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# The Tropical Nullstellensatz and Positivstellensatz for Sparse Polynomial Systems

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## ABSTRACT

Grigoriev and Podolskii (2018) have established a tropical analog of the effective Nullstellensatz, showing that a system of tropical polynomial equations is solvable if and only if a linearized system obtained from a truncated Macaulay matrix is solvable. They provided an upper bound of the minimal admissible truncation degree, as a function of the degrees of the tropical polynomials. We establish a tropical nullstellensatz adapted to *sparse* tropical polynomial systems. Our approach is inspired by a construction of Canny-Emiris (1993), refined by Sturmfels (1994). This leads to an improved bound of the truncation degree, which coincides with the classical Macaulay degree in the case of  $n + 1$  equations in  $n$  unknowns. We also establish a tropical positivstellensatz, allowing one to decide the inclusion of tropical basic semialgebraic sets. This allows one to reduce decision problems for tropical semi-algebraic sets to the solution of systems of tropical linear equalities and inequalities. The later systems are known to be reducible to mean payoff games, which can be solved in practice, in a scalable way, by value iteration methods. We illustrate this approach by examples.

## CCS CONCEPTS

• **Theory of computation** → *Algorithmic game theory*; • **Mathematics of computing** → *Computations on polynomials*.

## KEYWORDS

Tropical geometry, Polynomial systems, Zero-sum games, Algorithmic complexity

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## 1 INTRODUCTION

**Motivation.** A basic problem in computational tropical geometry consists in solving systems of equations or of inequations involving multivariate tropical polynomials.

Such systems arise in the study of *non-archimedean amoebas*, which are images by a non-archimedean valuation of an algebraic set [12]. A super-approximation of a non-archimedean amoeba can be obtained by translating the defining equations of the algebraic set in the tropical semifield. This leads to a system of tropical polynomial equations, the solution set of which is called a *tropical prevariety*. Moreover, when working over an algebraically closed non-archimedean field, this approximation is exact under appropriate genericity conditions, or if the defining set is sufficiently rich, see in particular the “fundamental theorem of tropical geometry” in [25], in which case the non-archimedean amoeba is called a *tropical variety*.

Similarly, the solution sets of systems of tropical polynomial inequalities provide upper approximations of images by a non-archimedean valuation of semi-algebraic sets over real closed non-archimedean fields, and these approximations are exact under genericity conditions [6, 20, 22]. Moreover, owing to their combinatorial nature, tropical polynomial systems are often easier to grasp than their classical analogues. These ideas, which can be traced back to works of Viro (see [33]), Gelfand, Kapranov, Zelevinsky [15], Sturmfels (see [18]), or Mikhalkin [28], are at the heart of tropical geometry, see [19, 25] for background.

Systems of tropical polynomial equations and inequations are also interesting in their own right, since they arise in specific applications, independently of the nonarchimedean interpretation. In particular, they arise in the computation of stationary behaviors of discrete event systems [10], see e.g. [3] for an application to performance evaluation of emergency call centers. Other motivations arise from auction theory [7].

Grigoriev and Podolskii established in [17] a *tropical Nullstellensatz*, which states that a system of tropical equations is solvable if and only if its linearization, gotten by truncating the Macaulay matrix up to appropriate degree, is solvable. Their results also apply to polynomial inequations. Since systems of tropical linear equations and inequations reduce to mean payoff games [2], this provides both theoretical tools (strong duality theorems) and algorithms. The proof of [17], confirming a conjecture made by Grigoriev, is based on ingenious geometric arguments. Moreover, they observed it leads to an estimate of the truncation degree which may not be optimal.

**Contribution.** We provide here a *tropical Nullstellensatz* for systems of tropical equations and inequations, taking into account the *sparse* structure. When specialized to full polynomials with given degrees, we recover the statement of Grigoriev and Podolskii, with an improved degree bound, matching the classical Macaulay degree bound for systems of  $n + 1$  equations in  $n$  unknowns. Our approach also leads to a *tropical Positivstellensatz*, allowing one to check the inclusion between two tropical basic semialgebraic sets, by reduction to a system with both strict and weak tropical linear inequalities.

To do so, we connect the tropical Nullstellensatz with a fundamental notion in classical polynomial system solving, the *Canny-Emiris sets*. In [9] and [13], Canny and Emiris developed an algorithm to compute sparse resultants, for systems of  $n + 1$  polynomial equations in  $n$  variables. This involves the construction of a submatrix of the Macaulay matrix, in which rows and columns are determined by considering the integer points of a generic perturbation of the Minkowski sum of the Newton polytopes of the polynomials. Their construction was generalized by Sturmfels in [31, §3], by considering a generic collection of polyhedral concave functions. Here, we apply this tool to a different problem, showing that any Canny-Emiris set leads to a tropically valid linearization. In a nutshell, we show that if a tropical system is unfeasible, then, the “row contents” arising in the Canny-Emiris construction provide a submatrix of the Macaulay matrix which is tropically nonsingular, and which therefore serves as an unfeasibility certificate. The case of inequalities is more delicate: it relies on additional ingredients, including the Shapley-Folkman theorem, and it leads to a bound which is generally not optimal. It also relies on a refined reduction to mean payoff games, allowing one to deal with strict inequalities [4].

A potential advantage of the linearization approach lies in the *scalability* of mean-payoff games algorithms. Whereas the existence of a polytime algorithm to solve a mean-payoff game is a long standing open problem, in practice, typical large sparse instances of mean-payoff games can be solved efficiently by value-iteration type algorithms. We illustrate our approach by numerical examples, obtained using a preliminary implementation.

For simplicity, we state and prove the present tropical Nullstellensätze and Positivstellensätze by working over the “vanilla” tropical semifield  $\mathbb{R}_\infty$ . More generally, a tropical semifield can be constructed over any divisible (totally) ordered group, and this is helpful to study “higher rank” tropicalizations. Then, using the completeness of the first order theory of divisible ordered groups, it is immediate to see that the present results carry over to this case.

**Related work.** As discussed above, the present work is inspired by [17]. The main contribution here is the handling of sparsity, with a new proof, leading to an improved truncation bound in the dense case. We also handle strict inequalities, which allows us to interpret the results in terms of “Positivstellensätze”. However, contrarily to what is done in [17], we limit our attention to the “toric case”, looking only for solutions in  $\mathbb{R}^n$ .

The standard approach to the computation of tropical prevarieties is to exploit the duality between arrangements of tropical hypersurfaces and mixed-polyhedral subdivisions. In that way, decision problems concerning tropical prevarieties can be reduced to the enumeration of mixed cells, see [23, 26]. A number of current

works deal with the efficient computation of tropical varieties and prevarieties, see [27], [16] and the references therein.

In these approaches, *all* solutions are typically constructed. In contrast, when applying the tropical Nullstellensatz or Positivstellensatz, we do not need to enumerate all the cells of a polyhedral complex (which amounts to testing all candidate solutions). Instead, we directly decide the feasibility or unfeasibility by reduction to a mean payoff game. In the special case of linear inequalities, the advantage of the second approach can be quantified: there are exponentially many tropically extreme solutions [5], hence exponentially many cells, making the enumeration unfeasible unless the dimension is only of a few dozens, whereas checking the feasibility reduces to solving a mean payoff game with a size linearly bounded in input size, which can be done for large systems with pseudo-polynomial complexity bounds. This advantage subsists for a significant class of higher degree instances, although the size of the game now becomes exponential in the input size. We expect the present approach to be especially useful in the “Positivstellensatz” case, in which the exhaustive cell enumeration is especially heavy, whereas only a single feasible cell, or an unfeasibility certificate, is looked for.

A different approach relies on the application of general-purpose SMT solving algorithms [24].

Finally, we note that tropical Nullstellensätze of different natures have been established in [8, 30].

## 2 THE SPARSE TROPICAL NULLSTELLENSATZ

### 2.1 Tropical polynomial systems and matrices

The tropical or max-plus semifield,  $\mathbb{R}_\infty$ , is the set  $\mathbb{R} \cup \{-\infty\}$  equipped with the addition  $a \oplus b := \max(a, b)$  and the multiplication  $a \odot b := a + b$ . We denote by  $\mathbb{0} := -\infty$  and  $\mathbb{1} := 0$  the zero and unit elements of this semifield. We shall often denote the tropical multiplication by concatenation, writing  $ab$  instead of  $a \odot b$ .

**Definition 2.1.** A *formal tropical (Laurent) polynomial*  $f$  in  $n$  variables is a map  $f : \mathbb{Z}^n \rightarrow \mathbb{R}_\infty$ ,  $\alpha \mapsto f_\alpha$ , such that  $f_\alpha = \mathbb{0}$  except for finitely many values of  $\alpha$ . We write  $\alpha = (\alpha_1, \dots, \alpha_n)$ , and use the following notation:

$$f = f(X) = \bigoplus_{\alpha \in \mathbb{Z}^n} f_\alpha X^\alpha \quad \text{with } X^\alpha := X_1^{\alpha_1} \dots X_n^{\alpha_n} .$$

We denote by  $\mathbb{R}_\infty[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  the semiring of formal tropical Laurent polynomials in  $n$  variables.

The *support* of  $f$  is the set

$$\text{supp}(f) := \{\alpha \in \mathbb{Z}^n : f_\alpha \neq \mathbb{0}\} .$$

We say that  $f$  is an *ordinary* polynomial if its support is included in  $\mathbb{Z}_{\geq 0}^n$ . The *polynomial function* associated to  $f$  is the partially defined function  $\hat{f}$  from  $\mathbb{R}_\infty^n$  to  $\mathbb{R}_\infty$  obtained by evaluating  $f$ .

