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Inferring RPO Symbol Ordering

(work in progress)

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Abstract

We study the recursive path ordering (RPO) in the context of string-rewriting systems. We are interested in finding a symbol ordering in RPO such that for every rule in the rewriting system, the left hand side is higher than the right hand side. We show that this SYMBOL-ORDER problem is NP-complete by a reduction from the 2-3-SAT problem. We also work on the one-rule case and show a polynomial time algorithm for such systems.

Keywords: recursive path ordering, string-rewriting systems, NP-complete

1 Introduction

Termination orderings for term rewriting systems is an important topic. A major breakthrough in this came when Nachum Dershowitz introduced *simplification orderings* and, in particular, the *recursive path ordering* (RPO) scheme [2]. The elegance of this scheme is that it lifts an ordering on the function symbols to an ordering on terms.

On strings, RPO can be defined in two ways: Let Σ be an alphabet and \succ be an ordering on Σ . Then $x \succ_{rpo} y$ if and only if one of the following conditions hold:

- (1) $y = \varepsilon$ and $|x| > 0$.
- (2) $x = au$, $y = av$, and $u \succ_{rpo} v$.
- (3) $x = au$, $y = bv$, and either
 - (3a) $u \succeq_{rpo} y$, or
 - (3b) $a \succ b$ and $x \succ_{rpo} v$.

We use $x \succ_{\exists rpo} y$ to denote “there is an ordering \succ on Σ such that $x \succ_{rpo} y$ ”. Note that we can assume that the ordering is total, since any partial ordering can be extended to a total ordering without affecting

¹Some of the results in this paper will likely be in the first author’s dissertation.

the string comparison.

In the case where the symbol ordering \succ is total, an alternative characterization was introduced in [5]. Let $\max(w, \Sigma)$ stand for the *maximal* symbol of Σ that occurs in w and let $\text{mul}(w, \Sigma)$ be the number of times *this* symbol occurs in w , i.e., $\#_{\max(w, \Sigma)}(w)$. Now $w >_{rpo} w'$ iff one of the following holds:

1. $\max(w, \Sigma) \succ \max(w', \Sigma)$
2. $\max(w, \Sigma) = \max(w', \Sigma)$ and $\text{mul}(w, \Sigma) > \text{mul}(w', \Sigma)$
3. $\mathbf{a} = \max(w, \Sigma) = \max(w', \Sigma)$, $\text{mul}(w, \Sigma) = \text{mul}(w', \Sigma)$,

$$w = w_0 \mathbf{a} w_1 \mathbf{a} w_2 \dots \mathbf{a} w_k$$

$$w' = u_0 \mathbf{a} u_1 \mathbf{a} u_2 \dots \mathbf{a} u_k$$

and there exists $0 \leq i \leq k$ such that $w_i >_{rpo} u_i$ and for all $j > i$ we have $w_j = u_j$.

The *Parikh vector* of a string w over an (ordered) alphabet $\{a_1, \dots, a_n\}$ is the n -tuple

$$\pi(w) = \left(\#_{a_1}(w), \dots, \#_{a_n}(w) \right)$$

The partial order \leq_v on integer vectors is defined as usual: $(c_1, \dots, c_n) \leq_v (b_1, \dots, b_n)$ if and only if $c_i \leq b_i$ for all i . Let $<_v$ be the strict ordering associated with \leq_v .

For a string w , let w_i denote the suffix of length i . For a symbol $a \in \Sigma$, let $\mu(a, w)$ be the longest suffix of w that does not contain a .

1.1 Some properties of *rpo* on strings

Lemma 1. For all x, y, z , $x >_{rpo} y$ if and only if $xz >_{rpo} yz$.

Lemma 2. For all x, y, z , $x >_{rpo} y$ if and only if $zx >_{rpo} zy$.

Lemma 3. If $x_1 >_{rpo} y_1$ and $x_2 >_{rpo} y_2$ then $x_1 x_2 >_{rpo} y_1 y_2$.

In the light of Lemma 1, we can prove the following for two strings *that share no common suffix*:

Lemma 4. Let w and w' be two strings that do not share a common suffix. Then $w >_{rpo} w'$ iff one of the following holds:

1. $\max(w, \Sigma) \succ \max(w', \Sigma)$
2. $\max(w, \Sigma) = \max(w', \Sigma)$ and $\text{mul}(w, \Sigma) > \text{mul}(w', \Sigma)$
3. $\mathbf{a} = \max(w, \Sigma) = \max(w', \Sigma)$, $\text{mul}(w, \Sigma) = \text{mul}(w', \Sigma)$, and $\mu(\mathbf{a}, w) >_{rpo} \mu(\mathbf{a}, w')$.

2 NP-completeness

We prove that the following problem which we call the SYMBOL-ORDER problem, is NP-complete:

Input: A string-rewriting system $\{l_i \rightarrow r_i \mid 1 \leq i \leq n\}$

Question: Is there a symbol ordering \succ such that $l_j >_{rpo} r_j$ for all j ?

