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Finite/nearly Fixed-Time stability of Nonlinear Impulsive Systems with Destabilizing Impulses and its Application to Neural Networks

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ABSTRACT

This paper investigated the finite-time stability (FTS) and *nearly* fixed-time stability (nFxTS) of nonlinear impulsive systems with *destabilizing* impulses. The concept of nFxTS is different from FxTS, as the former may be just asymptotically stable but the latter must be FTS. A Lyapunov inequality with linear terms has been used to derive sufficient conditions based on the dwell time (DT) and average dwell time (ADT) properties of impulsive sequences to ensure FTS/FxTS of the system. Theorems are constructed based on the initial state of the system, which belongs to either δ -neighbourhood (nbh) or outside δ -nbh of the origin. When the system starts within δ -nbh of the origin then FTS can be achieved by the system following conditions on DT and ADT of impulsive sequences. The same is true for nFxTS of the system when it starts outside δ -nbh of the origin. We investigated the condition on DT for which nFxTS is not possible, but FTS is possible only if $\delta > 0$ is bounded. The main results of this paper are applied to the synchronization problem of impulsive neural networks with *destabilizing* impulses. Finally, a numerical example of an impulsive neural network is given to illustrate the validity of the proposed theoretical results.

1. Introduction

Impulsive differential equations [1] are a suitable mathematical framework for the modeling and analysis of dynamical processes. Such systems are made up of three parts: a continuous-time differential equation that models the system's motion between impulsive points or resetting events; a difference equation that models the way the system reacts to sudden changes at impulsive points; and a criterion for the occurrence of the resetting events. Over the past few decades, investigation of the dynamical properties of impulsive systems has received much attention because of their rich applications in neural networks, network control systems, epidemiology, ecology, etc. See [2, 3, 4, 5, 6] and the references cited therein.

Impulsive effects on systems can be classified as either stabilizing or destabilizing. The study of the dynamical properties of systems with stabilizing and destabilizing impulses is called the "impulse control problem" (ICP) and "impulse perturbation problem" (IPP), respectively. If the systems' dynamics are controlled at discrete time points, it is known as ICP. The main idea behind the working mechanism of ICP is that the control protocol on a plant activates only at discrete moments. Thus, it has the advantage of saving control costs compared to the continuous control protocol. For instance, in impulsive synchronization of communication security systems [7, 8], impulsive control actions allow the synchronization signals to communicate with the terminal system only at discrete impulse instances, effectively reducing the dependence on transmitted signals and increasing robustness against disturbances. Because of the benefits mentioned above, ICP has been studied in a variety of real-world problems, including neural networks [9], population control [10], secure communication [11], and infectious disease models [12]. When the system demonstrates a complex quality of dynamical behavior such as bifurcation, stability, periodic solution, or almost periodic solution, the corresponding dynamics can be preserved when the system experiences sudden perturbation. Such a phenomenon is known as IPP. This type of problem has been investigated in many fields, such as neural networks [13], complex control systems [14], biological models [15], and financial market models [7].

Over the past few decades, many researchers have investigated the (Lyapunov) asymptotic or exponential stability of impulsive systems with stabilizing and destabilizing impulses, see [16, 17, 13, 18, 19] and the references cited therein. One of the typical properties of asymptotic stability is that the solution converges to an equilibrium state as

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time approaches infinity. Many physical and aeronautical problems, on the other hand, require the system's state to be converged in a finite time. For example, in the missile interception defense systems, the intercepting missiles have to track the coming ones at the right location in a finite time; see [20]. Furthermore, many coupled oscillator models in physics [21] must be stable in a finite time at a fixed location. The concept of FTS serves as a particular case of asymptotic stability in which there exists a finite time (a settling-time function involving an initial condition) for which the solution converges to an equilibrium state. Recently, many interesting results on FTS have been reported for different types of systems; see [20, 22, 23, 24] and the references cited therein. For instance, in [20], a Lyapunov based theory for the FTS of continuous autonomous systems has been developed, which is a basic technique to analyze the FTS of nonlinear systems. Many FTS results for different systems, such as the non-autonomous system [25], switched system [26, 27], delay system [22], impulsive system [28, 29, 30, 31, 32, 33], stochastic system [34], and hybrid system [35] have been investigated. Later, in [33], FTS conditions for a nonlinear impulsive system with state-dependent impulses were derived. To the best of the authors' knowledge, only a few FTS results of nonlinear impulsive systems with time-scheduled impulses have been studied in [28, 29, 30, 31, 32]. One of the main issues in the study of FTS of impulsive systems is investigating the effects of stabilizing and destabilizing impulses on settling-time functions and constructing suitable impulsive sequences. In [28], authors considered their effects and established Lyapunov-based FTS theorems that include estimation of the settling-time functions. They developed a technique to design suitable impulsive sequences such that the FTS system attains the desired estimation of settling time. Motivated by [28], the authors extended the existing results to more general Lyapunov-based FTS theorems in [29, 30, 32], and estimated the corresponding settling-time functions. However, in the mentioned works, there are two drawbacks: the first is that there must be an upper or lower bound on the impulsive interval, and the second is that impulsive strength depends on the continuous dynamics of the system. To unify the upper and lower bounds, the concept of ADT has been proposed in [36]. This concept can be used to deal with stabilizing and destabilizing impulses in a more effective way when studying their effects on the dynamics of impulsive systems. In [31], ADT property has been applied to obtain a unified FTS criteria for a discontinuous impulsive differential equation under the framework of differential inclusion.

A settling-time function in FTS results is dependent on an initial value, which greatly limits its applications in real practice. Therefore, a new definition of FxTS in which the settling-time function does not depend on an initial value was originally proposed in [37]. Later, FxTS of uncertain linear systems has been investigated in [38]. A novel concept of *implicit* Lyapunov function approach was proposed in [39] to study FTS/FxTS of nonlinear systems. Several authors have also expanded upon the FxTS results for fractional order impulsive differential equations, see [40, 41] and the references cited therein. The Lyapunov-based necessary and sufficient conditions for autonomous dynamical systems to be FxTS have been investigated in [42]. In [43], the authors proposed a new Lyapunov inequality with time-varying coefficient terms and derived sufficient conditions for FxTS of nonlinear impulsive systems. They applied the two-stage comparison principle method with the ADT property in derivation. In the previous results of FxTS/FTS, the authors

considered $\xi = \eta^{\frac{1}{(q_1-1)(1-q_2)}}$ in the discrete part of system (1), see [43, 28, 29, 30, 31, 32] and the references cited there. This implies that the impulse depends on the values of q_1 and q_2 . We can say that the dynamics of continuous part of system (1) are affecting the dynamics of discrete one. However, to the best of our knowledge, a few results [44, 45, 46, 47] on FxTS of impulsive systems have been published that deal with the impulsive gain independent of the system's continuous sub-part. For the first time, in [44], FxTS of impulsive dynamical systems with stabilizing impulses has been investigated, in which *impulsive strength* is independent of the continuous-time behaviors of the system. Unfortunately, the authors claimed that FxTS of the same system can not be possible for *destabilizing impulses*. In [45], the authors considered *destabilizing impulses with the systems and investigated FxTS results. Thus, one of the main contributions in this paper is that we have extended the FxTS results of [44] for destabilizing impulses.*

Motivated by the above-mentioned discussions, this paper deals with FTS/nFxTS of nonlinear impulsive systems with *destabilizing* impulses. The fixed-time stability to a ball is known as nFxTS [48]. The notion of nFxTS is conceptually different from FxTS as defined in [37]. The first one may be just asymptotically stable, but the second one must be FTS. The definitions of nFxTS and FxTS are given below. The main results of this paper have been classified into two cases depending on the initial state of the system: (i) the first case when the initial state belongs to δ -nbh of the origin, and (ii) the second case when the initial state belongs to the outside region of δ -nbh of the origin. For both cases, the results of FTS/nFxTS for nonlinear impulsive systems with impulse perturbation have been derived separately with DT and ADT properties of impulsive sequences based on the Lyapunov inequality with linear terms. The following results have been obtained for DT and ADT cases:

- For DT case (below ξ and m_3 are some parameters characterising discrete and continuous-time dynamics of the system, respectively, their precise definition is given later.)
 - If the system starts from δ -nbh of the origin, *for any* $\delta > 0$, and DT satisfies $\tau_{min} > \frac{\ln \xi}{m_3}$ then the system achieves FTS. On the other hand, if DT satisfies $0 < \tau_{min} < \frac{\ln \xi}{m_3}$ then $\delta > 0$ must have an upper bound to achieve FTS.
 - If the system starts from *outside* of δ -nbh of the origin, *for any* $\delta > 0$, and DT satisfies $\tau_{min} > \frac{\ln \xi}{m_3}$ then the system achieves nFxTS. On the other hand, if DT satisfies $0 < \tau_{min} < \frac{\ln \xi}{m_3}$ then $\delta > 0$ must have lower bound for the system to converge into the ball of the origin. Due to having the restriction on $\delta > 0$, the system will not be nFxTS according to the definition.
- For ADT case
 - If the system starts from δ -nbh of the origin, *for any* $\delta > 0$, and ADT satisfies $\tau_a > \frac{\ln \xi}{m_3}$ then the system achieves FTS.
 - If the system starts from *outside* δ -nbh of the origin, *for any* $\delta > 0$, and ADT satisfies $\tau_a > \frac{\ln \xi}{m_3}$ then the system achieves nFxTS.

The obtained results in this paper have been applied to the FTS/nFxTS synchronization of impulsive neural networks with *destabilizing* impulses. A sufficient condition has been derived to guarantee FTS/nFxTS synchronization between drive-response systems. Finally, the efficiency of the obtained results is numerically verified. The remaining parts of the paper are organized as follows.

The problem of this paper, with some essential preliminaries, is formulated in Section 2. The FTS/FxTS results with DT and ADT properties are derived in Section 3. Section 4 contains an application of the main results of this paper to the FTS/nFxTS synchronization of impulsive neural networks. Section 5 includes a numerical example to demonstrate the obtained results. Section 6 contains the conclusions of this paper.

2. Preliminaries and Problem Formulation

Let \mathbb{R} , \mathbb{R}_+ , and \mathbb{R}^n denote a set of real numbers, a set of positive real numbers, and a n -dimensional vector space with $\|\cdot\|$, respectively. Let \mathbb{N} denotes a set of natural numbers without zero. $\mathbb{PC}([t_0, +\infty), \mathbb{R}^n)$ is a class of piecewise continuous functions having at most a finite number of jumps defined in $[t_0, \infty)$. Let \mathcal{K} denotes a set of continuous and strictly increasing function $\omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which satisfies $\omega(0) = 0$. Moreover, \mathcal{K}_∞ is a subset of \mathcal{K} which contains unbounded functions, i.e., $\lim_{s \rightarrow +\infty} \omega(s) = +\infty$.

