is true.*

*Proof.* The conditions (14) imply  $P\tilde{B} - \tilde{C}^\top = \text{diag}(P_1 B_1 - C_1^\top, P_2 B_2 - C_2^\top)$ . The condition  $E_1 G_2 = -(E_2 G_1)^\top$  implies  $\tilde{D} + \tilde{D}^\top = \text{diag}(D_1 + D_1^\top, D_2 + D_2^\top)$ , while (17) guarantees  $-\tilde{A}P - P\tilde{A} = \text{diag}(Q_1, Q_2)$ . Thus the interconnection is decoupled, and since both LCS are passive, (11) (d) is true.  $\square$

Some comments on Lemma 4: The necessary conditions (21) may not be easy to check. If  $D_{s,1} = 0$  and  $D_{s,2} = 0$ , they simplify as follows: (i) always holds, (ii) implies  $\text{Im}(\bar{Q}_{12}) \subseteq \text{Im}(Q_1) + \text{Im}(P_{12}L_2H_1) + \text{Im}(P_{12}L_2H_1)^\top$ , (iii) implies  $\text{Im}(\bar{Q}_2) \subseteq \text{Im}(\bar{Q}_{12}^\top \bar{Q}_1 \bar{Q}_{12}) \Rightarrow \text{Im}(\bar{Q}_2) \subseteq \text{Im}(\bar{Q}_{12}^\top)$ , which furnish simpler necessary conditions. The conditions simplify also if  $\bar{Q}_1 \succ 0$  and  $\bar{Q}_2 \succ 0$ , as it may occur (see section 5), using Corollary 8 to analyze (21) (ii) and (iii) (then (i) is always true).

Similarly, condition (18) in item 4.1 in Lemma 4, is not very practical, and it is of interest to analyze the influence of (13) on its satisfaction. We have  $\text{Im}((\tilde{D} + \tilde{D}^\top)^\dagger)^{\frac{1}{2}} = \text{Im}(\tilde{D} + \tilde{D}^\top)$ , using [7, Proposition 6.1.6, Theorem 2.4.3], and  $\text{Im}(N) \subseteq \text{Im}\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$  [7, Lemma 2.4.1].

Assume now that (13) holds, which implies that (12) holds, equivalently  $\text{Im}\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \subseteq \text{Im}(\tilde{D} + \tilde{D}^\top)$  [8, Fact 8.4.3]. Thus if  $\text{Im}(\tilde{D} + \tilde{D}^\top) = \text{Im}\begin{pmatrix} D_1 + D_1^\top \\ E_2 G_1 + G_2^\top E_1^\top \end{pmatrix} + \text{Im}\begin{pmatrix} E_1 G_2 + G_1^\top E_2^\top \\ D_2 + D_2^\top \end{pmatrix} \subseteq$

$\text{Im}(\text{diag}(Q_1, Q_2) + \bar{Q})$ , it is inferred that the necessary condition (18) holds. To summarize: (13)  $\Rightarrow$  (12)  $\Leftrightarrow$   $\text{Im} \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \subseteq \text{Im}(\tilde{D} + \tilde{D}^\top) \Rightarrow \text{Im}(N) \subseteq \text{Im}(\tilde{D} + \tilde{D}^\top) \Rightarrow [\text{Im}(\tilde{D} + \tilde{D}^\top) \subseteq \text{Im}(\text{diag}(Q_1, Q_2) + \bar{Q}) \Rightarrow (18)]$ , while (18)  $\Leftarrow$  (11) **(d)**. The next proposition concatenates Lemmata 1 through 4

**Proposition 1.** (*passivity preservation*) *Let Assumption 1 hold, consider the quadruples  $(A_i, B_i, C_i, D_i)$  in (2),  $i = 1, 2$ , and the interconnection in (4). Let  $P = \begin{pmatrix} P_1 & P_{12} \\ P_{12}^\top & P_2 \end{pmatrix}$ . **(i)** Assume that  $\sigma_{\max}^2(P_{12}) < \lambda_{\min}(P_1)\lambda_{\min}(P_2)$  if  $P_1 \succ 0$ ,  $P_2 \succ 0$ , or  $P \succcurlyeq 0$  if  $P_1 \succcurlyeq 0$  and  $\text{Im}(P_{12}) \subseteq \text{Im}(P_1)$  and  $P_2 \succcurlyeq P_{12}^\top P_1^\dagger P_{12}$ . It is inferred that if the conditions in Lemmata 1 through 4 hold, then the quadruple  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is passive with storage function matrix  $P = \begin{pmatrix} P_1 & P_{12} \\ P_{12}^\top & P_2 \end{pmatrix}$ . **(ii)** If the conditions in items 4.1 or 4.2 in Lemma 4 do not hold, then it is not passive. **(iii)** It is passive with  $P = \text{diag}(P_1, P_2)$ , if  $E_1 G_2 = -G_1^\top E_2^\top$ ,  $P_1 L_1 G_2 - H_1^\top E_2^\top = 0$ ,  $P_2 L_2 G_1 - H_2^\top E_1^\top = 0$ , and  $P_1 L_1 H_2 + (P_2 L_2 H_1)^\top = 0$ .*

*Proof.* The conditions guarantee that (11) hold, and **(i)** secures that  $P \succ 0$  or  $P \succcurlyeq 0$  using Corollary 8 and [7, Proposition 8.2.4]. Assertion **(iii)** follows from Lemma 5. It is also possible to check that (18), (19) and (20) hold under these conditions, due to the passivity of each subsystem. □

## 2.2 Strong Passivity Preservation after Interconnection

Let us now state a result about the interconnection of strongly passive subsystems.

**Proposition 2.** (*preservation of strong passivity*) *Assume that both subsystems in (2) are strongly passive. Then the quadruple  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is strongly passive with  $P = \begin{pmatrix} P_1 & P_{12} \\ P_{12}^\top & P_2 \end{pmatrix} \succ 0$  where  $P_1 \succ 0$  and  $P_2 \succ 0$  are solutions of (5), if conditions (28), (29) and (30) are verified, where:*

$$\min(\lambda_{\min}(D_1 + D_1^\top), \lambda_{\min}(D_2 + D_2^\top)) \geq \frac{1}{2} \lambda_{\max}^{\frac{1}{2}}(E_1 G_2 G_2^\top E_1^\top) + \frac{1}{2} \lambda_{\max}^{\frac{1}{2}}(E_2 G_1 G_1^\top E_2^\top) \quad (28)$$

(then  $\tilde{D} + \tilde{D}^\top \succ 0$ ), and:

$$\sigma_{\max}(\bar{Q}) \leq \min(\lambda_{\min}(Q_1), \lambda_{\min}(Q_2)), \quad (29)$$

with  $\bar{Q}$  in (8),  $Q_1 \succ 0$  and  $Q_2 \succ 0$  (these conditions hold if  $P_{12} = 0$  and  $P_1 L_1 H_2 =$

$-(P_2L_2H_1)^\top$ , since then  $\bar{Q} = 0$  hence  $-\tilde{A}^\top P - P\tilde{A} \succ 0$ ), and:

$$\begin{aligned} & \sigma_{\max} \left\{ \text{diag}(P_1B_1 - C_1^\top, P_2B_2 - C_2^\top) \right. \\ & \quad \left. + \begin{pmatrix} P_{12}L_2G_1 & P_1L_1G_2 + P_{12}B_2 - H_1^\top E_2^\top \\ P_{12}^\top B_1 + P_2L_2G_1 - H_2^\top E_1^\top & P_{12}^\top L_1G_2 \end{pmatrix} \right\} \\ & \leq \left( \lambda_{\min}(\text{diag}(D_1 + D_1^\top, D_2 + D_2^\top)) + \lambda_{\min} \begin{pmatrix} 0 & E_1G_2 + G_1^\top E_2^\top \\ E_2G_1 + G_2^\top E_1^\top & 0 \end{pmatrix} \right)^{\frac{1}{2}} \\ & \quad \times (\lambda_{\min}(\text{diag}(Q_1, Q_2)) + \lambda_{\min}(\bar{Q}))^{\frac{1}{2}}, \end{aligned} \quad (30)$$

and the conditions in item (i) in Proposition 1 hold.

*Proof.* Strong passivity of both subsystems implies that  $D_1 \succ 0$  and  $D_2 \succ 0$ . Consider (9). We have

$$\tilde{D} + \tilde{D}^\top = \text{diag}(D_1 + D_1^\top, D_2 + D_2^\top) + \begin{pmatrix} 0 & E_1G_2 + G_1^\top E_2^\top \\ G_2^\top E_1^\top + E_2G_1 & 0 \end{pmatrix}$$

Using [7, Fact 9.14.15], it follows that

$$\begin{aligned} \sigma_{\max} \begin{pmatrix} 0 & E_1G_2 \\ E_2G_1 & 0 \end{pmatrix} & < \frac{1}{2\sigma_{\max}(\text{diag}((D_1 + D_1^\top)^{-1}, (D_2 + D_2^\top)^{-1}))} \\ & = \frac{1}{2}\lambda_{\min}(\text{diag}(D_1 + D_1^\top, D_2 + D_2^\top)) \end{aligned}$$

guarantees from Corollary 8 that  $\tilde{D} \succ 0$ . Still using [7, Fact 9.14.15] we have  $\sigma_{\max} \begin{pmatrix} 0 & E_1G_2 \\ E_2G_1 & 0 \end{pmatrix} = \sigma_{\max} \left( \begin{pmatrix} 0 & E_1G_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ E_2G_1 & 0 \end{pmatrix} \right) \leq \sigma_{\max} \begin{pmatrix} 0 & E_1G_2 \\ 0 & 0 \end{pmatrix} + \sigma_{\max} \begin{pmatrix} 0 & 0 \\ E_2G_1 & 0 \end{pmatrix}$ . Now using [7, Fact 4.10.12, Proposition 5.5.20 v)]:

$$\sigma_{\max} \begin{pmatrix} 0 & E_1G_2 \\ 0 & 0 \end{pmatrix} = \lambda_{\max}^{\frac{1}{2}} \begin{pmatrix} E_1G_2G_2^\top E_1^\top & 0 \\ 0 & 0 \end{pmatrix} = \lambda_{\max}^{\frac{1}{2}}(E_1G_2G_2^\top E_1^\top)$$

Similarly  $\sigma_{\max} \begin{pmatrix} 0 & 0 \\ E_2G_1 & 0 \end{pmatrix} = \lambda_{\max}^{\frac{1}{2}}(E_2G_1G_1^\top E_2^\top)$ . Continuing:

$$\lambda_{\min}(\text{diag}(D_1 + D_1^\top, D_2 + D_2^\top)) = \min(\lambda_{\min}(D_1 + D_1^\top), \lambda_{\min}(D_2 + D_2^\top)). \quad (31)$$

Hence (28) guarantees that  $\tilde{D} \succ 0$  ( $\Leftrightarrow \tilde{D} + \tilde{D}^\top \succ 0$ ), which guarantees in passing that (11) (a) holds trivially. Obviously this is a conservative criterion since  $E_1G_2 + G_1^\top E_2^\top = 0$  guarantees  $\tilde{D} + \tilde{D}^\top \succ 0$  also.

The second condition is a consequence of partitioning  $-\tilde{A}^\top P - P\tilde{A} = \text{diag}(Q_1, Q_2) + \bar{Q}$ , and applying Corollary 8. It follows that  $\sigma_{\max}(\bar{Q}) \leq \frac{1}{\sigma_{\max}(\text{diag}(Q_1^{-1}, Q_2^{-1}))}$ . Since  $Q_i = Q_i^\top \succ 0$ , we have

$$\sigma_{\max}(\text{diag}(Q_1^{-1}, Q_2^{-1})) = \lambda_{\max}(\text{diag}(Q_1^{-1}, Q_2^{-1})) = \lambda_{\min}^{-1}(\text{diag}(Q_1, Q_2)).$$

Then (29) follows which guarantees that  $-\tilde{A}^\top P - P\tilde{A} \succ 0$ .

Now using [7, Proposition 8.2.4] the inequality in (6) is satisfied strictly if and only if

$$-\tilde{A}^\top P - P\tilde{A} - (P\tilde{B} - \tilde{C}^\top)(\tilde{D} + \tilde{D}^\top)^{-1}(P\tilde{B} - \tilde{C}^\top)^\top \succ 0. \quad (32)$$

and  $-\tilde{A}^\top P + P\tilde{A} \succ 0$  (which is assured by the previous step). Using Corollary 8 it is inferred that (32) holds if  $\|P\tilde{B} - \tilde{C}^\top\|_{2,2}^2 \|(\tilde{D} + \tilde{D}^\top)^{-1}\|_{2,2} \leq \frac{1}{\sigma_{\max}((-\tilde{A}^\top P - P\tilde{A})^{-1})} = \frac{1}{\lambda_{\max}((-\tilde{A}^\top P - P\tilde{A})^{-1})} = \frac{1}{\lambda_{\min}(-\tilde{A}^\top P - P\tilde{A})}$  (where it is used that the  $\|\cdot\|_{2,2}$  norm is submultiplicative). From [8, Fact 10.12.17] this is equivalent to  $\sigma_{\max}((\tilde{D} + \tilde{D}^\top)^{-1}) = \lambda_{\max}((\tilde{D} + \tilde{D}^\top)^{-1}) = \frac{1}{\lambda_{\min}(\tilde{D} + \tilde{D}^\top)} \leq \frac{\lambda_{\min}(-\tilde{A}^\top P - P\tilde{A})}{\sigma_{\max}^2(P\tilde{B} - \tilde{C}^\top)}$ , which is equivalently rewritten as  $\lambda_{\min}(\tilde{D} + \tilde{D}^\top) \lambda_{\min}(-\tilde{A}^\top P - P\tilde{A}) \geq \sigma_{\max}^2(P\tilde{B} - \tilde{C}^\top)$ . Using [7, Fact 5.12.2], (10), (9), this inequality is implied by (30). The last condition (item **(i)** in Proposition 1) guarantees  $P \succ 0$  from [8, Fact 10.12.17].  $\square$

Notice that  $D_1 + D_1^\top \succ 0$  and  $D_2 + D_2^\top \succ 0$  imply that (13) is simplified since the last two inclusions hold trivially.

**Lemma 6.** *Let both subsystems in (2) be strongly passive. The interconnection quadruple is strongly passive with storage function matrix  $P = \text{diag}(P_1, P_2)$ , if: (i)  $E_1 G_2 = -(E_2 G_1)^\top$ , (ii)  $P_1 L_1 H_2 = -(P_2 L_2 H_1)^\top$ , (iii)  $P_1 L_1 G_2 = (E_2 H_1)^\top$  and  $P_2 L_2 G_1 = (E_1 H_2)^\top$ .*

*Proof.* (i) using the comment after (31):  $\tilde{D} + \tilde{D}^\top \succ 0$ , (ii) implies that  $\bar{Q}_{12} = 0$  in (8), thus (11) **(c)** holds with strict inequality, (iii) implies that  $P\tilde{B} - \tilde{C}^\top = \text{diag}(P_1 B_1 - C_1^\top, P_2 B_2 - C_2^\top)$ . From the strong passivity of both subsystems, it follows that  $\begin{pmatrix} -\tilde{A}^\top P - P\tilde{A} & -P\tilde{B} + \tilde{C}^\top \\ (-P\tilde{B} + \tilde{C}^\top)^\top & \tilde{D} + \tilde{D}^\top \end{pmatrix} \succ 0$ , which ends the proof (see section B).  $\square$

It is easily deduced from  $\tilde{D} + \tilde{D}^\top \succ 0$  (strict version of (11)**(a)**) and [15, Theorem A.65] that strong passivity of the interconnection implies that  $D_i + D_i^\top \succ 0 \Leftrightarrow D_i \succ 0$  for  $i = 1, 2$ .

