



**HAL**  
open science

## Modal unification step by step

Sam v. Gool, Johannes Marti

► **To cite this version:**

Sam v. Gool, Johannes Marti. Modal unification step by step. UNIF 2023 - 37th International Workshop on Unification, Veena Ravishankar; Christophe Ringeissen, Jul 2023, Rome, Italy. hal-04128087

**HAL Id: hal-04128087**

**<https://inria.hal.science/hal-04128087v1>**

Submitted on 14 Jun 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

# Modal unification step by step\*

Sam v. Gool<sup>1</sup> and Johannes Marti<sup>2</sup>

<sup>1</sup> IRIF, Université Paris Cité

<sup>2</sup> Department of Computer Science, University of Oxford

## Abstract

We propose an approach to unification in modal logics that combines duality and step-by-step constructions of terminal frames. We apply this approach to the modal logics  $\mathbf{K}$ ,  $\mathbf{Alt}_1$ , and a logic that we call  $\mathbf{KdB}$ . For  $\mathbf{K}$ , we show how our methods give an effective reformulation of the unifiability problem as a non-trivial decision problem involving neighborhood frames. For  $\mathbf{Alt}_1$ , we show that the problem is PSPACE-complete. For  $\mathbf{KdB}$ , we establish that unifiability is equivalent to a problem involving de Bruijn graphs.

**Introduction.** The decision problem of *unifiability* for a normal modal logic  $L$  asks, for a given formula  $\phi(p_1, \dots, p_n)$ , whether there exist formulas  $\psi_1, \dots, \psi_n$  such that the formula  $\phi(\psi_1/p_1, \dots, \psi_n/p_n)$ , that is, the result of substituting  $\psi_i$  for every occurrence of  $p_i$ , is a theorem of  $L$ . In this case, we call the substitution  $p_i \mapsto \psi_i$  a *unifier* for  $\phi$ , and a *ground unifier* if the  $\psi_i$  do not use variables. We only consider ground unifiers in this paper, which suffices for unifiability. In the case of the smallest normal modal logic,  $\mathbf{K}$ , it is a prominent open question whether or not unifiability is decidable, see e.g. [WZ08, Jer13]. Much more is known in *transitive* modal logics, that is, extensions of  $\mathbf{K}$  in which  $\diamond\diamond\phi \rightarrow \diamond\phi$  is a theorem for every formula  $\phi$ . For a survey of unification problems in modal logic, with connections to description logic, we refer to [BG11], and we also refer to the introductions of [BGM<sup>+</sup>19] and [BGRT22] for more information about the modal logic side.

The main idea we pursue in this paper is that a combination of *duality* and *step-by-step constructions* gives a useful perspective on unifiability problems for modal logics beyond the transitive. Both ingredients are present in the ground-breaking series of papers on unification in transitive modal and intuitionistic logics [Ghi99, Ghi00], and the duality perspective on unification has been made more explicit since then, for example in [CM15, Ghi18, BG22]. The step-by-step approach is an algebraic way of viewing normal forms in modal logic, systematically studied in [Ghi95] and giving rise to finite model results in [Mos07]. On the semantic side of the duality, our construction is similar to the *terminal sequence* for a coalgebra functor, see e.g. [Bar93, Wor99], which has also been applied to prove finite model properties [Pat03].

Until further notice, we work in a propositional language with a single unary operator  $\diamond$ . Recall that  $\mathbf{Alt}_1$  is defined as the extension of  $\mathbf{K}$  by the axiom  $\diamond\phi \rightarrow \Box\phi$ . Our main results are the following: (1) a general equivalence between unifiability in modal logic and a mapping problem between generalized graphs (Theorem 4), as corollary an effective reformulation of the decision problem for  $\mathbf{K}$  (Corollary 5); (2) using the general equivalence, PSPACE-completeness of unifiability for  $\mathbf{Alt}_1$  (Theorem 7); (3) an equivalence between unifiability in a logic  $\mathbf{KdB}$ , to be introduced below, and a mapping problem of *de Bruijn graphs* (Theorem 8).