In particular,  $\hat{f}(x)$  is well defined for all  $x \in \mathbb{R}^n$ , and if  $f$  is an ordinary polynomial, it is also well defined for all  $x \in \mathbb{R}_\infty^n$ . Then, we have, with the ordinary notation

$$\hat{f}(x) = \max_{\alpha \in \mathbb{Z}^n} (f_\alpha + \langle x, \alpha \rangle) , \quad (1)$$

with  $\langle x, \alpha \rangle = \sum_i x_i \cdot \alpha_i$  and the convention  $(-\infty) \cdot 0 = 0$ .

The map  $f \mapsto \hat{f}$  is a special case of the Legendre-Fenchel transform. In particular, it is not injective.

**Definition 2.2.** Let  $f = \bigoplus_{\alpha \in \mathbb{Z}^n} f_\alpha X^\alpha \in \mathbb{R}_\infty[X_1^\pm, \dots, X_n^\pm]$  be a tropical polynomial. An element  $x = (x_1, \dots, x_n)$  of  $\mathbb{R}_\infty^n$  is called a *tropical root* of  $f$  whenever the expression (1) is well defined and the maximum in this expression is achieved by at least two distinct values of  $\alpha \in \mathbb{Z}^n$ . This is denoted by  $f(x) \nabla 0$ . Whenever  $x \in \mathbb{R}^n$ , we say that  $x$  is a *finite tropical root*.

Note in particular that if  $\hat{f}(x) = 0$ , then  $x$  is a tropical root. The notation “ $\nabla$ ” (to be read “balances”) originates from [1, Section 4], in which it is defined in a setting of semirings with “symmetry”. Here, we use it only in a formal sense, thinking of  $f(x) \nabla 0$  as the tropical analog of an equation  $f(x) = 0$ .

More generally, given a collection  $f = (f_1, \dots, f_k)$  of polynomials, we denote by  $f(x) \nabla 0$  the system

$$f_i(x) \nabla 0 \quad \text{for all } 1 \leq i \leq k$$

of tropical polynomial equations with unknown  $x \in \mathbb{R}^n$ .

A matrix with entries in the tropical semifield  $\mathbb{R}_\infty$  is called a *tropical matrix*. The tropical addition  $\oplus$  and multiplication  $\odot$  on tropical matrices are defined by replacing the usual operations by their tropical version in the definition of the matrix operations. In particular, if  $A = (a_{ij})$  is a  $p \times m$  tropical matrix and if  $y = (y_j)$  is an element of  $\mathbb{R}_\infty^m$ , then we have  $(A \odot y)_i = \bigoplus_{1 \leq j \leq m} a_{ij} \odot y_j$ , for all  $1 \leq i \leq p$ , which allows us to identify every row of  $A$  to a formal tropical polynomial of degree 1 in the indeterminates  $y_1, \dots, y_m$ . Hence we will write  $A \odot y \nabla 0$  to mean that for all  $1 \leq j \leq m$ , the maximum in the expression  $\max_{1 \leq i \leq p} (a_{ij} + y_j)$  is achieved twice at least. The set of vectors  $y \in \mathbb{R}_\infty^m$  such that  $A \odot y \nabla 0$  is called the *tropical right null space* or *tropical kernel* of the matrix  $A$ .

## 2.2 Statement of the Nullstellensätze

We first recall the tropical analogue of the notion of Macaulay matrix, already considered in [17].

**Definition 2.3.** Given a collection of tropical polynomials  $f = (f_1, \dots, f_k)$ , we define the *Macaulay matrix*  $\mathcal{M}$  of  $f$  as follows: the rows of  $\mathcal{M}$  are indexed by pairs  $(i, \alpha)$  where  $1 \leq i \leq k$  and  $\alpha \in \mathbb{Z}^n$ , the columns of  $\mathcal{M}$  are indexed by integer vectors  $\beta \in \mathbb{Z}^n$ , and for given  $(i, \alpha)$  and  $\beta$ , the entry  $\mathcal{M}_{(i, \alpha), \beta}$  of  $\mathcal{M}$  is equal to the coefficient of the monomial  $X^\beta$  in the polynomial  $X^\alpha f_i(X)$ , or to  $-\infty$  if no such monomial exists.

Given a Macaulay matrix  $\mathcal{M}$  as above, a nonempty finite subset  $\mathcal{E}$  of  $\mathbb{Z}^n$ , and a collection  $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_k)$  of subsets of  $\mathbb{Z}^n$ , we denote by  $\mathcal{M}_{\mathcal{E}}^{\mathcal{A}}$  the submatrix of  $\mathcal{M}$  consisting only of the columns with indices  $\beta \in \mathcal{E}$ , and the rows indexed by pairs  $(i, \alpha)$  where  $1 \leq i \leq k$  and  $\alpha \in \mathbb{Z}^n$  such that  $\alpha + \mathcal{A}_i \subset \mathcal{E}$ . When the polynomials  $f_i$  have their support equal to  $\mathcal{A}_i$ , and  $\mathcal{M}$  is associated to  $f = (f_1, \dots, f_k)$ , we simply write  $\mathcal{M}_{\mathcal{E}}$  instead of  $\mathcal{M}_{\mathcal{E}}^{\mathcal{A}}$ .

Note that, equivalently,  $\mathcal{M}_{\mathcal{E}}$  is the submatrix of  $\mathcal{M}$  consisting of the columns with indices  $\beta \in \mathcal{E}$ , and the rows that have all their finite entries in these columns.

Given a collection  $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_k)$  of subsets of  $\mathbb{Z}^n$ , for all  $1 \leq i \leq k$ , let  $Q_i$  be the convex hull of  $\mathcal{A}_i$  and set  $Q := Q_1 + \dots + Q_k$  (the Minkowski sum of the sets  $Q_i$ ). Denote by  $V \subseteq \mathbb{R}^n$  the vector

space directing the affine hull of  $Q$ , and take a *generic vector*  $\delta \in V + \mathbb{Z}^n$  (meaning, the sum of an arbitrary element of  $\mathbb{Z}^n$  and of a generic vector of  $V$ ), and consider the set

$$\mathcal{E} := (Q + \delta) \cap \mathbb{Z}^n. \quad (2)$$

We will refer to nonempty sets of this form as *Canny-Emiris subsets* of  $\mathbb{Z}^n$  associated to the collection  $\mathcal{A}$ . Note that for  $\delta$  small enough, we always have the inclusion

$$\text{relint}(Q) \cap \mathbb{Z}^n \subseteq \mathcal{E} \subseteq Q \cap \mathbb{Z}^n,$$

where  $\text{relint}$  denotes the relative interior. Moreover, even when  $\text{relint}(Q) \cap \mathbb{Z}^n$  is empty, one can find a small generic  $\delta \in V$  such that  $\mathcal{E}$  is nonempty.

Now, for a collection  $f = (f_1, \dots, f_k)$  of tropical polynomials, we shall consider in particular the collection  $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_k)$ , where, for  $i = 1, \dots, k$ ,  $\mathcal{A}_i$  is the support of  $f_i$ . In that case, the set  $Q_i$  is referred to as the *Newton polytope* of  $f_i$ , and we shall also refer to  $Q$  as the *Newton polytope* of  $f$ . Also the Canny-Emiris subsets  $\mathcal{E}$  associated to the collection  $\mathcal{A}$  are referred to as the Canny-Emiris sets associated to  $f$ . We have the following result, showing that any superset of a Canny-Emiris set leads to a valid linearization of a tropical polynomial system.

**THEOREM 2.4 (NULLSTELLENSATZ FOR SPARSE TROPICAL POLYNOMIAL SYSTEMS).** *There exists a common root  $x \in \mathbb{R}^n$  to the system  $f(x) \nabla 0$  if and only if there exists a vector  $y \in \mathbb{R}^{\mathcal{E}'}$  in the tropical kernel of the submatrix  $\mathcal{M}_{\mathcal{E}'}$  of the Macaulay matrix  $\mathcal{M}$  associated to the collection  $f$  — i.e.  $\mathcal{M}_{\mathcal{E}'} \odot y \nabla 0$  — where  $\mathcal{E}'$  is any subset of  $\mathbb{Z}^n$  containing a Canny-Emiris subset  $\mathcal{E}$  of  $\mathbb{Z}^n$  associated to  $f$ . Moreover, if  $\mathcal{E}' = \mathcal{E}$ , these conditions are equivalent to the existence of a vector  $y \in \mathbb{R}_\infty^{\mathcal{E}} \setminus \{0\}$  such that  $\mathcal{M}_{\mathcal{E}} \odot y \nabla 0$ .*

When we have no particular information on the supports of the polynomials  $f_i$  besides that they are ordinary tropical polynomials with respective degrees  $d_i$ , we denote by  $\mathcal{M}_N$  the submatrix  $\mathcal{M}_{\bar{\mathcal{E}}}^{\mathcal{A}}$  of  $\mathcal{M}$  with  $\bar{\mathcal{E}} = N \Delta \cap \mathbb{Z}_{\geq 0}^n$ , and  $\mathcal{A}_i = d_i \Delta \cap \mathbb{Z}_{\geq 0}^n$ . Here  $\Delta$  denotes the unit simplex  $\{\alpha \in \mathbb{R}_{\geq 0}^n : |\alpha| = \alpha_1 + \dots + \alpha_n \leq 1\}$ . In this case, the integer  $N$  is called the *truncation degree* of the Macaulay submatrix  $\mathcal{M}_N$ .

More generally, if we only know, for all  $i = 1, \dots, k$ , that the support of  $f_i$  is included in  $\mathcal{A}_i$  (which plays now the role of an *a priori* support), we shall consider a bigger set

$$\bar{\mathcal{E}} := Q \cap \mathbb{Z}^n,$$

where  $Q$  is defined using the collection  $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_k)$ . Recall that for  $\delta$  small enough  $\mathcal{E} \subset \bar{\mathcal{E}}$ . In the following result we consider the matrices  $\mathcal{M}_{\mathcal{E}'}^{\mathcal{A}}$  with  $\mathcal{E}' \supset \bar{\mathcal{E}}$ .

