



HAL
open science

Comments on "Finite-time stability of discrete autonomous systems" [Automatica 122 (2020) 109282] ★

Bernard Brogliato

► To cite this version:

Bernard Brogliato. Comments on "Finite-time stability of discrete autonomous systems" [Automatica 122 (2020) 109282] ★. Automatica, 2023, 156, pp.111206:1-3. 10.1016/j.automatica.2023.111206 . hal-04121636

HAL Id: hal-04121636

<https://inria.hal.science/hal-04121636>

Submitted on 8 Jun 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

Comments on “Finite-time stability of discrete autonomous systems” [Automatica 122 (2020) 109282] [★]

Bernard Brogliato ^a,

^aUniv. Grenoble Alpes, INRIA, CNRS, Grenoble INP, LJK, 38000 Grenoble, France

Abstract

Interestingly enough, some controllers and stability results in Haddad and Lee (2020) can be shown to be equivalent to algorithms obtained from the implicit Euler discretization of a set-valued system, which is known to preserve many nice properties of the continuous-time dynamics after discretization, like the convergence towards the (discrete) sliding surface in a finite number of steps.

Key words: .

1 The first system

In [6, Equation (15)] the following discrete-time closed-loop system is considered:

$$x_{k+1} = x_k - \operatorname{sgn}(x_k) \min(|x_k|, c), \quad k \geq 0 \quad (1)$$

where $x_k \in \mathbb{R}$, $k \in \mathbb{N}$, $\operatorname{sgn}(x) = 1$ if $x > 0$, -1 if $x < 0$, 0 if $x = 0$, and $c > 0$. Quite similar systems are also studied in [9, Theorem 2] [7]. Let us notice that the right-hand side of (1) is equivalently rewritten as:

$$x_{k+1} = \begin{cases} 0 & \text{if } x_k \in [-c, c] \\ x_k + c & \text{if } x_k \leq -c \\ x_k - c & \text{if } x_k \geq c. \end{cases} \quad (2)$$

Consider now the Euler discretization of $\dot{x}(t) = u(t)$. Assume that we want to discretize the sliding-mode, set-valued input $u(x(t)) = \lambda(t)$, $\lambda(t) \in -\alpha \operatorname{sgn}(x(t))$, $\alpha > 0$, where $\operatorname{sgn}(x) = -1$ if $x < 0$, 1 if $x > 0$, $[-1, 1]$ if $x = 0$. The zero-order-hold (ZOH) method yields $x_{k+1} = x_k + hu_k$, $h > 0$ is

the timestep (or sampling time), with u_k constant on $[t_k, t_{k+1})$, $t_{k+1} - t_k = h$, $k \geq 0$. The implicit Euler method consists of taking $u_k = -\alpha \lambda_{k+1}$, with $\lambda_{k+1} \in \operatorname{sgn}(x_{k+1})$, see [1,2,5] for details. The calculation of λ_{k+1} , which is a selection of $\operatorname{sgn}(x_{k+1})$, is done as follows. First notice that we have:

$$\begin{aligned} x_{k+1} &\in x_k - h\alpha \operatorname{sgn}(x_{k+1}) \\ \Leftrightarrow x_{k+1} &\in (I_d + h\alpha \operatorname{sgn})^{-1}(x_k) \end{aligned} \quad (3)$$

where I_d is the identity mapping, and $x \mapsto (I_d + h\alpha \operatorname{sgn})^{-1}(x)$ is a single-valued Lipschitz continuous mapping (the resolvent of index $h\alpha$ the maximal monotone mapping $\operatorname{sgn}(\cdot)$ [3]). The right-hand side in (3) is the explicit form of the implicit Euler method. Second, we have $\lambda_{k+1} \in \operatorname{sgn}(x_{k+1}) \Leftrightarrow x_{k+1} \in \mathbf{N}_{[-1,1]}(\lambda_{k+1})$, where $\mathbf{N}_{[-1,1]}(\cdot)$ is the normal cone mapping to the set $[-1, 1]$. Therefore the left-hand side of (3) is equivalently rewritten as $x_k -$

[★] This paper was not presented at any IFAC meeting. Corresponding author Bernard Brogliato.