**A step-by-step approach: the general picture.** Throughout this paper,  $V$  will denote a finite set of (propositional) variables. We take  $\neg, \vee, \perp$  and  $\diamond$  as primitive connectives, and other common logical symbols are considered as abbreviations. It will be convenient to borrow

---

\*We gratefully acknowledge many useful discussions on the topics of this paper with Silvio Ghilardi, George Metcalfe and Leif Sabellek.

some notation from coalgebraic logic. Until further notice, we fix an arbitrary endofunctor  $\mathcal{T}$  on the category of sets and assume  $\mathcal{T}$  preserves finite sets. Recall that such an endofunctor gives rise to a *coalgebraic modal logic*  $L$  [Mos99], but we do not require the general definition in this paper. Instead, we will assume for now that the modal logic  $L$  associated to  $\mathcal{T}$  is described as a normal extension of  $\mathbf{K}$  by a set of axioms  $\Gamma$ , given as formulas with variables in  $V$ . A reader not familiar with coalgebra may think of  $L$  as being either  $\mathbf{K}$  or  $\mathbf{Alt}_1$ , and of the functor  $\mathcal{T}$  as being either  $\mathcal{P}$  or  $1 + (-)$ , respectively. A third instance,  $L = \mathbf{KdB}$ , will be examined later.

An  $L$ -algebra is a modal algebra  $B$  such that, for every interpretation  $f: V \rightarrow B$  of variables as elements of  $B$  and for every axiom  $\phi \in \Gamma$ , the evaluation  $\phi[v \mapsto f(v)]$  is equal to  $\top$  in  $B$ . We denote by  $F(V)$  the *Lindenbaum algebra over  $V$*  (for the logic  $L$ ), which can be defined on the set of  $L$ -equivalence classes of formulas with variables in  $V$ ; in algebraic terms,  $F(V)$  is the free  $L$ -algebra over the set  $V$ . By a common abuse of notation, we identify modal formulas with their corresponding class in  $F(V)$ . An algebraic formulation of the unifiability problem for  $L$  is: given as input a modal formula  $\phi$  with variables in  $V$ , determine whether or not there exists a homomorphism  $h: F(V) \rightarrow F(\emptyset)$  such that  $h(\phi) = \top$ . Such a homomorphism is also called a (ground) *unifier* for  $\phi$ ; its *depth* is the maximal modal depth of the formulas  $h(p)$ , for  $p \in V$ .

We will make use of duality and one-step frames; we only briefly recall what we need explicitly in this abstract, for more details, see [CG12, BGJ14]. When  $S$  is a subset of a Boolean algebra  $A$ , we will write  $\langle S \rangle_{\vee, \neg}$  for the Boolean subalgebra of  $A$  generated by  $S$ . We define a chain of subalgebras of  $F(V)$ . Let  $F_0(V) := \langle V \rangle_{\vee, \neg}$  and, for any  $n \in \mathbb{N}$ , let  $F_{n+1}(V) := \langle a, \diamond a \mid a \in F_n(V) \rangle_{\vee, \neg}$ . Thus,  $F_n(V)$  contains the classes of formulas of modal depth  $\leq n$ . Note that, for any  $n \in \mathbb{N}$ , we have an inclusion homomorphism  $e_n: F_n(V) \rightarrow F_{n+1}(V)$  and an operator  $\diamond_n: F_n(V) \rightarrow F_{n+1}(V)$ , the restriction of the operator  $\diamond$  on  $F(V)$  to the subalgebra  $F_n(V)$ . The main idea of the step-by-step approach is that, given, for some  $n \geq 1$ , a particular

‘level’  $F_{n-1}(V) \begin{array}{c} \xrightarrow{\diamond_{n-1}} \\ \xleftarrow{e_{n-1}} \end{array} F_n(V)$  of the approximating chain, it is possible to reconstruct the ‘next level’  $F_n(V) \begin{array}{c} \xrightarrow{\diamond_n} \\ \xleftarrow{e_n} \end{array} F_{n+1}(V)$  without needing to refer to the global free algebra  $F(V)$ .

We are thus dealing with a general construction on ‘levels’. Any such level can be conveniently encoded as a *one-step algebra* for the logic  $L$ . In the interest of space, we only recall the dual definition of *one-step frame* for the functor  $\mathcal{T}$ .