The reduction is from the following NP-complete problem which we call “2-3-SAT”:

Input: A set S consisting of 2-clauses and 3-clauses where each 2-clause only has negative literals and each 3-clause only has positive literals.

Question: Is S satisfiable?

Note that this is a small variant of the well-known *monotone 3SAT* problem.

Lemma 5. *The 2-3-SAT problem is NP-complete.*

Proof. We reduce the 3-SAT problem. Given a set of 3-clauses we can replace every negative literal $\neg a$ by a new positive literal a' and add the 2-clauses: $\neg a \vee \neg a'$ and $a \vee a'$. Then we introduce new positive literals z_1, z_2 and replace $a \vee a'$ by the clauses $a \vee a' \vee z_1, a \vee a' \vee z_2$ and $\neg z_1 \vee \neg z_2$. The resulting set of clauses can be obtained in polynomial time from the initial 3-SAT problem. The 3-SAT problem is satisfiable iff the resulting set is satisfiable. \square

Lemma 6. *The SYMBOL-ORDER problem is NP-complete.*

Proof. We reduce the 2-3-SAT problem to the SYMBOL-ORDER problem. Let Φ be any CNF formula consisting of 2-clauses and 3-clauses as in the statement of the 2-3-SAT problem. For each variable x_i in Φ , we introduce a symbol a_i . We also introduce a symbol d to simulate truth and falsehood of a variable: x_i is true if and only if $a_i \succ d$, and x_i is false if and only if $a_i \prec d$. For each 3-clause $(x_i \vee x_j \vee x_k)$ we introduce the rule

$$a_i a_j a_k \rightarrow d$$

and for each 2-clause $(\neg x_m \vee \neg x_n)$ we add the rules

$$\begin{aligned} da_m a_n d &\rightarrow a_n a_m \quad \text{and} \\ da_n a_m d &\rightarrow a_m a_n \end{aligned}$$

From 2-3-SAT to the SYMBOL-ORDER problem: Suppose $(x_i \vee x_j \vee x_k)$ is true, then at least one of these three variables must be true. Let x_i be true, then $a_i \succ d$. Based on the definition of RPO, clearly $a_i a_j a_k >_{rpo} d$. When x_j or x_k is true, we can get the same ordering by the definition.

Suppose $(\neg x_m \vee \neg x_n)$ is true, then at least one of these two variables must be false. If x_m is false, then $a_m \prec d$. By the definition of RPO, we can determine that $da_m a_n d >_{rpo} a_n a_m$ and $da_n a_m d >_{rpo} a_m a_n$.

The step is similar when x_n is false.

From the SYMBOL-ORDER problem to 2-3-SAT: Suppose $a_i a_j a_k \succ_{rpo} d$, then at least one of these three symbols is higher than d . We can prove this by contradiction: assume none of these three symbols is higher than d . Then $d \succ_{rpo} a_i a_j a_k$ by the definition of RPO. If any symbol is higher than d , we give the corresponding variable a truth value of true. Thus if $a_i \succ d$, then assign true to the variable x_i and the clause is satisfied.

Suppose $d a_m a_n \succ_{rpo} a_n a_m$ and $d a_n a_m \succ_{rpo} a_m a_n$, then at least one of these two symbols must be lower than d . This can also be proved by contradiction, if $a_m \succ a_n \succ d$, then the second rule will not be oriented correctly, and if $a_n \succ a_m \succ d$, then the first rule will not be oriented correctly. \square

3 The one-rule case

Let Σ be an alphabet and $\Delta \subseteq \Sigma$. A Δ -skeleton of a string w over Σ^* is defined as the homomorphic image $h(w)$ where h is defined as

$$\begin{aligned} h(a) &= a && \text{if } a \in \Delta \\ &= \varepsilon && \text{otherwise} \end{aligned}$$

In other words, we erase every symbol in w that is not in Δ .

Lemma 7. *Let x and y be distinct strings.*

- (1) *If $\#_a(x) > \#_a(y)$ for some $a \in \Sigma$, then $x \stackrel{\exists}{\succ}_{rpo} y$.*
- (2) *If $\#_a(y) > \#_a(x)$ for all $a \in \Sigma$, then $x \not\stackrel{\exists}{\succ}_{rpo} y$.*

Proof. For (1), let a be the max symbol in the ordering, then according to Lemma 4 of [5], whichever string has more copies of the max symbol will be higher. For (2), let a be the max symbol of some ordering, as y has more copies of a , y will be higher. This case is true for any ordering because y has more copies of every symbol. Therefore, x can't be higher. \square

Lemma 8. *Let x and y be two distinct strings whose Parikh vectors are the same. Then $x \stackrel{\exists}{\succ}_{rpo} y$.*

Proof. We can assume without loss of generality that x and y do not share a common suffix. Thus the rightmost symbols in x and y will be different. Let $x = x'a$ and $y = y'b$. Choose \succ such that b is the max symbol. Then $\mu(b, y) = \varepsilon$ and $\mu(b, x) \neq \varepsilon$. \square

Corollary 8.1. *Let x and y be two distinct strings of the same length. Then $x \stackrel{\exists}{\succ}_{rpo} y$.*

For two strings x, y , we define

$$\Delta_{x,y} = \{a \in \Sigma \mid \#_a(x) = \#_a(y)\}.$$

We omit the subscript when the strings under consideration are obvious from the context.