Consider a nonlinear impulsive system as

$$\begin{cases} \dot{x}(t) = f(t, x(t)), & t \neq t_k, \\ \Delta x(t) = I(t_k^-, x(t_k^-)), & k \in \mathbb{N}, \\ x(t_0) = x_0, \end{cases} \quad (1)$$

where $x(t) \in \mathbb{PC}([t_0, +\infty), \mathbb{R}^n)$ is the state of system (1), $t \geq t_0 \geq 0$. The function $f : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is quasi-bounded [49] in t and locally Lipschitz continuous with respect to $x(t)$. $I : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous function of both arguments. Also, $f(t, 0) = 0$ and $I(t, 0) = 0$ for any $t \in \mathbb{R}_+$. The set of impulsive sequences is denoted by $\mathcal{S} = \{t_k : k \in \mathbb{N}\}$, where $0 = t_0 < t_1 < t_2 < \dots < t_k < t_{k+1} \dots$ and $\lim_{k \rightarrow \infty} t_k = \infty$. $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^+) = \lim_{t \rightarrow t_k^+} x(t) = x(t_k)$ and $x(t_k^-) = \lim_{t \rightarrow t_k^-} x(t) \neq x(t_k)$, i.e., the state $x(t)$ is right-hand continuous but left-hand discontinuous at the impulsive instant t_k . On the basis of the solution existence theorem in [50], the solution of system (1) is defined as $x(t) = x(t, t_0, x_0)$ with initial value $x(t_0) = x_0$ for $t \in [t_0, t_1)$. After hitting the line of discontinuity at the impulsive instant $t_1 \in \mathcal{S}$, the solution will be $x(t_1) = x(t_1^-) + I(t_1^-, x(t_1^-))$ and $x(t) = x(t, t_1, x(t_1))$ for $t \in [t_1, t_2)$. By repeating the same process, one can obtain the solution of system (1) for any initial value $x_0 \in \mathbb{R}^n$. We assume that the solution exists globally under some appropriate conditions.

In order to formulate our main results, we need the following definitions.

Definition 2.1. The origin of system (1) is said to be locally FTS for the set of impulsive sequences \mathcal{S} if for any $\{t_k\} \in \mathcal{S}$:

- (i) the origin of system (1) is Lyapunov stable, and
- (ii) there exists an open neighborhood $U \subset \mathbb{R}^n$ of the origin such that any solution $x(t, t_0, x_0)$ starting from $(t_0, x_0) \in \mathbb{R}_+ \times U$ converges to zero in finite-time, i.e., there exists $T^{t_0, x_0, \{t_k\}} \in \mathbb{R}_+$ such that $\|x(t, t_0, x_0)\| = 0$ for all $t > t_0 + T^{t_0, x_0, \{t_k\}}$, where $T(t_0, x_0, \{t_k\}) = \inf\{T^{t_0, x_0, \{t_k\}} \geq 0 : \|x(t, t_0, x_0)\| = 0 \forall t > t_0 + T^{t_0, x_0, \{t_k\}}\}$ is the settling-time function for system (1). If $U = \mathbb{R}^n$ then the origin of system (1) is global FTS.

Definition 2.2. The origin of system (1) is said to be nFxTS for the set of impulsive sequences \mathcal{S} , if for any $\{t_k\} \in \mathcal{S}$ the origin of system (1) is Lyapunov stable, and for any $\delta > 0, t_0 \in \mathbb{R}_+$, there exists $T^{t_0, \delta, \{t_k\}} \in \mathbb{R}_+$ such that $\|x(t, t_0, x_0)\| \leq \delta$ for all $t > t_0 + T^{t_0, \delta, \{t_k\}}$ and $x_0 \in U$.

Definition 2.3. In the particular case of Definition 2.2, if there exists $T^{t_0, \{t_k\}} \in \mathbb{R}_+$ for any $t_0 \in \mathbb{R}_+$ such that $\|x(t, t_0, x_0)\| = 0$ for all $t > t_0 + T^{t_0, \{t_k\}}$ and $x_0 \in U$, then system (1) is called FxTS.

In Definitions 2.1, 2.2, and 2.3, if we restrict the settling-time to be independent of $t_0 \geq 0$, then the FTS, nFxTS, and FxTS are called *uniform*. In this paper, we consider uniform stability notion; otherwise, $t_0 > 0$.

From the definitions of nFxTS and FxTS, it is clear that both are conceptually different. The former one may be just asymptotically stable at the origin with $\sup_{\delta > 0} T^{t_0, \delta, \{t_k\}} \leq \infty$, but a FxTS system must be FTS at the origin.

Definition 2.4. [51] The upper Dini derivative of $W : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ is defined as

$$D^+ W(t, x) = \limsup_{h \rightarrow 0^+} \frac{W(t+h, x+hf(t, x)) - W(t, x)}{h} \text{ for all } t \in [t_k, t_{k+1}),$$

where derivative is taken along the solution of system (1).

As we know that the dynamical behaviors of impulsive systems are subject to the class of impulsive sequences. Therefore, the FTS, nFxTS, and FxTS of system (1) also depend on the properties of impulsive sequence belonging to the set \mathcal{S} . The following definition restricts the impulsive sequences contained in the set \mathcal{S} .

Definition 2.5. For any impulsive sequence $\{t_k\}$, satisfying $t_{k+1} - t_k \geq \tau_k > 0$ for all $k \in \mathbb{N} \cup \{0\}$ denote $\tau_{min} = \min_{k \in \mathbb{N} \cup \{0\}} \tau_k$ for $k \in \mathbb{N}$. If for a sequence $\{t_k\} \in \mathcal{S}$ there exist positive numbers $N_0 \geq 1$ and $\tau_a > 0$ such that

$$N_S(t_0, T) \leq N_0 + \frac{T - t_0}{\tau_a}, \quad (2)$$

where $N_S(t_0, T)$ is the number of impulsive points in $[t_0, T)$, then τ_a is called ADT of the impulsive sequence. If $N_0 = 1$ then τ_a is called the DT of the impulsive sequence.

Remark 1. According to Definition 2.5, the solution of system (1) has jump kind of discontinuities at the impulsive instants which are separated by intervals of time of length τ_k for $k \in \mathbb{N}$. For the case $N_0 = 1$ in inequality (2), $\tau_a = \tau_{min} \leq \tau_k, \forall k \in \mathbb{N} \cup \{0\}$, that is, the consecutive impulsive points of sequences satisfying inequality (2) for this case will be separated by at least τ_a for all $k \in \mathbb{N}$. Since system (1) will be experiencing only destabilizing impulses, the performance of the solution depends on how often or seldom the impulses occur. A small value of average (dwell time) implies often occurrence of impulses that affects the system's performance adversely. Therefore, one of our primary focus in this paper is to find the restrictions on τ_{min} and τ_a so that the system (1) be the FTS, nFxTS, and FxTS uniformly for the class of impulsive sequences belonging to the set \mathcal{S} .

3. FTS/nFxTS theorems of nonlinear impulsive systems with DT and ADT properties

In this section, we will be interested in deriving a sufficient condition for uniform FTS of system (1) for the class of impulsive sequences from the set \mathcal{S} possessing the DT property under two variants of restriction on τ_{min} .

Theorem 3.1. For any impulsive sequence of set \mathcal{S} with DT, if there exist two \mathcal{K}_∞ functions ω_1, ω_2 , and a locally Lipschitz Lyapunov function $W : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that

$$(A_1) \quad \omega_1(\|x\|) \leq W(t, x(t)) \leq w_2(\|x\|) \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n,$$

(A₂) when $t \neq t_k$,

$$D^+W(t, x) \leq -m_1W^{q_1}(t, x) - m_2W^{q_2}(t, x) - m_3W(t, x), \quad (3)$$

where $q_1 > 1, 0 < q_2 < 1$, and $m_i > 0$ for $i = 1, 2, 3$,

(A₃) when $t = t_k$,

$$W(t_k, x(t_k)) \leq \xi W(t_k^-, x(t_k^-)),$$

where $\xi > 1$,

then the origin of system (1) is locally FTS with $W(t_0, x_0) \leq \delta$ provided that

Case 1 If $\tau_{\min} > \frac{\ln \xi}{m_3}$, then there exists a maximal number of impulsive points N_1 , for any $\delta > 0$, after which the solution of system (1) converges to zero, i.e.,

$$\|x(t, t_0, x_0)\| = 0 \text{ for } t > t_{N_1}, \quad (4)$$

and the settling-time function is estimated as

$$T(t_0, x_0, \{t_k\}) \leq \frac{\tau_{\min} \ln \left(1 + \frac{\mu_3(\xi^{q_2-1} e^{\mu_3 \tau_{\min}} - 1) W^{1-q_2}(0)}{\mu_2(e^{\mu_3 \tau_{\min}} - 1)} \right)}{\ln(\xi^{q_2-1} e^{\mu_3 \tau_{\min}})},$$

and the number of impulsive points $N_1 = \left\lceil \frac{\ln \left(1 + \frac{\mu_3(\xi^{q_2-1} e^{\mu_3 \tau_{\min}} - 1) \delta^{1-q_2}}{\mu_2(e^{\mu_3 \tau_{\min}} - 1)} \right)}{\ln(\xi^{q_2-1} e^{\mu_3 \tau_{\min}})} \right\rceil$, where $\lceil \cdot \rceil$ is a ceiling function and $\mu_i = (1 - q_2)m_i > 0$ for $i = 2, 3$; $W(t) = W(t, x(t))$.

Case 2 If $0 < \tau_{\min} < \frac{\ln \xi}{m_3}$ and $\delta < M$ for $M = \left(\frac{\mu_2(e^{\mu_3 \tau_{\min}} - 1)}{\mu_3(1 - \xi^{q_2-1} e^{\mu_3 \tau_{\min}})} \right)^{\frac{1}{1-q_2}}$, then there exists a maximal finite number of impulsive points N_2 after which the solution of system (1) converges to zero, i.e., equality (4) holds for $t > t_{N_2}$, and the settling-time function is estimated as

$$T(t_0, x_0, \{t_k\}) \leq \frac{\tau_{\min} \ln \left(1 - \frac{\mu_3(1 - \xi^{q_2-1} e^{\mu_3 \tau_{\min}}) W^{1-q_2}(0)}{\mu_2(e^{\mu_3 \tau_{\min}} - 1)} \right)}{\ln(\xi^{q_2-1} e^{\mu_3 \tau_{\min}})}.$$

The number of impulsive points $N_2 = \left\lceil \frac{\ln \left(1 - \frac{\mu_3(1 - \xi^{q_2-1} e^{\mu_3 \tau_{\min}}) \delta^{1-q_2}}{\mu_2(e^{\mu_3 \tau_{\min}} - 1)} \right)}{\ln(\xi^{q_2-1} e^{\mu_3 \tau_{\min}})} \right\rceil$.

Proof. For any impulsive sequence of the set \mathcal{S} and $W(0)$ close to the origin, i.e., $W(0) \leq \delta$ for any $\delta > 0$, the dominating term in inequality (A₂) will be $-m_2W^{q_2}(t, x)$. Thus, we have

$$D^+W(t) \leq -m_2W^{q_2}(t) - m_3W(t), \text{ for } t \neq t_k,$$

and

$$W(t_k) \leq \xi W(t_k^-) \text{ for } t = t_k.$$

Using the transformation $U(t) = W^{1-q_2}(t)$, one can get the following impulsive linear differential equation as

$$\begin{cases} D^+U(t) \leq -(1 - q_2)(m_2 + m_3U(t)), & t \neq t_k, \\ U(t_k) \leq \xi^{1-q_2}U(t_k^-), \\ U(0) \leq \delta^{1-q_2}. \end{cases} \quad (5)$$

Solution of system (5) for $t \in [0, t_1)$ is

$$U(t) \leq e^{-\mu_3 t} U(0) - \mu_2 \int_0^t e^{\mu_3(\tau-t)} d\tau = e^{-\mu_3 t} U(0) - \frac{\mu_2(1 - e^{-\mu_3 t})}{\mu_3}.$$

For $t = t_1$,

$$U(t_1) \leq \xi^{1-q_2} U(t_1^-) \leq \xi^{1-q_2} \left[e^{-\mu_3 t_1} U(0) - \frac{\mu_2(1 - e^{-\mu_3 t_1})}{\mu_3} \right].$$