**THEOREM 2.5 (NULLSTELLENSATZ FOR SPARSE TROPICAL POLYNOMIAL SYSTEMS WITH A PRIORI SUPPORTS).** *There exists a common root  $x \in \mathbb{R}^n$  to the system  $f(x) \nabla 0$  if and only if there exists a vector  $y \in \mathbb{R}^{\mathcal{E}'}$  in the tropical kernel of the submatrix  $\mathcal{M}_{\mathcal{E}'}$  of the Macaulay matrix  $\mathcal{M}$  associated to the collection  $f$ , where  $\mathcal{E}'$  is any subset of  $\mathbb{Z}^n$  containing  $\bar{\mathcal{E}}$  and  $\mathcal{A}$  is a collection of a priori supports of the  $f_i$ . Moreover, when the Newton polytope of  $f$  has same dimension as  $Q$ , one can replace  $\bar{\mathcal{E}}$  by any Canny-Emiris set  $\mathcal{E}$  associated to  $\mathcal{A}$ .*

When  $f = (f_1, \dots, f_k)$  is a collection of ordinary tropical polynomials  $f_i$  with respective degree  $d_i$ , applying Theorem 2.5 with  $\mathcal{A}_i = d_i \Delta \cap \mathbb{Z}_{\geq 0}^n$ , we deduce the following result.

**COROLLARY 2.6.** *There exists a common root  $x \in \mathbb{R}^n$  to the system  $f(x) \nabla \mathbb{0}$  if and only if there exists a vector  $y \in \mathbb{R}^{\mathcal{E}'}$  such that  $M_N \odot y \nabla \mathbb{0}$ , where  $\mathcal{E}' = N \Delta \cap \mathbb{Z}_{\geq 0}^n$  and  $N \geq d_1 + \dots + d_k$ . Moreover, when the Newton polytope of  $f$  has full dimension, one can replace the above lower bound on  $N$  by  $N \geq d_1 + \dots + d_k - n$ .*

**Remark 2.7.** This improves on Grigoriev and Podolskii's Tropical Dual Nullstellensatz from [17, Theorem 3.3 (i)], which requires  $N \geq (n+2)(d_1 + \dots + d_k)$ . Moreover, under the condition of the second assertion, and when  $k = n+1$ , we obtain the classical Macaulay bound  $N \geq d_1 + \dots + d_{n+1} - n$ .

However, in other cases, the lower bound in Corollary 2.6 is not necessarily optimal. For instance for degree one polynomials, the bound is  $N \geq k$  for general such polynomials, although  $N = 1$  is sufficient. Even the bound of the second assertion, which is  $N \geq k - n$ , is not optimal for  $k > n+1$  as well.

### 2.3 Preliminary results

As said in the introduction, the proof of Theorem 2.4 is obtained by applying the tools of [9, 13, 31] in a different context. Therefore, we are giving the main steps. In particular, we state a number of lemmas which translate the properties of Cannis-Emiris construction in the tropical framework. These are used in the proof of Theorem 2.4.

**2.3.1 Nonsingular and diagonally dominant tropical matrices.** Let  $A = (a_{ij})$  be a  $d \times d$  square tropical matrix. We will say that  $A$  is *tropically nonsingular* whenever the tropical kernel of  $A$  is reduced to the set  $\{0\}$  (see for instance [11, 21], [2] for equivalent properties). Moreover,  $A$  is said to be *diagonally dominant* in the tropical sense whenever we have the inequalities

$$a_{ii} > a_{ij} \quad \text{for all } 1 \leq i, j \leq d \text{ such that } i \neq j,$$

The following properties of tropical matrices are elementary.

**LEMMA 2.8.** *Any diagonally dominant tropical matrix is necessarily tropically nonsingular.*

**LEMMA 2.9.** *Let  $A = (a_{ij})$  be a  $d \times d$  tropical matrix, and let  $\varepsilon = (\varepsilon_j) \in \mathbb{R}^d$ . Set  $\tilde{A} = (\tilde{a}_{ij})$  with  $\tilde{a}_{ij} = a_{ij} + \varepsilon_j$  for all  $1 \leq i, j \leq d$ . Then  $A$  is tropically nonsingular if and only if  $\tilde{A}$  is tropically nonsingular.*

**LEMMA 2.10.** *Let  $A$  be a  $p \times m$  tropical matrix, and assume that  $A$  can be written up to permutations of rows and columns as a block lower-triangular matrix*

$$A = \begin{pmatrix} A^{(d)} & \mathbb{0} \\ * & * \end{pmatrix}$$

with  $A^{(d)}$  a  $d \times d$  tropically nonsingular submatrix. Then the equation  $A \odot y \nabla \mathbb{0}$  of unknown  $y$  has no solution in  $\mathbb{R}^m$ .

**2.3.2 Generalities on sup-convolution and Minkowski sums.** The following notions and results of convex analysis will also be used in the proof of Theorem 2.4, and can be found for instance in [29].

The *Minkowski sum* of two subsets  $E, F \subset \mathbb{R}^n$  is defined as  $E + F = \{x + y : x \in E, y \in F\}$ . Moreover, the *sup-convolution* is the binary operator  $\square$  defined on the set of functions from  $\mathbb{R}^n$  to  $\mathbb{R} \cup \{\pm\infty\}$

by  $(g \square h)(x) = \sup_{y+z=x} g(y) + h(z)$ , with the convention  $(-\infty) + (+\infty) = -\infty$ . Recall also that the operations of sup-convolution and Minkowski sum are commutative and associative.

For any function  $h : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\pm\infty\}$ , we define the *support* of  $h$  as  $\text{supp}(h) = \{x \in \mathbb{R}^n : h(x) > -\infty\}$ , and its *hypograph* as the set  $\text{hypo}(h) = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : t \leq h(x)\}$ . In particular, if  $g$  and  $h$  are upper semicontinuous, take values in  $\mathbb{R}_\infty$ , and have compact support, then, the supremum in the expression of  $(g \square h)(x)$  is achieved and  $g \square h$  also has compact support equal to the Minkowski sum of the supports of  $g$  and  $h$ , and the hypograph of  $g \square h$  is also the Minkowski sum of the hypographs of  $g$  and  $h$ .

We shall extensively use the notion of (convex) polyhedron of  $\mathbb{R}^n$  (or  $\mathbb{R}^n \times \mathbb{R}$ ), that is an intersection of a finite number of affine half spaces, and of concave polyhedral function  $h : \mathbb{R}^n \rightarrow \mathbb{R}_\infty$ , that is a function whose hypograph is a closed (convex) polyhedron. The *normal cone*  $N_x(P)$  to a polyhedron  $P \subset \mathbb{R}^n$  at a point  $x \in P$  is defined by

$$N_x(P) = \{y \in \mathbb{R}^n : \langle y, z - x \rangle \leq 0, \forall z \in P\}.$$

One can then define the notions of normal cone  $N_F(P)$  at a face  $F$  of  $P$  and of normal fan of  $P$ , which is in correspondance with the set of faces of  $P$ , see for instance [32, Section 7]. This correspondance implies in particular that if  $F$  is a face of  $P$ , and  $W$  is the vector subspace of  $\mathbb{R}^n$  directing the affine hull of  $P$ , then

$$\dim(N_F(P) \cap W) = \dim(P) - \dim(F). \quad (3)$$

We also say that a convex subset  $F$  of  $\mathbb{R}^n \times \mathbb{R}$  is *vertical* if the orthogonal of the vector space directing the affine hull of  $F$  is included in  $\mathbb{R}^n \times \{0\}$ . In particular if  $F$  is a face of a polyhedron of  $\mathbb{R}^n \times \mathbb{R}$ , this face is vertical if its normal cone is included in  $\mathbb{R}^n \times \{0\}$ .

For all functions  $h : \mathbb{R}^n \rightarrow \mathbb{R}_\infty$ , and for all  $x \in \mathbb{R}^n$ , we set

$$C(x, h) := \arg \max_{q \in \mathbb{R}^n} (\langle q, x \rangle + h(q))$$

$$\text{and } \mathcal{F}(x, h) := \{(q, h(q)) : q \in C(x, h)\}.$$

We make the following crucial observations.

**OBSERVATION 2.11.** (a) *Let  $h$  be a concave polyhedral function non-identically  $-\infty$ . Then  $\mathcal{F}(x, h)$  is a face of the hypograph of  $h$ , which is obtained as the intersection of this hypograph with a supporting hyperplane of outer normal vector  $(x, 1)$ . In particular, this face is non-vertical, and hence it is a proper face of  $\text{hypo}(h)$ , i.e.  $\mathcal{F}(x, h) \subsetneq \text{hypo}(h)$ .*

(b) *For such a function  $h$ , if  $F$  is a non-vertical face of  $\text{hypo}(h)$ , then  $F = \mathcal{F}(x, h)$  if and only if  $(x, 1)$  is in the relative interior of  $N_F(\text{hypo}(h))$ . In particular, if  $F$  is a facet of  $\text{hypo}(h)$ , then there exists a unique vector  $x \in \mathbb{R}^n$  such that  $F = \mathcal{F}(x, h)$  and  $x$  is in the vector space  $V$  directing the affine hull of  $\text{supp}(h)$ . Indeed, from (3), we have  $\dim(N_F(\text{hypo}(h)) \cap W) = (m+1) - m = 1$  where  $W = V \times \mathbb{R}$  and  $m$  is the dimension of  $V$ , and thus if  $(x, 1)$  and  $(x', 1)$  are both in the half-line  $N_F(\text{hypo}(h)) \cap W$ , then it follows that  $x' = x$ .*

We now state a useful lemma on convex polyhedra, written in the context of hypographs.

