



**HAL**  
open science

## Confluence as a cut elimination property

Gilles Dowek

► **To cite this version:**

| Gilles Dowek. Confluence as a cut elimination property. 2003. hal-04103228

**HAL Id: hal-04103228**

**<https://inria.hal.science/hal-04103228v1>**

Preprint submitted on 23 May 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Confluence as a cut elimination property

Gilles Dowek

École polytechnique and INRIA

LIX, École polytechnique, 91128 Palaiseau Cedex, France.

Gilles.Dowek@polytechnique.fr, <http://www.lix.polytechnique.fr/~dowek>

**Abstract.** The goal of this note is to compare two notions, one coming from the theory of rewrite systems and the other from proof theory: confluence and cut elimination. We show that to each rewrite system on terms, we can associate a logical system: *asymmetric deduction modulo this rewrite system* and that the confluence property of the rewrite system is equivalent to the cut elimination property of the associated logical system. This equivalence, however, does not extend to rewrite systems directly rewriting atomic propositions.

The goal of this note is to compare two notions, one coming from the theory of rewrite systems and the other from proof theory: confluence and cut elimination.

The confluence a rewrite system permits to reduce the search space when we want to establish that two terms are convertible. Similarly, the cut elimination property of a logical system permits to reduce the search space, when we want to establish that some proposition is provable.

Moreover, both properties can be used to prove the decidability of convertibility or provability, when this reduction yields a finite search space. Finally, both properties can be used to prove independence results (i.e. that two terms are not convertible or that a proposition is not provable), and in particular consistency results, when this reduction yields an empty search space.

The goal of this note is to show that this similarity between confluence and cut elimination can be seen as a consequence of the fact that to each rewrite system  $\mathcal{R}$  rewriting terms, we can associate a logical system: *asymmetric deduction modulo  $\mathcal{R}$* , a variant of deduction modulo introduced in [5], and that the confluence property of the rewrite system is equivalent to the cut elimination property of the associated logical system. More precisely, we establish a parallel between

- an equality  $t = u$  and a sequent  $P(t) \vdash P(u)$ ,
- the notion of conversion sequence and that of proof,
- the notion of peak and that of cut, and
- the notion of valley sequence and that of cut free proof.

Both valley sequences and cut free proofs may be called *analytic* as they exploit the information present in their conclusion and its sub-parts but no other information.

Finally, we relate a method used to prove cut elimination by defining an algorithm transforming proofs step by step until all cuts are removed (see, for instance, [8]) and a method used to prove confluence by defining an algorithm transforming rewrite sequences step by step until all peaks are removed (see, for instance, [2, 9, 1]). As an example, we reformulate Newman's confluence theorem [9] as a cut elimination theorem.

Asymmetric deduction modulo can be extended by allowing not only rules rewriting terms in propositions, but also directly atomic propositions. With such rules, confluence and cut elimination do not coincide anymore and confluence is not a sufficient analyticity condition: it must be replaced by cut elimination.

## 1 Asymmetric deduction modulo

In deduction modulo [5], the notions of language, term and proposition are that of first-order predicate logic. But a theory is formed with a set of axioms  $\Gamma$  and a congruence  $\equiv$  defined on propositions. Deduction rules are modified to take this congruence into account. For instance, the right rule of conjunction is not stated as usual

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}$$

as the conclusion need not be exactly  $A \wedge B$  but may be only convertible to this proposition, hence it is stated

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash C, \Delta} \text{ if } C \equiv A \wedge B$$

All rules of sequent calculus, or natural deduction, may be defined in a similar way.

In this note, we consider only congruences defined by a rewrite system on terms. A *rewrite rule* is a pair of terms  $\langle l, r \rangle$ , written  $l \rightarrow r$ , such that  $l$  is not a variable. A *rewrite system* is a set of rules. Given such a system, the relation  $\rightarrow^1$  is the smallest relation defined on terms and on propositions compatible with the structure of terms and propositions such that for all substitutions  $\theta$  and all rewrite rules  $l \rightarrow r$  of the rewrite system  $\theta l \rightarrow^1 \theta r$ . The relation  $\rightarrow^+$  is the transitive closure of  $\rightarrow^1$ , the relation  $\rightarrow^*$  is its reflexive-transitive closure and the relation  $\equiv$  its reflexive-symmetric-transitive closure. Notice that rewriting does not change the logical structure of a proposition, in particular an atomic proposition only rewrites to an atomic proposition.

A *conversion sequence* is a finite sequence of terms or propositions  $C_1, \dots, C_n$  such that for each  $i$  either  $C_i \rightarrow^1 C_{i+1}$  or  $C_i \leftarrow^1 C_{i+1}$ . Obviously two terms or two propositions  $A$  and  $B$  are convertible if there is a conversion sequence whose first element is  $A$  and last element is  $B$ . A *peak* in such a sequence is an index  $i$  such that  $C_{i-1} \leftarrow^1 C_i \rightarrow^1 C_{i+1}$ . A sequence is called a *valley sequence* if it contains no peak, i.e. if it has the form  $A \rightarrow^1 \dots \rightarrow^1 \leftarrow^1 \dots \leftarrow^1 B$ .