Solution of system (5) propagates further for $t \in [t_1, t_2)$ as

$$\begin{aligned} U(t) &\leq e^{-\mu_3(t-t_1)} U(t_1) - \frac{\mu_2(1 - e^{\mu_3(t_1-t)})}{\mu_3} \\ &\leq e^{-\mu_3(t-t_1)} \left[\xi^{1-q_2} \left(e^{-\mu_3 t_1} U(0) - \frac{\mu_2(1 - e^{-\mu_3 t_1})}{\mu_3} \right) - \frac{\mu_2(1 - e^{\mu_3(t_1-t)})}{\mu_3} \right]. \end{aligned}$$

For $t = t_2$, we get

$$U(t_2) \leq \xi^{1-q_2} U(t_2^-) \leq \xi^{1-q_2} \left[\xi^{1-q_2} \left(e^{-\mu_3 t_2} U(0) - \frac{\mu_2 e^{\mu_3(t_1-t_2)}(1 - e^{-\mu_3 t_1})}{\mu_3} \right) - \frac{\mu_2(1 - e^{\mu_3(t_1-t_2)})}{\mu_3} \right].$$

By using mathematical induction, for $t = t_k$, we obtain

$$U(t_k) \leq \xi^{1-q_2} U(t_k^-) \leq \xi^{k(1-q_2)} e^{-\mu_3 t_k} U(0) - \frac{\mu_2}{\mu_3} \sum_{j=1}^k [\xi^{j(1-q_2)} e^{\mu_3(t_{k-j+1}-t_k)} (1 - e^{\mu_3(t_{k-j}-t_{k-j+1})})]. \quad (6)$$

To simplify inequality (6), we use

$$e^{\mu_3(k-j+1)\tau_{min}} \leq e^{\mu_3 t_{k-j+1}}, \quad 1 - e^{-\mu_3 \tau_{min}} \leq 1 - e^{\mu_3(t_{k-j}-t_{k-j+1})}, \quad \text{and} \quad e^{-\mu_3 t_k} \leq e^{-\mu_3 k \tau_{min}}, \quad (7)$$

to obtain

$$U(t_k) \leq e^{-k\mu_3 \tau_{min}} \left[\xi^{k(1-q_2)} U(0) - \frac{\mu_2}{\mu_3} (1 - e^{-\mu_3 \tau_{min}}) \sum_{j=1}^k (\xi^{j(1-q_2)} e^{\mu_3(k-j+1)\tau_{min}}) \right].$$

The inequality for $t \in [t_k, t_{k+1})$, $k \in \mathbb{N} \cup \{0\}$, is as follows

$$U(t) \leq e^{-\mu_3(t-t_k)} U(t_k) - \frac{\mu_2(1 - e^{\mu_3(t_k-t)})}{\mu_3} \leq e^{-\mu_3(t-t_k)} U(t_k). \quad (8)$$

By using $U(t) = W^{1-q_2}(t)$, the inequalities (6) and (8) can be written in the form of $W(t)$ as

$$W(t) \leq e^{-\frac{\mu_3}{1-q_2}(t-t_k)} W(t_k), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N} \cup \{0\}. \quad (9)$$

For $t = t_k$, $k \geq 1$, we have

$$W(t_k) \leq e^{-\frac{k\mu_3 \tau_{min}}{1-q_2}} \xi^k \left[W^{1-q_2}(0) - \frac{\mu_2}{\mu_3} (e^{\mu_3 \tau_{min}} - 1) \sum_{j=1}^k (\xi^{q_2-1} e^{\mu_3 \tau_{min}})^{k-j} \right]^{\frac{1}{1-q_2}}. \quad (10)$$

In order to show $W(t_k)$ is converging as k increases, we have to simplify the right-hand side of inequality (10).

Case 1 If $\tau_{min} > \frac{\ln \xi}{m_3}$, then inequality (10) can be written as

$$W(t_k) \leq \underbrace{e^{-\frac{k\mu_3 \tau_{min}}{1-q_2}} \xi^k}_{\text{first factor}} \left[\underbrace{W^{1-q_2}(0) - \frac{\mu_2}{\mu_3} (e^{\mu_3 \tau_{min}} - 1) \frac{(\xi^{q_2-1} e^{\mu_3 \tau_{min}})^k - 1}{\xi^{q_2-1} e^{\mu_3 \tau_{min}} - 1}}_{\text{second factor}} \right]^{\frac{1}{1-q_2}}, \quad k \geq 1, \quad (11)$$

where $\sum_{j=1}^k (\xi^{q_2-1} e^{\mu_3 \tau_{min}})^{k-j} = \frac{(\xi^{q_2-1} e^{\mu_3 \tau_{min}})^k - 1}{\xi^{q_2-1} e^{\mu_3 \tau_{min}} - 1}$. It can be observed from inequality (11) that the convergence of $W(t_k)$ depends on two factors for $k \geq 1$: the first factor decreases exponentially as k increases, and the second factor contains a fixed term $W^{1-q_2}(0)$ and an expression dependent on k . There exists k such that the second factor of inequality (11) becomes negative, implying that $W(t_k)$ eventually converges to zero. For the continuous-time behavior of system (1), substitute inequality (10) in (9) then we get

$$\begin{aligned} W(t) &\leq e^{-\frac{k\mu_3\tau_{min}}{1-q_2}} \xi^k \left[W^{1-q_2}(0) - \frac{\mu_2}{\mu_3} (e^{\mu_3\tau_{min}} - 1) \sum_{j=1}^k (\xi^{q_2-1} e^{\mu_3\tau_{min}})^{k-j} \right]^{\frac{1}{1-q_2}} e^{-\frac{\mu_3}{1-q_2}(t-t_k)} \\ &\leq e^{-\frac{k\mu_3\tau_{min}}{1-q_2}} \xi^k W(0) e^{-\frac{\mu_3}{1-q_2}(t-t_k)} \\ &\leq W(0), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N} \cup \{0\}. \end{aligned} \quad (12)$$

From assumption (A₁) and inequality (12), we can obtain the following: $\|x(t)\| \leq \omega_1^{-1}(W(0))$ for all $t \in [t_0, \infty)$; hence, the solution of system (1) is Lyapunov stable. Now, we will proceed to derive settling-time function for system (1) to be FTS. From inequality (12), we can obtain that for any $\delta > 0$, $\mu_2 > 0$, $\mu_3 > 0$, $\xi > 1$, $0 < q_2 < 1$, there exists k such that whenever

$$\delta^{1-q_2} \leq \frac{\mu_2}{\mu_3} (e^{\mu_3\tau_{min}} - 1) \frac{(\xi^{q_2-1} e^{\mu_3\tau_{min}})^k - 1}{\xi^{q_2-1} e^{\mu_3\tau_{min}} - 1}, \quad (13)$$

holds, this implies $W(t) = 0$ for $t > t_k$. Where k is estimated as

$$k \geq \frac{\ln \left(1 + \frac{\mu_3(\xi^{q_2-1} e^{\mu_3\tau_{min}} - 1)\delta^{1-q_2}}{\mu_2(e^{\mu_3\tau_{min}} - 1)} \right)}{\ln(\xi^{q_2-1} e^{\mu_3\tau_{min}})}.$$

For integer value of k , obtain $N_1 = \left\lceil \frac{\ln \left(1 + \frac{\mu_3(\xi^{q_2-1} e^{\mu_3\tau_{min}} - 1)\delta^{1-q_2}}{\mu_2(e^{\mu_3\tau_{min}} - 1)} \right)}{\ln(\xi^{q_2-1} e^{\mu_3\tau_{min}})} \right\rceil$. Thus, after having N_1 number of impulsive jumps, we have

$$W(t) = 0, \quad \text{for } t \geq T(t_0, x_0, \{t_k\}) \geq t_{N_1},$$

where $T(t_0, x_0, \{t_k\})$ is estimated as

$$T(t_0, x_0, \{t_k\}) \leq \frac{\tau_{min} \ln \left(1 + \frac{\mu_3(\xi^{q_2-1} e^{\mu_3\tau_{min}} - 1)W^{1-q_2}(0)}{\mu_2(e^{\mu_3\tau_{min}} - 1)} \right)}{\ln(\xi^{q_2-1} e^{\mu_3\tau_{min}})}.$$

Case 2 If $0 < \tau_{min} < \frac{\ln \xi}{m_3}$, then inequality (10) can be written as

$$W(t_k) \leq \underbrace{e^{-\frac{k\mu_3\tau_{min}}{1-q_2}} \xi^k}_{\text{first factor}} \left[\underbrace{W^{1-q_2}(0) - \frac{\mu_2}{\mu_3} (e^{\mu_3\tau_{min}} - 1) \frac{1 - (\xi^{q_2-1} e^{\mu_3\tau_{min}})^k}{1 - \xi^{q_2-1} e^{\mu_3\tau_{min}}}}_{\text{second factor}} \right]^{\frac{1}{1-q_2}}, \quad k \geq 1, \quad (14)$$

$$\leq \Lambda \left[W^{1-q_2}(0) - \frac{\mu_2}{\mu_3} (e^{\mu_3\tau_{min}} - 1) \frac{1 - (\xi^{q_2-1} e^{\mu_3\tau_{min}})^k}{1 - \xi^{q_2-1} e^{\mu_3\tau_{min}}} \right]^{\frac{1}{1-q_2}}, \quad (15)$$

where $\sum_{j=1}^k (\xi^{q_2-1} e^{\mu_3\tau_{min}})^{k-j} = \frac{1 - (\xi^{q_2-1} e^{\mu_3\tau_{min}})^k}{1 - \xi^{q_2-1} e^{\mu_3\tau_{min}}}$ and $e^{-\frac{k\mu_3\tau_{min}}{1-q_2}} \xi^k \leq \Lambda$ for some $k \in \mathbb{N}$. As in Case 1, inequality (14) shows that the convergence of sequence $W(t_k)$ in Case 2 also depends on two factors. Contrary to inequality (11), the first factor in the right-hand side of inequality (14) increases as k increases, but there exists a finite value of $k \in \mathbb{N}$ such

that the second factor becomes negative, then the first factor will have an upper bound Λ . Therefore, sequence $W(t_k)$ must converge eventually to zero.

For the continuous-time behavior of system (1), substituting inequality (15) in (9) then we get

$$W(t) \leq \Lambda \left[W^{1-q_2}(0) - \frac{\mu_2}{\mu_3} (e^{\mu_3 \tau_{min}} - 1) \sum_{j=1}^k (\xi^{q_2-1} e^{\mu_3 \tau_{min}})^{k-j} \right]^{\frac{1}{1-q_2}} e^{-\frac{\mu_3}{1-q_2}(t-t_k)} \quad (16)$$

$$\leq \Lambda W(0), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{N} \cup \{0\}. \quad (17)$$

From assumption (A_1) and inequality (17), the solution of system (1) is Lyapunov stable. We will proceed to derive the settling-time function for system (1) to be FTS under Case 2. Returning to inequality (15), there exists k such that whenever

$$\delta^{1-q_2} \leq \frac{\mu_2}{\mu_3} (e^{\mu_3 \tau_{min}} - 1) \frac{1 - (\xi^{q_2-1} e^{\mu_3 \tau_{min}})^k}{1 - \xi^{q_2-1} e^{\mu_3 \tau_{min}}}, \quad (18)$$

holds, this implies $W(t) = 0$ for $t > t_k$. It can be observed that the right-hand side of inequality (18) is bounded, i.e.,

there exists $M > 0$ such that $\delta < M = \left(\frac{\mu_2(e^{\mu_3 \tau_{min}} - 1)}{\mu_3(1 - \xi^{q_2-1} e^{\mu_3 \tau_{min}})} \right)^{\frac{1}{1-q_2}}$ for which k is estimated as

$$k \geq \frac{\ln \left(1 - \frac{\mu_3(1 - \xi^{q_2-1} e^{\mu_3 \tau_{min}}) \delta^{1-q_2}}{\mu_2(e^{\mu_3 \tau_{min}} - 1)} \right)}{\ln(\xi^{q_2-1} e^{\mu_3 \tau_{min}})}.$$

For integer value k , obtain $N_2 = \left\lceil \frac{\ln \left(1 - \frac{\mu_3(1 - \xi^{q_2-1} e^{\mu_3 \tau_{min}}) \delta^{1-q_2}}{\mu_2(e^{\mu_3 \tau_{min}} - 1)} \right)}{\ln(\xi^{q_2-1} e^{\mu_3 \tau_{min}})} \right\rceil$. The settling-time function $T(t_0, x_0, \{t_k\})$ is estimated as

$$T(t_0, x_0, \{t_k\}) \leq \frac{\tau_{min} \ln \left(1 - \frac{\mu_3(1 - \xi^{q_2-1} e^{\mu_3 \tau_{min}}) W^{1-q_2}(0)}{\mu_2(e^{\mu_3 \tau_{min}} - 1)} \right)}{\ln(\xi^{q_2-1} e^{\mu_3 \tau_{min}})}.$$

Hence, the proof is completed. \square

Remark 2. It is worth to note that in Theorem 3.1, the domain for initial value of system (1) depends on the restrictions of τ_{min} . Specifically, system (1) can start from a δ -nbh for any $\delta > 0$ and achieve FTS when τ_{min} is large, i.e., $\tau_{min} > \frac{\ln \xi}{m_3}$ whereas $\delta > 0$ must be bounded when τ_{min} is small, i.e., $0 < \tau_{min} < \frac{\ln \xi}{m_3}$.