**LEMMA 2.12.** *Let  $h_1, \dots, h_\ell$  denote concave non-identically  $-\infty$  polyhedral functions from  $\mathbb{R}^n$  to  $\mathbb{R}_\infty$ , and set  $h = h_1 \square \dots \square h_\ell$ . Let  $F$  be a non-vertical face of  $\text{hypo}(h)$ , and let  $x \in \mathbb{R}^n$  be such that  $(x, 1)$*

is in the relative interior of the normal cone of  $\text{hypo}(h)$  at this face. Consider a point  $(q, \lambda) \in F$  such that

$$(q, \lambda) = (q_1, \lambda_1) + \cdots + (q_\ell, \lambda_\ell)$$

with  $(q_i, \lambda_i) \in \text{hypo}(h_i)$  for all  $1 \leq i \leq \ell$ . Then, for all  $1 \leq i \leq \ell$ ,  $(q_i, \lambda_i) \in \mathcal{F}(x, h_i)$ , and  $(x, 1)$  is in the normal cone of  $\text{hypo}(h_i)$  at point  $(q_i, \lambda_i)$ . In particular

$$F = \mathcal{F}(x, h) = \mathcal{F}(x, h_1) + \cdots + \mathcal{F}(x, h_\ell) .$$

## 2.4 Proving the Tropical Nullstellensatz

**2.4.1 The Canny-Emiris construction.** Here, we assume given a collection of polynomials  $f = (f_1, \dots, f_k)$ , a Canny-Emiris subset  $\mathcal{E}$  associated to  $f$ , and a subset  $\mathcal{E}'$  of  $\mathbb{Z}^n$  containing  $\mathcal{E}$ . We also assume in this section that the polynomials  $f_i$  do not share a common (finite) root, so the system  $f(x) \nabla \mathbb{0}$  has no solution  $x \in \mathbb{R}^n$ . We can now describe the construction of Canny and Emiris from [9] and [13], which was generalized by Sturmfels in [31, §3]. We will apply it to the collection  $f$  under the previous assumption.

We shall also consider, for all  $i = 1, \dots, k$ , the coefficient map of  $f_i$ , defined from  $\mathbb{R}^n$  to  $\mathbb{R}_\infty$ , by  $\alpha \mapsto f_{i,\alpha}$  if  $\alpha \in \mathcal{A}_i$  and  $\mathbb{0}$  otherwise, and its concave envelope  $h_i : \mathbb{R}^n \rightarrow \mathbb{R}_\infty$ . We set  $h := h_1 \square \cdots \square h_k$ . Then,  $\text{hypo}(h_i)$  and  $\text{hypo}(h)$  correspond to the sets  $Q_{i,\omega}$  and  $Q_\omega$  in the construction of [31], and are referred to as *extended Newton polytopes* in [17].

**OBSERVATION 2.13.** For all  $1 \leq i \leq k$ , by construction of the maps  $h_i$ , we have  $h_i(\alpha) \geq f_{i,\alpha} \forall \alpha \in \mathcal{A}_i$ , with equality whenever the monomial  $f_{i,\alpha} X^\alpha$  is essential in  $f_i$ , that is whenever the point  $(\alpha, f_{i,\alpha})$ , or the point  $(\alpha, h_i(\alpha))$ , is a vertex of  $\text{hypo}(h_i)$ .

We shall use the notations given in Section 2.2. Then, the projection of  $\text{hypo}(h)$  onto  $Q$  induces a mixed coherent subdivision of  $Q$ , given by the points of non-differentiability of  $h$ . The following observation follows readily from the genericity of  $\delta$ .

**OBSERVATION 2.14.** Given  $p \in \mathcal{E}$ , the couple  $(p - \delta, h(p - \delta))$  lies in the relative interior of a unique non-vertical facet  $F$  of  $\text{hypo}(h)$ , or equivalently,  $p - \delta$  lies in the relative interior of the associated cell  $C$  of  $Q$ , obtained as the image of  $F$  by the projection mapping. Moreover, there is a unique  $x \in V$  such that  $F = \mathcal{F}(x, h)$  and  $C = C(x, h)$ .

Let  $p \in \mathcal{E}$ , and consider the associated  $x \in V$ . Since  $f(x) \not\prec \mathbb{0}$ , we know that at least one of the  $C(x, h_i)$  is a singleton, and we let  $j$  be the maximal index such that  $C(x, h_j) = \{a_j\}$  for some  $a_j \in \mathcal{A}_j$ . We call the couple  $(j, a_j)$  the *row content* of  $p$ . Then, since  $p - \delta \in C(x, h)$ , using Lemma 2.12, we obtain that

$$p - \delta = q_1 + \cdots + q_j + \cdots + q_k \quad (4)$$

with  $q_j = a_j$  and  $q_i \in C(x, h_i)$  for all  $1 \leq i \neq j \leq k$ .

We then construct the matrix  $\mathcal{M}_{\mathcal{E}\mathcal{E}'}$  indexed by  $\mathcal{E} \times \mathcal{E}'$  as follows : for every  $p \in \mathcal{E}$ , we associate to  $p$  its row content  $(j, a_j)$ , and then we set the row  $p$  of  $\mathcal{M}_{\mathcal{E}\mathcal{E}'}$  to be the row  $(j, p - a_j)$  of the matrix  $\mathcal{M}_{\mathcal{E}'}$ . Note that, the latter row exists in  $\mathcal{M}_{\mathcal{E}'}$ , since, using (4), we have that  $p - a_j + \mathcal{A}_j \subset \mathcal{E} \subset \mathcal{E}'$ , or equivalently all the exponents appearing in the support of  $X^{p-a_j} f_j(X)$ , belong to  $\mathcal{E}$  and thus to  $\mathcal{E}'$ . Thus, we end up with a matrix  $\mathcal{M}_{\mathcal{E}\mathcal{E}'} = (m_{pp'})_{(p,p') \in \mathcal{E} \times \mathcal{E}'}$  which can be written as a block matrix

$$\mathcal{M}_{\mathcal{E}\mathcal{E}'} = \begin{pmatrix} \mathcal{M}_{\mathcal{E}\mathcal{E}} & \mathbb{0} \\ * & * \end{pmatrix} \quad (5)$$

where  $\mathcal{M}_{\mathcal{E}\mathcal{E}} = (m_{pp'})_{(p,p') \in \mathcal{E} \times \mathcal{E}}$  is a square matrix.

**2.4.2 The proof of Theorem 2.4.** The following result is deduced using Lemma 2.12 and Observation 2.11.

**LEMMA 2.15.** Assume that the system  $f(x) \nabla \mathbb{0}$  has no solution  $x \in \mathbb{R}^n$ . Let  $p \in \mathcal{E}$ , let  $(j, a_j)$  be its row content, and denote  $h_j^\vee = h_1 \square \cdots \square h_{j-1} \square h_{j+1} \square \cdots \square h_k$ . Then for all  $p' \in \mathcal{E}$  and  $a'_j \in \mathbb{Z}^n$  such that  $p' = p - a_j + a'_j$ , we have

$$h(p' - \delta) \geq h_j(a'_j) + h_j^\vee(p - \delta - a_j), \quad (6)$$

with equality if and only if  $p' = p$  and  $a'_j = a_j$ .

**PROOF OF THEOREM 2.4.** The ‘only if’ implication is straightforward, for if  $x \in \mathbb{R}^n$  is solution of the system  $f(x) \nabla \mathbb{0}$ , then its image by the Veronese embedding  $\mathbb{R}^n \rightarrow \mathbb{R}^{\mathcal{E}'}$ ,  $x \mapsto \text{ver}(x) := (x^v)_{v \in \mathcal{E}'}$  yields a finite vector in the tropical kernel of the matrix  $\mathcal{M}_{\mathcal{E}'}$ . Note that by convention, this also holds when  $\mathcal{M}_{\mathcal{E}'}$  is empty.

For the converse implication, we rather show the contrapositive. Assume that the tropical polynomial system  $f(x) \nabla \mathbb{0}$  does not have a solution, and consider the matrix  $\mathcal{M}_{\mathcal{E}\mathcal{E}'}$  obtained in Section 2.4.1. Note that, since  $\mathcal{E}$  is nonempty, this construction implies that the matrix  $\mathcal{M}_{\mathcal{E}\mathcal{E}'}$  and a fortiori the matrix  $\mathcal{M}_{\mathcal{E}'}$  is nonempty. Let  $\widetilde{\mathcal{M}}_{\mathcal{E}\mathcal{E}} = (\widetilde{m}_{pp'})_{(p,p') \in \mathcal{E} \times \mathcal{E}}$  with  $\widetilde{m}_{pp'} = m_{pp'} - h(p' - \delta)$ . We show that  $\widetilde{\mathcal{M}}_{\mathcal{E}\mathcal{E}}$  is tropically diagonally dominant. Indeed, let  $p, p' \in \mathcal{E}$ , and let  $(j, a_j)$  be the row content of  $p$ . Then, we have for  $a'_j \in \mathbb{Z}^n$  such that  $p' = p - a_j + a'_j$

$$\widetilde{m}_{pp'} = f_{j,a'_j} - h(p' - \delta) \text{ if } a'_j \in \mathcal{A}_j \text{ and } m_{pp'} = \mathbb{0} \text{ otherwise.}$$