A *one-step  $\mathcal{T}$ -frame* is a tuple  $(X, Y, p, \alpha)$ , where  $X, Y$  are sets and  $p: Y \rightarrow X$  and  $\alpha: Y \rightarrow \mathcal{T}X$  are functions, and a *bounded morphism*  $(X, Y, p, \alpha) \rightarrow (X', Y', p', \alpha')$  is a pair of functions  $f: X \rightarrow X'$ ,  $g: Y \rightarrow Y'$  such that  $f \circ p = p' \circ g$  and  $\mathcal{T}f \circ \alpha = \alpha' \circ g$ . We define a chain of one-step  $\mathcal{T}$ -frames. Let  $T_0(V) := 2^V$  and, for any  $n \in \mathbb{N}$ , let  $T_{n+1}(V) := 2^V \times \mathcal{T}T_n(V)$ . Define  $p_0: T_1(V) \rightarrow T_0(V)$  to be the projection onto the first coordinate, and for any  $n \geq 1$ , define  $p_n: T_{n+1}(V) \rightarrow T_n(V)$  to be  $\text{id}_{2^V} \times \mathcal{T}p_{n-1}$ . Also write, for any  $n \in \mathbb{N}$ ,  $\alpha_n: T_{n+1}(V) \rightarrow \mathcal{T}T_n(V)$  for the projection onto the second coordinate. We will use the notation  $C_n(V) := (T_n(V), T_{n+1}(V), p_n, \alpha_n)$  and call this the *canonical one-step frame at level  $n$* . In the cases that will interest us in this paper, the one-step  $\mathcal{T}$ -frame  $(T_n(V), T_{n+1}(V), p_n, \alpha_n)$  is *dual* to  $(F_n(V), F_{n+1}(V), e_n, \diamond_n)$ . What this means concretely is that one may exhibit, for every  $n \in \mathbb{N}$ , a bijection  $\phi_n$  between  $T_n(V)$  and the set of atoms of  $F_n(V)$ , in such a way that the function  $p_n$  corresponds to  $e_n$ , and the function  $\alpha_n$  corresponds to the operator  $\diamond_n$ , under standard dual categorical equivalences between finite Boolean algebras and finite sets.

For example, in the case where  $L$  is the modal logic  $\mathbf{K}$ , we have  $\mathcal{T} = \mathcal{P}$ , and it is well-known [Ghi95, Mos07] that the bijections may be described as follows. Given  $a \in T_0(V) = 2^V$ , define the modal depth 0 formula  $\phi_0(a) := \bigwedge_{\substack{p \in V \\ a(p)=1}} p \wedge \bigwedge_{\substack{p \in V \\ a(p)=0}} \neg p$ . Then  $\phi_0$  gives a bijection between  $T_0(V)$  and the set of atoms of  $F_0(V)$ . Inductively, assuming a bijection  $\phi_n$  between  $T_n(V)$  and

the atoms of  $F_n(V)$  has already been defined, we may define, for  $a \in 2^V$  and  $S \in \mathcal{PT}_n(V)$ , the formula  $\phi_{n+1}(a, S)$  as  $\phi_0(a) \wedge \square(\bigvee \phi_n[S]) \wedge \bigwedge_{s \in S} \diamond \phi_n(s)$ .

Towards formulating the unifiability problem for  $L$  using one-step frames, we now observe that we may restrict to formulas of depth at most 1, at the expense of a polynomial increase in the number of propositional variables; this is known as *flattening*, see e.g. [CG12, Remark 3.1]. If  $\phi$  is a formula with variables in  $V$ , let  $V'$  be the disjoint union of  $V$  with a set of fresh proposition letters  $q_\psi$ , where  $\psi$  ranges over the set  $S(\phi)$  of subformulas of  $\phi$ . For every  $\psi \in S(\phi)$ , define a formula  $\chi_\psi$  with variables in  $V'$  as follows. If  $\psi = \perp$  or  $\psi = p$  then  $\chi_\psi$  is  $q_\psi \leftrightarrow \psi$ . If  $\psi = \psi_1 \wedge \psi_2$  then  $\chi_\psi$  is  $q_\psi \leftrightarrow q_{\psi_1} \wedge q_{\psi_2}$ . If  $\psi = \neg\theta$  then  $\chi_\psi$  is  $q_\psi \leftrightarrow \neg q_\theta$ . Finally, if  $\psi = \diamond\theta$  then  $\chi_\psi$  is  $q_\psi \leftrightarrow \diamond q_\theta$ . Now define  $\phi' := q_\phi \wedge \bigwedge_{\psi \in S(\phi)} \chi_\psi$ . Then  $\phi'$  has modal depth at most 1, is computable in time polynomial in the size of  $\phi$ , and is unifiable for  $L$  if and only if  $\phi$  is.