## 2.3 Strict State Passivity Preservation after Interconnection

Finally a third result concerns the interconnection of two strictly state passive subsystems.

**Proposition 3.** *(preservation of strict state passivity) Assume that both subsystems in (2) are strictly state passive with constants  $\mu_1 > 0$ ,  $\mu_2 > 0$ , with  $P_i \succ 0$  the solution of the strict state passivity LMI,  $i = 1, 2$ . Let  $P = \begin{pmatrix} P_1 & P_{12} \\ P_{12}^\top & P_2 \end{pmatrix}$ . Suppose that: **(1)** (12) holds, **(2)** conditions in Proposition 1 **(i)** and in Lemma 2 hold, and:*

**(3)** the next conditions hold:

$$\begin{aligned}
\text{i)} & (\mu_1 - \mu)P_1 + P_{12}L_2H_1 + H_1^\top L_2^\top P_{12}^\top \succcurlyeq 0 \quad (\text{true if: } \mathbf{i1} \mu_1 > \mu, \sigma_{\max}(P_{12}L_2H_1) \\
& < \frac{(\mu_1 - \mu)}{2} \lambda_{\max}(P_1) \text{ or if: } \mathbf{i2} \mu_1 \geq \mu, P_{12} = H_1^\top L_2^\top), \\
\text{ii)} & (\mu_2 - \mu)P_2 + P_{12}^\top L_1H_2 + H_2^\top L_1^\top P_{12} \succcurlyeq 0 \quad (\text{true if: } \mathbf{ii1} \mu_2 > \mu, \sigma_{\max}(P_{12}L_1H_2) \\
& < \frac{(\mu_2 - \mu)}{2} \lambda_{\max}(P_2) \text{ or if: } \mathbf{ii2} \mu_2 \geq \mu, P_{12} = H_2^\top L_1^\top), \\
\text{iii)} & \text{Im}(\bar{Q}_{12}^\top - \mu P_{12}^\top) \subseteq \text{Im}((\mu_2 - \mu)P_2 + H_2^\top L_1^\top P_{12} + P_{12}^\top L_1H_2) \quad (\text{true if } \mathbf{ii1} \text{ holds}), \\
\text{iv)} & (\mu_1 - \mu)P_1 + P_{12}L_2H_1 + H_1^\top L_2^\top P_{12}^\top \succcurlyeq (\bar{Q}_{12} - \mu P_{12})((\mu_2 - \mu)P_2 \\
& + H_2^\top L_1^\top P_{12} + P_{12}^\top L_1H_2)^\dagger (\bar{Q}_{12}^\top - \mu P_{12}^\top) \\
& (\text{true if: } \mathbf{i1}, \mathbf{ii1} \text{ hold, and:} \\
& \sigma_{\max}^2(\bar{Q}_{12} - \mu P_{12}) < \lambda_{\min}((\mu_2 - \mu)P_2 + H_2^\top L_1^\top P_{12} + P_{12}^\top L_1H_2) \\
& \quad \times \lambda_{\max}((\mu_1 - \mu)P_1 + P_{12}L_2H_1 + H_1^\top L_2^\top P_{12}^\top)), \tag{33}
\end{aligned}$$

**(4)** the next conditions hold:

**(d'1):**  $\text{rank}(\tilde{D} + \tilde{D}^\top) = d$ ,  $\tilde{D} + \tilde{D}^\top \neq 0$  and:

$$\begin{aligned}
& \sigma_{\max} \left\{ \text{diag}(P_1B_1 - C_1^\top, P_2B_2 - C_2^\top) \right. \\
& \quad \left. + \begin{pmatrix} P_{12}L_2G_1 & P_1L_1G_2 + P_{12}B_2 - H_1^\top E_2^\top \\ P_{12}^\top L_1G_2 & P_{12}^\top L_1G_2 \end{pmatrix} \right\} \\
& < \lambda_d^{\frac{1}{2}} \left( \text{diag}(D_1 + D_1^\top, D_2 + D_2^\top) + \begin{pmatrix} 0 & E_1G_2 + G_1^\top E_2^\top \\ E_2G_1 + G_2^\top E_1^\top & 0 \end{pmatrix} \right) \\
& \quad \times (\sigma_{\max}(\text{diag}(Q_1, Q_2) + \bar{Q}) - \mu \lambda_{\max}(P))^{\frac{1}{2}} \tag{34}
\end{aligned}$$

with  $\bar{Q}$  in (8).

**(d'2):**  $\tilde{D} + \tilde{D}^\top = 0$ , equivalently:  $D_1 + D_1^\top = 0$  and  $D_2 + D_2^\top = 0$  and  $E_2G_1 + G_2^\top E_1^\top = 0$ . Then the quadruple  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is strictly state passive with storage function matrix  $P$  and dissipation constant  $\mu > 0$ .

*Proof.* The LMI for strict state passivity is  $P = P^\top \succ 0$ , (11) **(a)** **(b)** and

$$\text{(c')} \quad -\tilde{A}^\top P - P\tilde{A} - \mu P \succcurlyeq 0 \tag{35}$$

$$\text{(d')} \quad -\tilde{A}^\top P - P\tilde{A} - \mu P \succcurlyeq (P\tilde{B} - \tilde{C}^\top)(\tilde{D} + \tilde{D}^\top)^\dagger (\tilde{B}^\top P - \tilde{C}) \tag{36}$$

for some  $\mu > 0$ .

**(a)** Equality (11) **(a)** holds under (12), hence under **(1)**.

**(b)** The necessary and sufficient conditions such that condition (11) **(b)** holds are the same as in Lemma 2.

**(c')** Condition (35) is analysed as follows. Notice that  $P \succ 0$  since item **(i)** in Proposition 1 holds. We have:  $-\tilde{A}^\top P - P\tilde{A} = \text{diag}(Q_1, Q_2) + \bar{Q}$  with  $\bar{Q}$  in (8), with

$Q_1 - \mu_1 P_1 \succcurlyeq 0$  and  $Q_2 - \mu_2 P_2 \succcurlyeq 0$ . Thus  $-\tilde{A}^\top P - P\tilde{A} \succcurlyeq \text{diag}(\mu_1 P_1, \mu_2 P_2) + \bar{Q}$ . Let us study the conditions such that  $\text{diag}(\mu_1 P_1, \mu_2 P_2) + \bar{Q} \succcurlyeq \mu P \Leftrightarrow \text{diag}(\mu_1 P_1, \mu_2 P_2) + \bar{Q} - \mu P \succcurlyeq 0$  for some  $\mu > 0$ . To this aim let us analyse the inequality:

$$\begin{pmatrix} (\mu_1 - \mu)P_1 + P_{12}L_2H_1 + H_1^\top L_2^\top P_{12}^\top & \bar{Q}_{12} - \mu P_{12} \\ \bar{Q}_{12}^\top - \mu P_{12}^\top & (\mu_2 - \mu)P_2 + P_{12}^\top L_1H_2 + H_2^\top L_1^\top P_{12} \end{pmatrix} \succcurlyeq 0. \quad (37)$$

Recall that  $P_{12}L_2H_1 \in \mathbb{R}^{n_1 \times n_1}$ , and  $P_{12}^\top L_1H_2 \in \mathbb{R}^{n_2 \times n_2}$ . The necessary and sufficient conditions for (37) to hold are in [8, Proposition 10.2.5, Fact 8.4.3] and are given in **(3 i)–iv)** (these are the counterparts of (11) for (37)). Let us examine the conditions given in (33) **i)** to **iv)** one by one. **i)** This matrix inequality is satisfied strictly if  $\mu_1 > \mu$ ,  $\sigma_{\max}(P_{12}L_2H_1 + H_1^\top L_2^\top P_{12}^\top) \leq \sigma_{\max}(P_{12}L_2H_1) + \sigma_{\max}(H_1^\top L_2^\top P_{12}^\top) = 2\sigma_{\max}(P_{12}L_2H_1) < (\mu_1 - \mu)\sigma_{\max}(P_1) = (\mu_1 - \mu)\lambda_{\max}(P_1)$  (where we used [7, Fact 9.14.15] and Corollary 8); or it is satisfied if:  $\mu_1 \geq \mu$ ,  $P_{12}L_2H_1 \succcurlyeq 0$ , which holds if  $P_{12} = H_1^\top L_2^\top$  (this secures the positive semidefiniteness). **ii)** Similarly this matrix inequality is satisfied strictly if  $\mu_2 > \mu$ ,  $\sigma_{\max}(P_{12}^\top L_1H_2 + H_2^\top L_1^\top P_{12}) \leq \sigma_{\max}(P_{12}^\top L_1H_2) + \sigma_{\max}(H_2^\top L_1^\top P_{12}) = 2\sigma_{\max}(P_{12}^\top L_1H_2) < (\mu_2 - \mu)\sigma_{\max}(P_2) = (\mu_2 - \mu)\lambda_{\max}(P_2)$ ; or is satisfied if  $\mu_2 \geq \mu$ ,  $P_{12}^\top L_1H_2 \succcurlyeq 0$ , which holds if  $P_{12} = L_1H_2$  (this secures the positive semidefiniteness). **iii)** this condition uses [8, Fact 8.4.3]; the first set of sufficient conditions **ii1** in (33) **ii)** guarantees strict inequalities, consequently **iii)** holds since the right-hand side matrix in the inclusion is full-rank. **iv)** stems from [8, Proposition 10.2.5, iii)]; let conditions **i1** and **ii1** be true (this secures that both matrices in **i)** and **ii)** are  $\succ 0$ , hence both matrices in the left-hand and the right-hand sides in inequality **(3 iv)** are  $\succ 0$ ); we have:

$$\begin{aligned} & \sigma_{\max}((\bar{Q}_{12} - \mu P_{12})((\mu_2 - \mu)P_2 + H_2^\top L_1^\top P_{12} + P_{12}^\top L_1H_2)^{-1}(\bar{Q}_{12}^\top - \mu P_{12}^\top)) \\ & \leq \sigma_{\max}^2(\bar{Q}_{12} - \mu P_{12})\sigma_{\max}((\mu_2 - \mu)P_2 + H_2^\top L_1^\top P_{12} + P_{12}^\top L_1H_2)^{-1} \\ & = \sigma_{\max}^2(\bar{Q}_{12} - \mu P_{12})\lambda_{\max}((\mu_2 - \mu)P_2 + H_2^\top L_1^\top P_{12} + P_{12}^\top L_1H_2)^{-1} \\ & = \sigma_{\max}^2(\bar{Q}_{12} - \mu P_{12})\lambda_{\min}^{-1}((\mu_2 - \mu)P_2 + H_2^\top L_1^\top P_{12} + P_{12}^\top L_1H_2) \end{aligned}$$

where [7, Corollary 9.6.5] is used. Then we can apply Corollary 8 to obtain the sufficient condition stated in **(3 iv)**.

**(d')** If (36) holds for  $\mu > 0$ , then it holds strictly for some  $\mu' < \mu$  since  $P \succ 0$ . Hence without loss of generality, we consider a strict inequality. Applying Corollary 8, (36) holds if  $\sigma_{\max}((P\tilde{B} - \tilde{C}^\top)(\tilde{D} + \tilde{D}^\top)^\dagger(\tilde{B}^\top P - \tilde{C}) + \mu P) < \sigma_{\max}(-\tilde{A}^\top P - P\tilde{A})$ . Using [7, Facts 9.14.15, Corollary 9.6.5],  $\sigma_{\max}((P\tilde{B} - \tilde{C}^\top)(\tilde{D} + \tilde{D}^\top)^\dagger(\tilde{B}^\top P - \tilde{C}) + \mu P) \leq \sigma_{\max}^2((P\tilde{B} - \tilde{C}^\top)\sigma_{\max}((\tilde{D} + \tilde{D}^\top)^\dagger) + \sigma_{\max}(\mu P))$ . **(d'1)** If  $\text{rank}(\tilde{D} + \tilde{D}^\top) = d$  and  $\tilde{D} + \tilde{D}^\top \neq 0$ , then  $\sigma_{\max}((\tilde{D} + \tilde{D}^\top)^\dagger) = \sigma_d^{-1}(\tilde{D} + \tilde{D}^\top)$  [7, Fact 6.3.28]. Consequently (36) holds if  $\sigma_{\max}^2((P\tilde{B} - \tilde{C}^\top)\sigma_d^{-1}(\tilde{D} + \tilde{D}^\top) + \mu\lambda_{\max}(P)) < \sigma_{\max}(\text{diag}(Q_1, Q_2) + \bar{Q})$ . Since  $\tilde{D} + \tilde{D}^\top \succcurlyeq 0$  by item **(b)**,  $\sigma_d(\tilde{D} + \tilde{D}^\top) = \lambda_d(\tilde{D} + \tilde{D}^\top)$ . Thus we get equivalently  $\sigma_{\max}^2(P\tilde{B} - \tilde{C}^\top) < \lambda_d(\tilde{D} + \tilde{D}^\top)(\sigma_{\max}(\text{diag}(Q_1, Q_2) + \bar{Q}) - \mu\lambda_{\max}(P))$ . Using (9) and (10) yields (34). **(d'2)** We have  $\tilde{D} + \tilde{D}^\top = 0 \iff D_1 + D_1^\top = 0, D_2 + D_2^\top = 0, E_1G_2 + G_1^\top E_2^\top = 0$ . From item **(a)** and (12), it follows that  $P\tilde{B} - \tilde{C}^\top = 0$ . □



## 2.4 Comments

Some comments on the foregoing results arise:

- Many conditions in the above three propositions imply some "smallness" of the interconnections, in the following sense. Consider Lemma 2: all the conditions are satisfied if  $E_1G_2 = -(E_2G_1)^\top$ , which is a kind of "skew-symmetry" condition. The conditions of Lemma 2 state how much this skew-symmetry property can be deteriorated. Similar conclusions are drawn for Lemma 1 in (13).
- LMIs can be solved numerically efficiently. However it is unclear how to compute the interconnections terms from scratch so that the interconnected system's matrix inequality is solvable. Notice in particular that the matrix inequalities are nonlinear if the interconnection matrices  $H_1$ ,  $H_2$ ,  $G_1$  and  $G_2$  (and possibly in some design problems  $L_1$ ,  $L_2$ ,  $E_1$  and  $E_2$ ) are considered as unknowns in addition to  $P_{12}$ . The usefulness of the above results is that they provide analytical conditions which pave the way towards a criterion guaranteeing that (6) possesses at least a solution for  $H_1$ ,  $H_2$ ,  $G_1$ ,  $G_2$  and  $P_{12}$ . In an even more general setting  $P_1$  and  $P_2$  could be considered unknowns of the problem also.
- The decomposition in item 4.1 in Lemma 4 can be done with  $N = (P\tilde{B} - \tilde{C}^\top)((\tilde{D} + \tilde{D}^\top)^\dagger)^{\frac{1}{2}}$ . The Cholesky decomposition [7, Fact 8.9.38], or Gram matrix [7, Fact 8.9.37] are possible choices, also in items 4.2 and 4.4. Similar condition as in (23) can be applied to the inequality (25) **(iii)**, following a reasoning as in the proof of item 4.4 in Lemma 4. Similarly the conditions  $\bar{Q}_1 \succcurlyeq 0$  and  $\bar{Q}_2 \succcurlyeq 0$  in Lemma 3, can be studied more deeply, still using the decompositions of  $Q_1$ ,  $Q_2$ , and of  $-P_{12}L_2H_1 - (P_{12}L_2H_1)^\top$  and of  $-P_{12}^\top L_1H_2 - (P_{12}^\top L_1H_2)^\top$ .
- In case  $\tilde{D} + \tilde{D}^\top = 0$ , then (11) reduces to conditions **(a)** and **(c)**. Thus it is sufficient to look at Lemmata 1 and 3.
- Item 4.4 in Lemma 4 yields an algorithm for checking that (11) **(d)** is satisfied.
- The storage matrices  $P_1$  and  $P_2$  may be nonunique. Therefore conditions involving these matrices (like (14) and (17)) have to be true for at least one pair of storage functions. Thus the interconnection matrices may vary while keeping the same property of the interconnection.
- The sufficient conditions proposed in the case  $P_{12} = 0$  imply that the interconnections terms cancel the coupling terms between both subsystems.
- The condition in (28) can be replaced by the diagonal dominance theorem.
- The conditions in Proposition 2 guarantee that the inequality in (6) is strict, by imposing small enough perturbation to the matrix

$$\begin{pmatrix} Q_1 & 0 & P_1B_1 - C_1^\top & 0 \\ 0 & Q_2 & 0 & P_2B_2 - C_2^\top \\ (P_1B_1 - C_1^\top)^\top & 0 & D_1 + D_1^\top & 0 \\ 0 & (P_2B_2 - C_2^\top)^\top & 0 & D_2 + D_2^\top \end{pmatrix}. \quad (38)$$

## 2.5 Lack/excess of passivity

It is often of interest to investigate cases with lack/excess of passivity, *i.e.*, can the excess of passivity in one subsystem, compensate for the lack of passivity in the other one.

**Proposition 4.** *Assume that  $(A_1, B_1, C_1, D_1)$  is strongly passive, while system 2 is marginally passive with  $D_2 + D_2^\top = 0$ , and with  $A_2^\top P_2 + P_2 A_2 = -Q_2 = 0$ . Then the interconnection (2) (3) is strictly passive with respect to  $x_1$  and  $\lambda_1$ , with  $P = \text{blockdiag}(P_1, P_2)$ , if:*

$$\begin{aligned} P_1 L_1 H_2 + (P_2 L_2 H_1)^\top &= 0, & P_1 L_1 G_2 - H_1^\top E_2^\top &= 0 \\ P_2 L_2 G_1 - H_2^\top E_1^\top &= 0, & E_1 G_2 + G_1^\top E_2^\top &= 0. \end{aligned} \quad (39)$$

In this case (assuming AC solutions exist for all  $x(0) = x_0 \in \text{Dom}(\mathcal{F})$  with  $\mathcal{F}(x) = -\tilde{A}x + \tilde{B}(\tilde{D} + \mathcal{N}_{K^*}^{-1})^{-1}(\tilde{C}x)$ ) the dissipation inequality:

$$\dot{V}(x(t)) \leq -\alpha_1 \|x_1\|^2 - \alpha_2 \|\lambda_1\|^2, \quad \alpha_1 > 0, \quad \alpha_2 > 0, \quad (40)$$

holds along the interconnected system's trajectories, with  $V(x) = x_1^\top P_1 x_1 + x_2^\top P_2 x_2$ .

*Proof.* We have  $\begin{pmatrix} -A_1^\top P_1 - P_1 A_1 & P_1 B_1 - C_1^\top \\ (P_1 B_1 - C_1^\top)^\top & D_1 + D_1^\top \end{pmatrix} \succ 0$ ,  $P_1 = P_1^\top \succ 0$ , and also  $-A_2^\top P_2 - P_2 A_2 = 0$ ,  $P_2 = P_2^\top \succcurlyeq 0$ ,  $D_2 + D_2^\top = 0 \Rightarrow P_2 B_2 = C_2^\top$ . The inequality in (6) is composed of the matrix in (38) which is equal to:

$$\begin{pmatrix} Q_1 & 0 & P_1 B_1 - C_1^\top & 0 \\ 0 & 0 & 0 & 0 \\ (P_1 B_1 - C_1^\top)^\top & 0 & D_1 + D_1^\top & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (41)$$

where  $Q_1 = -A_1^\top P_1 - P_1 A_1$ , with the perturbation:

$$\begin{pmatrix} 0 & P_1 L_1 H_2 + (P_2 L_2 H_1)^\top & 0 & P_1 L_1 G_2 - H_1^\top E_2^\top \\ (P_1 L_1 H_2)^\top + P_2 L_2 H_1 & 0 & P_2 L_2 G_1 - H_2^\top E_1^\top & 0 \\ 0 & (P_2 L_2 G_1 - H_2^\top E_1^\top)^\top & 0 & E_1 G_2 + G_1^\top E_2^\top \\ (P_1 L_1 G_2 - H_1^\top E_2^\top)^\top & 0 & (E_1 G_2 + G_1^\top E_2^\top)^\top & 0 \end{pmatrix} \quad (42)$$

Permuting the third and second columns, then the third and second rows, these matrices are transformed into:

$$\begin{pmatrix} Q_1 & P_1 B_1 - C_1^\top & 0 & 0 \\ (P_1 B_1 - C_1^\top)^\top & D_1 + D_1^\top & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (43)$$

and

$$\begin{pmatrix} 0 & 0 & P_1 L_1 H_2 + (P_2 L_2 H_1)^\top & P_1 L_1 G_2 - H_1^\top E_2^\top \\ 0 & 0 & (P_2 L_2 G_1 - H_2^\top E_1^\top)^\top & E_1 G_2 + G_1^\top E_2^\top \\ (P_1 L_1 H_2)^\top + P_2 L_2 H_1 & P_2 L_2 G_1 - H_2^\top E_1^\top & 0 & 0 \\ (P_1 L_1 G_2 - H_1^\top E_2^\top)^\top & (E_1 G_2 + G_1^\top E_2^\top)^\top & 0 & 0 \end{pmatrix} \quad (44)$$

From [7, Proposition 8.2.4] [15, Lemmata A.69, A.70], the sum of the matrices in (43) and (44) is positive semidefinite, if and only if:

$$\begin{pmatrix} P_1 L_1 H_2 + (P_2 L_2 H_1)^\top & P_1 L_1 G_2 - H_1^\top E_2^\top \\ (P_2 L_2 G_1 - H_2^\top E_1^\top)^\top & E_1 G_2 + G_1^\top E_2^\top \end{pmatrix} = 0. \quad (45)$$

The proof of the passivity is complete. To end the proof, let us take into account (39), calculations yield  $\dot{V}(t) = (x_1^\top \ \lambda_1^\top) \begin{pmatrix} A_1^\top P_1 + P_1 A_1 & P_1 B_1 - C_1^\top \\ (P_1 B_1 - C_1^\top)^\top & -D_1 - D_1^\top \end{pmatrix} \begin{pmatrix} x_1 \\ \lambda_1 \end{pmatrix}$  and the result follows from strong passivity of  $(A_1, B_1, C_1, D_1)$  and [7, Corollary 8.4.2].  $\square$

The conditions for asymptotic stability of this interconnection will be studied in section 4.1.

### 3 Maximal Monotonicity and Incremental Passivity

Under some basic conditions (see Theorem 3), the negative feedback interconnection of a passive system with a maximal monotone static nonlinearity (possibly set-valued) defines a DI:  $\dot{x} \in -\mathcal{F}(x) = \tilde{A}x - \tilde{B}(\tilde{D} + \mathcal{N}_{K^*}^{-1})^{-1}(\tilde{C}x)$  with  $\mathcal{F}(\cdot)$  a maximal monotone operator with domain  $\{x \in \mathbb{R}^n \mid \tilde{C}x \in \text{dom}(\tilde{D} + \mathcal{N}_{K^*}^{-1})^{-1} = \text{Im}(\tilde{D} + \mathcal{N}_{K^*}^{-1})\} = \tilde{C}^{-1}(\text{Im}(\tilde{D} + \mathcal{N}_{K^*}^{-1}))$ . This motivates the next result.

**Corollary 3.** *Consider subsystems in (2) under Assumption 1, and their interconnection in (4). Let  $\text{Im}(\tilde{C}) \cap \text{rint}(\text{Im}(\mathcal{N}_{K^*}^{-1} + \tilde{D})) \neq \emptyset$ . If the conditions in Proposition 1 (resp. 2, resp. 3) are satisfied, then the LCS in (4) has a maximal monotone (resp. single valued Lipschitz strongly monotone; resp. strongly monotone) right-hand side mapping  $\mathcal{F}(\cdot)$ .*

*Proof.* If the conditions of Proposition 1 hold, then  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  is passive, hence under the stated condition, the mapping  $\mathcal{F}(\cdot)$  is maximal monotone, see Theorem 3. Let the quadruple  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  be strongly passive, then there exists  $\mu > 0$  such that  $(\tilde{A} + \mu I_n, \tilde{B}, \tilde{C}, \tilde{D})$  is passive (this holds for all  $0 < \mu < \lambda_{\min}(\tilde{Q})$  with  $\tilde{Q}$  in (6), applying for instance Corollary 8). Therefore the mapping  $\tilde{\mathcal{F}} : x \mapsto -(\tilde{A} + \mu I_n)x + \tilde{B}(\tilde{D} + \mathcal{N}_{K^*}^{-1})^{-1}(\tilde{C}x)$  is maximal monotone, and  $\langle x_1 - x_2, y_1 - y_2 \rangle \geq \mu \|x_1 - x_2\|^2$  for all  $x_i \in \text{dom}(\tilde{\mathcal{F}})$ ,  $y_i \in \tilde{\mathcal{F}}(x_i)$ ,  $i = 1, 2$ . In addition strong passivity implies  $\tilde{D} \succ 0$ , hence applying [14, Proposition 1] the mapping  $(\tilde{D} + \mathcal{N}_{K^*}^{-1})^{-1}$  is single-valued, well-defined and Lipschitz continuous with constant  $\frac{1}{\lambda_{\min}(\tilde{D} + \tilde{D}^\top)}$ . Let now  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  be strictly state passive, i.e.,  $\tilde{Q} \succ 0$  and  $-\tilde{A}^\top P + P\tilde{A} - \mu P \succ 0$  for some  $\mu > 0$ . Then the quadruple  $(\tilde{A} + \frac{\mu}{2} I_n, \tilde{B}, \tilde{C}, \tilde{D})$  is passive. Thus the mapping  $x \mapsto -(\tilde{A} + \mu I_n)x + \tilde{B}(\tilde{D} + \mathcal{N}_{K^*}^{-1})^{-1}(\tilde{C}x)$  is maximal monotone. Hence  $\mathcal{F}(\cdot)$  is  $\mu$ -strongly monotone (but not necessarily single valued).  $\square$

Let us analyze the condition in Theorem 3 and in Corollary 3.

**Lemma 7.** *Let the quadruple  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  be passive. Assume that  $\tilde{D} = \tilde{D}^\top$  ( $\Leftrightarrow D_1 = D_1^\top, D_2 = D_2^\top, E_2G_1 = (E_1G_2)^\top$ ). Assume that  $\{\text{Im} \begin{pmatrix} C_1 \\ E_2H_1 \end{pmatrix} + \text{Im} \begin{pmatrix} E_1H_2 \\ C_2 \end{pmatrix}\} \cap \{\text{rint}(K^*) + \text{rint}(\text{Im}(\tilde{D}))\} \neq \emptyset$ . Then  $\text{Im}(\tilde{C}) \cap \text{rint}(\text{Im}(\mathcal{N}_{K^*}^{-1} + \tilde{D})) \neq \emptyset$ .*

*Proof.* Let us characterize  $\text{Im}(\mathcal{N}_{K^*}^{-1} + \tilde{D})$ . Both mappings  $x \mapsto \tilde{D}x$  and  $\mathcal{N}_{K^*}^{-1}(\cdot)$  are maximal monotone, and they are both the subdifferential of proper, convex lower semi-continuous functions, because  $\tilde{D}$  is symmetric, and  $\mathcal{N}_{K^*}^{-1}(\cdot) = \partial\sigma_{K^*}(\cdot)$ . Hence both mappings satisfy the property (\*) stated in [10, section 1]. From [6, Corollary 24.4 (ii)], the sum is maximal monotone. Using [10, Theorems 3, 4], we have  $\overline{\text{Im}(\mathcal{N}_{K^*}^{-1} + \tilde{D})} = \overline{\text{Im}(\mathcal{N}_{K^*}^{-1}) + \text{Im}(\tilde{D})} = \mathbb{R}_+^m + \text{Im}(\tilde{D})$ , and  $\text{int}(\text{Im}(\mathcal{N}_{K^*}^{-1} + \tilde{D})) = \text{int}(\text{Im}(\mathcal{N}_{K^*}^{-1}) + \text{Im}(\tilde{D})) = \text{int}(K^* + \text{Im}(\tilde{D}))$ . Therefore  $\text{Im}(\mathcal{N}_{K^*}^{-1} + \tilde{D}) = K^* + \text{Im}(\tilde{D})$ . Using [30, Corollary 6.6.2], we have  $\text{rint}(K^* + \text{Im}(\tilde{D})) = \text{rint}(K^*) + \text{rint}(\text{Im}(\tilde{D}))$ . The proof finishes by applying [8, Fact 3.14.8] to  $\text{Im}(\tilde{C})$ .  $\square$

Monotonicity and incremental passivity are closely related one to each other [15]. Let us consider the system  $\dot{x} \in -\mathcal{F}(x) + u$ , with  $\mathcal{F}(\cdot)$  maximal monotone. Then the mapping  $u \mapsto x$  is incrementally passive. Indeed, let  $\dot{x}_1 \in -\mathcal{F}(x_1) + u_1$  and  $\dot{x}_2 \in -\mathcal{F}(x_2) + u_2$ , with  $u_i$  such that the system is well-posed with AC solutions. Then  $s(x_1, x_2, u_1, u_2) \triangleq \langle x_1 - x_2, u_1 - u_2 \rangle = \langle x_1 - x_2, \dot{x}_1 - \dot{x}_2 \rangle + \langle x_1 - x_2, \xi_1 - \xi_2 \rangle$ , where  $\xi_i \in \mathcal{F}(x_i)$ . Thus defining the storage function  $V(x_1, x_2) = \frac{1}{2}\langle x_1 - x_2, x_1 - x_2 \rangle$ , it follows that  $\dot{V} = s(x_1, x_2, u_1, u_2) - \langle x_1 - x_2, \xi_1 - \xi_2 \rangle \leq s(x_1, x_2, u_1, u_2)$ . This motivates us to enlarge the interconnection (4) to

$$\begin{aligned} \dot{x} &= \tilde{A}x + \tilde{B}\lambda + Lv \\ K \ni \lambda \perp w &= \tilde{C}x + \tilde{D}\lambda \in K^*, \end{aligned} \quad (46)$$

for some admissible control input  $u$ . The system (46) is rewritten equivalently as:

$$\dot{x} \in \tilde{A}x - \tilde{B}(\tilde{D} + \mathcal{N}_{K^*}^{-1})^{-1}(\tilde{C}x) + Lv = -\mathcal{F}(x) + Lv. \quad (47)$$

The system in (47) has exactly the same form as the subsystems in (2), indicating that further interconnections of interconnections can be made. This is an interesting feature, similar to nested negative feedback interconnections of passive systems. In the next proposition an output  $z$  is considered for the interconnected system.