By definition of the row content,  $\mathcal{F}(x, h_j) = \{(a_j, h_j(a_j))\}$ . Hence,  $(a_j, h_j(a_j))$  is a vertex of  $\text{hypo}(h_j)$ , which implies, by Observation 2.13, that the monomial  $f_{j,a_j} X^{a_j}$  is essential in  $f_j$ , and that we have  $h_j(a_j) = f_{j,a_j}$ . From the equality case of Lemma 2.15, we obtain that  $\widetilde{m}_{pp} = -h_j^\vee(p - \delta - a_j)$ . Consider now  $p' \neq p$ . Since  $h_j(a'_j) \geq f_{j,a'_j}$ , by rearranging the inequality of Lemma 2.15, we obtain that

$$\widetilde{m}_{pp} = -h_j^\vee(p - \delta - a_j) \geq f_{j,a'_j} - h(p' - \delta) = \widetilde{m}_{pp'} ,$$

with equality if and only if  $p' = p$  and  $h_j(a'_j) = f_{j,a'_j}$ . This shows that the matrix  $\widetilde{\mathcal{M}}_{\mathcal{E}\mathcal{E}}$  is tropically diagonally dominant. In particular, it follows that any two rows of  $\mathcal{M}_{\mathcal{E}\mathcal{E}}$  are distinct, and thus the matrix  $\mathcal{M}_{\mathcal{E}\mathcal{E}'}$  is indeed a submatrix of  $\mathcal{M}_{\mathcal{E}'}$ . Therefore, the matrix  $\mathcal{M}_{\mathcal{E}'}$  has the following form

$$\mathcal{M}_{\mathcal{E}'} = \begin{pmatrix} \mathcal{M}_{\mathcal{E}\mathcal{E}} & \mathbb{0} \\ * & * \end{pmatrix} .$$

Applying Lemma 2.8, Lemma 2.9 and Lemma 2.10, we obtain that there cannot exist a finite vector in the tropical kernel of  $\mathcal{M}_{\mathcal{E}'}$ .  $\square$

## 3 THE TROPICAL POSITIVSTELLENSATZ FOR MIXED SYSTEMS OF EQUALITIES AND WEAK AND STRICT INEQUALITIES

### 3.1 Statement of the theorem

Now we formulate a result similar to Theorem 2.4, for systems mixing equalities and weak and strict inequalities between tropical polynomial functions (also called ‘two sided’ systems). More precisely, consider  $f^+ = (f_1^+, \dots, f_k^+)$  and  $f^- = (f_1^-, \dots, f_k^-)$  two

collections of  $k$  tropical polynomials. For  $1 \leq i \leq k$ , we denote by  $\mathcal{A}_i^+$  and  $\mathcal{A}_i^-$  respectively the supports of  $f_i^+$  and  $f_i^-$ , and set  $\mathcal{A}_i = \mathcal{A}_i^+ \cup \mathcal{A}_i^-$ , and  $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_k)$ . Equivalently,  $\mathcal{A}$  is the collection of supports of the polynomials  $f_i := f_i^+ \oplus f_i^-$ . Consider also a collection  $\triangleright = (\triangleright_1, \dots, \triangleright_k)$  of relations  $\triangleright_i \in \{\geq, =, >\}$ , for  $1 \leq i \leq k$ . We denote by  $f^+(x) \triangleright f^-(x)$  the system

$$\max_{\alpha \in \mathcal{A}_i^+} (f_{i,\alpha}^+ + \langle \alpha, x \rangle) \triangleright_i \max_{\alpha \in \mathcal{A}_i^-} (f_{i,\alpha}^- + \langle \alpha, x \rangle) \text{ for all } 1 \leq i \leq k, \quad (7)$$

of unknown  $x \in \mathbb{R}_\infty^n$ . Note that this is equivalent to the expressions  $\hat{f}_i^+(x) \triangleright_i \hat{f}_i^-(x)$ ,  $i = 1, \dots, k$ . Finally, we denote by  $M^+$  and  $M^-$  the Macaulay matrices associated to  $f^+$  and  $f^-$  respectively, so with entries  $M_{(i,\alpha),\beta}^\pm = f_{i,\beta-\alpha}^\pm$ . Then, for any subset  $\mathcal{E}$  of  $\mathbb{Z}^n$ , we denote by  $M_{\mathcal{E}}^+$  and  $M_{\mathcal{E}}^-$  the submatrices associated to  $\mathcal{E}$  and the collection  $\mathcal{A}$  defined above, that is  $M_{\mathcal{E}}^+ = (M^+)_{\mathcal{E}}^{\mathcal{A}}$  and  $M_{\mathcal{E}}^- = (M^-)_{\mathcal{E}}^{\mathcal{A}}$ . Moreover, we denote by  $M_{\mathcal{E}}^+ \odot y \triangleright M_{\mathcal{E}}^- \odot y$  the following system of tropical linear inequalities:

$$\max_{\beta \in \mathcal{E}} (M_{(i,\alpha),\beta}^+ + y_\beta) \triangleright_i \max_{\beta \in \mathcal{E}} (M_{(i,\alpha),\beta}^- + y_\beta) .$$

Let  $Q$  be the Newton polytope of the collection  $f = (f_1, \dots, f_k)$ , that is  $Q = Q_1 + \dots + Q_k$ , where  $Q_i$  is the convex hull of  $\mathcal{A}_i$ . We now call *Canny-Emiris subsets* of  $\mathbb{Z}^n$  associated to the pair of collections  $(f^+, f^-)$  any set  $\mathcal{E}$  of the form

$$\mathcal{E} := ((n+1)Q + \delta) \cap \mathbb{Z}^n ,$$

where  $\delta$  is a generic vector in  $V + \mathbb{Z}^n$ , with  $V$  the direction of the affine hull of  $Q$  – notice the introduction of the factor  $(n+1)$ , by comparison with (2).

**THEOREM 3.1 (TROPICAL POSITIVESTELLENSATZ).** *There exists a solution  $x \in \mathbb{R}^n$  to the system  $f^+(x) \triangleright f^-(x)$  if and only if there exists a vector  $y \in \mathbb{R}^{\mathcal{E}'}$  satisfying  $M_{\mathcal{E}'}^+ \odot y \triangleright M_{\mathcal{E}'}^- \odot y$ , where  $\mathcal{E}'$  is any subset of  $\mathbb{Z}^n$  containing a nonempty Canny-Emiris subset  $\mathcal{E}$  of  $\mathbb{Z}^n$  associated to the pair  $(f^+, f^-)$ .*

The following immediate corollary justifies the terminology of *tropical Positivstellensatz* for Theorem 3.1.

**COROLLARY 3.2.** *Let  $f_0^\pm, \dots, f_k^\pm$  be a collection of pairs of tropical polynomials. Then, the following implication holds for all  $x \in \mathbb{R}^n$*

$$f_i^+(x) \geq f_i^-(x) \quad \forall i = 1, \dots, k \quad \implies \quad f_0^+(x) \geq f_0^-(x) ,$$

iff the Macaulay linearization  $M_{\mathcal{E}'}^+ \odot y \triangleright M_{\mathcal{E}'}^- \odot y$  associated to the relations  $f_i^+(x) \geq f_i^-(x)$  for  $i = 1, \dots, k$  and  $f_0^+(x) < f_0^-(x)$ , where  $\mathcal{E}'$  is as in Theorem 3.1, has no finite solution  $y$ .

We define a *tropical basic semi-algebraic set* to be a set vectors  $x \in \mathbb{R}^n$  satisfying the relations (7). Then, using the same type of reduction as in Corollary 3.2, we see that Theorem 3.1 allows us to decide the inclusion of tropical basic semi-algebraic sets.

Now, as for Theorem 2.4, the ‘only if’ implication in Theorem 3.1 is immediate by choosing  $y$  to be the image  $\text{ver}(x)$  of  $x$  by the Veronese embedding. Therefore, we will only focus on the converse implication in what follows, or rather on its contrapositive.

## 3.2 Further preliminary results

To prove this theorem, we need to adapt the notion of tropically diagonally dominant matrices to the two sided case.

**Definition 3.3.** Let  $A = (a_{ij})$  and  $B = (b_{ij})$  be a pair of  $d \times d$  tropical matrices. One says that  $B$  *diagonally dominates*  $A$ , or that  $A$  is *diagonally dominated* by  $B$  whenever

$$b_{ii} > a_{ij} \quad \text{for all } 1 \leq i, j \leq d .$$

**LEMMA 3.4.** *Let  $A$  and  $B$  be a pair of  $d \times d$  tropical matrices such that  $A$  is diagonally dominated by  $B$ . Then the only solution to the inequation  $A \odot y \geq B \odot y$  of unknown  $y \in \mathbb{R}_\infty^d$  is  $y = 0$ .*

We note that Lemma 2.9 and Lemma 2.10 readily carry over to the two-sided case.

In order to define a notion of row content adapted to the two-sided case, we shall make use of the following classical result of discrete convexity, which can be found in [29, Theorem 3.1.2].