Email address: bernard.brogliato@inria.fr (Bernard Brogliato).

$\alpha h \lambda_{k+1} \in \mathbf{N}_{[-1,1]}(\lambda_{k+1})$ and we have:

$$\begin{aligned}
& x_k - h\alpha \lambda_{k+1} \in \mathbf{N}_{[-1,1]}(\lambda_{k+1}) \\
& \Leftrightarrow h\alpha \lambda_{k+1} - x_k \in -\mathbf{N}_{[-1,1]}(\lambda_{k+1}) \\
& \Leftrightarrow \lambda_{k+1} - \frac{x_k}{h\alpha} \in -\mathbf{N}_{[-1,1]}(\lambda_{k+1}) \\
& \Leftrightarrow \lambda_{k+1} = \text{proj}[-1, 1]; \frac{x_k}{h\alpha} \\
& \Leftrightarrow u_k = -\alpha \text{proj}[-1, 1]; \frac{x_k}{h\alpha} \\
& \Leftrightarrow \text{find } \lambda_{k+1} \in [-1, 1] \text{ such that:} \\
& \langle \lambda_{k+1} - \frac{x_k}{h\alpha}, \lambda - \lambda_{k+1} \rangle \geq 0, \text{ for all } \lambda \in [-1, 1],
\end{aligned} \tag{4}$$

where $\mathbf{N}_{[-1,1]}(x)$ is the normal cone to $[-1, 1]$ at $x \in [-1, 1]$, $\text{proj}[-1, 1]; x$ is the projection of x onto $[-1, 1]$. These simple manipulations (recalled here for convenience) show that this input yields the right-hand side in (3). It is also checked that inserting u_k in (4) into $x_{k+1} = x_k + hu_k$ yields the right-hand side in (2) with $c = h\alpha$.

Lemma 1 *The mapping $x \mapsto (I_d + h\alpha \text{sgn})^{-1}(x)$ is identical to $x \mapsto x - \text{sgn}(x) \min(|x|, c)$ when $c = h\alpha$. Hence the difference equation in (1) is identical to the difference inclusion in (3) when $c = h\alpha$. In other words, (1) is an explicit formulation of the implicit Euler discretization of the set-valued controller $u \in -\alpha \text{sgn}(x)$ applied to the integrator after a ZOH discretization.*

Consequently the finite-time convergence and stability properties of (1) follow from the results stated in [1,2,5,8,11], where the results are shown for systems of arbitrary dimensions, in the presence of unknown disturbances.

2 The second system

Let us now examine the system in [6, Equation (17)]:

$$x_{k+1} = x_k - c \text{sgn}(x_k) \min\left(\frac{|x_k|}{c}, |x_k|^\gamma\right), \quad k \geq 0, \tag{5}$$

with $0 < \gamma < 1$, $c > 0$ (when $\gamma = 0$, (5) is equal to (1)). Few manipulations show that (5) can be rewritten equivalently as $x_{k+1} = f(x_k)$, with:

$$f(x_k) = \begin{cases} 0 & \text{if } x_k \in [-c^{\frac{1}{1-\gamma}}, c^{\frac{1}{1-\gamma}}] \\ x_k - cx_k^\gamma & \text{if } x_k \geq c^{\frac{1}{1-\gamma}} \\ x_k + c(-x_k)^\gamma & \text{if } x_k \leq -c^{\frac{1}{1-\gamma}}. \end{cases} \tag{6}$$