We may thus assume that  $\phi \in F_1(V)$ . By duality, the finite Boolean algebra  $F_1(V)$  is isomorphic to the algebra on  $\mathcal{PT}_1(V)$ ; write  $\widehat{\phi} \subseteq T_1(V)$  for the subset corresponding to  $\phi$ . Define the one-step frame  $P(\phi) := (T_0(V), \widehat{\phi}, p', r')$ , where  $r'$  and  $p'$  are the restrictions of  $r_0$  and  $p_0$ , respectively, to  $\widehat{\phi}$ . Then duality between one-step algebras and one-step frames gives the following.

**Proposition 1.** *Let  $\phi \in F_1(V)$ . For any  $n \geq 0$ , the set of functions  $f: V \rightarrow F_n(\emptyset)$  such that  $\phi(f(p)/p) = \top$  is in bijection with the set of bounded morphisms  $C_n(\emptyset) \rightarrow P(\phi)$ .*

To analyze unification problems combinatorially, it is useful to replace one-step frames by single-sorted structures, which in several concrete cases below will become *directed graphs* with some additional structure. We define a  $\mathcal{T}$ -graph to be a pair  $(X, \rho)$ , where  $X$  is a set and  $\rho: X \rightarrow \mathcal{PT}X$  is a function. A  $\mathcal{T}$ -graph homomorphism from  $(X, \rho)$  to  $(X', \rho')$  is a function  $f: X \rightarrow X'$  such that  $\mathcal{P}(\mathcal{T}f)(\rho(x)) \subseteq \rho'(f(x))$  for all  $x \in X$ .<sup>1</sup> Note that, when  $\mathcal{T}$  is the identity functor,  $\mathcal{T}$ -graphs are just directed graphs, and when  $\mathcal{T} = \mathcal{P}$ ,  $\mathcal{T}$ -graphs are known as *neighborhood frames*, but our morphisms are not the same as the usual ones in that context.

We will now show that we can toggle between  $\mathcal{T}$ -graphs and one-step  $\mathcal{T}$ -frames by means of an adjoint pair of functors, as we shall briefly describe now. When  $(X', \rho')$  is a  $\mathcal{T}$ -graph, define the *associated one-step  $\mathcal{T}$ -frame*  $\Phi(X', \rho') := (X', R', \pi_1, \pi_2)$ , where  $R' := \{(w, v) \in X' \times \mathcal{T}(X') \mid v \in \rho'(w)\}$ , and  $\pi_1$  and  $\pi_2$  are the restrictions to  $R'$  of the projection maps on  $X'$  and  $\mathcal{T}(X')$ , respectively. Conversely, when  $\mathcal{F} = (X, Y, p, \alpha)$  is a one-step  $\mathcal{T}$ -frame, the *associated  $\mathcal{T}$ -graph* is defined as  $\Gamma\mathcal{F} := (X, \mathcal{P}\alpha \circ p^{-1})$ . Note that if  $(f, g): \mathcal{F} \rightarrow \Phi(X', \rho')$  is a bounded morphism, then in particular  $f$  is a  $\mathcal{T}$ -graph homomorphism  $\Gamma\mathcal{F} \rightarrow (X', \rho')$ .