**Proposition 5.** *Consider subsystems in (2) under Assumption 1, and their interconnection in (4), and the input/output mapping  $\Sigma_{vz} : v \mapsto z$ , with  $z = Hx + Gv$ , in (46). Consider the storage function  $V(x_1, x_2) = \frac{1}{2}\langle x_1 - x_2, P(x_1 - x_2) \rangle$ , for some  $P = P^\top \succcurlyeq 0$ .*

1. *Let the mapping  $P\mathcal{F}(\cdot)$  be maximal monotone. Then  $\Sigma_{vz}$  is incrementally passive if  $G + G^\top \succcurlyeq 0$  and  $PL = H^\top$ .*

2. Let the mapping  $P\mathcal{F}(\cdot)$  be maximal  $\alpha$ -strongly monotone,  $\alpha > 0$ . Then  $\Sigma_{vz}$  is incrementally passive if  $\begin{pmatrix} 2\alpha I_n & PL - H^\top \\ (PL - H^\top)^\top & G + G^\top \end{pmatrix} \succcurlyeq 0$ . If this inequality is strict then  $\Sigma_{vz}$  is incrementally strictly state and input passive. If  $G + G^\top = 0$  then incremental strict state passivity holds if  $PL = H^\top$ .

*Proof.* From (47) we have  $\dot{x} \in -\mathcal{F}(x) + Lv$ ,  $z = Hx + Gv$ . For two admissible inputs  $v_1$  and  $v_2$  it follows that  $\dot{x}_i \in -\mathcal{F}(x_i) + Lv_i$ ,  $z_i = Hx_i + Gv_i$ ,  $i = 1, 2$ . Let  $\xi_i \in \mathcal{F}(x_i)$ , and  $\omega = \langle z_1 - z_2, v_1 - v_2 \rangle$ .

1. From  $P\dot{x}_i \in -P\mathcal{F}(x_i) + PLv_i$ :

$$\begin{aligned} \dot{V} &= -(x_1 - x_2)^\top P(\xi_1 - \xi_2) + (x_1 - x_2)^\top PL(v_1 - v_2) \\ &\quad \pm (z_1 - z_2)^\top (v_1 - v_2) \\ &= -(x_1 - x_2)^\top P(\xi_1 - \xi_2) + (x_1 - x_2)^\top (PL - H^\top)(v_1 - v_2) \\ &\quad + \frac{1}{2}(v_1 - v_2)^\top (G + G^\top)(v_1 - v_2) + \omega. \end{aligned} \quad (48)$$

If  $P\mathcal{F}(\cdot)$  is monotone, then  $-(x_1 - x_2)^\top P(\xi_1 - \xi_2) \leq 0$ . Then incremental passivity holds if and only if  $-(x_1 - x_2)^\top P(\xi_1 - \xi_2) + (x_1 - x_2)^\top (PL - H^\top)(v_1 - v_2) + \frac{1}{2}(v_1 - v_2)^\top (G + G^\top)(v_1 - v_2) \leq 0$  for all  $v_1, v_2, x_1, x_2$  (see, e.g., [29, Definition 1]). The result is proved.

2. This time we get:

$$\begin{aligned} \dot{V} &\leq -\alpha(x_1 - x_2)^\top P(x_1 - x_2) + (x_1 - x_2)^\top (PL - H^\top)(v_1 - v_2) \\ &\quad + \frac{1}{2}(v_1 - v_2)^\top (G + G^\top)(v_1 - v_2) + \omega \\ &= (x_1^\top - x_2^\top, v_1^\top - v_2^\top) \begin{pmatrix} 2\alpha I_n & PL - H^\top \\ (PL - H^\top)^\top & G + G^\top \end{pmatrix} \begin{pmatrix} x_1 - x_2 \\ v_1 - v_2 \end{pmatrix} + \omega. \end{aligned} \quad (49)$$

The last result follows using [15, Lemma A.69] or [7, Proposition 8.2.4].

□

Proposition 5 shows that if the conditions in Propositions 1,2 and 3 hold, then the interconnection (4) defines incrementally passive systems.

**Corollary 4.** *Let the conditions of item 2 in Proposition 5 be satisfied. Then  $\Sigma_{vz}$  is incrementally passive if  $\sigma_{\max}^2(PL - H^\top) \leq 2\alpha\lambda_{\min}(G + G^\top)$ .*

*Proof.* Directly from [8, Fact 10.12.17].

□

**Remark 1.** *Assumption 1 plus  $\text{Im}(C_i) \cap \text{rint}(\text{Im}(\mathcal{N}_{K_i}^{-1} + D_i)) \neq \emptyset$ ,  $i = 1, 2$ , allow us to interpret each subsystem with  $u_i \equiv 0$ , as a maximal monotone mapping, using Theorem 3. Under conditions similar to those in Proposition 5, the interconnection is that of two incrementally passive systems (with inputs  $\lambda_i$  and outputs  $w_i$ ,  $i = 1, 2$ ). This topic has been studied for a long time [37, 29, 36], however with different interconnections.*

## 4 Asymptotic Stability

In this section some asymptotic stability results are given for various cases of interconnections of LCS, *i.e.*,  $K_1 = \mathbb{R}_+^{m_1}$  and  $K_2 = \mathbb{R}_+^{m_2}$ . Since the systems we deal with are usually set-valued, stability and invariance principles have to rely on specific results for DIs [16, 5]. As a prerequisite it has to be assumed that the system (4), equivalently (7), is well-posed with AC solutions and uniqueness. The maximal monotonicity (deduced from Corollary 3) and the fact that  $x_1^* = 0$ ,  $x_2^* = 0$ ,  $\lambda_1^* = 0$ ,  $\lambda_2^* = 0$  is an equilibrium point of (4), guarantee that it satisfies the conditions 1, 2, 3 in [16, Section 6.2.1]. In this section it is assumed that  $x(0) = x_0 \in \text{Dom}(\mathcal{F})$ , where the set-valued right-hand side of (7) is denoted as above as  $-\mathcal{F}(x) = \tilde{A}x - \tilde{B}(\tilde{D} + \mathcal{N}_{\mathbb{R}_+^m}^{-1})^{-1}(\tilde{C}x)$ . If conditions in Corollary 3 hold, then the interconnection in (4) has bounded AC solutions from Theorems 3 and 2.

### 4.1 Passive Interconnection

Let the conditions in Proposition 4 be satisfied. The dissipation inequality in (40) is important because it paves the way to the application of Krasovskii-LaSalle's invariance principle. Notice using (7) that  $\lambda_1 = \lambda_1(x_1, x_2)$ , hence (40) can be rewritten as  $\dot{V}(x) \leq -W(x_1, x_2)$ ,  $W : \mathbb{R}^n \mapsto \mathbb{R}_+$ . The application of the Krasovskii-LaSalle invariance principle as in [16, sections 6.2, 6.3] [5], requires that  $W(\cdot)$  be continuous. We may therefore split the analysis into two parts: **(i)**  $W(x) \triangleq \alpha_1 \|x_1\|^2 + \alpha_2 \|\lambda_1(x_1, x_2)\|^2$  if  $\lambda_1(x_1, x_2)$  is continuous, and **(ii)**  $W(x) \triangleq \alpha_1 \|x_1\|^2$  if it is not. Since  $\lambda_1 = -(D_1 + \mathcal{N}_{\mathbb{R}_+^{m_1}}^{-1})^{-1}(C_1 x_1 + E_1 H_2 x_2 + E_1 G_2 \lambda_2)$  and  $D_1 \succ 0$  by assumption, then if  $E_1 G_2 = 0$ , continuity of  $\lambda_1(x_1, x_2)$  is guaranteed. One basic assumption for the invariance principle is that  $\text{Dom}(-\mathcal{F})$  contains a positively invariant set  $\Psi$ . Here we have  $\text{Dom}(-\mathcal{F}) = \text{dom}(\tilde{A}) \cap \text{Dom}(-\tilde{B}(\tilde{D} + \mathcal{N}_{\mathbb{R}_+^m}^{-1})^{-1}(\tilde{C}\cdot)) = \text{Dom}(-\tilde{B}(\tilde{D} + \mathcal{N}_{\mathbb{R}_+^m}^{-1})^{-1}(\tilde{C}\cdot)) = \text{Dom}((\tilde{D} + \mathcal{N}_{\mathbb{R}_+^m}^{-1})^{-1}(\tilde{C}\cdot)) = \{x \in \mathbb{R}^n \mid \tilde{C}x \in \text{Dom}((\tilde{D} + \mathcal{N}_{\mathbb{R}_+^m}^{-1})^{-1})\} = \text{Im}(\tilde{D} + \mathcal{N}_{\mathbb{R}_+^m}^{-1})\} = \tilde{C}^{-1}(\text{Im}(\tilde{D} + \mathcal{N}_{\mathbb{R}_+^m}^{-1})) = \tilde{C}^{-1}(\text{Im}(\tilde{D}) + \mathbb{R}_+^m) \neq \emptyset$  from Lemma 7 and following the reasoning in its proof. This guarantees the existence and uniqueness of solutions, and from (40) the existence of  $\Psi$ .

#### 4.1.1 Case (i)

So we assume that  $E_1 G_2 = 0$ . Our goal is to characterize the largest invariant set  $\mathcal{S}_{lis}$  of the interconnection inside  $\mathcal{Z} = \{x \in \text{Dom}(\mathcal{F}) \mid x_1 = 0 \text{ and } \lambda_1(x_1, x_2) = 0\}$ . In  $\mathcal{Z}$  it follows that  $L_1(H_2 x_2 + G_2 \lambda_2) = 0$ , and  $0 \leq 0 \perp E_1 H_2 x_2 \geq 0$ . We therefore end up with the nonsmooth constrained dynamics on  $\mathcal{Z}$ :

$$\begin{aligned}
 (a) \quad & \dot{x}_2 = A_2 x_2 + B_2 \lambda_2 \\
 (b) \quad & 0 \leq \lambda_2 \perp w_2 = C_2 x_2 + D_2 \lambda_2 \geq 0 \\
 (c) \quad & L_1(H_2 x_2 + G_2 \lambda_2) = 0 \\
 (d) \quad & E_1 H_2 x_2 \geq 0.
 \end{aligned} \tag{50}$$

This system is an LCS with equality and inequality constraints. The constraint in (50) (b) is equivalent to:  $\lambda_2 \in -(D_2 + \mathcal{N}_{\mathbb{R}_+^{m_2}}^{-1})^{-1}(C_2x_2)$ , hence (50) is equivalent to:

$$\begin{aligned} (a) \quad & \dot{x}_2 \in -\mathcal{F}_2(x_2) \stackrel{\Delta}{=} A_2x_2 - B_2(D_2 + \mathcal{N}_{\mathbb{R}_+^{m_2}}^{-1})^{-1}(C_2x_2) \\ (b) \quad & 0 \in L_1\mathcal{G}(x_2) \\ (c) \quad & E_1H_2x_2 \geq 0. \end{aligned} \tag{51}$$

where  $\mathcal{G}(x_2) \stackrel{\Delta}{=} H_2x_2 - G_2(D_2 + \mathcal{N}_{\mathbb{R}_+^{m_2}}^{-1})^{-1}(C_2x_2)$ . Notice that  $x_2 = 0$  and  $\lambda_2 = 0$  is always a solution of (50) or of (51), because  $0 \in \mathcal{G}(0)$  since  $\text{Im}(D_2 + \mathcal{N}_{\mathbb{R}_+^{m_2}}^{-1}) = \text{Im}(D_2) + \mathbb{R}_+^{m_2} \ni 0$  (proceeding as in the proof of Lemma 7). This equilibrium is not an asymptotically stable equilibrium of (50) (a) (b) in view of the assumption in Proposition 4, which implies that  $\dot{V}_2(t) = \frac{d}{dt}(x_2^\top P_2x_2) = 0$  along the trajectories of (50)(a) (b) (or of (51) (a)),  $V_2(x_2) = \frac{1}{2}x_2^\top P_2x_2$ .

**The case  $G_2 = 0$  (passivity and Nagumo's Theorem approach)** (51) is a DI with constraints, and the problem is a viability problem. However the right-hand side of (51) (a) may not be compact, hence the material in [4, Chapter 4] does not apply. If  $\{0\} \subset \mathcal{S}_c \stackrel{\Delta}{=} -E_1H_2\text{Ker}(L_1H_2) \cap \mathbb{R}_+^{m_1}$ , the set of solutions to (51) (b) (c) is nontrivial (*i.e.*, not reduced to  $x_2 = 0$  and  $\lambda_2 = 0$ ). Both sets of the intersection are nonempty convex pointed cones (since the image of a subspace by a linear map is a subspace [7, Fact 2.9.26]). Thus  $\mathcal{S}_c$  is a convex nonempty polyhedral cone [7, Fact 2.9.9], pointed since it contains the origin. Hence the following is true:

**Lemma 8.** *Let  $G_2 = 0$ , the conditions of Proposition 4 hold, and  $\text{Im}(C_2) \cap \text{rint}(\text{Im}(D_2 + \mathcal{N}_{\mathbb{R}_+^{m_2}}^{-1})) \neq \emptyset$ . Then (51) is a maximal monotone DI subjected to the convex conic polyhedral constraint  $x_2 \in \mathcal{S}_c$ .*

The maximal monotonicity of  $\mathcal{F}_2(\cdot)$  in (50) (a) follows from Theorem 3 and the assumptions of Proposition 4, and we infer from Theorem 2 that  $x_2(\cdot)$  as a solution of (51) (a) is AC and bounded, with uniqueness.