**LEMMA 3.5 (SHAPLEY-FOLKMAN).** *Let  $A_1, \dots, A_k \subseteq \mathbb{R}^n$ , and let  $x \in \sum_{i=1}^k \text{conv}(A_i)$ . Then there is an index set  $I \subseteq \{1, \dots, k\}$  with  $|I| \leq n$  such that  $x \in \sum_{i \in I} \text{conv}(A_i) + \sum_{i \in \{1, \dots, k\} \setminus I} A_i$ .*

## 3.3 Proving the Tropical Positivstellensatz

**3.3.1 A row content adapted to inequalities.** We consider the collection  $f = (f_1, \dots, f_k)$  of tropical polynomials, and the associated Newton polytopes  $Q_i, Q$  as in Section 3.1. We also consider the maps  $h_1, \dots, h_k, h$  defined as in Section 2.4.1. Recall that now  $\mathcal{E}$  is a Canny-Emiris set associated to the pair  $(f^+, f^-)$ , that is  $\mathcal{E} = ((n+1)Q + \delta) \cap \mathbb{Z}^n$ , where  $\delta$  is a generic vector in  $V + \mathbb{Z}^n$  and  $V$  is the direction of the affine hull of  $Q$ . Assume that there is no solution  $x \in \mathbb{R}^n$  to inequation  $f^+(x) \geq f^-(x)$ , i.e. that for all  $x \in \mathbb{R}^n$ , there exists  $1 \leq j \leq k$  such that

$$f_j^-(x) > f_j^+(x) . \quad (8)$$

Then for  $p \in \mathcal{E}$ ,  $(p - \delta, h^{\square n+1}(p - \delta))$  is in the relative interior of a facet  $G$  of  $\text{hypo}(h^{\square n+1}) = (n+1) \text{hypo}(h)$ . This facet satisfies  $G = \mathcal{F}(x, h^{\square n+1})$  for some  $x \in V$ , and then

$$G = (n+1)(F_1 + \dots + F_k) = (n+1)F_1 + \dots + (n+1)F_k ,$$

with  $F_i = \mathcal{F}(x, h_i)$ . This means that we can write  $p - \delta = (n+1)q_1 + \dots + (n+1)q_k$  with  $(q_i, h_i(q_i)) \in F_i$  for all  $1 \leq i \leq k$ . Set  $j$  to be the maximal index such that (8) is satisfied. Then since

$$(n+1)(q_j, h_j(q_j)) \in (n+1)F_j = F_j + \dots + F_j$$

where the sum has  $n+1$  terms, and  $F_j$  is isomorphic to its projection  $C_j := C(x, h_j)$  on  $\mathbb{R}^n$ , and is the convex hull of the finite set of its extremal points, it follows from the Shapley-Folkman lemma (that is Lemma 3.5) that there exists  $(q'_j, h_j(q'_j)) \in F_j$  and  $a_j$  an extremal point of  $C_j$  such that  $(n+1)(q_j, h_j(q_j)) = n(q'_j, h_j(q'_j)) + (a_j, h_j(a_j))$ . Moreover, since  $C_j$  is already the convex hull of  $\mathcal{A}_j \cap C_j$ , we have that  $a_j \in \mathcal{A}_j \cap C_j$ . We will define the *row content* of  $p$  in this context to be equal to a couple  $(j, a_j)$  satisfying the above properties. Note that the element  $a_j \in \mathcal{A}_j$  satisfying the above condition need not be unique, but we will see in the proof that it does not matter. Since  $a_j$  is an extremal point of  $C(x, h_j)$ , we

have that  $(a_j, h_j(a_j))$  is a vertex of  $\text{hypo}(h_j)$ , which implies that  $h_j(a_j) = f_{j,a_j}$ , by Observation 2.13, therefore

$$\begin{aligned} \hat{f}_j(x) &= \max_{q_j \in Q_j} \langle (q_j, h_j(q_j)), (x, 1) \rangle = \langle (a_j, h_j(a_j)), (x, 1) \rangle \\ &= \langle a_j, x \rangle + f_{j,a_j} . \end{aligned}$$

Moreover, by assumption  $j$  and  $x$  satisfy (8), hence we must have  $a_j \in \mathcal{A}_j^-$  because the maximum in the expression  $\hat{f}_j(x)$  cannot be attained by a monomial of  $f_j^+$ .

Now for any subset  $\mathcal{E}'$  of  $\mathbb{Z}^n$  containing  $\mathcal{E}$ , we can construct the matrices  $M_{\mathcal{E}\mathcal{E}'}^\pm = (m_{pp'}^\pm)_{(p,p') \in \mathcal{E} \times \mathcal{E}'}$  similarly to Section 2.4.1, by setting the row  $p$  of  $M_{\mathcal{E}\mathcal{E}'}^\pm$  to be the row  $(j, p - a_j)$  of  $M_{\mathcal{E}'}^\pm$ , if  $(j, a_j)$  is the row content of  $p$ .

**3.3.2 The proof of Theorem 3.1 for systems of inequalities.** We first look at the case of systems in the form  $f^+(x) \geq f^-(x)$ . Similarly to Lemma 2.15, we obtain the following result using Lemma 2.12, Observation 2.11 and Observation 2.13.

**LEMMA 3.6.** *Consider a pair  $(f^+, f^-)$ , and an associated Canny-Emiris set  $\mathcal{E}$ . Assume that the system  $f^+(x) \geq f^-(x)$  has no solution  $x \in \mathbb{R}^n$ . Let  $p \in \mathcal{E}$  and let  $(j, a_j)$  be its row content with respect to the pair  $(f^+, f^-)$ . Then, for all  $p' \in \mathcal{E}$  and  $a'_j \in \mathbb{Z}^n$  such that  $p' = p - a_j + a'_j$ , we have*

$$h^{\square n+1}(p' - \delta) \geq h^{\square n+1}(p - \delta) - h_j(a_j) + h_j(a'_j) \quad (9a)$$

$$\geq h^{\square n+1}(p - \delta) - f_{j,a_j}^- + f_{j,a'_j}^+ , \quad (9b)$$

and moreover at least one of the two inequalities is strict.

Now, similarly to Section 2.4.2, we define the matrices  $\widetilde{M}_{\mathcal{E}\mathcal{E}}^\pm = (\widetilde{m}_{pp'}^\pm)_{(p,p') \in \mathcal{E} \times \mathcal{E}}$  by setting

$$\widetilde{m}_{pp'}^\pm = m_{pp'}^\pm - h^{\square n+1}(p' - \delta)$$

for all  $p, p' \in \mathcal{E}$ . Applying Lemma 3.6, we obtain that the matrix  $\widetilde{M}_{\mathcal{E}\mathcal{E}}^+$  is diagonally dominated by  $\widetilde{M}_{\mathcal{E}\mathcal{E}}^-$ . Therefore, thanks to Lemma 3.4, and two sided versions of Lemma 2.9 and Lemma 2.10 this shows that there cannot exist a vector  $y \in \mathbb{R}^{\mathcal{E}'}$  such that  $M_{\mathcal{E}'}^+ \odot y \geq M_{\mathcal{E}'}^- \odot y$ .

**3.3.3 The proof of Theorem 3.1 for mixed systems.** Assume now that the system  $f^+(x) \triangleright f^-(x)$  contains both relations of the type  $f_i^+(x) = f_i^-(x)$  and  $f_i^+(x) \geq f_i^-(x)$ . Assume likewise that there is no solution to the system  $f^+(x) \triangleright f^-(x)$ . This means that for all  $x \in \mathbb{R}^d$ , there exists  $1 \leq j \leq k$  such that either  $\triangleright_j$  is  $\geq$  and (8) is satisfied, or  $\triangleright_j$  is the equality and

$$f_j^+(x) > f_j^-(x) \quad \text{or} \quad f_j^+(x) < f_j^-(x) . \quad (10)$$

We adapt the construction of the row content of Section 3.3.1 to this situation, by replacing (8) by (10) when necessary. In the latter case, one may have  $a_j \in \mathcal{A}_j^+$  or  $a_j \in \mathcal{A}_j^-$ , however one always have that  $a_j$  is an extremal point of  $C(x, h_j)$ .

We can then reiterate the previous construction, and obtain square matrices  $M_{\mathcal{E}\mathcal{E}}^\pm$  and  $\widetilde{M}_{\mathcal{E}\mathcal{E}}^\pm$ . Then, adapting Lemma 3.6, we obtain this time that if  $(j, a_j)$  is the row content of  $p$  and  $\triangleright_j$  is the equality, then  $\widetilde{m}_{pp}^- > \max_{p' \in \mathcal{E}} \widetilde{m}_{pp'}^+$  whenever  $a_j \in \mathcal{A}_j^-$  and  $\widetilde{m}_{pp}^+ > \max_{p' \in \mathcal{E}} \widetilde{m}_{pp'}^-$  whenever  $a_j \in \mathcal{A}_j^+$ . One can then construct

new matrices  $\overline{M}_{\mathcal{E}\mathcal{E}}^\pm$  obtained by exchanging the rows  $p$  of the matrices  $\widetilde{M}_{\mathcal{E}\mathcal{E}}^\pm$  when  $a_j \in \mathcal{A}_j^+$ . These new matrices are such that  $\overline{M}_{\mathcal{E}\mathcal{E}}^+$  is diagonally dominated by  $\overline{M}_{\mathcal{E}\mathcal{E}}^-$ , hence, by Lemma 3.4, the only solution to the system of inequalities  $\overline{M}_{\mathcal{E}\mathcal{E}}^+ \odot z \geq \overline{M}_{\mathcal{E}\mathcal{E}}^- \odot z$  is  $z = 0$ . This implies (by using two sided versions of Lemma 2.9 and Lemma 2.10) that there cannot exist a vector  $y \in \mathbb{R}^{\mathcal{E}'}$  such that  $M_{\mathcal{E}'}^+ \odot y \triangleright M_{\mathcal{E}'}^- \odot y$ .

Finally, suppose the system  $f^+(x) \triangleright f^-(x)$  comprises in addition relations of the form  $f_i^+(x) > f_i^-(x)$ , with  $i \in I \subset \{1, \dots, k\}$ . Let  $\lambda > 1$ , and let us consider the transformed system,  $\mathcal{S}(\lambda)$ , in which every relation  $f_i^+(x) > f_i^-(x)$  is replaced by  $f_i^+(x) \geq \lambda f_i^-(x)$ , let  $M_{\mathcal{E}'}^-(\lambda)$  denote the negative Macaulay matrix associated to  $\mathcal{S}(\lambda)$ , and let us take  $\triangleright'_j$  to be equal to " $\geq$ " for  $j \in I$ , and to  $\triangleright_j$  otherwise. Then, the relation  $M_{\mathcal{E}'}^+ \odot y \triangleright M_{\mathcal{E}'}^- \odot y$  holds iff  $M_{\mathcal{E}'}^+ \odot y \triangleright' M_{\mathcal{E}'}^-(\lambda) \odot y$  holds for some  $\lambda > 1$ . Then, applying the statement of the theorem, which is already established for systems without strict inequalities, we get that this is equivalent to the solvability of  $\mathcal{S}(\lambda)$ , for the same value of  $\lambda$ . Therefore, the solvability of  $f^+(x) \triangleright f^-(x)$  with  $x \in \mathbb{R}^n$  is equivalent to the solvability of  $M_{\mathcal{E}'}^+ \odot y \triangleright M_{\mathcal{E}'}^- \odot y$  with  $y \in \mathbb{R}^{\mathcal{E}'}$ .