We have $f^{-1}(x) = c^{\frac{1}{1-\gamma}} \text{sgn}(x) + l^{-1}(x)$, $\text{sgn}(\cdot)$ is the set-valued signum function, $l^{-1}(\cdot)$ is single-valued, differentiable and strictly monotone (see Fig. 1), its inverse function being given by:

$$l(x) = \begin{cases} (x + c^{\frac{1}{1-\gamma}}) - c(x + c^{\frac{1}{1-\gamma}})^\gamma & \text{if } x \geq 0 \\ (x - c^{\frac{1}{1-\gamma}}) + c(-x + c^{\frac{1}{1-\gamma}})^\gamma & \text{if } x \leq 0. \end{cases} \tag{7}$$

When $\gamma = 0$, $l^{-1}(x) = l(x) = x$, $f^{-1}(x) = c \text{sgn}(x) + x$. In general $l^{-1}(\cdot)$ is not easy to calculate. If $\gamma = \frac{1}{2}$ we obtain:

$$l^{-1}(y) = \begin{cases} \left(\frac{c + \sqrt{c^2 + 4y}}{2}\right)^2 - c^2 & \text{if } y \geq 0 \\ -\left(\frac{c + \sqrt{c^2 - 4y}}{2}\right)^2 + c^2 & \text{if } y \leq 0 \end{cases} \tag{8}$$

Both $f(\cdot)$ and $f^{-1}(\cdot)$ are maximal monotone ($f^{-1}(\cdot)$

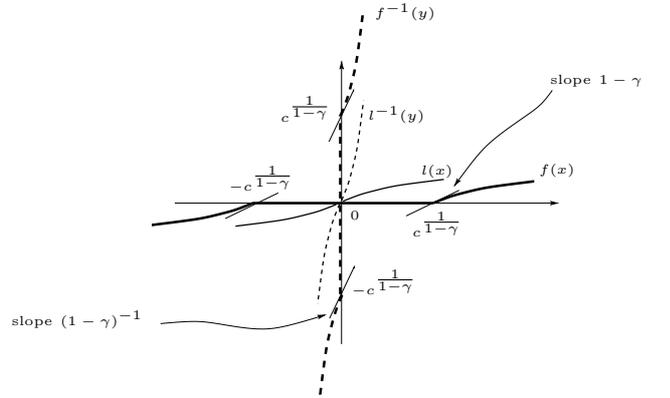


Fig. 1. The functions $f(\cdot)$, $f^{-1}(\cdot)$, $l(\cdot)$, $l^{-1}(\cdot)$ with $\gamma = \frac{1}{2}$.

is set-valued while $f(\cdot)$ is single-valued) and have their domains equal to \mathbb{R} . Let us now consider the implicit difference equation:

$$x_{k+1} \in x_k - \alpha \beta(x_{k+1}) \Leftrightarrow x_{k+1} = (I_d + \alpha \beta)^{-1}(x_k), \tag{9}$$

where $\beta(\cdot)$ is a set-valued mapping, $\alpha > 0$. Let us define:

$$\beta(x) = \frac{1}{\alpha} (c^{\frac{1}{1-\gamma}} \text{sgn}(x) + l^{-1}(x) - x). \tag{10}$$

Then we have for all $x \in \mathbb{R}$:

$$\alpha \beta(x) = f^{-1}(x) - x = c^{\frac{1}{1-\gamma}} \text{sgn}(x) + l^{-1}(x) - x, \tag{11}$$

and $f(\cdot) = (I_d + \alpha \beta)^{-1}(\cdot) \Leftrightarrow f^{-1}(\cdot) = (I_d + \alpha \beta)(\cdot)$. Notice that the mapping $x \mapsto (I_d + \alpha \beta)(x) = x +$

$\alpha\beta(x) = c^{\frac{1}{1-\gamma}} \text{sgn}(x) + l^{-1}(x)$ and its inverse mapping are both maximal monotone. Therefore we can write the left-hand side of (9) equivalently as:

$$\begin{cases} x_{k+1} = x_k + u_k \\ u_k = -\alpha\lambda_{k+1}, \lambda_{k+1} \in \beta(x_{k+1}) \end{cases} \quad (12)$$