**Corollary 5.** *Let the conditions of Lemma 8 hold. If  $\text{rank}(L_1H_2) = n_2$  (*i.e.*,  $L_1H_2$  is full-column rank), the unique solution of (51) is  $x_2^* = 0$  with multiplier  $\lambda_2^* \in \text{SOLLCP}(D_2, 0) \cap \text{Ker}(B_2)$ . Hence the origin of the interconnection is asymptotically stable.*

Notice that (51) (b) implies  $x_2 = 0$  under the rank condition. Then the largest invariant set  $\mathcal{S}_{lis} = \{0\}$  and the proof follows from [16, Theorem 6.5]. Matrix  $D_2$  is not nondegenerate [24, Definition 3.6.1], so it is not an  $\mathbf{R}_0$  matrix [24, Definition 3.8.7] and the LCP in (50) (b) with  $x_2 = 0$  may have other solutions than  $\lambda_2^* = 0$ . The results in [3] extend Nagumo's Theorem to maximal monotone DI, and apply to (51) with  $G_2 = 0$ . Let Lemma 8 apply, the fact that  $x_2 \mapsto \mathcal{F}_2(x_2)$  is maximal monotone secures that [3, Corollaries 6, 8]

can be applied. A basic condition in [3] is that an invariant set  $\mathcal{S}$  satisfies  $\mathcal{S} \subseteq \overline{\text{Dom}(\mathcal{F}_2)}$ . Assume that  $D_2 = 0$ , then  $\text{Dom}(\mathcal{F}_2) = \mathbb{R}^{n_2} \cap \text{Dom}(B_2 \mathcal{N}_{\mathbb{R}_+^{m_2}}(C_2 \cdot)) = C_2^{-1}(\mathbb{R}_+^{m_2})$ , thus the requirement is that  $\mathcal{S} \subseteq C_2^{-1}(\mathbb{R}_+^{m_2})$ . For instance, for  $\mathcal{S}_c$  to be an invariant set of (50) (a), it is necessary and sufficient (because solutions are strong solutions in finite dimension [3, Section 3]) that, when  $D_2 = 0$ :

$$\mathcal{S}_c = -E_1 H_2 \text{Ker}(L_1 H_2) \cap \mathbb{R}_+^{m_1} \subseteq C_2^{-1}(\mathbb{R}_+^{m_2}). \quad (52)$$

Let us define as in [3, Lemma 3]:  $\mathcal{S}_m = \{x_2 \in \mathcal{S} \cap \text{Dom}(\mathcal{F}_2) \mid \|\mathcal{F}_2^0(x_2)\| \leq m\}$ , for some  $m \geq 0$ , where  $\mathcal{F}_2^0(x_2) = \{y \in \mathcal{F}_2(x_2) \mid \|y\| = \min_{z \in \mathcal{F}_2(x_2)} \|z\|\} = \text{proj}[0; \mathcal{F}_2(x_2)]$  is the closed set of least-norm elements of  $\mathcal{F}_2(x_2)$ . Thus  $\mathcal{F}_2^0(x_2) = \{y \in \mathbb{R}^{n_2} \mid y = -A_2 x_2 + B_2 z, z = \text{argmin}_{z \in \mathcal{N}_{\mathbb{R}_+^{m_2}}(C_2 x_2)} \|-A_2 x_2 + B_2 z\|\}$ . Assume that  $C_2 x_2 > 0$ , then the set  $B_2 \mathcal{N}_{\mathbb{R}_+^{m_2}}(C_2 x_2) = \{0\}$ , and  $\mathcal{F}_2^0(x_2) = \{-A_2 x_2\}$ . If  $C_2 x_2 = 0$ , it is inferred that  $\mathcal{N}_{\mathbb{R}_+^{m_2}}(0) = \mathbb{R}_+^{m_2}$ ,  $\mathcal{F}_2(x_2) = -A_2 x_2 + B_2 \mathbb{R}_+^{m_2}$ , and  $\mathcal{F}_2^0(x_2) = \{y \in \mathbb{R}^{n_2} \mid y = -A_2 x_2 + B_2 z, z = \text{argmin}_{z \leq 0} \|-A_2 x_2 + B_2 z\|\}$ . Using [24, Theorem 3.8.6] (or [24, Proposition 3.7.14, Corollary 3.7.12]) and KKT conditions for the associated quadratic problem, it follows that the set of such  $z$  is nonempty convex polyhedral [24, Theorem 3.1.7]. Therefore  $\mathcal{F}_2^0(x_2)$  is nonempty convex polyhedral. Let us apply [3, Corollary 8].

**Lemma 9.** *Let  $D_2 = 0$  and the conditions of Lemma 8 hold. Then the nonempty set  $\mathcal{S}_c$  of solutions to (51) (b)(c) is an invariant set of (51) (a) if and only if (52) holds and*

$$\begin{aligned} -\mathcal{F}_2^0(x_2) \in \mathcal{T}_{\mathcal{S}_{c,m}}(x_2), \text{ for all } m \geq \|\mathcal{F}_2^0(x_2)\| \\ \Downarrow \\ \text{for each } m \geq \|\mathcal{F}_2^0(x_2)\|, \sup_{\xi \in \mathcal{N}_{\mathcal{S}_{c,m}}(x_2)} \langle \xi, -\mathcal{F}_2^0(x_2) \rangle \leq 0, \end{aligned} \quad (53)$$

where  $\mathcal{S}_{c,m} = \{x_2 \in \mathcal{S}_c \cap \text{Dom}(\mathcal{F}_2) \mid \|\mathcal{F}_2^0(x_2)\| \leq m\}$ ,  $\mathcal{T}_{\mathcal{S}_{c,m}}(x_2)$  is the Bouligand tangent cone,  $\mathcal{N}_{\mathcal{S}_{c,m}}(x_2)$  is the Fréchet or the proximal normal cone to  $\mathcal{S}_{c,m}$  at  $x_2$ .

In both above cases ( $C_2 x_2 > 0$  and  $C_2 x_2 = 0$ ),  $\mathcal{S}_{c,m}$  is convex polyhedral so the tangent and normal cones are the polyhedral linearization cones [4, 16]. Notice that  $\mathcal{S}_c \cap \text{Dom}(\mathcal{F}_2) = \mathcal{S}_c$  from (52). If  $\mathcal{S}_c = \{0\}$  ( $\Rightarrow \mathcal{S}_{c,m} = \{0\}$ ) then (52) and (53) hold.

**Corollary 6.** *Let  $D_2 = 0$  and the conditions of Lemma 8 hold. (i) If (52) (53) hold and  $\mathcal{S}_c$  is nontrivial, trajectories of the interconnection converge to  $\mathcal{S}_c$ . (ii) If either (52) or (53) do not hold and  $\rho \mathbb{B}_{n_2} \subset \mathcal{S}_c$  for some  $\rho > 0$ , then the origin of the interconnection is not asymptotically stable. (iii) If such a  $\rho$  does not exist, the origin  $(x_1, x_2) = (0, 0)$  is asymptotically stable. (iv) If  $\mathcal{S}_c = \{0\}$  the origin  $(x_1, x_2) = (0, 0)$  is asymptotically stable.*

*Proof.* (i) The first result is a consequence of the fact that  $\mathcal{S}_{lis} \subseteq \mathcal{S}_c$ , and using [16, Theorem 6.5] trajectories converge asymptotically to  $\mathcal{S}_{lis}$ . However  $\mathcal{S}_c$  is an invariant set of (51) (a) from Lemma 9, and it must be the largest invariant set of (51) (a) (b) (c). (ii) The level sets of  $V_2(\cdot)$  satisfy  $V_2^{-1}(l) \subseteq \mathcal{S}_c$  for all  $l \in [0, l^*]$  and some  $l^* > 0$ . As seen



above these level sets are invariant sets of (51) (a). (iii) In this case  $l^* = 0$  and the only invariant set is  $\mathcal{S}_{lis} = \{0\}$ . (iv) If  $\mathcal{S}_c = \{0\}$  then the only solution of (51) (a)(b)(c) is the origin, and the conclusion follows from [16, Theorem 6.5].  $\square$

Since  $\mathcal{S}_c$  is a convex pointed polyhedral cone, the condition in item (ii) is restrictive. Conditions such that it holds are not pursued here for the sake of brevity.

**The case  $G_2 \neq 0$  (complementarity approach)** In a more general setting, the constrained system (50) can be studied as follows, noticing that (50) (a)(b)(c) is an LCS with extended equality constraint. If  $\text{Im}(G_2) \subseteq \text{Im}(H_2)$ , the equality (50) (c) yields  $x_2 = -(L_1 H_2)^\dagger L_1 G_2 \lambda_2 + (I_{n_2} - (L_1 H_2)^\dagger L_1 H_2) x'_2$ , for some  $x'_2 \in \mathbb{R}^{n_2}$  [8, Proposition 8.1.9], where  $(I_{n_2} - (L_1 H_2)^\dagger L_1 H_2)$  is the projector onto  $\text{Ker}(L_1 H_2)$  [8, Proposition 8.1.7 xiv), equ. (4.8.1)]. Inserting this into (50) (b) yields the LCP:  $0 \leq \lambda_2 \perp \bar{D}_2 \lambda_2 + q_2(x'_2) \geq 0$ , with  $q_2(x'_2) \triangleq C_2(I_{n_2} - (L_1 H_2)^\dagger L_1 H_2) x'_2$  and  $\bar{D}_2 \triangleq -C_2(L_1 H_2)^\dagger L_1 G_2 + D_2 \in \mathbb{R}^{m_2 \times m_2}$ . The inequality (50) (d) yields  $-E_1 H_2 (L_1 H_2)^\dagger L_1 G_2 \lambda_2 + E_1 H_2 (I_{n_2} - (L_1 H_2)^\dagger L_1 H_2) x'_2 \geq 0$ . Therefore the following is proved:

**Lemma 10.** *Let  $\text{Im}(G_2) \subseteq \text{Im}(H_2)$ . System (50) is rewritten equivalently as:*

$$\begin{aligned} (a) \quad & \dot{x}_2 = (-A_2(L_1 H_2)^\dagger L_1 G_2 + B_2) \lambda_2 + A_2(I_{n_2} - (L_1 H_2)^\dagger L_1 H_2) x'_2 \\ (b) \quad & x_2 = -(L_1 H_2)^\dagger L_1 G_2 \lambda_2 + (I_{n_2} - (L_1 H_2)^\dagger L_1 H_2) x'_2 \\ (c) \quad & -E_1 H_2 (L_1 H_2)^\dagger L_1 G_2 \lambda_2 + E_1 H_2 (I_{n_2} - (L_1 H_2)^\dagger L_1 H_2) x'_2 \geq 0 \\ (d) \quad & 0 \leq \lambda_2 \perp \bar{D}_2 \lambda_2 + q_2(x'_2) \geq 0 \end{aligned} \tag{54}$$

for some  $x'_2$ .

Thus (54) (d) is a distorted LCP that incorporates (50) (c) (equivalently (54) (b)), and (54) (c) is (50) (d) incorporating (50) (c). From (54) a necessary condition for the existence of solutions of (50) is  $\text{SOLLCP}(\bar{D}_2, q_2(x'_2)) \neq \emptyset$ . From [24, Theorem 3.8.6], a sufficient condition for the solvability of (54) (d) when  $\bar{D}_2$  is copositive [24, Definition 3.8.1] is:  $q_2(x'_2) \in \text{SOLLCP}(\bar{D}_2, 0)^*$ . This result needs the computation of  $\text{SOLLCP}(\bar{D}_2, 0)$ , with  $D_2 = -D_2^\top$  and (39), to be constructive, as well as criteria which guarantee the copositiveness [25]. Assume that  $\bar{D}_2$  is a P-matrix. The LCP( $\bar{D}_2, q_2(x'_2)$ ) in (54) (d) has a solution  $\lambda_2(x'_2)$ , which is continuous piecewise linear [24], and  $\lambda_2(0) = 0$ . A necessary and sufficient condition for (54) (a) (b) (d) to be well-posed, is that (a) and (b) be compatible. This yields that  $x'_2$ , possibly a time-function, has to be a solution of:

$$\begin{aligned} & [-(L_1 H_2)^\dagger L_1 G_2 \frac{\partial \lambda_2}{\partial x'_2} + (I_{n_2} - (L_1 H_2)^\dagger L_1 H_2)] \dot{x}'_2 + (A_2(L_1 H_2)^\dagger L_1 G_2 - B_2) \lambda_2(x'_2) \\ & - A_2(I_{n_2} - (L_1 H_2)^\dagger L_1 H_2) x'_2(t) = 0 \end{aligned} \tag{55}$$

when  $\frac{\partial \lambda_2}{\partial x'_2}$  exists, in which case it is a constant matrix (*i.e.*, (55) holds almost everywhere). Therefore, we can rewrite (55) as the switching system:

$$M_1^{\sigma(x'_2(t))} (x'_2(t)) \dot{x}'_2 + M_2^{\sigma(x'_2(t))} (x'_2(t)) x'_2 = 0, \tag{56}$$

with the index step function  $\sigma \circ x'_2 : \mathbb{R}_+ \rightarrow \{0, \dots, 2^{m_2} - 1\} \subseteq \mathbb{N}$ ,  $M_i^{\sigma \circ x'_2(t)}(x'_2(t))$  are constant matrices for each  $t$ . If  $x'_2$  is constant, then  $\sigma \circ x'_2(t) = 0$  for all  $t \geq 0$ , otherwise it may switch depending on the LCP modes. Each  $\sigma(t)$  corresponds to a partitioning of the index set  $\{1, \dots, m_2\}$  into two subsets  $\alpha$  and  $\bar{\alpha} = \{1, \dots, m_2\} \setminus \alpha$ , such that the solution of LCP( $\bar{D}_2, q_2(x'_2)$ ) is equal to:  $\lambda_2^\alpha(x'_2) = -(\bar{D}_2^{\alpha\alpha})^{-1}q_2^\alpha(x'_2)$ ,  $\lambda_2^{\bar{\alpha}}(x'_2) = 0$ , where  $M^{\alpha\beta}$  denotes the submatrix constructed from  $M \in \mathbb{R}^{m \times m}$  by taking rows indexed in  $\alpha$  and columns indexed in  $\beta$ ,  $\alpha$  and  $\beta$  two index subsets of  $\bar{m} = \{1, \dots, m\}$  [20]. Therefore  $q_2^\alpha = (I_{n_2} - (L_1 H_2)^\dagger L_1 H_2)_{\alpha\bullet} x'_2 = (I_{n_2} - (L_1 H_2)^\dagger L_1 H_2)_{\alpha\bar{m}} x'_2$ . Assuming that indices are ordered such that  $\lambda_2 = \begin{pmatrix} \lambda_2^\alpha \\ \lambda_2^{\bar{\alpha}} \end{pmatrix}$ , we have  $\frac{\partial \lambda_2}{\partial x'_2} = \begin{pmatrix} -(\bar{D}_2^{\alpha\alpha})^{-1}(I_{n_2} - (L_1 H_2)^\dagger L_1 H_2)_{\alpha\bar{m}} \\ 0 \end{pmatrix}$ , which

shows the structure of  $M_1^{\sigma(x'_2(t))}$  and  $M_2^{\sigma(x'_2(t))}$ , and  $M_3^{\sigma(x'_2(t))}$  shown below. Let  $x'_2(\cdot)$  be a solution of (56),  $\lambda_2(x'_2(\cdot))$  the solution of LCP( $\bar{D}_2, q_2(x'_2)$ ). This means that  $x_2(\cdot)$  as above is a solution of (50) (a)(b)(c). Thus it is a solution of (50) if and only if in addition (54) (c) holds for  $x'_2$  and  $\lambda_2(x'_2)$ . Notice that (54) (c) is, for each  $\sigma(t)$ , a linear inequality of the form  $M_3^{\sigma(x'_2(t))} x'_2 \geq 0$ , which defines a convex nonempty cone. We are therefore led to characterize the set of solutions to (56) subject to  $M_3^{\sigma(x'_2(t))} x'_2 \geq 0$ , keeping in mind that solutions  $x_2(\cdot)$  to (50) are AC and bounded since (40) holds. Assume that  $\text{rank}(M_1^{\sigma(x'_2(t))}(x'_2(t))) = n_2$  for all  $t \geq 0$ , and let us analyse possibly time-varying  $x'_2(\cdot)$ , solutions to the constrained piecewise linear ODE:

$$\begin{aligned} \dot{x}'_2 &= -(M_1^{\sigma(x'_2(t))}(x'_2(t)))^{-1} M_2^{\sigma(x'_2(t))}(x'_2(t)) x'_2, \\ M_3^{\sigma(x'_2(t))}(x'_2(t)) x'_2 &\geq 0. \end{aligned} \tag{57}$$