**Proposition 2.** *Let  $(X, Y, p, \alpha)$  be a one-step  $\mathcal{T}$ -frame and let  $(X', \rho')$  be a  $\mathcal{T}$ -graph. For any  $\mathcal{T}$ -graph homomorphism  $f: \Gamma(X, Y, p, \alpha) \rightarrow (X', \rho')$ , there is a unique function  $g$  such that  $(f, g): (X, Y, p, \alpha) \rightarrow \Phi(X', \rho')$  is a bounded morphism.*

For any one-step  $\mathcal{T}$ -frame  $\mathcal{F} = (X, Y, p, \alpha)$ , we have that  $\Phi\Gamma\mathcal{F} = (X, R, \pi_1, \pi_2)$ , where  $R = \{(x, \alpha(y)) \mid x \in X, y \in p^{-1}(x)\}$ . We thus get a natural one-step morphism  $\eta_{\mathcal{F}} := (\text{id}_X, g): \mathcal{F} \rightarrow \Phi\Gamma\mathcal{F}$ , where  $g(y) := (p(x), \alpha(y))$  for every  $y \in Y$ .

**Proposition 3.** *For any modal formula  $\phi \in F_1(V)$ , the one-step morphism  $\eta_{P(\phi)}: P(\phi) \rightarrow \Phi\Gamma(P(\phi))$  is an isomorphism of one-step  $\mathcal{T}$ -frames.*

Combining Propositions 1, 2, and 3, we obtain the following.

**Theorem 4.** *Let  $\phi$  be a modal formula of depth  $\leq 1$ . For any  $n \geq 0$ , there is a bijection between the set of ground  $L$ -unifiers of  $\phi$  of depth  $\leq n$  and the set of  $\mathcal{T}$ -graph homomorphisms  $f: \Gamma(C_n(\emptyset)) \rightarrow \Gamma(P(\phi))$ .*

<sup>1</sup>For any function  $g: Y \rightarrow Y'$  between sets, we use the notation  $\mathcal{P}(g)$  for the direct image map  $\mathcal{P}(Y) \rightarrow \mathcal{P}(Y')$ .

**Corollary 5.** *The unifiability problem for  $\mathbf{K}$  is equivalent to the problem which asks, given a modal formula  $\phi \in F_1(V)$ , whether or not there exist  $n \geq 0$  and a  $\mathcal{P}$ -graph homomorphism  $f: \Gamma(C_n(\emptyset)) \rightarrow \Gamma(P(\phi))$ .*

**Unification for  $\mathbf{Alt}_1$  step by step.** In this section we apply the step-by-step approach to the modal logic  $\mathbf{Alt}_1$ . This logic can be described as the coalgebraic modal logic for the endofunctor  $\mathcal{T} = 1 + (-)$  on the category of sets, which sends any set  $X$  to the set  $1 + X := \{*\} \sqcup X$ . One can show that the canonical one-step frame at level  $n$ , when constructed from this functor  $\mathcal{T}$  as described above, is dual to the one-step algebras of formulas of depth  $n$  in  $\mathbf{Alt}_1$ . A  $\mathcal{T}$ -graph for this functor  $\mathcal{T}$  is a pair  $(X, \rho)$  with  $\rho: X \rightarrow \mathcal{P}(1 + X)$ . Because  $\mathcal{P}(1 + (-)) \cong 2 \times \mathcal{P}(-)$ , these can be viewed concretely as graphs  $(X, S, E)$  consisting of a set  $X$  with a binary relation  $S \subseteq X \times X$  and a set  $E \subseteq X$ .  $\mathcal{T}$ -graph homomorphisms are then structure homomorphisms in the usual sense, meaning functions that preserve the relation  $S$  and the labeling  $E$ .