The results in Theorem 3.1 are derived for the solution of system (1) that starts *within* a δ -nbh of the origin, where $\delta > 0$ can be any value or there exists an upper bound of δ such that system (1) achieves FTS. Now, we proceed to obtain the results for nFxTS when the solution of system (1) starts from *outside* of δ -nbh. In such a case, we are looking for a finite-time for which the solution converges within δ -nbh of the origin uniformly with respect to initial value.

Theorem 3.2. Assume that Assumptions (A_1) , (A_2) , and (A_3) of Theorem 3.1 are true. Then, for any impulsive sequence of set S with DT, the origin of system (1) is nFxTS with $W(t_0, x_0) \geq \delta$, provided that

Case 1 If $\tau_{min} > \frac{\ln \xi}{m_3}$ then for any $\delta > 0$ there exists $M_1 \in \mathbb{N}$ such that the solution of system (1) converges eventually uniformly into a δ -nbh of the origin after having M_1 impulsive jumps, i.e.,

$$\|x(t, t_0, x_0)\| \leq \delta \text{ for } t > t_{M_1},$$

where $M_1 = \left\lceil \frac{\ln \left(1 + \frac{v_3(\xi^{1-q_1} e^{v_3 \tau_{min}} - 1) \delta^{1-q_1}}{v_1(1 - e^{-v_3 \tau_{min}})} \right)}{\ln(\xi^{1-q_1} e^{v_3 \tau_{min}})} \right\rceil$. The settling-time function for system (1) to be nFxTS is estimated as

$$T(t_0, \delta, \{t_k\}) \leq \frac{\tau_{min} \ln \left(1 + \frac{v_3(\xi^{1-q_1} e^{v_3 \tau_{min}} - 1) \delta^{1-q_1}}{v_1(1 - e^{-v_3 \tau_{min}})} \right)}{\ln(\xi^{1-q_1} e^{v_3 \tau_{min}})},$$

where $v_i = (q_1 - 1)m_i$ for $i = 1, 2, 3$.

Case 2 If $0 < \tau_{min} < \frac{\ln \xi}{m_3}$ and $\delta > \bar{M}$ for $\bar{M} = \left(\frac{v_1(1 - e^{-v_3 \tau_{min}})}{v_3(1 - \xi^{1-q_1} e^{v_3 \tau_{min}})} \right)^{\frac{1}{1-q_1}}$, then there exists $M_2 \in \mathbb{N}$ such that the solution of system (1) converges into a ball of radius bigger than \bar{M} in a fixed time after having M_2 number of impulsive jumps. The settling-time function is estimated as

$$T(t_0, \delta, \{t_k\}) \leq \frac{\tau_{min} \ln \left(1 - \frac{v_3(1 - \xi^{1-q_1} e^{v_3 \tau_{min}}) \delta^{1-q_1}}{v_1(1 - e^{-v_3 \tau_{min}})} \right)}{\ln(\xi^{1-q_1} e^{v_3 \tau_{min}})},$$

$$\text{and } M_2 = \left\lceil \frac{\ln \left(1 - \frac{v_3(1 - \xi^{1-q_1} e^{v_3 \tau_{min}}) \delta^{1-q_1}}{v_1(1 - e^{-v_3 \tau_{min}})} \right)}{\ln(\xi^{1-q_1} e^{v_3 \tau_{min}})} \right\rceil.$$

Proof. When the initial value of differential inequality (3) be far from the origin, i.e., $W(0) \geq \delta$ for any $\delta > 0$ then $-m_1 W^{q_1}(t)$ will be a dominating term in inequality (3). Thus, we get

$$D^+ W(t) \leq -m_1 W^{q_1}(t) - m_3 W(t), \text{ for } t \neq t_k,$$

and

$$W(t_k) \leq \xi W(t_k^-), \text{ for } t = t_k, \quad (19)$$

where inequality (19) is true for any impulsive sequence of set S . Suppose $U(t) = -W^{1-q_1}(t)$ then we obtain

$$\begin{cases} D^+ U(t) \leq (q_1 - 1)(m_3 U(t) - m_1), & t \neq t_k, \\ U(t_k) \leq \xi^{1-q_1} U(t_k^-), & k \geq 1, \\ U(0) \geq -\delta^{1-q_1}. \end{cases} \quad (20)$$

In the similar way as in Theorem 3.1, the solution of system (20) can be estimated for $t \in [t_k, t_{k+1})$ as

$$U(t) \leq e^{v_3(t-t_k)} U(t_k) + \frac{v_1}{v_3} [1 - e^{v_3(t-t_k)}] \leq e^{v_3(t-t_k)} U(t_k). \quad (21)$$

For $t = t_k$, we have

$$U(t_k) \leq \xi^{k(1-q_1)} e^{v_3 t_k} U(0) + \frac{v_1}{v_3} \sum_{j=1}^k \left[\xi^{j(1-q_1)} e^{v_3(t_k - t_{k-j+1})} [1 - e^{v_3(t_{k-j+1} - t_{k-j})}] \right]. \quad (22)$$

Transforming inequality (22) in terms of $W(t_k)$ by using $U(t_k) = -W^{1-q_1}(t_k)$ for $k \geq 1$, we get

$$W^{q_1-1}(t_k) \leq \left[\xi^{k(1-q_1)} e^{v_3 t_k} W^{1-q_1}(0) + \frac{v_1}{v_3} \sum_{j=1}^k \left[\xi^{j(1-q_1)} e^{v_3(t_k - t_{k-j+1})} (e^{v_3(t_{k-j+1} - t_{k-j})} - 1) \right] \right]^{-1}.$$

Using the simplification (7) of Theorem 3.1, we obtain

$$W^{q_1-1}(t_k) \leq \left[\xi^{k(1-q_1)} e^{v_3 k \tau_{min}} W^{1-q_1}(0) + \frac{v_1}{v_3} (1 - e^{-v_3 \tau_{min}}) \sum_{j=1}^k (\xi^{1-q_1} e^{v_3 \tau_{min}})^j \right]^{-1}. \quad (23)$$

In order to show the uniform convergence of $W(t_k)$, for $k \geq 1$, we have to simplify inequality (23) for two cases.

Case 1 If $\tau_{min} > \frac{\ln \xi}{m_3}$, then inequality (23) can be written as

$$W^{q_1-1}(t_k) \leq \left[\xi^{k(1-q_1)} e^{v_3 k \tau_{min}} W^{1-q_1}(0) + \frac{v_1}{v_3} (1 - e^{-v_3 \tau_{min}}) \frac{(\xi^{1-q_1} e^{v_3 \tau_{min}})^k - 1}{\xi^{1-q_1} e^{v_3 \tau_{min}} - 1} \right]^{-1}$$

where $\sum_{j=1}^k (\xi^{1-q_1} e^{v_3 \tau_{min}})^j = \frac{(\xi^{1-q_1} e^{v_3 \tau_{min}})^{k+1} - 1}{\xi^{1-q_1} e^{v_3 \tau_{min}} - 1}$. For any $\delta > 0$, there exists $M_1 \in \mathbb{N}$ such that whenever

$$1 + \frac{v_3 \delta^{1-q_1}}{v_1 (1 - e^{-v_3 \tau_{min}})} \leq \sum_{j=0}^k (\xi^{1-q_1} e^{v_3 \tau_{min}})^j = \frac{(\xi^{1-q_1} e^{v_3 \tau_{min}})^{k+1} - 1}{\xi^{1-q_1} e^{v_3 \tau_{min}} - 1} \quad (24)$$

holds, this implies

$$W(t_k) \leq \delta, \quad \forall k \geq M_1. \quad (25)$$

Inequality (25) shows that sequence $W(t_k)$ is converging eventually uniformly into δ -nbh of the origin. For the continuous-time behavior of $W(t)$, we have the following from inequality (21)

$$W^{q_1-1}(t) \leq e^{-v_3(t-t_k)} W^{q_1-1}(t_k), \quad t \in [t_k, t_{k+1}). \quad (26)$$

Substituting inequality (23) in (26), we obtain

$$W^{q_1-1}(t) \leq e^{-v_3(t-t_k)} \left[\xi^{k(1-q_1)} e^{v_3 k \tau_{min}} W^{1-q_1}(0) + \frac{v_1}{v_3} (1 - e^{-v_3 \tau_{min}}) \frac{(\xi^{1-q_1} e^{v_3 \tau_{min}})^k - 1}{\xi^{1-q_1} e^{v_3 \tau_{min}} - 1} \right]^{-1} \quad (27)$$

$$\leq e^{-v_3(t-t_k)} \xi^{k(q_1-1)} e^{-v_3 k \tau_{min}} W^{q_1-1}(0). \quad (28)$$

Using the lower bound of τ_{min} , inequality (28) can be estimated as follows:

$$W(t) \leq W(0), \quad t \in [t_k, t_{k+1}), k \in \mathbb{N} \cup \{0\}. \quad (29)$$

From inequality (29) and assumption (A₁), the solution of system (1) is Lyapunov stable. It is concluded from inequalities (25) and (29) that the solution of system (1) is converging eventually uniformly into the domain of radius $\delta > 0$ containing the origin in finite-time. Hence, from Definition 2.2, system (1) is nFxTS. The number of impulsive jumps after which the solution of system (1) eventually enters into a δ -nbh of the origin can be estimated from inequality (24) as follows:

$$k \geq \frac{\ln \left(1 + \frac{v_3 (\xi^{1-q_1} e^{v_3 \tau_{min}} - 1) \delta^{1-q_1}}{v_1 (1 - e^{-v_3 \tau_{min}})} \right)}{\ln(\xi^{1-q_1} e^{v_3 \tau_{min}})}.$$

For integer value, obtain $M_1 = \left\lceil \frac{\ln \left(1 + \frac{v_3 (\xi^{1-q_1} e^{v_3 \tau_{min}} - 1) \delta^{1-q_1}}{v_1 (1 - e^{-v_3 \tau_{min}})} \right)}{\ln(\xi^{1-q_1} e^{v_3 \tau_{min}})} \right\rceil$. The settling-time function for nFxTS of system (1) is estimated as

$$T(t_0, \delta, \{t_k\}) \leq \frac{\tau_{min} \ln \left(1 + \frac{v_3 (\xi^{1-q_1} e^{v_3 \tau_{min}} - 1) \delta^{1-q_1}}{v_1 (1 - e^{-v_3 \tau_{min}})} \right)}{\ln(\xi^{1-q_1} e^{v_3 \tau_{min}})}.$$