System (57) is a switching system with switching inequality constraints, where the inequality represents a switching polyhedral set (that has to be positively invariant for the dynamical part). Solutions to (57) are AC. All bounded solutions to (57) yield a solution  $x_2(\cdot)$  to (54), hence to (50). Notice that the LCP( $\bar{D}_2, q_2(x'_2)$ ) has  $2^{m_2}$  modes. Thus all possible matrices  $M_i^{\sigma(x'_2(t))}(x'_2(t))$  can, in principle, be calculated. In the following  $M_i^\sigma$  denotes a matrix in the set  $\{M_i^0, \dots, M_i^{2^{m_2}}\}$ . Basically there are two situations for the solutions to (57): **(a)** either  $x'_2(\cdot)$  is such that there exists  $T < +\infty$  such that for all  $t \geq T$ ,  $\sigma \circ x'_2(t) = \sigma_f$  is constant; or **(b)**  $x'_2(\cdot)$  is such that  $\sigma \circ x'_2(\cdot)$  never attains a constant integer inside  $\{0, \dots, 2^{m_2} - 1\}$ , and the matrices  $M_i^{\sigma(t)}$  keep on switching indefinitely. Let us analyze **(a)**.

**Lemma 11.** *Let  $\text{Im}(G_2) \subseteq \text{Im}(H_2)$  and  $\bar{D}_2$  be a P-matrix. Situation **(a)** occurs if and only if there exists an essentially nonnegative matrix  $M$  such that  $M M_3^{\sigma_f} = -M_3^{\sigma_f} (M_1^{\sigma_f})^{-1} M_2^{\sigma_f}$ , which is true only if  $\text{Ker}(M_3^{\sigma_f})$  is a  $-(M_1^{\sigma_f})^{-1} M_2^{\sigma_f}$ -invariant subspace. Then  $x_2(\cdot)$  is given by  $x_2(t) = (I_{n_2} - (L_1 H_2)^\dagger L_1 H_2) x'_2(t)$  for all  $t \geq T$ . If the above conditions are not satisfied, then the only global solution to (57) is  $x'_2 = 0$ , so that the only solution to (54) is  $x_2(t) = x_2^* = 0$ .*

*Proof.* It follows from [22, Proposition 1, Lemma 1]. The given condition is necessary for positive invariance of the set  $\{z \in \mathbb{R}^{n_2} \mid M_3^{\sigma_f} z \geq 0\}$  with respect to the dynamics  $\dot{x}'_2 = -(M_1^{\sigma_f})^{-1} M_2^{\sigma_f} x'_2$ . In this case, the projection of  $x'_2(t)$  onto  $\text{Ker}(L_1 H_2)$  is bounded for all  $t \geq T$ ,  $x'_2(t) = \exp(-(M_1^{\sigma_f})^{-1} M_2^{\sigma_f})(t - T)x'_2(T)$ , for all  $t \geq T$ , and  $x_2(\cdot)$  is a (bounded) solution to (54) (a):  $x_2(t) = x_2(T) + (-A_2(L_1 H_2)^\dagger L_1 G_2 + B_2)\lambda_2^f(t - T) + A_2(I_{n_2} - (L_1 H_2)^\dagger L_1 H_2) \int_t^T x'_2(s) ds$ . Boundedness of the state implies  $\lambda_2^f \in \text{Ker}(-A_2(L_1 H_2)^\dagger L_1 G_2 + B_2)$  and from (51) (b)  $\lambda_2^f \in \text{Ker}((L_1 H_2)^\dagger L_1 G_2)$ , which are compatible since (54)(a) and (b) are compatible from (55). Thus the result for  $x_2(t)$  follows. The second result follows from the fact that if the condition on positive invariance is not satisfied, the only solution is the null solution.  $\square$

The following holds:

**Corollary 7.** *Assume that there is no index  $\sigma \in \{0, \dots, 2^{m_2} - 1\}$  such that  $\text{Ker}(M_3^\sigma)$  is a  $-(M_1^\sigma)^{-1} M_2^\sigma$ -invariant subspace. Then  $\mathcal{S}_{lis}$  reduces to  $x_2^* = 0$ .*

*Proof.* The unique global solution to (57) is  $x'_2 = 0 \Rightarrow q_2(x'_2) = 0 \Rightarrow \lambda_2 = 0$  since  $\bar{D}_2$  is a P-matrix. From the boundedness of  $x_2(\cdot)$ , and from (54)(b),  $x_2 = 0$ .  $\square$

**The case  $G_2 \neq 0$  (passivity and VI approach)** Checking the above conditions may not be easy in general if  $m_2$  is too large. Let us provide an alternative. First let us study the set of solutions to (51) (b), relying on maximal monotonicity properties of the mappings. In the following we shall assume that  $D_2 = D_2^\top$ , which in view of  $D_2 + D_2^\top = 0$  in Proposition 4 implies  $D_2 = 0$ .

**Lemma 12.** *Let  $D_2 = 0$ . Assume that the quadruple  $(L_1 H_2, L_1 G_2, C_2, 0)$  is passive ( $\Rightarrow n_1 = n_2$ ), and that  $\text{Im}(C_2) \cap \mathbb{R}_+^{m_2}$  is not reduced to  $\{0\}$ . Then the set  $\mathcal{S}$  of solutions to (51) (b) (which is (50) (b) (c)), is closed and convex. If in addition this quadruple is strictly state passive, then  $\mathcal{S} = \{0\} = \mathcal{S}_{lis}$ .*

*Proof.* Let us first check that  $\text{Im}(C_2) \cap \text{rint}(\text{Im}(D_2 + \mathcal{N}_{\mathbb{R}_+^{m_2}}^{-1})) \neq \emptyset$ , for any  $D_2 = D_2^\top \succcurlyeq 0$ . Both mappings  $z \mapsto D_2 z$  and  $\mathcal{N}_{\mathbb{R}_+^{m_2}}(\cdot)$  are maximal monotone, and they verify property (\*) in [10, section 1], since they are subdifferentials of convex proper lower semi-continuous functions [10, p.167]. The mapping  $(D_2 + \mathcal{N}_{\mathbb{R}_+^{m_2}})(\cdot)$  is maximal monotone from [6, Corollary 24 (ii)]. Thus using [10, Theorems 3, 4], one finds  $\overline{\text{Im}(D_2 + \mathcal{N}_{\mathbb{R}_+^{m_2}}^{-1})} = \overline{\text{Im}(D_2) + \text{Im}(\mathcal{N}_{\mathbb{R}_+^{m_2}}^{-1})} = \overline{\text{Im}(D_2) + \mathbb{R}_+^{m_2}} = \mathbb{R}_+^{m_2}$  in our case since  $D_2 = 0$ . Also  $\text{int}(\text{Im}(D_2 + \mathcal{N}_{\mathbb{R}_+^{m_2}}^{-1})) = \text{int}(\text{Im}(D_2) + \text{Im}(\mathcal{N}_{\mathbb{R}_+^{m_2}}^{-1})) = \text{int}(\text{Im}(D_2) + \mathbb{R}_+^{m_2})$ . Thus  $\text{Im}(D_2 + \mathcal{N}_{\mathbb{R}_+^{m_2}}^{-1}) = \text{Im}(D_2) + \mathbb{R}_+^{m_2} \supseteq \text{Im}(D_2) \cup \mathbb{R}_+^{m_2}$  [7, Fact 2.9.11]. Since  $C_2 \in \mathbb{R}^{m_2 \times n_2}$ ,  $\text{Im}(C_2) \subseteq \mathbb{R}^{m_2}$ , and  $\text{Im}(C_2) \cap \text{rint}(\text{Im}(D_2 + \mathcal{N}_{\mathbb{R}_+^{m_2}}^{-1})) = \text{Im}(C_2) \cap \text{rint}(\text{Im}(D_2) + \mathbb{R}_+^{m_2})$ , which in our case ( $D_2 = 0$ ) reduces to  $\text{Im}(C_2) \cap \text{int}(\mathbb{R}_+^{m_2}) \neq \emptyset$  from the assumption. Then passivity guarantees that the map  $L_1 \mathcal{G}(\cdot)$  is maximal monotone using Theorem 3. From [6, Proposition 23.39] the

first result is proved. Using Corollary 3, strong monotonicity follows from strict state passivity. From [6, Proposition 23.37] it is inferred that  $\mathcal{S} = \{0\}$ , the unique solution to (51) is  $x_2^* = 0$ .  $\square$

#### 4.1.2 Case (ii)

The goal is to characterize the largest invariant set inside  $\mathcal{Z} = \{x \in \mathbb{R}^n \mid x_1 = 0\}$ . On the set  $\mathcal{Z}$ , the interconnection (4) has the dynamics:

$$\begin{aligned} (a) \quad & \dot{x}_2 = A_2 x_2 + (L_2 G_1 \ B_2) \lambda \\ (b) \quad & L_1 H_2 x_2 + (B_1 \ L_1 G_2) \lambda = 0 \\ (c) \quad & 0 \leq \lambda \perp \tilde{D} \lambda + \begin{pmatrix} E_1 H_2 \\ C_2 \end{pmatrix} x_2 \geq 0, \end{aligned} \tag{58}$$

which is a mixed LCS (MLCS), and  $x_2 = 0$ ,  $\lambda_2 = 0$  is an equilibrium pair. This MLCS can be studied along the same lines as (50).

## 4.2 Strictly State and Strongly Passive Interconnection

Let us place ourselves in the framework of Proposition 3. The strict state passivity of the interconnection guarantees that along its trajectories:  $\dot{V}(x_1, x_2) = x^\top P \dot{x} \leq -\epsilon x^\top P x$ . On the subspace  $x_1 = 0$  and  $x_2 = 0$ , the subsystems in (2) reduce to:

$$\begin{aligned} 0 &= B_1 \lambda_1^* + L_1 G_2 \lambda_2^* \\ 0 &= B_2 \lambda_2^* + L_2 G_1 \lambda_1^* \\ (a) \quad & 0 \leq \lambda_1^* \perp D_1 \lambda_1^* + E_1 G_2 \lambda_2^* \geq 0 \\ (b) \quad & 0 \leq \lambda_2^* \perp D_2 \lambda_2^* + E_2 G_1 \lambda_1^* \geq 0, \end{aligned} \tag{59}$$

equivalently:  $\lambda^* \in \text{Ker}(\tilde{B}) \cap \text{SOLLCP}(\tilde{D}, 0)$ , which is nonempty since  $\lambda^* = 0$  is a solution. Since  $\tilde{D} \succcurlyeq 0$ , it follows that the set of solutions to the LCP (59) (a) (b), is convex and contains a least norm solution, which is  $\lambda^* = 0$ . We have  $\text{SOLLCP}(\tilde{D}, 0) = \{0\}$  if and only if  $\tilde{D}$  is an  $R_0$ -matrix [24, Definition 3.8.7], equivalently (since  $\tilde{D}$  is a  $P_0$ -matrix) if and only if it is an  $R$ - or a  $Q$ -matrix [24, Theorem 3.9.22]. There exists finite algorithm to check the Qness of a matrix [28, Exercise 3.87].

**Proposition 6.** *Let the conditions of Proposition 3 (resp. Proposition 2) be satisfied. Then the interconnection (4) possesses the equilibrium point  $x^* = 0$  and multipliers  $\lambda^* \in \text{Ker}(\tilde{B}) \cap \text{SOLLCP}(\tilde{D}, 0)$ , and  $x^*$  is asymptotically stable (resp. has a unique equilibrium point  $x^* = 0$  and multiplier  $\lambda^* = 0$ ,  $x^*$  is globally asymptotically stable and  $\lambda(t) \rightarrow 0$  as  $t \rightarrow +\infty$ ).*

*Proof.* Follows using [16, Theorem 6.3] (resp. Theorem 2 and Corollary 3).  $\square$