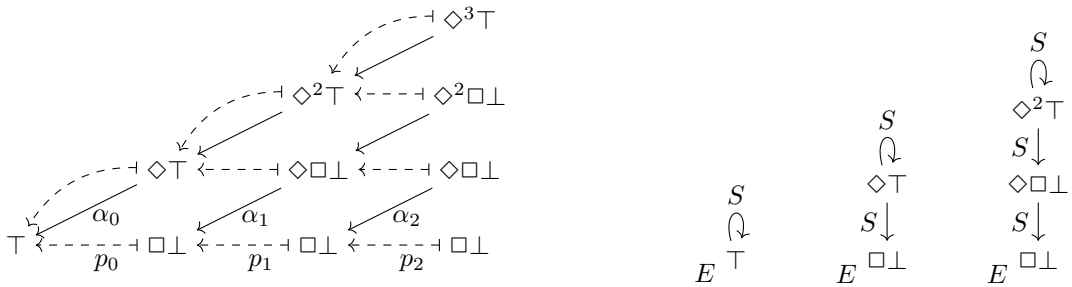
To get an idea of how our approach applies, note that the set  $T_n(V)$  is in bijection with the set of atoms in  $F_n(V)$ . In  $\mathbf{Alt}_1$  the corresponding formulas can be described concretely by the same bijection  $\phi_0$  as in the case of  $\mathbf{K}$  above, and then, for any  $n \in \mathbb{N}$ ,  $a \in 2^V$ ,  $s \in 1 + T_n(V)$ ,  $\phi_{n+1}(a, s) := \phi_0(a) \wedge \Box \perp$  if  $s = *$  and  $\phi_{n+1}(a, s) := \phi_0(a) \wedge \Diamond \phi_n(s)$  if  $s \in T_n(V)$ . In Figure 1a, we depict the initial part of the sequence of sets  $T_n(\emptyset)$ , with each element labeled by the corresponding formula, and the functions  $p_n: T_{n+1}(\emptyset) \rightarrow T_n(\emptyset)$  and  $\alpha_n: T_{n+1}(\emptyset) \rightarrow \mathcal{T}T_n(\emptyset)$ . Figure 1b shows the graphs  $\Gamma(C_n)$  that are derived from the canonical frames as described in the previous section. For  $\mathbf{Alt}_1$ , for any  $n \in \mathbb{N}$ ,  $\Gamma(C_n)$  is an  $S$ -chain of length  $n$ , where the first vertex is  $S$ -reflexive and the last vertex is the only vertex that satisfies  $E$ . Combining this observation with Theorem 4 leads to the following characterization of unification in  $\mathbf{Alt}_1$ :

**Theorem 6.** *For every formula  $\phi$  of depth  $\leq 1$  and  $n \geq 0$  the ground  $\mathbf{Alt}_1$ -unifiers of  $\phi$  with depth  $\leq n$  are in bijection with paths in the graph  $\Gamma(P(\phi))$  that are of the form  $v_0 S v_1 S \dots S v_n$ , where  $v_0 S v_0$  and  $v_n \in E$ .*

Using the characterization from Theorem 6 we obtain the following complexity result.

**Theorem 7.** *The unifiability problem for  $\mathbf{Alt}_1$  is PSPACE-complete.*

From Proposition 6.1 in [BT16] it is already known that unifiability for  $\mathbf{Alt}_1$  is in PSPACE. Our analysis shows that finding a unifier corresponds to finding a particular kind of path in the



(a) Sets  $T_0(\emptyset), \dots, T_3(\emptyset)$  with the labels given by  $\phi_n$ , and the functions  $p_n$  (dashed) and  $\alpha_n$ . Nodes with no solid outgoing arrow are sent to  $*$  by  $\alpha_n$ .

(b) Graphs  $\Gamma(C_0(\emptyset)), \Gamma(C_1(\emptyset))$  and  $\Gamma(C_2(\emptyset))$ .

Figure 1: Construction of  $C_n(\emptyset) = (T_n(\emptyset), T_{n+1}(\emptyset), p_n, \alpha_n)$  and  $\Gamma(C_n(\emptyset))$  for small  $n$ .

graph  $\Gamma(P(\phi))$ . The vertices of the graph  $\Gamma(P(\phi))$ , on input  $\phi$ , can be represented in polynomial space and the relation  $S$  and subset  $E$  can be computed in polynomial time. The existence of such a path in  $\Gamma(P(\phi))$  can be determined with a nondeterministic polynomial space algorithm that successively guesses the vertices on the path. Because  $\text{PSPACE} = \text{NPSPACE}$  this yields the  $\text{PSPACE}$  upper bound.