Case 2 If $0 < \tau_{min} < \frac{\ln \xi}{m_3}$ the inequality (23) can be written as

$$W^{q_1-1}(t_k) \leq \left[\xi^{k(1-q_1)} e^{v_3 k \tau_{min}} W^{1-q_1}(0) + \frac{v_1}{v_3} (1 - e^{-v_3 \tau_{min}}) \frac{1 - (\xi^{1-q_1} e^{v_3 \tau_{min}})^k}{1 - \xi^{1-q_1} e^{v_3 \tau_{min}}} \right]^{-1},$$

where $\sum_{j=1}^k (\xi^{1-q_1} e^{v_3 \tau_{min}})^j = \frac{1 - (\xi^{1-q_1} e^{v_3 \tau_{min}})^k}{1 - \xi^{1-q_1} e^{v_3 \tau_{min}}}$. For sufficiently big $\delta > \bar{M}$ there exists $M_2 \in \mathbb{N}$ such that whenever

$$\delta^{1-q_1} \leq \frac{v_1}{v_3} (1 - e^{-v_3 \tau_{min}}) \frac{1 - (\xi^{1-q_1} e^{v_3 \tau_{min}})^k}{1 - \xi^{1-q_1} e^{v_3 \tau_{min}}}, \quad (30)$$

holds then $W(t_k) \leq \delta$ for all $k \geq M_2$, where $\bar{M} = \left(\frac{v_1(1 - e^{-v_3 \tau_{min}})}{v_3(1 - \xi^{1-q_1} e^{v_3 \tau_{min}})} \right)^{\frac{1}{1-q_1}}$. This implies $W(t_k)$ is converging eventually uniformly into δ -neighborhood of the origin. For the continuous-time behavior of $W(t)$, similar to Case 1, we have

$$W^{q_1-1}(t) \leq e^{-v_3(t-t_k)} \left[\xi^{k(1-q_1)} e^{v_3 k \tau_{min}} W^{1-q_1}(0) + \frac{v_1}{v_3} (1 - e^{-v_3 \tau_{min}}) \frac{1 - (\xi^{1-q_1} e^{v_3 \tau_{min}})^k}{1 - \xi^{1-q_1} e^{v_3 \tau_{min}}} \right]^{-1}.$$

From inequality (30), we obtain

$$W^{q_1-1}(t) \leq e^{-v_3(t-t_k)} \xi^{k(q_1-1)} e^{-v_3 k \tau_{min}} W^{q_1-1}(0),$$

this implies

$$W(t) \leq \Omega W(0), \quad \forall t \in [t_k, t_{k+1}), \quad k \in \mathbb{N} \cup \{0\}. \quad (31)$$

For a suitably defined Ω dependent on M_2 , from inequality (31) and assumption (A_1) , it follows that the solution of system (1) is Lyapunov stable. The finite-time convergence of $W(t_k)$ and inequality (31) conclude that the solution of system (1) is converging to a ball of radius bigger than \bar{M} . Using a similar approach as in Case 1, we can estimate the number of impulsive jumps as

$$k \geq \frac{\ln \left(1 - \frac{v_3(1 - \xi^{1-q_1} e^{v_3 \tau_{min}}) \delta^{1-q_1}}{v_1(1 - e^{-v_3 \tau_{min}})} \right)}{\ln(\xi^{1-q_1} e^{v_3 \tau_{min}})},$$

for integer value, obtain $M_2 = \left\lceil \frac{\ln \left(1 - \frac{v_3(1 - \xi^{1-q_1} e^{v_3 \tau_{min}}) \delta^{1-q_1}}{v_1(1 - e^{-v_3 \tau_{min}})} \right)}{\ln(\xi^{1-q_1} e^{v_3 \tau_{min}})} \right\rceil$. The settling-time function for fixed time convergence of the solution to a ball of the radius bigger than \bar{M} is estimated as

$$T(t_0, \delta, \{t_k\}) \leq \frac{\tau_{min} \ln \left(1 - \frac{v_3(1 - \xi^{1-q_1} e^{v_3 \tau_{min}}) \delta^{1-q_1}}{v_1(1 - e^{-v_3 \tau_{min}})} \right)}{\ln(\xi^{1-q_1} e^{v_3 \tau_{min}})}.$$

□

Remark 3. In the second case of Theorem 3.2, when the initial solution of system (1) starts from outside of a sufficiently big vicinity of the origin then the forward solution with a finite number of impulsive jumps converges uniformly to a ball of system (1) in a fixed time, provided that DT is chosen to be sufficiently small, i.e., $0 < \tau_{min} < \frac{\ln \xi}{m_3}$. This property is a weak version of nFxTS property.

Remark 4. The results obtained for any $\delta > 0$ in Case 1 of Theorem 3.1 and Theorem 3.2 show FTS and nFxTS of system (1) having impulses with DT property. In particular, for $\delta = 1$, the combined results of Theorem 3.1 and

Theorem 3.2 obtained in Case 1 will give us the result for FxTS of system (1). This implies that if $W(t_0, x_0) \geq 1$, then from inequality (24), one can obtain the fixed-time

$$T_1 = \frac{\tau_{min} \ln \left(1 + \frac{v_3(\xi^{1-q_1} e^{v_3 \tau_{min}} - 1)}{v_1(1 - e^{-v_3 \tau_{min}})} \right)}{\ln(\xi^{1-q_1} e^{v_3 \tau_{min}})},$$

such that the solution of system (1) will converge into the neighborhood of the origin with radius one, i.e., $W(t) \leq 1$ for $t \geq T_1$. Further, from inequality (13), one can obtain the fixed-time

$$T_2 = \frac{\tau_{min} \ln \left(1 + \frac{\mu_3(\xi^{q_2-1} e^{\mu_3 \tau_{min}} - 1)}{\mu_2(e^{\mu_3 \tau_{min}} - 1)} \right)}{\ln(\xi^{q_2-1} e^{\mu_3 \tau_{min}})},$$

such that $W(t) = 0$ for all $t \geq T_1 + T_2$. That is the fixed-settling time for FxTS of system (1), which is estimated as $T(t_0, \{t_k\}) = T_1 + T_2$.

The FTS/nFxTS results obtained in Theorems 3.1 and 3.2 are based on the class of impulsive sequences with DT property, i.e., the impulsive sequences defined in Definition 2.5 for $N_0 = 1$. In order to obtain the results based on ADT property as defined in Definition 2.5, we will use the result of the theorem given in [1] for a nonlinear impulsive system. This theorem is stated as follows:

Theorem 3.3. [1] Assume that the sequence $\{t_k\}$ is given, and $m(t) \in \mathbb{PC}([0, +\infty), \mathbb{R})$ is right-continuous at t_k for $k = 1, 2, \dots$ such that

$$\begin{cases} D^+ m(t) \leq p(t)m(t) + q(t), & t \neq t_k \geq t_0, \\ m(t_k) \leq d_k m(t_k^-) + b_k, & t = t_k, \end{cases}$$

where $p, q \in C([0, +\infty), \mathbb{R})$, $d_k \geq 0$ and $b_k \in \mathbb{R}$ are constants. Then,

$$\begin{aligned} m(t) \leq m(t_0) \prod_{t_0 < t_k < t} d_k \exp\left(\int_{t_0}^t p(s) ds\right) + \sum_{t_0 < t < t_k} \left(\prod_{t_0 < t_j < t} d_j \exp\left(\int_{t_k}^t p(s) ds\right) \right) b_k \\ + \int_{t_0}^t \prod_{s < t_k < t} d_k \exp\left(\int_s^t p(\sigma) d\sigma\right) q(s) ds, \quad t \geq t_0. \end{aligned}$$

Theorem 3.4. Assume that Assumptions (A₁), (A₂), and (A₃) of Theorem 3.1 are satisfied and $W(t_0, x_0) \leq \delta$ for any $\delta > 0$. The origin of system (1) is FTS for any impulsive sequence of set S with ADT property satisfying

$$\tau_a > \frac{\ln \xi}{m_3}. \quad (32)$$

The settling-time function for system (1) to be FTS is estimated as

$$T(t_0, x_0, \{t_k\}) \leq \frac{\ln \left[1 + \frac{m_3 \xi^{(1-q_2)N_0} W^{1-q_2}(0)}{m_2} \right]}{(1 - q_2) \left(m_3 - \frac{\ln \xi}{\tau_a} \right)}.$$

Proof. For any impulsive sequence of the set S with ADT property, if we replace $p(t) = -(1 - q_2)m_3$, $q(t) = -(1 - q_2)m_2$, $b_k = 0$, and $d_k = \xi^{1-q_2}$ in Theorem 3.3 then the solution of impulsive system (5) can be estimated as

$$U(t) \leq U(0)M(t, 0) - (1 - q_2)m_2 \int_0^t M(t, s) ds, \quad t \geq 0, \quad (33)$$

where $U(t) = W^{1-q_2}(t)$ as in Theorem 3.1, and

$$M(t, s) = \prod_{s < t_k < t} \xi^{1-q_2} e^{-(1-q_2)m_3(t-s)} = e^{-(1-q_2)m_3(t-s)} \xi^{(1-q_2)N_S(s,t)}. \quad (34)$$

Substituting (34) in inequality (33) and using Definition 2.5, we get

$$U(t) \leq U(0) \xi^{(1-q_2)N_S(0,t)} e^{-(1-q_2)m_3 t} - (1-q_2)m_2 \int_0^t e^{-(1-q_2)m_3(t-s)} \xi^{(1-q_2)N_S(s,t)} ds \quad (35)$$

$$\leq e^{-(1-q_2)m_3 t} \left[U(0) \xi^{(1-q_2)\left(N_0 + \frac{t}{\tau_a}\right)} - (1-q_2)m_2 \int_0^t e^{(1-q_2)m_3 s} ds \right]. \quad (36)$$

Using the transformation $U(t) = W^{1-q_2}(t)$ in inequality (36), we get

$$W(t) \leq e^{-m_3 t} \left[W^{1-q_2}(0) \xi^{(1-q_2)\left(N_0 + \frac{t}{\tau_a}\right)} - (1-q_2)m_2 \int_0^t e^{(1-q_2)m_3 s} ds \right]^{\frac{1}{1-q_2}} \quad (37)$$

$$\leq W(0) \xi^{N_0} e^{-\left(m_3 - \frac{\ln \xi}{\tau_a}\right)t}.$$

From inequality (32), we obtain

$$W(t) \leq \xi^{N_0} W(0), \quad t \geq 0. \quad (38)$$

From Assumption (A₁) and inequality (38), the solution of system (1) is Lyapunov stable. In order to demonstrate FTS of system (1), the right-hand side of inequality (37) must be negative after a finite-time. Hence, $W(t) = 0$ for all $t \geq T(t_0, x_0, \{t_k\})$, where the settling-time function $T(t_0, x_0, \{t_k\})$ is estimated by the following inequality.

$$W^{1-q_2}(0) \xi^{(1-q_2)\left(N_0 + \frac{t}{\tau_a}\right)} \leq (1-q_2)m_2 \int_0^t e^{(1-q_2)m_3 s} ds,$$

$$W^{1-q_2}(0) \xi^{(1-q_2)N_0} \xi^{(1-q_2)\frac{t}{\tau_a}} \leq (1-q_2)m_2 \int_0^t e^{(1-q_2)m_3 s} ds,$$

$$W^{1-q_2}(0) \xi^{(1-q_2)N_0} \leq \frac{m_2}{m_3} \left[e^{(1-q_2)\left(m_3 - \frac{\ln \xi}{\tau_a}\right)t} - 1 \right]. \quad (39)$$