## 5 Examples

Let us analyse the interconnection of two circuits as in Fig. 2 (b). The dynamics is given by the LCS:

$$\begin{cases} \dot{x}_{11} = \frac{-1}{\mathbf{R}_1\mathbf{C}_1}x_{11} + x_{12} + \frac{1}{\mathbf{R}_1}\lambda_1 + \left(\frac{1}{\mathbf{R}_1} \ 0\right)u_1 \\ \dot{x}_{12} = \frac{-1}{\mathbf{L}_1\mathbf{C}_1}x_{11} + \frac{1}{\mathbf{L}_1}\lambda_1 + \left(\frac{1}{\mathbf{L}_1} \ \frac{1}{\mathbf{L}_1}\right)u_1 \\ 0 \leq \lambda_1 \perp w_1 = \frac{-1}{\mathbf{R}_1\mathbf{C}_1}x_{11} + x_{12} + \frac{1}{\mathbf{R}_1}\lambda_1 + \left(\frac{1}{\mathbf{R}_1} \ 0\right)u_1 \geq 0, \end{cases} \quad (60)$$

where  $x_{11}$  is the capacitor's voltage ( $\mathbf{C}_1\dot{x}_{11} = i_{c1}$ ),  $x_{12} = i_{l1}$  (the current through the inductance),  $u_1 = (u_{11} \ u_{12})^\top$  the voltage sources,  $\lambda_1$  the voltage across the diode,  $w_1$  the current through the diode. We have  $x_1 = (x_{11} \ x_{12})^\top$ ,  $A_1 = \begin{pmatrix} \frac{-1}{\mathbf{R}_1\mathbf{C}_1} & 1 \\ \frac{-1}{\mathbf{L}_1\mathbf{C}_1} & 0 \end{pmatrix}$ ,  $B_1 = \begin{pmatrix} \frac{1}{\mathbf{R}_1} \\ \frac{1}{\mathbf{L}_1} \end{pmatrix}$ ,  $C_1 = \begin{pmatrix} \frac{-1}{\mathbf{R}_1\mathbf{C}_1} & 1 \end{pmatrix}$ ,  $D_1 = \frac{1}{\mathbf{R}_1}$ ,  $L_1 = \begin{pmatrix} \frac{1}{\mathbf{R}_1} & 0 \\ \frac{1}{\mathbf{L}_1} & \frac{1}{\mathbf{L}_1} \end{pmatrix}$ ,  $E_1 = \begin{pmatrix} \frac{1}{\mathbf{R}_1} & 0 \end{pmatrix}$ . We shall consider the interconnection

of two identical circuits, the interconnection matrices in (3) are:  $H_i = \begin{pmatrix} H_i^{11} & H_i^{12} \\ H_i^{21} & H_i^{22} \end{pmatrix}$ ,  $G_i = \begin{pmatrix} G_{i1} \\ G_{i2} \end{pmatrix}$ ,  $i = 1, 2$ . The transfer function  $\mathbf{H}_i(s) = \frac{w_i(s)}{\lambda_i(s)} = \frac{s\mathbf{C}_i(s\mathbf{L}_i + \mathbf{R}_i)}{s^2\mathbf{L}_i\mathbf{R}_i\mathbf{C}_i + s\mathbf{L}_i + \mathbf{R}_i}$ ,  $i = 1, 2$ , is PR but not SPR. The conditions in Lemma 1, (12) are given by ( $P_{12} \in \mathbb{R}^{2 \times 2}$ ):

$$\begin{aligned} & \text{Im} \left( \begin{pmatrix} \left(\frac{1}{\mathbf{R}_1} \ \frac{1}{\mathbf{L}_1}\right)P_1 + \left(\frac{1}{\mathbf{R}_1\mathbf{C}_1} \ -1\right) + \left(\frac{G_{11}}{\mathbf{R}_2} \ \frac{G_{11}+G_{12}}{\mathbf{L}_2}\right)P_{12}^\top \\ \left(\frac{G_{21}}{\mathbf{R}_1} \ \frac{G_{22}+G_{21}}{\mathbf{L}_1}\right)P_1 + \left(\frac{1}{\mathbf{R}_2} \ \frac{1}{\mathbf{L}_2}\right)P_{12}^\top - \left(\frac{H_1^{21}}{\mathbf{R}_2} \ \frac{H_1^{22}}{\mathbf{R}_2}\right) \end{pmatrix} \right) \\ & + \text{Im} \left( \begin{pmatrix} \left(\frac{1}{\mathbf{R}_1} \ \frac{1}{\mathbf{L}_1}\right)P_{12} + \left(\frac{G_{11}}{\mathbf{R}_2} \ \frac{G_{11}+G_{12}}{\mathbf{L}_2}\right)P_2 - \left(\frac{H_2^{21}}{\mathbf{R}_1} \ \frac{H_2^{22}}{\mathbf{R}_1}\right) \\ \left(\frac{1}{\mathbf{R}_2} \ \frac{1}{\mathbf{L}_2}\right)P_2 + \left(\frac{1}{\mathbf{R}_2\mathbf{C}_2} \ -1\right) + \left(\frac{G_{21}}{\mathbf{R}_1} \ \frac{G_{21}+G_{22}}{\mathbf{L}_1}\right)P_{12} \end{pmatrix} \right) \quad (61) \\ & \subseteq \text{Im} \left( \begin{pmatrix} \frac{2}{\mathbf{R}_1} \\ \frac{G_{11}}{\mathbf{R}_2} + \frac{G_{21}}{\mathbf{R}_1} \end{pmatrix} \right) + \text{Im} \left( \begin{pmatrix} \frac{G_{11}}{\mathbf{R}_2} + \frac{G_{21}}{\mathbf{R}_1} \\ \frac{2}{\mathbf{R}_2} \end{pmatrix} \right) \end{aligned}$$

which holds always if  $|\frac{G_{21}}{\mathbf{R}_1} + \frac{G_{11}}{\mathbf{R}_2}| \neq \frac{2}{\sqrt{\mathbf{R}_1\mathbf{R}_2}}$ . Corollary 1 gives  $\left(\frac{G_{21}}{\mathbf{R}_1} \ \frac{G_{22}+G_{21}}{\mathbf{L}_1}\right)P_1 - \left(\frac{H_1^{11}}{\mathbf{R}_2} \ \frac{H_1^{12}}{\mathbf{R}_2}\right) = 0$  and  $\left(\frac{G_{11}}{\mathbf{R}_2} \ \frac{G_{11}+G_{12}}{\mathbf{L}_2}\right)P_2 - \left(\frac{H_2^{11}}{\mathbf{R}_1} \ \frac{H_2^{12}}{\mathbf{R}_1}\right) = 0$ . The conditions in Lemma 2 boil down to:  $\text{Im}\left(\frac{G_{21}}{\mathbf{R}_1} + \frac{G_{11}}{\mathbf{R}_2}\right) \subseteq \text{Im}\left(\frac{2}{\mathbf{R}_1}\right) = \mathbb{R}$ ,  $|\frac{G_{21}}{\mathbf{R}_1} + \frac{G_{11}}{\mathbf{R}_2}| < \frac{4}{\mathbf{R}_1\mathbf{R}_2}$ . Let us now study the conditions in Lemma 3, which involve the matrices (see section 2 for definitions):

$$\bar{Q}_1 = Q_1 + P_{12} \overbrace{\begin{pmatrix} \frac{H_1^{11}}{\mathbf{R}_2} & \frac{H_1^{12}}{\mathbf{R}_2} \\ \frac{H_1^{11}+H_1^{21}}{\mathbf{L}_2} & \frac{H_1^{12}+H_1^{22}}{\mathbf{L}_2} \end{pmatrix}}^{L_2H_1} + \begin{pmatrix} \frac{H_1^{11}}{\mathbf{R}_2} & \frac{H_1^{12}}{\mathbf{R}_2} \\ \frac{H_1^{11}+H_1^{21}}{\mathbf{L}_2} & \frac{H_1^{12}+H_1^{22}}{\mathbf{L}_2} \end{pmatrix}^\top P_{12}^\top \quad (62)$$

and similarly for  $\bar{Q}_2$  and  $\bar{Q}_{12}$ . We have  $P_i = \begin{pmatrix} \frac{1}{\mathbf{C}_i} & 0 \\ 0 & \mathbf{L}_i \end{pmatrix}$ , so  $Q_i = \begin{pmatrix} \frac{2}{\mathbf{R}_i \mathbf{C}_i} & 0 \\ 0 & 0 \end{pmatrix}$ ,  $i = 1, 2$ .

There are 12 parameters to be tuned in (62) in order to satisfy the conditions of Lemma 3. It is noteworthy that when  $Q_1 \succcurlyeq 0$  or  $Q_2 \succcurlyeq 0$ , conditions of Lemmata 3 and 4 may not be easy to check analytically. This is greatly simplified if  $Q_1 \succ 0$  and  $Q_2 \succ 0$ , because it is then always possible to choose a small enough  $P_{12}$  so that  $\bar{Q}_1 \succ 0$ ,  $\bar{Q}_2 \succ 0$  (using Corollary 8), and  $H_1, P_{12}$  such that  $P_{12} L_2 H_1 + H_1^\top L_2^\top P_{12}^\top$  has rank  $n_1$  so that the inclusion in Lemma 3 holds. This is the case if circuit in Fig. 2 (b) has an additional resistance  $\mathbf{R}'_1$  in series with the inductance  $\mathbf{L}_1$  (and same for the second circuit). Then  $Q_i = \begin{pmatrix} \frac{2}{\mathbf{R}_i \mathbf{C}_i} & 0 \\ 0 & 2\mathbf{R}'_i \end{pmatrix}$ ,

$\mathbf{H}_i(s) = \frac{1}{\mathbf{R}_i} \frac{s^2 + \frac{\mathbf{R}_i + \mathbf{R}'_i}{\mathbf{L}_i} s}{s^2 + \left(\frac{1}{\mathbf{R}_i \mathbf{C}_i} + \frac{\mathbf{R}'_i}{\mathbf{L}_i}\right) s + \frac{1}{\mathbf{L}_i \mathbf{C}_i} + \frac{\mathbf{R}'_i}{\mathbf{R}_i \mathbf{C}_i \mathbf{L}_i}}$ ,  $i = 1, 2$ , which is not SPR and thus not SSPR, so the systems are neither strongly nor strictly state passive (the material in sections 2.2 and 2.3 does not apply). See comments after Lemma 5 for the application of Lemma 4's

conditions. The equality (17) is  $P_1 \begin{pmatrix} \frac{H_2^{11}}{\mathbf{R}_1} & \frac{H_2^{12}}{\mathbf{R}_1} \\ \frac{H_2^{11} + H_2^{21}}{\mathbf{L}_1} & \frac{H_2^{12} + H_2^{22}}{\mathbf{L}_1} \end{pmatrix} = - \begin{pmatrix} \frac{H_1^{11}}{\mathbf{R}_2} & \frac{H_1^{12}}{\mathbf{R}_2} \\ \frac{H_1^{11} + H_1^{21}}{\mathbf{L}_2} & \frac{H_1^{12} + H_1^{22}}{\mathbf{L}_2} \end{pmatrix} P_2$ ,

which can be easily satisfied with suitable choice of  $H_1$  and  $H_2$ . The circuit in Fig. 2 (a) can be analysed the same way, setting  $\mathbf{R}_1 = \mathbf{R}_2 = +\infty$ , and a single voltage source. Proposition 4 and Lemma 12 potentially apply to the interconnection of two circuits as in Fig. 2 (a) and (b).

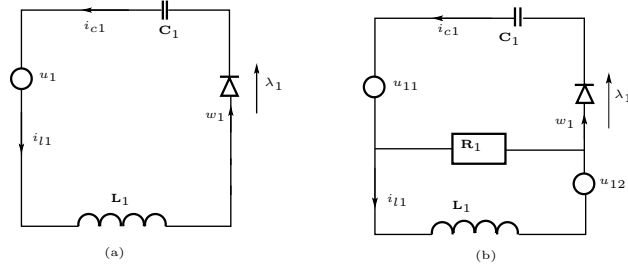


Figure 2: Passive circuits with ideal diode.

Switching DAEs [12, 34, 33, 32] can also be studied within the above framework. Let us consider:

$$\begin{cases} \dot{x}_{i1} = a_{i1}x_{i1} + b_{i1}x_{i2} + c_{i1}z_i + u_{i1}(t) \\ \dot{x}_{i2} = a_{i2}x_{i1} + b_{i2}x_{i2} + c_{i2}z_i + u_{i2}(t) \\ 0 = a_{i3}x_{i1} + b_{i3}x_{i2} + c_{i3}z_i + u_{i4}(t) \text{ if } x_{i1} + u_{i3}(t) < 0 \\ 0 = a_{i4}x_{i1} + b_{i3}x_{i2} + c_{i3}z_i + u_{i4}(t) \text{ if } x_{i1} + u_{i3}(t) > 0, \end{cases} \quad (63)$$

for  $i = 1, 2$ , where  $x_{i1}, x_{i2}$  are the differential state variables,  $z_i$  is the algebraic state variable,  $u_{i1}, u_{i2}, u_{i3}, u_{i4}$  are inputs. Assuming  $c_{i3} \neq 0$  (index 1 DAEs), (63) is rewritten

equivalently as the LCS, for each  $i = 1, 2$ :

$$\begin{cases} \dot{x}_{i1} = a_{i1}x_{i1} + \left(b_{i1} - \frac{c_{i1}}{c_{i3}}b_{i3}\right)x_{i2} + \frac{c_{i1}}{c_{i3}}a_{i3}(\lambda_{i1} - u_{i3}) - \frac{c_{i1}}{c_{i3}}a_{i4}(\lambda_{i2} - u_{i3}) + \bar{u}_{i1} \\ \dot{x}_{i2} = a_{i2}x_{i1} + \left(b_{i2} - \frac{c_{i2}}{c_{i3}}b_{i3}\right)x_{i2} + \frac{c_{i2}}{c_{i3}}a_{i3}(\lambda_{i1} - u_{i3}) - \frac{c_{i2}}{c_{i3}}a_{i4}(\lambda_{i2} - u_{i3}) + \bar{u}_{i2} \\ 0 \leq \lambda_{i1} \perp \lambda_{i1} + x_{i1} + u_{i3}(t) \geq 0, \quad 0 \leq \lambda_{i2} \perp \lambda_{i2} - x_{i1} - u_{i3}(t) \geq 0, \end{cases} \quad (64)$$

with  $\bar{u}_{i1} = u_{i1} - \frac{c_{i1}}{c_{i3}}u_{i4}$  and  $\bar{u}_{i2} = u_{i2} - \frac{c_{i2}}{c_{i3}}u_{i4}$ . The transformation of (63) into (64) allows us to see clearly which operator plays a role for passivity analysis, and which role is played by the various controllers. Given the passivity of  $(A_i, B_i, C_i, D_i)$  defined from (64), the ma-

terial in foregoing sections applies to study (63). We have:  $A_i = \begin{pmatrix} a_{i1} & b_{i1} - \frac{c_{i1}}{c_{i3}}b_{i3} \\ a_{i2} & b_{i2} - \frac{c_{i2}}{c_{i3}}b_{i3} \end{pmatrix}$ ,

$$B_i = \begin{pmatrix} \frac{c_{i1}}{c_{i3}}a_{i3} & -\frac{c_{i1}}{c_{i3}}a_{i4} \\ \frac{c_{i2}}{c_{i3}}a_{i3} & -\frac{c_{i2}}{c_{i3}}a_{i4} \end{pmatrix}, \quad C_i = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \quad D_i = I_2, \quad L_i = \begin{pmatrix} 1 & 0 & 0 & -\frac{c_{i1}}{c_{i3}} \\ 0 & 1 & 0 & -\frac{c_{i2}}{c_{i3}} \end{pmatrix}, \quad E_i = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad H_{i1} \in \mathbb{R}^{4 \times 2}, \quad H_{i2} \in \mathbb{R}^{4 \times 2}, \quad G_{i1} \in \mathbb{R}^{4 \times 2}, \quad G_{i2} \in \mathbb{R}^{4 \times 2}. \quad \text{Once } P_i \succcurlyeq 0 \text{ has been calculated, one can use the first four inclusions in (13) to check (12). Conditions in Lemma 2 reduce to assuring a "small enough" } E_1G_2 + G_1^TE_2^T.$$

## 6 Conclusions

In this article the interconnection of passive linear cone complementarity systems is analysed. Conditions which guarantee that the interconnected system defines a maximal monotone mapping are given. The asymptotic stability is studied with an invariance principle. Examples illustrate the theoretical findings. Future research could concern the extension of complementarity towards general maximal monotone mappings.