Thus the following is proved:

Lemma 2 *The difference equation in (5) is the explicit form of the implicit difference inclusion (12).*

It is possible to further develop the implicit scheme in (12) to show that it relies on the solvability of generalized equations (GE) for the input calculation (which is a common feature of implicit Euler schemes [5]). Indeed (12) is equivalently rewritten as:

$$\begin{aligned} (a) \quad & \begin{cases} x_{k+1} = l\left(x_k - \frac{c^{\frac{1}{1-\gamma}}}{\alpha} \zeta_{k+1}\right) \Leftrightarrow \\ x_{k+1} \in \mathbf{N}_{[-1,1]}(\zeta_{k+1}) \end{cases} \\ (b) \quad & \begin{cases} \frac{c^{\frac{1}{1-\gamma}}}{\alpha} \zeta_{k+1} = x_k - l^{-1}(x_{k+1}) \Leftrightarrow \\ \zeta_{k+1} \in \text{sgn}(x_{k+1}) \end{cases} \\ (c) \quad & \text{Find } \zeta_{k+1} \in [-1, 1] \text{ such that:} \\ & \langle l\left(x_k - \frac{c^{\frac{1}{1-\gamma}}}{\alpha} \zeta_{k+1}\right), \zeta - \zeta_{k+1} \rangle \leq 0 \text{ for all } \zeta \in [-1, 1]. \\ & \Updownarrow \\ (d) \quad & \text{Find } x_{k+1} \in \mathbb{R} \text{ such that:} \\ & \langle \frac{\alpha}{c^{\frac{1}{1-\gamma}}}(x_k - l^{-1}(x_{k+1})), x - x_{k+1} \rangle + |x| - |x_{k+1}| \leq 0 \\ & \text{for all } x \in \mathbb{R}. \end{aligned} \quad (13)$$

The problem in (13) (a) is a GE with unknown ζ_{k+1} (whose solution $\zeta_{k+1}(x_k)$ is at the intersection between the graph of the normal cone and the graph of the single-valued function $\zeta_{k+1} \mapsto l\left(x_k - \frac{c^{\frac{1}{1-\gamma}}}{\alpha} \zeta_{k+1}\right)$), the problem in (13) (b) is a GE with unknown x_{k+1} (whose solution $x_{k+1}(x_k)$ is at the intersection between the graph of the set-valued signum function and the single-valued function $x_{k+1} \mapsto \frac{\alpha}{c^{\frac{1}{1-\gamma}}}(x_k - l^{-1}(x_{k+1}))$). The problems in (13) (c) and (d) are variational inequalities (the inclusions in (4) can also be written as VIs). Due to the strict monotonicity of $l(\cdot)$ and the maximal monotonicity of both set-valued functions, it

follows that the solutions are unique. Consequently:

$$u_k = -l^{-1}(x_{k+1}) + l\left(x_k - \frac{c^{\frac{1}{1-\gamma}}}{\alpha} \zeta_{k+1}\right) - c^{\frac{1}{1-\gamma}} \zeta_{k+1}. \quad (14)$$

where ζ_{k+1} and x_{k+1} are solutions of the VIs (13) (c) and (d), yields (5). Therefore the underlying structure of (5) is that of an implicit Euler time-discretization which involves VIs to be solved (numerically or analytically as it is the case for this scalar system) at each timestep.