From inequality (39), we get the upper bound on the settling-time function for system (1) to be FTS:

$$T(t_0, x_0, \{t_k\}) \leq \frac{\ln \left[1 + \frac{m_3 \xi^{(1-q_2)N_0} W^{1-q_2}(0)}{m_2} \right]}{(1-q_2)\left(m_3 - \frac{\ln \xi}{\tau_a}\right)}.$$

□

Theorem 3.5. Assume that the assumptions of Theorem 3.4 hold except $W(t_0, x_0) \geq \delta$ for any $\delta > 0$. If inequality (32) is true then the origin of system (1) is nFxTS for any impulsive sequence of set S with ADT property. The settling-time function for system (1) to be nFxTS is estimated as

$$T(t_0, \delta, \{t_k\}) \leq \frac{\ln \left[1 + \frac{\delta^{1-q_1} \xi^{(q_1-1)N_0} \left(m_3 - \frac{\ln \xi}{\tau_a}\right)}{m_1} \right]}{(q_1 - 1)\left(m_3 - \frac{\ln \xi}{\tau_a}\right)}.$$

Proof. Similar to Theorem 3.4, if we replace $p(t) = (q_1 - 1)m_3$, $q(t) = -(q_1 - 1)m_1$, $b_k = 0$, and $d_k = \xi^{1-q_1}$ in Theorem 3.3 then the solution of impulsive system (20) can be estimated as

$$U(t) \leq U(0) \tilde{M}(t, 0) - (q_1 - 1)m_1 \int_0^t \tilde{M}(t, s) ds, \quad t \geq 0, \quad (40)$$

where $U(t) = -W^{1-q_1}(t)$ as in Theorem 3.2, and

$$\tilde{M}(t, s) = e^{(q_1-1)m_3(t-s)} \prod_{s < t_k < t} \xi^{1-q_1} = e^{(q_1-1)m_3(t-s)} \xi^{(1-q_1)N_S(t,s)}. \quad (41)$$

Substituting (41) in inequality (40), we obtain

$$U(t) \leq U(0) e^{(q_1-1)m_3 t} \xi^{(1-q_1)\left(N_0 + \frac{t}{\tau_a}\right)} - (q_1 - 1)m_1 \int_0^t \xi^{(1-q_1)N_S(t,s)} e^{(q_1-1)m_3(t-s)} ds. \quad (42)$$

Using the transformation $U(t) = -W^{1-q_1}(t)$ in inequality (42), we get

$$W^{q_1-1}(t) \leq \left[W^{1-q_1}(0) e^{(q_1-1)m_3 t} \xi^{(1-q_1)\left(N_0 + \frac{t}{\tau_a}\right)} + (q_1 - 1)m_1 \int_0^t \xi^{(1-q_1)N_S(t,s)} e^{(q_1-1)m_3(t-s)} ds \right]^{-1} \quad (43)$$

$$\begin{aligned} &\leq W^{q_1-1}(0) e^{-(q_1-1)m_3 t} \xi^{(q_1-1)\left(N_0 + \frac{t}{\tau_a}\right)} \\ &\leq \xi^{(q_1-1)N_0} W^{q_1-1}(0), \quad t \geq 0. \end{aligned} \quad (44)$$

From inequality (44) and Assumption (A₁), the solution of system (1) will be Lyapunov stable. In order to show nFxTS, the following inequality should be true:

$$\delta^{1-q_1} \leq (q_1 - 1)m_1 \int_0^t \xi^{(1-q_1)N_S(t,s)} e^{(q_1-1)m_3(t-s)} ds. \quad (45)$$

Using inequality (45) in (43), we get

$$W(t) \leq \delta, \quad t \geq T(t_0, \delta, \{t_k\}),$$

where settling-time function $T(t_0, \delta, \{t_k\})$ is estimated from inequality (45) as follows:

$$\begin{aligned} \delta^{1-q_1} &\leq (q_1 - 1)m_1 \xi^{(1-q_1)N_0} e^{(q_1-1)\left(m_3 - \frac{\ln \xi}{\tau_a}\right)t} \int_0^t e^{-(q_1-1)\left(m_3 - \frac{\ln \xi}{\tau_a}\right)s} ds, \\ \delta^{1-q_1} &\leq \frac{m_1 \xi^{(1-q_1)N_0}}{m_3 - \frac{\ln \xi}{\tau_a}} \left(e^{(q_1-1)\left(m_3 - \frac{\ln \xi}{\tau_a}\right)t} - 1 \right), \\ e^{(q_1-1)\left(m_3 - \frac{\ln \xi}{\tau_a}\right)t} &\geq 1 + \frac{\delta^{1-q_1} \xi^{(q_1-1)N_0} \left(m_3 - \frac{\ln \xi}{\tau_a}\right)}{m_1}. \end{aligned} \quad (46)$$

From inequality (46), the upper bound on the settling-time function can be obtained:

$$T(t_0, \delta, \{t_k\}) \leq \frac{\ln \left[1 + \frac{\delta^{1-q_1} \xi^{(q_1-1)N_0} \left(m_3 - \frac{\ln \xi}{\tau_a}\right)}{m_1} \right]}{(q_1 - 1)\left(m_3 - \frac{\ln \xi}{\tau_a}\right)}.$$

□

Remark 5. In a similar manner to Remark 4, by setting $\delta = 1$, we can combine the results obtained with ADT property in Theorem 3.5 and Theorem 3.4 to obtain the FxTS of system (1). The fixed-settling time for system (1) to be FxTS is estimated as

$$T(t_0, \{t_k\}) = \frac{\ln \left[1 + \frac{m_3 \xi^{(1-q_2)N_0}}{m_2} \right]}{(1 - q_2)\left(m_3 - \frac{\ln \xi}{\tau_a}\right)} + \frac{\ln \left[1 + \frac{\xi^{(q_1-1)N_0} \left(m_3 - \frac{\ln \xi}{\tau_a}\right)}{m_1} \right]}{(q_1 - 1)\left(m_3 - \frac{\ln \xi}{\tau_a}\right)}.$$

:

Remark 6. In the previous results [43, 28, 29, 30, 31, 32] on FxTS/FTS, the authors considered $\xi = \frac{1}{\eta^{(q_1-1)(1-q_2)}}$ in the discrete part of system (1). This implies that the impulse depends on the values of q_1 and q_2 . Intuitively, the dynamics of continuous part of system (1) are affecting the dynamics of discrete one. In our case, as generalized version of the existing results, the continuous and discrete-time subsystems of system (1) are evolving independent of each other. In another work [44], the authors claimed that FxTS of system (1) is not possible in the case of *destabilizing impulses*. However, in this paper, a condition on ADT as $\tau_a > \frac{\ln \xi}{m_3}$ is derived such that system (1) can achieve FxTS for destabilizing impulses. It is worth noting that inequality (32) is dependent on m_3 which is a coefficient of the third term of Lyapunov inequality (A_2). Contrary to our case, the Lyapunov inequality used in [44] consists of only two terms $-m_1 W^{q_1}(t, x)$ and $-m_2 W^{q_2}(t, x)$. Therefore, the results obtained in this paper relax the conservativeness of previous results.

4. Application to FTS/nFxTS Synchronization of Impulsive Neural Networks

In this section, we will proceed to apply the theoretical results obtained in the previous sections to the synchronization problem of impulsive neural networks.

Consider a Hopfield neural network with impulsive perturbation as a drive system, which is described as follows:

$$\begin{cases} \dot{x}_i(t) = -\alpha_i x_i(t) + \sum_{j=1}^n w_{ij} f_j(x_j(t)) + I_i, & t \neq t_k, \\ x_i(t_k) = \bar{\xi} x_i(t_k^-), & i = 1, 2, \dots, n \end{cases} \quad (47)$$

where $x_i(t) \in \mathbb{R}$ is the state of the i th neuron; $\alpha_i > 0$ is the rate of neuron's self inhibition; $f_j(\cdot)$ is a nonlinear activation function; w_{ij} denotes the connection strength from the j th neuron to the i th neuron; $I_i \in \mathbb{R}$ denotes an external stimulus to the neuron; $\bar{\xi} > 1$ is the impulsive gain.

The corresponding response system is described as follows:

$$\begin{cases} \dot{y}_i(t) = -\alpha_i y_i(t) + \sum_{j=1}^n w_{ij} f_j(y_j(t)) + I_i + u_i(t), & t \neq t_k \\ y_i(t_k) = \bar{\xi} y_i(t_k^-), \end{cases} \quad (48)$$

where $y_i(t) \in \mathbb{R}$ is the state and $u_i(t) \in \mathbb{R}$, $i = 1, 2, \dots, n$, denotes controller.

The synchronization error is defined as $e_i(t) = y_i(t) - x_i(t)$, $i = 1, 2, \dots, n$. From the drive-response systems (47) and (48), one can obtain

$$\begin{cases} \dot{e}_i(t) = -\alpha_i e_i(t) + \sum_{j=1}^n w_{ij} (f_j(y_j(t)) - f_j(x_j(t))) + u_i(t), & t \neq t_k, \\ e_i(t_k) = \bar{\xi} e_i(t_k^-), \end{cases} \quad (49)$$

let $e(t) = (e_1(t), e_2(t), \dots, e_n(t))^T$.

Assumption 4.1. *Activation functions satisfy Lipschitz continuity condition, i.e., there exists $l_j > 0$ such that for any $x, y \in \mathbb{R}$*

$$|f_j(y) - f_j(x)| \leq l_j |y - x|, \quad j = 1, 2, \dots, n. \quad (50)$$

Lemma 4.1. [52] *If $k_1, k_2, \dots, k_n \geq 0$, $\eta_1 > 1$ and $0 < \eta_2 < 1$, we have*

$$\sum_{i=1}^n k_i^{\eta_2} \geq \left(\sum_{i=1}^n k_i \right)^{\eta_2}, \quad \sum_{i=1}^n k_i^{\eta_1} \geq n^{1-\eta_1} \left(\sum_{i=1}^n k_i \right)^{\eta_1} \quad (51)$$

In order to achieve FTS/nFxTS synchronization between drive-response systems (47) and (48), we design the controller as

$$u_i(t) = -m_{1i} e_i(t) - m_{2i} \text{sign}(e_i(t)) |e_i(t)|^{\eta_2} - m_{3i} \text{sign}(e_i(t)) |e_i(t)|^{\eta_1}, \quad (52)$$

where $\eta_1 > 1$, $0 < \eta_2 < 1$ and $m_{1i}, m_{2i}, m_{3i} > 0$ are control gains for $i = 1, 2, \dots, n$.

Theorem 4.2. If Assumption 4.1 and $m_{1i} > \sum_{j=1}^n \frac{|w_{ij}l_j + |w_{ji}l_i}{2} - \alpha_i$, $i = 1, 2, \dots, n$, are true then drive-response systems (47) and (48) can achieve FTS/nFxTS synchronization with the same settling-time functions as given in Theorems 3.1, 3.2, 3.4 and 3.5 for DT and ADT cases, where parameters are $m_1 = \min_i \{m_{3i}\} \cdot n^{\frac{1-\eta_1}{2}} \cdot 2^{\frac{\eta_1+1}{2}}$, $m_2 = \min_i \{m_{2i}\} \cdot 2^{\frac{\eta_2+1}{2}}$, $m_3 = 2 \cdot \min_i \{m_{1i} - \sum_{j=1}^n \frac{|w_{ij}l_j + |w_{ji}l_i}{2} + \alpha_i\}$, $\xi = \bar{\xi}^2$, $q_1 = \frac{\eta_1+1}{2}$, $q_2 = \frac{\eta_2+1}{2}$.