**3.3.4 The case where all polynomials  $f_i^-$  are monomials.** The special case where all polynomials in the collection  $f^-$  are monomial is of particular interest. Indeed, in that case, we can reduce the Canny-Emiris set  $\mathcal{E}$  in Theorem 3.1 as follows.

**PROPOSITION 3.7.** *Assume that for all  $1 \leq i \leq k$ ,  $f_i^-$  is a monomial. Then there exists a solution  $x \in \mathbb{R}^n$  to the system  $f^+(x) \geq f^-(x)$  if and only if there exists a vector  $y \in \mathbb{R}^{\mathcal{E}'}$  satisfying  $M_{\mathcal{E}'}^+ \odot y \geq M_{\mathcal{E}'}^- \odot y$ , where  $\mathcal{E}'$  is any subset of  $\mathbb{Z}^n$  containing a nonempty Canny-Emiris subset  $\mathcal{E}$  associated to  $f$ , that is of the form  $\mathcal{E} = (Q + \delta) \cap \mathbb{Z}^n$ , where  $\delta$  is a generic vector in  $V + \mathbb{Z}^n$ .*

A notable consequence of this particular case is that it allows one to reduce the size of the Canny-Emiris type subsets that need to be considered in order to obtain a witness submatrix for the Positivstellensatz in the more general cases where the supports of the polynomials of  $f^-$  and  $f^+$  are small enough. More precisely, if for  $1 \leq i \leq k$ , we denote by  $r_i^\pm$  the number of nonzero monomials of the polynomial  $f_i^\pm$ , then the system of inequalities  $f_i^+(x) \geq f_i^-(x)$  for  $i = 1, \dots, k$  is equivalent to the system  $f_i^+(x) \geq f_{i,a_i}^- \odot x^{\odot a_i}$  for  $i = 1, \dots, k$  and for all  $a_i \in \mathcal{A}_i^-$ , and the righthandside of this last system consists only of monomials. Thus as a corollary of Proposition 3.7, one can take as a Canny-Emiris set for this system the set

$$\mathcal{E} = ((r_1^- Q_1 + \dots + r_k^- Q_k) + \delta) \cap \mathbb{Z}^n ,$$

and in particular if all  $r_i^-$  are less than or equal to  $n + 1$ , we obtain a smaller Canny-Emiris set than the one used in Theorem 3.1. Note that we can further reduce the numbers  $r_i^-$  by considering only essential monomials in  $f_i^-$ , see Observation 2.13.

The previous works identically if we replace the weak inequalities by strict inequalities, and it can also be adapted to equalities as the system  $f_i^+(x) = f_i^-(x)$  for  $i = 1, \dots, k$  is equivalent to the system  $f_i^+(x) \geq f_i^-(x)$  and  $f_i^+(x) \leq f_i^-(x)$  for  $i = 1, \dots, k$ , which can be dealt with in a similar fashion, and therefore by setting  $r_i = r_i^+ + r_i^-$  for all  $1 \leq i \leq k$ , we can consider the Canny-Emiris set

$$\mathcal{E} = ((r_1 Q_1 + \dots + r_k Q_k) + \delta) \cap \mathbb{Z}^n ,$$



which also yields a smaller submatrix of the Macaulay matrix if all  $r_i$  are less than or equal to  $n + 1$ .

#### 4 ALGORITHMIC ASPECTS

It is shown in [2] that the feasibility problems for systems of linear tropical equalities, inequalities, as well as the non-emptiness of tropical linear prevarieties, reduce to mean-payoff games. This reduction was extended in [4] to systems allowing a mixture of strict and weak linear tropical inequalities, like the one arising in Corollary 3.2.

Consider the system of inequalities  $A \odot y \leq B \odot y$ , where  $A$  and  $B$  are two  $p \times m$  tropical matrices. To the matrix  $B$ , we associate the tropically linear operator also denoted  $B$  from  $(\mathbb{R} \cup \{\pm\infty\})^m$  to  $(\mathbb{R} \cup \{\pm\infty\})^p$ , such that  $(By)_i = \max_j (B_{ij} + y_j)$ , with the convention  $-\infty + (+\infty) = -\infty$ . To the matrix  $A$ , we associate the dual tropically linear operator  $A^\sharp$  from  $(\mathbb{R} \cup \{\pm\infty\})^p$  to  $(\mathbb{R} \cup \{\pm\infty\})^m$ , such that  $(A^\sharp(z))_j := \min_i (-A_{ij} + z_i)$ , now with the convention  $+\infty + (-\infty) = +\infty$ . Then, we define the operator  $T$  to be the map  $A^\sharp \circ B$ . When every row of  $B$  has at least one finite entry, this map preserves  $(\mathbb{R} \cup \{\pm\infty\})^m$ .  $T$  is the *Shapley operator* of a zero-sum game.

It is shown in [2] that the system  $A \odot y \leq B \odot y$  has a finite solution iff and only if all the initial positions of a mean-payoff game associated with the operator  $T$  are winning.

For two vectors  $u, v \in (\mathbb{R} \cup \{\pm\infty\})^n$ , we write  $v \ll u$  if for all  $i$  such that  $u_i < +\infty$ , we have  $v_i < u_i$ . Moreover, for  $\lambda \in \mathbb{R}$ , we denote  $\lambda + u$  the vector with coordinates  $\lambda + u_i$ .

---

##### Algorithm 1: Value iteration algorithm with widening.

---

```

input:  $T$  a Shapley operator from  $(\mathbb{R} \cup \{\pm\infty\})^m$  to  $(\mathbb{R} \cup \{\pm\infty\})^m$ 
          $\varepsilon > 0$  the approximation error for comparisons
output: Decides the feasibility of the system  $A \odot y \leq B \odot y$  in  $\mathbb{R}^m$ 
1 procedure VALUEITERATION( $T, \varepsilon$ ):
2    $u := 0 \in \mathbb{R}^m$ 
3    $v := 0 \in \mathbb{R}^m$ 
4 repeat
5     /* value iteration step */
6      $u := v$ 
7      $v := u \wedge T(u)$ 
8     /* widening step */
9      $I := \{i : v_i \geq -\varepsilon + u_i\}$ 
10     $\tilde{u} := (\tilde{u}_i \in (\mathbb{R} \cup \{\pm\infty\})^m \text{ with } \tilde{u}_i = +\infty \text{ if } i \in I$ 
11            $\text{and } \tilde{u}_i = u_i \text{ otherwise}$ 
12     $\tilde{v} := T(\tilde{u})$ 
13 until  $v \geq -\varepsilon + u$  or  $v \ll -\varepsilon + u$  or  $\tilde{v} \ll -\varepsilon + \tilde{u}$ 
14 if  $v \ll -\varepsilon + u$  or  $\tilde{v} \ll -\varepsilon + \tilde{u}$  then
15   /* No finite vector  $y$  satisfies  $T(y) \geq -\varepsilon + y$ . */
16   return "Unfeasible"
17 else
18   /* The vector  $u$  satisfies  $T(u) \geq -\varepsilon + u$ . */
19   return "Feasible"

```

---

Algorithm 1 exploits the classical idea of value iteration algorithm for mean-payoff games (see [34]), an originality is the introduction of a *widening* step, i.e., the construction of the vector  $\tilde{u}$  at line 7, which allows for a quick test of unfeasibility.

When considering the Shapley operator  $T = A^\sharp \circ B$ , with  $A$  and  $B$  having integer coefficients,  $T$  preserves  $(\mathbb{Z} \cup \{+\infty\})^{\mathcal{E}'}$ , and Algorithm 1 can be executed with  $\varepsilon = 0$  in exact (integer) arithmetics. In that case, if it terminates, it is correct. In particular the condition  $T(\tilde{u}) \ll \tilde{u}$  involving the “widened” vector  $\tilde{u}$  implies the unfeasibility; this can be established by applying the Collatz-Wielandt theorem of [2] to the operator  $y \mapsto -T(-y)$ .

One can show that Algorithm 1 applied to the Krasnoselskii-Mann damped Shapley operator  $T^{\text{KM}}(y) = \frac{1}{2}(y + T(y))$  does terminate for all sufficiently small positive values of  $\varepsilon$ , following an idea in [14]. Note also that taking  $\varepsilon < 1/(2m)$  suffices, then, the last condition  $T^{\text{KM}}(u) \geq -\varepsilon + u$  actually implies that there exists  $y$  finite such that  $T(y) \geq y$ , and so  $Ay \leq By$ .

One can then check if the system of tropical polynomial equations  $f^+(x) \geq f^-(x)$  has a finite solution, by applying the above algorithm to the Shapley operators  $T = (\mathcal{M}_{\mathcal{E}'}^-)^\sharp \circ \mathcal{M}_{\mathcal{E}'}^+$ , and  $T^{\text{KM}}$ , with  $\mathcal{E}'$  as in Theorem 3.1.