To prove  $\text{PSPACE}$ -hardness, we show that the language of an arbitrary polynomial space Turing machine  $M$  reduces to the unification problem. Fix  $M$ . For every input  $x$  of  $M$ , we introduce a set  $V$  of size polynomial in the size of  $x$ , and we will compute in polynomial time a formula  $\phi_x \in F_1(V)$  such that the accepting computations of  $M$  on input  $x$  correspond to paths in the graph  $\Gamma(P(\phi_x))$  of the kind described in Theorem 6. We briefly describe the construction of  $\phi_x$ . The idea is to let the vertices of  $\Gamma(P(\phi_x))$ , which are by definition elements of  $2^V$ , encode configurations of  $M$ , whose size is bounded by a polynomial in the size of the input  $x$ . The formula  $\phi_x$  can be defined in such a way that the following hold in  $\Gamma(P(\phi_x))$  for all distinct  $a, b \in 2^V$ : (1)  $aSa$  iff  $a$  encodes the initial configuration of  $M$  on input  $x$ . (2)  $aSb$  iff  $b$  encodes a successor configuration of the configuration encoded by  $a$ . (3)  $a \in E$  iff  $a$  encodes an accepting configuration. Requirement (2) can be expressed by a modal formula of polynomial size because it suffices to describe for each of the polynomially many individual cells on the tape how its content is updated when moving to the successor configuration.

**Unification for KdB step by step.** The modal logic **KdB** is an extension of the modal logic **KD!**, where we recall that **KD!** is defined by extending **Alt**<sub>1</sub> with the axiom  $\Box\phi \rightarrow \Diamond\phi$ . Every ground formula in **KD!** is equivalent to either  $\top$  or  $\perp$ , so **KD!**-unification is trivially decidable. To make the unification problem more interesting we add a single constant **c** to the language, which acts syntactically as a propositional letter but cannot be replaced by substitutions. The modal depth of the formula **c** is stipulated to be 1. We define **KdB** to be the logic resulting from adding the constant **c** to **KD!**, without any additional axioms. The notions of one-step algebra and one-step frame can be extended to provide an interpretation for **c**.

One can view **KdB** as the coalgebraic modal logic for the functor  $\mathcal{T} = 2 \times (-)$ . The notions of one-step  $\mathcal{T}$ -frame and  $\mathcal{T}$ -graph are adequate for our framework to apply. Because of the natural isomorphism  $\mathcal{P}(2 \times X) \cong \mathcal{P}(X + X) \cong \mathcal{P}X \times \mathcal{P}X$ , one can represent  $\mathcal{T}$ -graphs as structures  $(X, R_0, R_1)$  for a set  $X$  and two relations  $R_0, R_1 \subseteq X \times X$ . Graph homomorphisms are required to preserve the label  $i \in \{0, 1\}$  of the binary relation  $R_i$ .

The graph  $\Gamma(C_n(\emptyset))$ , for  $n \geq 1$ , is isomorphic to the so-called *de Bruijn graph*  $B(n) = (2^n, S_0, S_1)$  of dimension  $n$  over the alphabet  $2 = \{0, 1\}$  [dB46]. It is defined such that for  $b \in 2$  and  $w, v \in 2^n$  it holds that  $wS_bv$  iff  $w = bu$  and  $v = ua$  for some  $u \in 2^{n-1}$  and  $a \in 2$ . Following Theorem 4, we obtain the following characterization of the unification problem:

**Theorem 8.** *For every formula  $\phi$  of depth  $\leq 1$  and  $n \geq 0$  the ground **KdB**-unifiers of  $\phi$  with depth  $\leq n$  are in bijection with the graph homomorphisms from  $B(n)$  to  $\Gamma(P(\phi))$ .*

We have not yet been able to determine whether, given a  $\mathcal{T}$ -graph of the form  $\Gamma(P(\phi))$ , it is decidable if there exist  $n \geq 0$  and a homomorphism from  $B(n)$  to  $\Gamma(P(\phi))$ .

**Conclusion.** We have shown that unifiability problems in non-transitive modal logics are equivalent to the existence of homomorphisms between certain graph-like structures. In the case of **Alt**<sub>1</sub> this analysis allowed us to prove the new result that the unifiability problem is  $\text{PSPACE}$ -hard. We also exhibit a new logic **KdB**, seemingly less complex than **K**, but for which unifiability is still not easily seen to be decidable.

While duality has previously been used in the context of unification problems, the main novelty here is to combine it with step by step approaches, which, as one reviewer noted, in particular avoids having to make explicit use of the profinite topology inherent to these problems (although we took crucial inspiration from the existing literature that is framed in the infinite topological setting).

As future work, we are interested in investigating how our approach can be adapted to normal modal logics that are not coalgebraic logics for some endofunctor. We also note that for the purposes of this paper we decided to focus only on the question of *unifiability*, but, as one of the reviewers suggested, it is natural to ask whether the methods outlined here might also be useful for determining *unification type*.