Proof. Consider the following Lyapunov function:

$$W(e(t)) = \frac{1}{2} e^T(t)e(t) = \frac{1}{2} \sum_{i=1}^n (e_i(t))^2. \quad (53)$$

For $t \neq t_k$, differentiate Lyapunov function along the trajectory of system (49) then we get

$$\begin{aligned} \dot{W}(e(t)) &= \sum_{i=1}^n e_i(t) \dot{e}_i(t) \\ &= \sum_{i=1}^n e_i(t) \left[-\alpha_i e_i(t) + \sum_{j=1}^n w_{ij} (f_j(y_j(t)) - f_j(x_j(t))) + u_i(t) \right]. \end{aligned} \quad (54)$$

From Assumption 4.1, we have

$$|f_j(y_j(t)) - f_j(x_j(t))| \leq l_j |y_j(t) - x_j(t)| = l_j |e_j(t)|,$$

then we drive

$$\begin{aligned} &\sum_{i=1}^n e_i(t) \sum_{j=1}^n w_{ij} (f_j(y_j(t)) - f_j(x_j(t))) \\ &\leq \sum_{i=1}^n |e_i(t)| \sum_{j=1}^n |w_{ij}| \cdot l_j |e_j(t)| \\ &\leq \sum_{i=1}^n \sum_{j=1}^n \frac{|w_{ij}l_j + |w_{ji}l_i}{2} (e_i(t))^2. \end{aligned} \quad (55)$$

From controller (52), we get

$$\sum_{i=1}^n e_i(t) u_i(t) = - \sum_{i=1}^n m_{1i} (e_i(t))^2 - \sum_{i=1}^n m_{2i} |e_i(t)|^{\eta_2+1} - \sum_{i=1}^n m_{3i} |e_i(t)|^{\eta_1+1}. \quad (56)$$

Using inequality (55) and equation (56) in (54), we obtain

$$\dot{W}(e(t)) \leq \sum_{i=1}^n \left[\sum_{j=1}^n \frac{|w_{ij}l_j + |w_{ji}l_i}{2} - \alpha_i - m_{1i} \right] (e_i(t))^2 - \sum_{i=1}^n m_{2i} |e_i(t)|^{\eta_2+1} - \sum_{i=1}^n m_{3i} |e_i(t)|^{\eta_1+1}.$$

If $m_{1i} > \sum_{j=1}^n \frac{|w_{ij}l_j + |w_{ji}l_i}{2} - \alpha_i$, $i = 1, 2, \dots, n$, then we can obtain

$$\dot{W}(e(t)) \leq - \sum_{i=1}^n \gamma_i (e_i(t))^2 - \sum_{i=1}^n m_{2i} |e_i(t)|^{\eta_2+1} - \sum_{i=1}^n m_{3i} |e_i(t)|^{\eta_1+1},$$

where $\gamma_i = m_{1i} - \sum_{j=1}^n \frac{|w_{ij}l_j + |w_{ji}l_i}{2} + \alpha_i > 0$.

$$\dot{W}(e(t)) \leq - \min_i \{\gamma_i\} \sum_{i=1}^n (e_i(t))^2 - \min_i \{m_{2i}\} \sum_{i=1}^n (|e_i(t)|^2)^{\frac{\eta_2+1}{2}} - \min_i \{m_{3i}\} \sum_{i=1}^n (|e_i(t)|^2)^{\frac{\eta_1+1}{2}}. \quad (57)$$

Since $0 < \frac{\eta_2+1}{2} < 1$ and $\frac{\eta_1+1}{2} > 1$, we can apply Lemma 4.1 and obtain the following inequalities:

$$\sum_{i=1}^n (|e_i(t)|^2)^{\frac{\eta_2+1}{2}} \geq \left(\sum_{i=1}^n |e_i(t)|^2 \right)^{\frac{\eta_2+1}{2}} \quad \text{and} \quad \sum_{i=1}^n (|e_i(t)|^2)^{\frac{q_1+1}{2}} \geq n^{\frac{1-q_1}{2}} \left(\sum_{i=1}^n |e_i(t)|^2 \right)^{\frac{q_1+1}{2}}. \quad (58)$$

Using inequality (58) in (57), we get

$$\begin{aligned} \dot{W}(e(t)) &\leq -\min_i \{\gamma_i\} \sum_{i=1}^n (e_i(t))^2 - \min_i \{m_{2i}\} \left(\sum_{i=1}^n |e_i(t)|^2 \right)^{\frac{\eta_2+1}{2}} - \min_i \{m_{3i}\} \cdot n^{\frac{1-\eta_1}{2}} \left(\sum_{i=1}^n |e_i(t)|^2 \right)^{\frac{\eta_1+1}{2}} \\ &= -2 \min_i \{\gamma_i\} W(e(t)) - \min_i \{m_{2i}\} \cdot 2^{\frac{\eta_2+1}{2}} (W(e(t)))^{\frac{\eta_2+1}{2}} - \min_i \{m_{3i}\} \cdot n^{\frac{1-\eta_1}{2}} \cdot 2^{\frac{\eta_1+1}{2}} (W(e(t)))^{\frac{\eta_1+1}{2}}. \end{aligned}$$

From Theorem 4.2, we get the following for $t \neq t_k$

$$\dot{W}(e(t)) \leq -m_1 (W(e(t)))^{q_1} - m_2 (W(e(t)))^{q_2} - m_3 W(e(t)).$$

For $t = t_k$, we have

$$W(e(t_k)) = \frac{\bar{\xi}^2}{2} \sum_{i=1}^n (e(t_k^-))^2 = \xi W(e(t_k^-)).$$

Assumptions (A_1) , (A_2) and (A_3) of Theorem 3.1 are satisfied. Hence, FTS/nFxTS synchronization between drive-response systems (47) and (48) can achieve with the settling time functions given in Theorems 3.1, 3.2, 3.4 and 3.5 for DT and ADT cases. \square

5. Numerical Simulations

In this section, we will verify the obtained results with the help of a numerical example.

Example 1. Numerical values for the parameters of drive-response systems (47) and (48) are given as follows:

$$\alpha_1 = 0.2, \alpha_2 = 1.1, w_{11} = 2, w_{12} = -0.1, w_{21} = -5, w_{22} = 4.5, I_1 = \sin(t), I_2 = \cos(t), \bar{\xi} = 1.3$$

and $i = 1, 2$. The activation functions $f_1(a) = f_2(a) = 0.5(|a+1| - |a-1|)$ with Lipschitz constants $l_1 = l_2 = 1$. We proceed to show FTS/nFxTS stability of (49) to demonstrate synchronization between systems (47) and (48). For this, the parameters of the controller (52), for $i = 1, 2$, are chosen as $m_{11} = 6$, $m_{12} = 8$, $m_{21} = m_{22} = 1$, $m_{31} = m_{32} = 1$, and $\eta_1 = 1.5$, $\eta_2 = 0.5$. Sufficient condition

$$\begin{aligned} m_{11} - \sum_{j=1}^2 \frac{|w_{1j}|l_j + |w_{j1}|l_1}{2} + \alpha_1 &= 6.78 > 0, \\ m_{12} - \sum_{j=1}^2 \frac{|w_{2j}|l_j + |w_{j2}|l_2}{2} + \alpha_2 &= 2.05 > 0, \end{aligned}$$

is satisfied. Hence, according to Theorem 4.2, FTS/nFxTS synchronization of drive-response systems (47) and (48) can be achieved. The parameters' values to estimate the settling-time functions are: $m_1 = 1.4142$, $m_2 = 1.6818$, $m_3 = 4.10$, $\xi = 1.69$, $q_1 = 1.25$, $\frac{\ln(\bar{\xi})}{m_3} = 0.1280$, and $q_2 = 0.75$.

Impulsive sequence is taken from [53] that can be described as

$$t_k - t_{k-1} = \begin{cases} \theta, & \text{if } \text{mod}(k, N_0) \neq 0 \\ N_0(\Gamma_a - \theta) + \theta, & \text{if } \text{mod}(k, N_0) = 0, \end{cases} \quad (59)$$

where $N_0 \in \mathbb{N}$ and Γ_a are positive numbers known as elasticity number and ADT, respectively; θ is a positive number such that $\theta \leq \Gamma_a$. Choose a Lyapunov functional as $W(e(t)) = \frac{1}{2} \sum_{i=1}^2 (e_i(t))^2$ which satisfies Assumptions (A_1) , (A_2) , and (A_3) , of Theorem 3.1.

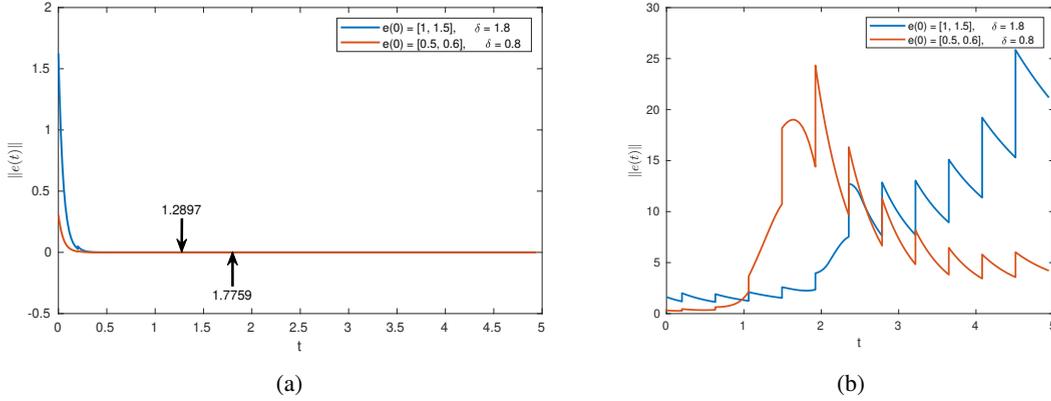


Figure 1: (a) Trajectories of system (49) with controller (52) for two different set of initial values. (b) The same trajectories as in (a) without controller (52).

5.1. For DT cases

For $N_0 = 1$ in the impulsive sequence (59), we have $\tau_{min} = \theta < t_k - t_{k-1}$. Calculate $\mu_2 = 0.4204$, $\mu_3 = 1.0250$. The results obtained in Theorem 3.1 can be validated as follows:

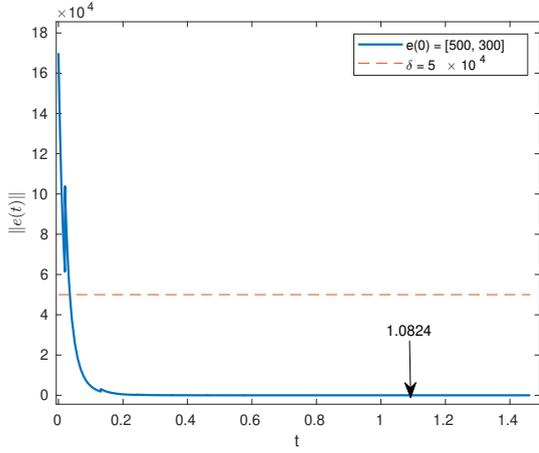
Case 1: Choose $\theta = 0.2$, $\tau_a = 0.431$ then $\tau_{min} = 0.2 > \frac{\ln(\xi)}{m_3}$. In order to simulate the results for any $\delta > 0$, we have taken two random values: $\delta = 1.8$ and $\delta = 0.8$ correspondingly we obtain $N_1 = 9$ and $N_1 = 8$. That is, if system (49) starts from a larger domain of its origin then the trajectory experiences more number of impulsive jumps before achieving FTS synchronization. For two different initial conditions $e_1(0) = 1, e_2(0) = 1.5$ and $e_1(0) = 0.5, e_2(0) = 0.6$, the settling-time functions are estimated as $T \leq 1.7759$ and $T \leq 1.2897$. From Fig. (1a), it can be seen that the error system (49), for two different initial conditions, is stabilized in a finite settling-time which is less than the theoretical upper bounds of settling-time functions. On the other hand, in Fig. (1b), the same trajectories of system (49) are destabilized without controller (52) which demonstrate the effectiveness of controller (52) under the influence of impulsive perturbation. Hence, the Case 1 of Theorem 3.1 is verified.