#### 5 CONCLUDING REMARKS

We considered here the feasibility problem for systems of polynomial equations and inequations over  $\mathbb{R}^n$ . A natural extension is to handle the case of solutions over  $\mathbb{R}_\infty^n$ , see [17] for results in the dense case. Another natural extension would be to develop a tropical analogue of “eigenvalue methods”, allowing one to compute a solution when there is one (instead of just deciding feasibility). Finally, iterative versions of the method (gradually increasing the set  $\mathcal{E}'$  instead of starting from a superset of a Canny-Emiris set), remain to be developed.

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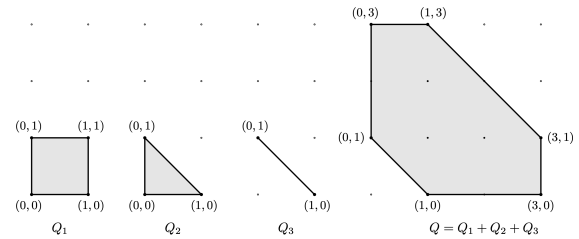
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## A SOME EXAMPLES

Let us illustrate Theorem 2.4 with some explicit examples. Consider the following collection:

$$f : \begin{cases} f_1 &= 1 \oplus 2x_1 \oplus 1x_2 \oplus 1x_1x_2 \\ f_2 &= 0 \oplus 0x_1 \oplus 1x_2 \\ f_3 &= 2x_1 \oplus 0x_2 \end{cases}$$

The Newton polytopes  $Q_i$  and  $Q$  of  $f_i$  and  $f$ , respectively, are shown in the figure below.



For this collection of supports, one can take  $\delta = (-1 + \varepsilon, -1 + \varepsilon)$  with  $\varepsilon > 0$  sufficiently small, which gives us the Canny-Emiris set

$$\mathcal{E} := (Q + \delta) \cap \mathbb{Z}^n = \{(0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2)\},$$

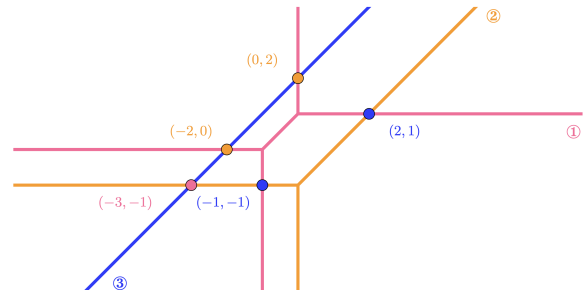
corresponding to the set of monomials  $\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}$ .

We obtain a  $7 \times 6$  Canny-Emiris matrix  $\mathcal{M}_{\mathcal{E}}$ :

$$\mathcal{M}_{\mathcal{E}} = \begin{matrix} & \begin{matrix} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{matrix} \\ \begin{matrix} f_1 \\ f_2 \\ x_1f_2 \\ x_2f_2 \\ f_3 \\ x_1f_3 \\ x_2f_3 \end{matrix} & \begin{pmatrix} 1 & 2 & 1 & & & \\ 0 & 0 & 1 & & & \\ 0 & & 0 & 1 & & \\ & 0 & 0 & 0 & 1 & \\ 2 & 0 & & 2 & 0 & \\ & & & & 2 & 0 \end{pmatrix} \end{matrix}$$

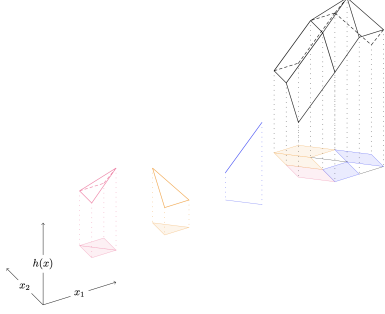
One can check that the system  $f(x) \nabla 0$  does not have a solution, as illustrated on Figure 1.

Figure 1: The arrangement of tropical varieties of the polynomials  $f_1, f_2, f_3$ .

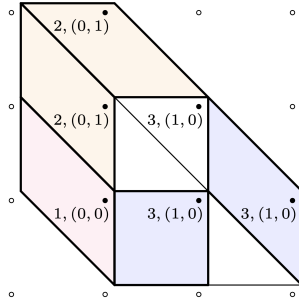


We can therefore apply the Canny-Emiris construction. First, we obtain a subdivision of  $Q$  by projecting the non-vertical faces of  $\text{hypo}(h)$  onto  $\mathbb{R}^n$  as shown in Figure 2.

**Figure 2: The subdivision of  $Q$  arises from the projection of  $\text{hypo}(h)$ .**



With this subdivision, we can associate to every point  $p \in \mathcal{E}$  its row content, which is determined by the maximal-dimensional cell of the decomposition of  $Q$  to which  $p - \delta$  belongs. This process is illustrated in Figure 3.



**Figure 3: The decomposition of the polytope  $Q + \delta$  into cells, with row contents of integer points of  $Q + \delta$ .**

This configuration yields the following nonsingular square matrix

$$M_{\mathcal{E}\mathcal{E}} = \begin{matrix} (0, 0) \rightarrow f_1 \\ (1, 0) \rightarrow f_3 \\ (0, 1) \rightarrow f_2 \\ (2, 0) \rightarrow x_1 f_3 \\ (1, 1) \rightarrow x_2 f_3 \\ (0, 2) \rightarrow x_2 f_2 \end{matrix} \begin{pmatrix} 1 & x_1 & x_2 & x_1^2 & x_1 x_2 & x_2^2 \\ 1 & 2 & 1 & & & \\ & 2 & 0 & & & \\ 0 & 0 & 1 & & & \\ & & & 2 & 0 & \\ & & & & 2 & 0 \\ & & 0 & & 0 & 1 \end{pmatrix}.$$

Thus, there is no solution  $y \in \mathbb{R}^{\mathcal{E}}$  to the equation  $M_{\mathcal{E}} \odot y \nabla \mathbb{0}$ .

## B NUMERICAL RESULTS

We illustrate here the method presented in Section 4. We consider systems of tropical polynomial inequations  $f^+(x) \geq f^-(x)$  in which  $f^\pm$  are collections of  $k$  tropical polynomials in  $n$  variables which are sparse, that is with a given maximal number  $r_i^\pm$  of non zero coefficients of  $f_i^\pm$ . We implemented Algorithm 1 in Python in the case of the Shapley operator  $T = (M_{\mathcal{E}^-})^\# \circ M_{\mathcal{E}^+}$ , taking into account the sparsity of the polynomials. In particular, the operator

$T$  has been computed using the sparse submatrices  $M_{\mathcal{E}'}^\pm$ , of  $M^\pm$  relying on the Canny-Emiris construction — instead of full submatrices of the form  $M_N^\pm$ , which are simply obtained by truncation to a certain degree bound  $N$ . This help reduce the size of the matrices at work, as illustrated in Table 1. This was implemented thanks to the Python package `polytope` which allows to compute the convex hull  $Q$  of the Minkowski sum of the  $\mathcal{A}_i$ , as well as to enumerate the integer points of  $Q$ . The integer points of  $Q$  were used as the superset  $\mathcal{E}'$  of Canny-Emiris  $\mathcal{E}$  to construct the matrices  $M_{\mathcal{E}'}^\pm$ .

For given values of  $k$  and  $n$ , we draw 100 random sparse polynomial systems with integer coefficients, such that for all  $i = 1, \dots, k$ ,  $f_i^\pm$  has a degree and a sparsity  $r_i^\pm$  both equal to 4, and such that  $Q$  has dimension  $n$ . We force Algorithm 1 to stop after 10000 iterations (this upper bound was never reached in practice). In Table 2, we give for each  $k$  and  $n$ , the number of feasible instances among the 100 draws, that is the number of systems of polynomial inequalities returning a solution  $u$ . In Table 3, we give the execution time on a single core of a 12 cores Intel(R) Xeon(R) CPU X5675 @3.07GHz with 148GB of RAM. The code from which these benchmarks were obtained can be found at <https://gitlab.inria.fr/abereau/tropical-polynomial-system-solving>. Empirically, we notice that the algorithm finishes significantly faster on unfeasible instances, compared to feasible instances (in only a few dozens iterations compared to up to several thousand iterations in the worst cases for feasible instances).

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**Table 1: Average number of columns in the Macaulay matrices in the sparse case (right) for random systems of  $k$  inequations in  $n$  variables among 100 samples, compared to the number of columns in the full case (left).**

		$k$									
		2	3	4	5	6					
$n$	2	45	- 35	91	- 73	153	- 128	231	- 193	325	- 276
	3	165	- 85	455	- 265	969	- 611	1771	- 1156	2925	- 1987
	4	495	- 138	1820	- 651	4845	- 2079	10626	- 5044	20475	- 10418
	5	1287	- 163	6188	- 1268	20349	- 5165	53130	- ×	118755	- ×
	6	3003	- 154	18564	- ~1300	74613	- ×	230230	- ×	593775	- ×

**Table 2: Number of feasible instances for random systems of  $k$  inequations in  $n$  variables among 100 samples.**

		$k$								
		2	3	4	5	6	7	8	9	10
$n$	2	97	91	62	48	38	28	12	12	5
	3	100	98	97	79	74	62	48	32	17
	4	100	100	100	100	93	92	80	×	×
	5	100	100	100	×	×	×	×	×	×
	6	100	100	×	×	×	×	×	×	×

**Table 3: Average runtime in seconds to solve an instance of a random system of  $k$  inequations in  $n$  variables among 100 samples.**

		$k$								
		2	3	4	5	6	7	8	9	10
$n$	2	0.04	0.16	0.67	1.58	2.91	3.73	2.43	4.55	2.53
	3	0.08	0.71	3.82	8.80	33.45	84.87	183.26	180.58	154.43
	4	0.74	2.95	11.54	48.47	266.95	654.06	1952.40	×	×
	5	5.09	64.94	312.66	×	×	×	×	×	×
	6	67.60	2427.67	×	×	×	×	×	×	×