We hope that this paper can serve as a first step towards new methods for unification for non-transitive modal logics.

## References

- [Bar93] Michael Barr. Terminal coalgebras in well-founded set theory. *Theoretical Computer Science*, 114(2):299–315, 1993.
- [BG11] Franz Baader and Silvio Ghilardi. Unification in modal and description logics. *Logic Journal of IGPL*, 19(6):705–730, 2011.
- [BG22] Philippe Balbiani and Quentin Gougeon. Projective unification through duality. In David Fernández-Duque, Alessandra Palmigiano, and Sophie Pinchinat, editors, *Advances in Modal Logic, AiML 2022, Rennes, France, August 22-25, 2022*, pages 119–134. College Publications, 2022.
- [BGJ14] Nick Bezhanishvili, Silvio Ghilardi, and Mamuka Jibladze. Free modal algebras revisited: the step-by-step method. In Guram Bezhanishvili, editor, *Leo Esakia on Duality in Modal and Intuitionistic Logics*, pages 43–62. Springer, 2014.
- [BGM<sup>+</sup>19] Philippe Balbiani, Çiğdem Gencer, Mojtaba Mojtahedi, Maryam Rostamigiv, and Tinko Tinchev. A gentle introduction to unification in modal logics. In *13èmes Journées d’Intelligence Artificielle Fondamentale (JIAF 2019)*, 2019.
- [BGRT22] Philippe Balbiani, Çiğdem Gencer, Maryam Rostamigiv, and Tinko Tinchev. Remarks about the unification types of some locally tabular normal modal logics. *Logic Journal of the IGPL*, 31(1):115–139, 2022.
- [BT16] Philippe Balbiani and Tinko Tinchev. Unification in modal logic  $Alt_1$ . In Lev D. Beklemishev, Stéphane Demri, and András Maté, editors, *Advances in Modal Logic 11*, pages 117–134. College Publications, 2016.
- [CG12] Dion Coumans and Sam v. Gool. On generalizing free algebras for a functor. *Journal of Logic and Computation*, 23(3):645–672, 2012. Available at <https://samvangool.net/assets/pdf/coumansvangool21112011.pdf>.
- [CM15] Leonardo Cabrer and George Metcalfe. Admissibility via natural dualities. *Journal of Pure and Applied Algebra*, 219(9):4229–4253, 2015.
- [dB46] Nicolaas Govert de Bruijn. A combinatorial problem. *Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen*, 49(7):758–764, 1946.
- [Ghi95] Silvio Ghilardi. An algebraic theory of normal forms. *Annals of Pure and Applied Logic*, 71(3):189–245, 1995.
- [Ghi99] Silvio Ghilardi. Unification in intuitionistic logic. *The Journal of Symbolic Logic*, 64(2):859–880, 1999.
- [Ghi00] Silvio Ghilardi. Best solving modal equations. *Annals of Pure and Applied Logic*, 102(3):183–198, 2000.

- [Ghi18] Silvio Ghilardi. Handling substitutions via duality. In *UNIF workshop*, 2018. Available at <https://easychair.org/smart-slide/slide/Bn7h#>.
- [Jeř13] Emil Jeřábek. Blending margins: the modal logic K has nullary unification type. *Journal of Logic and Computation*, 25(5):1231–1240, 2013.
- [Mos99] Lawrence S. Moss. Coalgebraic logic. *Annals of Pure and Applied Logic*, 96(1):277–317, 1999.
- [Mos07] Lawrence S. Moss. Finite models constructed from canonical formulas. *Journal of Philosophical Logic*, 36:605–640, 2007.
- [Pat03] Dirk Pattinson. Coalgebraic modal logic: soundness, completeness and decidability of local consequence. *Theoretical Computer Science*, 309(1):177–193, 2003.
- [Wor99] James Worrell. Terminal sequences for accessible endofunctors. *Electronic Notes in Theoretical Computer Science*, 19:24–38, 1999.
- [WZ08] Frank Wolter and Michael Zakharyashev. Undecidability of the unification and admissibility problems for modal and description logics. *ACM Trans. Comput. Logic*, 9(4), 2008.