Case 2: Since $\frac{\ln \xi}{m_3} = 0.1280$ so we consider an impulsive sequence with $\tau_a = 0.111$ and $\tau_{min} = \theta = 0.02$ such that $\tau_{min} < \frac{\ln \xi}{m_3}$. For this case, the error system (49) is not finite-time stable because settling-time function does not exist for any $\delta > 0$. According to Theorem 3.1, there exists an upper bound of $\delta < 4.3047 \times 10^{-5}$ such that system (49) converges to its origin in a finite-time where the settling-time function is estimated as $T < 0.1220$ and the maximal number of impulsive points $N_2 = 6$. This can be seen in the Fig. (3a) for an initial value $e(0) = [0.001, 0.002]$. In Fig. (3a), there is only one impulsive point occurs at $t_1 = 0.02$ because the system (49) converged to its origin before the second impulsive point $t_2 = 0.131 > 0.1220$ (upper bound of settling-time function). The similar scenario occurred in the Fig. (1a). Hence, Case 2 of Theorem 3.1 is verified.

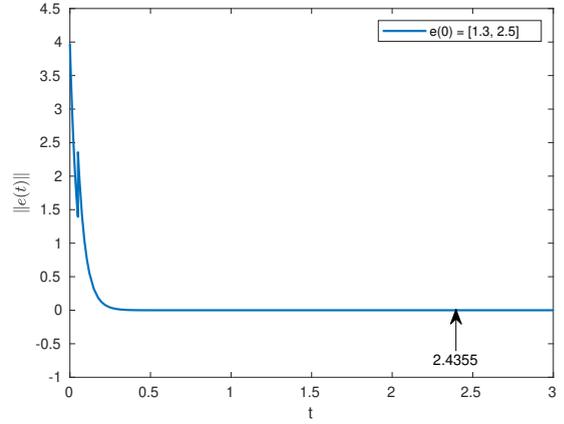
Next, we will proceed to verify the nFxTS results obtained in Theorem 3.2 when system (49) starts from outside of δ -nbh of the origin. By a simple calculation, we obtain $v_1 = 0.3535$ and $v_3 = 1.0250$.

Case 1: Choose $\delta = 3.5$ and consider the same impulsive sequence as in the Case 1 of Theorem 3.1. For an initial value $e(0) = [5.5, 8]$, we get $M_1 = 9$ and an upper bound of settling-time function $T \leq 1.7052$. In Fig. (3b), the trajectory of system (49) converge into the region of $\delta = 3.5$ in fixed-time. Hence, nFxTS synchronization between drive-response systems (47) and (48) is achieved.

Case 2: Using similar arguments as in the Case 2 of Theorem 3.1, if $\tau_{min} = 0.02 < \frac{\ln \xi}{m_3}$ then nFxTS is not possible because the settling-time function does not exist for every $\delta > 0$. There exists a lower bound of $\delta > 4.9625 \times 10^4$ such that if the system (49) starts from outside of δ -nbh then the solution converges to a ball of radius greater

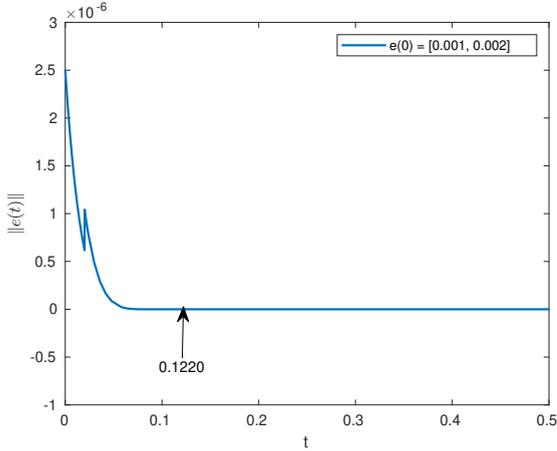


(a)

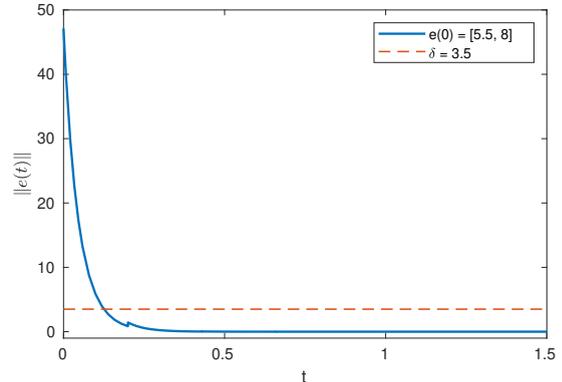


(b)

Figure 2: (a) Nearly Fixed-time convergence of system (49) to a δ -nbh of the origin.(b) FTS of system (49) with ADT condition.



(a)



(b)

Figure 3: (a) Finite-time convergence of system (49). (b) nFxTS of system (49) with DT condition.

than 4.9625×10^4 in a fixed-time. For an initial value $e(0) = [500, 300]$, the fixed-time convergence of system (49) can be seen in Fig. 2a, where the settling-time is estimated by $T \leq 1.0824$.

5.2. For ADT cases

In this subsection, we proceed to verify the results obtained in Theorem 3.4 and 3.5. In order to verify Theorem 3.4, choose $\delta = 9.8$ and initial condition $e(0) = [1.3, 2.5]$, i.e., $w(e(0)) \leq 9.8$. Consider an impulsive sequence $\tau_a = 0.341 > 0.1280$, $\theta = 0.05$, $N_0 = 2$ then the drive-response system (47) and (48) is FTS with the settling-time estimated by $T \leq 2.4355$. FTS synchronization of the drive-response system (47) and (48) with ADT condition is shown in Fig. 2b.

In order to verify Theorem 3.5, keep all parameters' values same as in Theorem 3.4 except the initial condition $e(0) = [3.5, 4.5]$ such that $w(e(0)) \geq 9.8$. Then the system (49) will be nFxTS with the settling-time $T \leq 1.3216$ that can be seen in Fig.4.

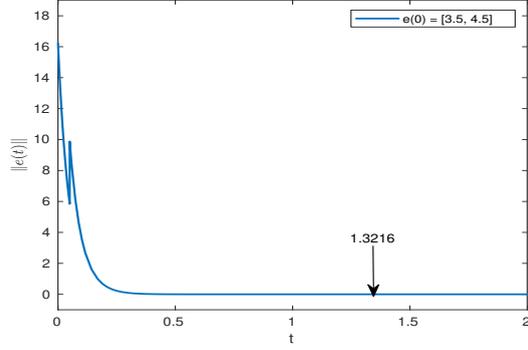


Figure 4: nFxTs of system (49) with ADT condition.

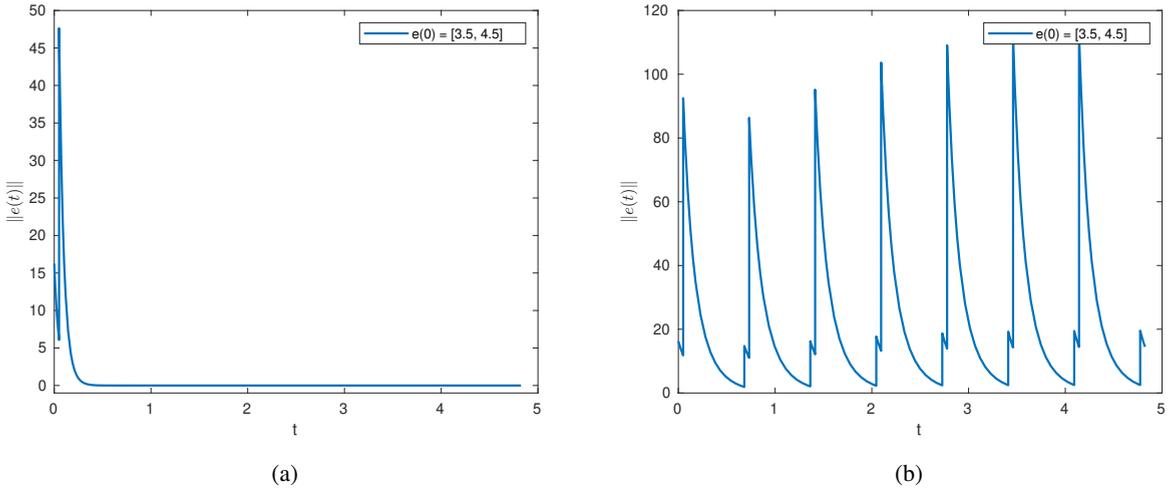


Figure 5: (a) FxTs of system (49) with controller (52) and $\xi = 2.8$. (b) System (49) with controller (52) without linear term and $\xi = 2.8$.

Remark 7. It is worth noting that if $\delta = 1$, then the nFxTs results presented in this paper are equivalent to the FxTs results obtained in [44, 45, 46, 47] for impulsive systems. Therefore, the existing results can be considered a specific case of this paper. Furthermore, the controller proposed in [44, 46, 47] only includes two terms (without a linear term), which does not enable impulsive systems to achieve FxTs under destabilizing impulses. In contrast, the controller designed in this paper fills this gap. To demonstrate this advantage numerically, let us consider $\tau_a = 0.341$, $\theta = 0.05$, $N_0 = 2$, $\xi = 2.8$, and $\delta = 1$. As shown in Figs. 5a and 5b, it can be observed that the controller *without a linear term* is not effective in achieving FxTs when the impulsive perturbation is $\xi = 2.8$.

6. Conclusions

This paper deals with FTS/nFxTs of nonlinear impulsive systems with *destabilizing* impulses. The concept of nFxTs is more general than FxTs because the former may be just asymptotically stable, but the latter must be FTS. A Lyapunov inequality with a linear term is used to derive the main results of this paper, separately for DT and ADT cases of impulsive sequence. For the DT case, when starting from within δ -nbh of the origin, FTS can be achieved for any $\delta > 0$, satisfying $\tau_{min} > \frac{\ln \xi}{m_3}$. However, if DT satisfies $0 < \tau_{min} < \frac{\ln \xi}{m_3}$, then $\delta > 0$ must be bounded in order for the system to be FTS. This means that the frequent occurrence of *destabilizing* impulse moments in the impulsive sequence restricts the neighbourhood of the origin to which the system's initial state belongs. When the system begins

from *outside* δ -nbh of the origin, nFxTS is possible, satisfying $\tau_{min} > \frac{\ln \xi}{m_3}$. According to the definition, nFxTS is not possible if $0 < \tau_{min} < \frac{\ln \xi}{m_3}$, because $\delta > 0$ is constrained by a nonzero lower bound. In ADT case, if $\tau_a > \frac{\ln \xi}{m_3}$, the system achieves FTS starting from within δ -nbh of the origin. Similarly, nFxTS can be achieved when the initial state starts from outside δ -nbh of the origin. The proposed theoretical results have been applied to the FTS/nFxTS synchronization of impulsive neural networks with destabilizing impulses by designing a continuous controller based on the Lyapunov inequality. Finally, a numerical example of an impulsive neural network is provided to illustrate the validity of the obtained results. Extending the proposed theoretical results of this paper to a nonlinear impulsive system with *time-delays* would be our future research.

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