Lemma 9. *Let x, y be two strings with no common suffix. Then $x \succ_{rpo}^{\exists} y$ iff either (a) there exists a such that $\#_a(x) > \#_a(y)$ or (b) there exists a in $\Delta_{x,y}$ and $\mu(a,x) \succ_{rpo}^{\exists} \mu(a,y)$.*

The following lemma establishes a base case for our algorithm.

Lemma 10. *Let $y \in \Sigma^*$, $a \in \Sigma$ and $m > 0$. Then $a^m \succ_{rpo}^{\exists} y$ iff $\#_a(y) < m$.*

Sketch of the algorithm:

As a preprocessing step remove common suffixes first. Then compute Parikh vectors of all suffixes of x and y .

Let $\$$ be a new symbol that occurs only once at the beginning of x and y . The algorithm builds a list L of length $\leq |\Sigma| + 1$ containing pairs of suffixes (x_i, y_j) such that $x_i \succ_{rpo}^{\exists} y_j$.

- Initialize list L to empty;
- For $i = 1$ to $|x|$:
 1. If $x_i = \mu(a,x)$ for some a , then continue else go to the next i .
 2. Let $y_j = \mu(a,y)$.
 3. If $\#_b(x_i) > \#_b(y_j)$ for any symbol b , then add (x_i, y_j) to L .
 4. Otherwise compute Δ_{x_i, y_j} and if $(\mu(c, x_i), \mu(c, y_j)) \in L$ for any symbol $c \in \Delta_{x_i, y_j}$ then add (x_i, y_j) to L .
 5. If $(\mu(\$, x), \mu(\$, y)) \in L$ then **True** else **False**

The algorithm runs in quadratic time: The sets of vectors $\pi(x_i)$ and $\pi(y_j)$ can be computed in quadratic time. Then we can compute a table $(i, j) \mapsto \Delta_{x_i, y_j}$ in quadratic time. Therefore we can conclude:

Theorem 1. *There is a polynomial time algorithm solving the SYMBOL-ORDER problem for one-rule string-rewriting systems.*

We add a few more results which may help in improving the performance of the algorithm.

Lemma 11. *Let x and y be two strings. If the Δ -skeletons of x and y are different, then $x \succ_{rpo}^{\exists} y$.*

Proof. Let x' and y' be the Δ -skeletons of x and y respectively. Since x' and y' are different but have same Parikh vectors, there is a symbol ordering \succ on Δ such that $x' \succ_{rpo} y'$. Let us extend the symbol ordering on Δ to Σ in such a way that for all $a \in \Delta$ and all $b \in \Sigma \setminus \Delta$ we have $b < a$.

Claim: If $u', v', z \in \Delta^*$, $w' \in (\Sigma \setminus \Delta)^*$ and $z \succ_{rpo} u'v'$ then $z \succ_{rpo} u'w'v'$.

We can write $x = w_0 u_1 w_1 \dots u_n w_n$ (resp., $y = w'_0 u'_1 w'_1 \dots u'_n w'_n$) where $u_1, u'_1, u_2, u'_2, \dots, u_n, u'_n \in \Delta$ and $w_0, w'_0, w_1, w'_1, \dots, w_n, w'_n \in (\Sigma \setminus \Delta)^*$. Note that $x \succ_{rpo} x'$ since \succ_{rpo} contains the embedding relation [2]. Then $x \succ_{rpo} y$ by iterated applications of the claim to x, y' . \square

Lemma 12. *Let x and y be two distinct strings such that $\#_a(x) \leq \#_a(y)$ for all $a \in \Sigma$. If the Δ -skeletons of x and y are the same, then $x \stackrel{\exists}{>}_{rpo} y$ if and only if there is a symbol $c \in \Delta$ such that $\mu(c, x) >_{rpo} \mu(c, y)$ for some symbol ordering on $\Sigma \setminus \{c\}$.*

Lemma 13. *Let x and y be two distinct non-suffix-sharing strings such that (a) $\#_a(x) \leq \#_a(y)$ for all $a \in \Sigma$, (b) $\Delta \neq \emptyset$, and (c) the Δ -skeletons of x and y are the same. Then $x \stackrel{\exists}{>}_{rpo} y$ if $\pi(\mu(c, x)) \not\leq_V \pi(\mu(c, y))$ for some $c \in \Delta$.*

This condition is the same as $\exists c \in \Delta \exists a \in (\Sigma \setminus \Delta) : \#_a(\mu(c, x)) > \#_a(\mu(c, y))$.

Note that the other direction does not work: consider $x = cad$ and $y = adcda$. We get $x >_{rpo} y$ if we choose $c \succ a \succ d$. But $\pi(\mu(c, x)) = (1, 1) \leq (1, 1) = \pi(\mu(c, y))$.

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