Let us interpret the above in terms of the implicit discretization of a continuous-time control system. Let $\dot{x}(t) = u(t)$. The implicit discretization writes $x_{k+1} = x_k + hu_k$, $h > 0$. Let $u(x) = -\alpha\lambda$, with $\lambda \in \beta(x)$, $\beta(\cdot)$ is in (10). This yields the continuous-time closed-loop system: $\dot{x}(t) \in -\alpha\beta(x(t)) = c^{\frac{1}{1-\gamma}} \text{sgn}(x) + l^{-1}(x) - x$. Its implicit Euler discretization writes as $x_{k+1} = x_k + hu_k$, $u_k = -\alpha\lambda_{k+1}$, $\lambda_{k+1} \in \beta(x_{k+1})$, which yields $x_{k+1} = (I_d + h\alpha\beta)^{-1}(x_k)$: this is (9), replacing α by $h\alpha$. Thus the set-valued function is obtained as in (10), and the GEs as in (13), replacing α by $h\alpha$ in these expressions.

3 Conclusion

This comment article sheds new light on two algorithms published in the above article. The above developments show that two finite-time-convergent difference equations studied in [6] fit within the class of implicit Euler discretization of set-valued systems, which have been the object of many studies and experimental validations in the past years (see [5] for a survey). The first algorithm [6, Equation (15)] is shown to fit with already analyzed schemes, while the second one [6, Equation (17)] seems to be new in the class of implicit Euler schemes. This study may pave the way to extensions of the proposed scheme towards higher dimension systems, using generalized equations to calculate the controller and suitable numerical solvers. Motivated by [4], other algorithms with fixed-time properties which have been studied in [10,12] could be the object of future analysis in the same framework.

References

- [1] V. Acary and B. Brogliato. Implicit Euler numerical scheme and chattering-free implementation of sliding mode systems. *Systems and Control Letters*, 59(5):284–293, 2010.

- [2] V. Acary, B. Brogliato, and Y. Orlov. Chattering-free digital sliding-mode control with state observer and disturbance rejection. *IEEE Transactions on Automatic Control*, 57(5):1087–1101, 2012.
- [3] H.H. Bauschke and P.L. Combettes. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*. Canadian Mathematical Society, Société mathématique du Canada. Springer Science+Business Media, 2011.
- [4] B. Brogliato and A. Polyakov. Globally stable implicit Euler time-discretization of a nonlinear single-input sliding-mode control system. In *Proc. 54th IEEE Conference on Decision and Control (CDC)*, pages 5426–5431, Osaka, Japan, 2015.
- [5] B. Brogliato and A. Polyakov. Digital implementation of sliding-mode control via the implicit method: A tutorial. *International Journal of Robust and Nonlinear Control*, 31(9):3528–3586, 2021.
- [6] W.M. Haddad and J. Lee. Finite-time stability of discrete autonomous systems. *Automatica*, 122:109282, 2020.
- [7] W.M. Haddad and J. Lee. Finite-time stabilization and optimal feedback control for nonlinear discrete-time systems. *IEEE Transactions on Automatic Control*, 68(3):1685–1691, 2023.
- [8] O. Huber, V. Acary, and B. Brogliato. Lyapunov stability and performance analysis of the implicit discrete sliding mode control. *IEEE Transactions on Automatic Control*, 61(10):3016–3030, 2016.
- [9] S. Ibrir. Finite-time stabilization procedures for discrete-time nonlinear systems in feedback form. *IEEE Access*, 9:27226–27238, 2021.
- [10] J. Lee and W.M. Haddad. Fixed time stability and optimal stabilisation of discrete autonomous systems. *International Journal of Control*, 0(0):1–15, 2022. <https://doi.org/10.1080/00207179.2022.2092557>.
- [11] F. Miranda-Villatoro, B. Brogliato, and F. Castanos. Set-valued sliding-mode control of uncertain linear systems: continuous and discrete-time analysis. *SIAM Journal on Control and Optimization*, 56(3):1756–1793, 2018.
- [12] F. Tatari and H. Modares. Deterministic and stochastic fixed-time stability of discrete-time autonomous systems. *IEEE/CAA Journal of Automatica Sinica*, 10(JAS-2022-1546):1, 2023. doi: 10.1109/JAS.2023.123405.