## A Positive Definite Matrices under Perturbations

Let us provide a corollary of [23, Theorem 2.11]. For  $M \in \mathbb{R}^{n \times n}$ ,  $\|M\|_{2,2}$  is the induced matricial norm such that  $\|M\|_{2,2} = \sigma_{\max}(M)$  (the largest singular value) [7, Proposition 9.4.9], it is a submultiplicative norm [7, Corollary 9.4.12]. The following is an easy consequence of [23].

**Theorem 1.** [23] *Let  $M \in \mathbb{R}^{n \times n}$  be a positive definite matrix. Then every matrix*

$$A \in \left\{ A \in \mathbb{R}^{n \times n} \mid \left\| \left( \frac{M+M^T}{2} \right)^{-1} \right\|_{2,2} \|M - A\|_{2,2} < 1 \right\}$$

*is positive definite.*

**Corollary 8.** Let  $D = P + N$ , where  $D$ ,  $P$  and  $N$  are  $n \times n$  real matrices, and  $P \succ 0$ , not necessarily symmetric. If  $\|N\|_{2,2} < \frac{1}{\|(P+P^\top)^{-1}\|_{2,2}}$ , then  $D \succ 0$ .

## B Passive Linear Systems

A quadruple  $(A, B, C, D)$  is said passive if there exists  $P = P^\top \succcurlyeq 0$  such that:

$$\begin{pmatrix} -A^\top P - PA & -PB + C^\top \\ (-PB + C^\top)^\top & D + D^\top \end{pmatrix} \succcurlyeq 0. \quad (65)$$

This inequality implies that  $\text{Im}(B^\top P - C^\top) \subseteq \text{Im}(D + D^\top)$ , and  $\text{Im}(PB - C^\top) \subseteq \text{Im}(A^\top P + PA)$ , using [7, Proposition 8.2.4] [8, Fact 8.4.3], or [18, Lemma 1 iv] and [7, Theorem 2.4.3, Fact 2.9.14]. The observability of the pair  $(C, A)$  guarantees that  $P \succ 0$  [18] [15, p.116]. The LMI (65) is equivalent to the dissipation equality:

$$\begin{aligned} V(x(t_1)) - V(x(t_2)) = & \int_{t_1}^{t_2} x^\top(s)(A^\top P + PA)x(s)ds \\ & + \int_{t_1}^{t_2} x^\top(s)(PB - C^\top)u(s)ds \\ & - \int_{t_1}^{t_2} u^\top(s)(D + D^\top)u(s)ds + \int_{t_1}^{t_2} u^\top(s)y(s)ds \end{aligned} \quad (66)$$

with  $V(x) = \frac{1}{2}x^\top P x$ , along the trajectories of the system  $\dot{x} = Ax + Bu$ ,  $y = Cx + Du$ , and for all  $t_1 \leq t_2$ . The system is said *strictly state passive* if in addition  $P \succ 0$  and there exists  $\mu > 0$  such that (65) holds with  $-A^\top P - PA - \mu P$  (hence (65) implies  $-A^\top P - PA \succcurlyeq \mu P \succ 0$ ). It is said *strongly passive* if  $P \succ 0$  and the inequality in (65) is strict. Positive real (PR) transfer matrices with minimal state space realization are passive. Strictly positive real (SPR) transfer matrices with minimal state space realization are strictly state passive, and *vice-versa* [27] [15, Theorem 4.73]. Strongly SPR transfer matrices with minimal state space realization are strongly passive [15, p.162], and output and input strictly passive [26]. Incremental passivity is defined in [15, Definition 4.62] [29, 36].

## C Maximal Monotone Mappings and DIs

Let  $\mathcal{M} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  be a set-valued mapping,  $\alpha > 0$ . Is is monotone (resp.  $\alpha$ -strongly monotone) if for any  $x_1, x_2 \in \text{Dom}(\mathcal{M})$ , for any  $y_1 \in \mathcal{M}(x_1)$ ,  $y_2 \in \mathcal{M}(x_2)$ , it holds  $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$  (resp.  $\geq \alpha \|x_1 - x_2\|^2$ ). It is maximal if its graph cannot be enlarged without destroying monotonicity. The following result appeared in [9, Theorem 3.1].

**Theorem 2.** Consider the system  $\dot{x} \in -\mathcal{M}(x)$  with  $\mathcal{M}(\cdot)$  maximal monotone. For ever  $x_0 \in \text{Dom}(\mathcal{M})$ , there exists a unique right-differentiable Lipschitz continuous (hence AC)



solution  $x : [0, \infty) \rightarrow \text{Dom}(\mathcal{M})$ , with  $x(0) = x_0$ , satisfying  $\|x(t)\| \leq \|x(0)\|$  and  $\|\dot{x}(t)\| \leq \|\mathcal{M}^0(x(0))\|$ . Moreover, if  $x_1, x_2$  represent two solutions of the system, then

$$\|x_2(t) - x_1(t)\|^2 \leq e^{-\alpha t} \|x_2(0) - x_1(0)\|^2. \quad (67)$$

Inequality (2) means that the system is incrementally stable. If the system has an equilibrium point then it is Lyapunov stable. In this case, if  $\mathcal{M}(\cdot)$  is strongly maximal monotone, the equilibrium point is exponentially stable, solutions are bounded and depend continuously on initial data. Using some identities from Convex Analysis, it is possible to rewrite system (1) in the form of a differential inclusion (DI) as:

$$\dot{x} \in -\mathcal{F}(x, u) \triangleq Ax + Lu - B(D + \mathcal{N}_{K^*}^{-1})^{-1}(Cx + Eu), \quad (68)$$

The next result follows from [21], see also [14, 11, 35] and [31, Theorem 12.43]. It uses passivity of  $(A, B, C, D)$  (as defined in Appendix B) to rewrite the LCCS as a DI with a maximal monotone operator, and the fact that the normal cone to a closed convex nonempty set defines a maximal monotone mapping, and so does its inverse mapping [31, Exercise 12.8].

**Theorem 3.** *Suppose that  $(A, B, C, D)$  in (1) is passive with storage function  $x \mapsto x^\top Px$ ,  $P = P^\top \succ 0$ ,  $K \subseteq \mathbb{R}^m$  a constant closed convex nonempty cone,  $u \equiv 0$  and  $\text{Im}(C) \cap \text{rint}(\text{Im}(\mathcal{N}_{K^*}^{-1} + D)) \neq \emptyset$ . Then, the mapping  $\mathcal{F} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is maximal monotone.*

## D Useful Matrix Algebra Results

Some of the results from [7, 8] and used at several places in the article are recalled for convenience.

**Proposition 7.** [7, Proposition 8.2.4] *Same as Proposition 10 below.*

**Corollary 9.** [7, Corollary 9.6.5] *Let  $A \in \mathbb{R}^{n \times m}$  and  $B \in \mathbb{R}^{m \times l}$ . Then  $\sigma_{\max}(AB) \leq \sigma_{\max}(A)\sigma_{\max}(B)$ .*

**Fact 1.** [8, Fact 8.4.3] *Let  $A \in \mathbb{R}^{n \times m}$  and  $B \in \mathbb{R}^{n \times l}$ . Then,  $\text{Im}(A) \subseteq \text{Im}(B) \Leftrightarrow BB^\dagger A = A$ .*

**Fact 2.** [8, Fact 3.13.15] *Let  $A, B \in \mathbb{R}^{n \times m}$ . Then  $\text{Im}(A + B) \subseteq \text{Im}(A) + \text{Im}(B)$ . Furthermore  $\text{Im}(A) \subseteq \text{Im}(A + B) \Leftrightarrow \text{Im}(B) \subseteq \text{Im}(A + B) \Leftrightarrow \text{Im}(A + B) = \text{Im}(A) + \text{Im}(B)$ .*

**Fact 3.** [8, Fact 7.17.24] *Let  $A \in \mathbb{R}^{n \times n}$ , assume that  $A = A^\top \succcurlyeq 0$ , and  $\text{rank}(A) = r$ . Then there exists  $B \in \mathbb{R}^{n \times r}$  such that  $A = BB^\top$ .*

**Theorem 4.** [7, Theorem 8.6.2] Let  $A \in \mathbb{R}^{n \times m}$ ,  $B \in \mathbb{R}^{n \times l}$ . Then: There exists  $C \in \mathbb{R}^{l \times m}$  such that  $A = BC \Leftrightarrow$  there exists  $\alpha > 0$  such that  $AA^\top \preceq \alpha BB^\top \Leftrightarrow \text{Im}(A) \subseteq \text{Im}(B)$ .

**Theorem 5.** [7, Theorem 2.4.3] Let  $A \in \mathbb{R}^{n \times m}$ . Then  $\text{Im}^\perp(A) = \text{Ker}(A^\top)$ ,  $\text{Im}(A) = \text{Im}AA^\top$ ,  $\text{Ker}(A) = \text{Ker}(A^\top A)$ .

**Theorem 6.** [7, Theorem 8.4.9] Let  $A = A^\top \in \mathbb{R}^{n \times n}$ ,  $B = B^\top \in \mathbb{R}^{n \times n}$ , with  $A \preceq B$ . Then for all  $i \in \{1, \dots, n\}$ :  $\lambda_i(A) \leq \lambda_i(B)$ . If  $A \neq B$ , there exists  $i \in \{1, \dots, n\}$  such that:  $\lambda_i(A) < \lambda_i(B)$ . If  $A \prec B$ , then  $\lambda_i(A) < \lambda_i(B)$  for all  $i \in \{1, \dots, n\}$ .

**Corollary 10.** [7, Corollary 8.4.10] Let  $A = A^\top \in \mathbb{R}^{n \times n}$ ,  $B = B^\top \in \mathbb{R}^{n \times n}$ . If  $0 \preceq A \preceq B$  then  $0 \leq \text{tr}(A) \leq \text{tr}(B)$ .

(Notice that the trace of a positive semidefinite matrix is nonnegative always).

**Fact 4.** [7, Fact 6.5.21] Let  $A \in \mathbb{R}^{n \times m}$  and  $B \in \mathbb{R}^{k \times l}$ . Then  $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^\dagger = \begin{pmatrix} A^\dagger & 0 \\ 0 & B^\dagger \end{pmatrix}$ .

**Proposition 8.** [8, Proposition 8.1.7 xxx] Let  $A = A^\top \in \mathbb{R}^{n \times m}$ , then  $A \succ 0 \Leftrightarrow A^\dagger \succ 0$ ,  $A \succeq 0 \Leftrightarrow A^\dagger \succeq 0$ .

**Fact 5.** [7, Fact 8.19.2] Let  $0 \prec A = A^\top \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $0 \prec C = C^\top \in \mathbb{R}^{m \times m}$ . If  $\sigma_{\max}^2(B) < \sigma_{\min}(A)\sigma_{\min}(C)$  then  $\begin{pmatrix} A & B \\ B^\top & C \end{pmatrix} \succ 0$ .

**Theorem 7.** [7, Theorem 5.6.3] Let  $A \in \mathbb{R}^{n \times m}$ ,  $A \neq 0$ , let  $\text{rank}(A) = r$ , define  $B = \text{diag}(\sigma_1(A), \dots, \sigma_r(A))$ . Then, there exist unitary matrices  $S_1 \in \mathbb{R}^{n \times n}$  and  $S_2 \in \mathbb{R}^{m \times m}$  such that  $A = S_1 \begin{pmatrix} B & 0_{r \times (m-r)} \\ 0_{(n-r) \times r} & 0_{(n-r) \times (m-r)} \end{pmatrix} S_2$ .

**Corollary 11.** [8, Corollary 11.6.8] Let  $A, B \in \mathbb{R}^{n \times n}$ , then, for all  $i \in \{1, \dots, n\}$ :  $\sigma_i(A)\sigma_{\min}(B) \leq \sigma_i(AB) \leq \sigma_i(A)\sigma_{\max}(B)$ ,  $\sigma_{\min}(A)\sigma_{\min}(B) \leq \sigma_{\min}(AB) \leq \sigma_{\min}(A)\sigma_{\max}(B)$ ,  $\sigma_{\max}(A)\sigma_{\min}(B) \leq \sigma_{\max}(AB) \leq \sigma_{\max}(A)\sigma_{\max}(B)$ .

**Proposition 9.** [7, Proposition 6.1.6 vi) and vii)] Let  $A \in \mathbb{R}^{n \times m}$ . Then  $\text{Im}(A) = \text{Im}(A^{\dagger\top})$  and  $\text{Im}(A^\top) = \text{Im}(A^\dagger)$ .

**Lemma 13.** [7, Lemma 2.4.1] Let  $A \in \mathbb{R}^{n \times m}$ ,  $B \in \mathbb{R}^{m \times l}$ ,  $C \in \mathbb{R}^{k \times n}$ . Then  $\text{Im}(AB) \subseteq \text{Im}(A)$  and  $\text{Ker}(A) \subseteq \text{Ker}(CA)$ .

**Fact 6.** [7, Fact 9.14.15] Let  $A, B \in \mathbb{R}^{n \times n}$ , then  $\sigma_{\max}(A+B) \leq \sigma_{\max}(A) + \sigma_{\max}(B)$ .

**Fact 7.** [8, Fact 10.12.17] Let  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $C \in \mathbb{R}^{m \times m}$ , assume that  $A = A^\top \succ 0$ ,  $C = C^\top \succ 0$ , and assume that  $\sigma_{\max}^2(B) \leq \sigma_{\min}(A)\sigma_{\min}(C)$ . Then  $\begin{pmatrix} A & B \\ B^\top & C \end{pmatrix} \succeq 0$ .

Assume that in addition  $\sigma_{\max}^2(B) < \sigma_{\min}(A)\sigma_{\min}(C)$ . Then  $\begin{pmatrix} A & B \\ B^\top & C \end{pmatrix} \succ 0$ .

**Proposition 10.** [8, Proposition 10.2.5] Let  $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^\top & A_{22} \end{pmatrix}$ ,  $A_{11} = A_{11}^\top$ ,  $A_{22} = A_{22}^\top$ . Then  $A \succcurlyeq 0 \Leftrightarrow A_{11} \succcurlyeq 0$  and  $A_{12} = A_{11}A_{11}^\dagger A_{12}$  and  $A_{12}^\top A_{11}^\dagger A_{12} \preccurlyeq A_{22} \Leftrightarrow A_{22} \succcurlyeq 0$  and  $A_{12} = A_{12}A_{22}^\dagger A_{22}$  and  $A_{12}A_{22}^\dagger A_{12}^\top \preccurlyeq A_{11}$ . Also  $A \succ 0 \Leftrightarrow A_{11} \succ 0$  and  $A_{12}^\top A_{11}^{-1} A_{12} \prec A_{22} \Leftrightarrow A_{22} \succ 0$  and  $A_{12}A_{22}^{-1} A_{12}^\top \prec A_{11}$ .

(**warning:** there is a mistake in [8, Fact 10.24.25] and in [7, Fact 8.21.22], copied in [15, Proposition A.68], see <https://inria.hal.science/hal-03364924v3/document>: so unfortunately these criterion cannot be used to characterize positive semidefiniteness.)

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