For example, in arithmetic, we can define a congruence with the following rules

$$0 + y \rightarrow y$$

$$\begin{aligned}
& S(x) + y \rightarrow S(x + y) \\
& 0 \times y \rightarrow 0 \\
& S(x) \times y \rightarrow x \times y + y
\end{aligned}$$

In the theory formed with the axiom  $\forall x x = x$  and this congruence, we can prove, in sequent calculus modulo, that the number 4 is even

$$\frac{\frac{\frac{4 = 4 \vdash 2 \times 2 = 4}{\forall x x = x \vdash 2 \times 2 = 4} \text{Axiom}}{\forall x x = x \vdash \exists x 2 \times x = 4} (x, 2 \times x = 4, 2) \exists\text{-right}}{\forall x x = x \vdash 2 \times 2 = 4} (x, x = x, 4) \forall\text{-left}$$

The sequent  $4 = 4 \vdash 2 \times 2 = 4$ , that requires a tedious proof in usual formulations of arithmetic, can simply be proved with the axiom rule here, as  $(4 = 4) \equiv (2 \times 2 = 4)$ .

Deduction modulo a congruence defined by a rewrite system on terms uses this rewrite system only through the congruence it generates. The way these congruences are established is not taken into consideration. Thus, cut free proofs are analytic in the sense that they do not use the cut rule but not in the sense they establish congruences with valley sequences. Hence, we introduce a weaker system, *asymmetric deduction modulo*, where propositions can only be reduced. The rules of asymmetric sequent calculus modulo are given in figure 1.

## 2 Cut elimination in atomic symmetric deduction modulo

We first consider a fragment of symmetric deduction modulo where all propositions are atomic. This system contains only the axiom rule, the cut rule and the structural rules (weakening and contraction).

To relate proofs in atomic deduction modulo and rewrite sequences, we prove that the sequent  $P(t) \vdash P(u)$  is provable in atomic deduction modulo if and only if  $t \equiv u$ . This is a direct consequence of the following proposition.

**Proposition 1.** *In atomic deduction modulo, the sequent  $\Gamma \vdash \Delta$  is provable if and only if  $\Gamma$  contains a proposition  $A$  and  $\Delta$  a proposition  $B$  such that  $A \equiv B$ .*

*Proof.* If  $\Gamma$  contains a proposition  $A$  and  $\Delta$  a proposition  $B$  such that  $A \equiv B$ , then the sequent  $\Gamma \vdash \Delta$  can be proved with the axiom rule.

Conversely, we prove by induction over proof structure that if the sequent  $\Gamma \vdash \Delta$  is provable, then  $\Gamma$  contains a proposition  $A$  and  $\Delta$  a proposition  $B$  such that  $A \equiv B$ .

This is obvious if the last rule of the proof is the axiom rule. If the last rule is a structural rule, we simply apply the induction hypothesis. Finally, if the last rule is the cut rule, then the proof has the form

$$\frac{\frac{\pi_1}{\Gamma \vdash C_1, \Delta} \quad \frac{\pi_2}{\Gamma, C_2 \vdash \Delta}}{\Gamma \vdash \Delta} \text{Cut}$$

With  $C_1 \equiv C_2$ . By induction hypothesis

$$\begin{array}{c}
\overline{\Gamma, A_1 \vdash A_2, \Delta} \text{ (A) Axiom if } A_1 \rightarrow^* A \leftarrow^* A_2 \\
\frac{\Gamma \vdash C_1, \Delta \quad \Gamma, C_2 \vdash \Delta}{\Gamma \vdash \Delta} \text{ (C) Cut if } C_1 \leftarrow^* C \rightarrow^* C_2 \\
\frac{\Gamma, A_1, A_2 \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ contr-left if } A_1 \leftarrow^* A \rightarrow^* A_2 \\
\frac{\Gamma \vdash A_1, A_2, \Delta}{\Gamma \vdash A, \Delta} \text{ contr-right if } A_1 \leftarrow^* A \rightarrow^* A_2 \\
\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ weak-left} \\
\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \text{ weak-right} \\
\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, C \vdash \Delta} \Rightarrow\text{-left if } C \rightarrow^* (A \Rightarrow B) \\
\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash C, \Delta} \Rightarrow\text{-right if } C \rightarrow^* (A \Rightarrow B) \\
\frac{\Gamma, A, B \vdash \Delta}{\Gamma, C \vdash \Delta} \wedge\text{-left if } C \rightarrow^* (A \wedge B) \\
\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash C, \Delta} \wedge\text{-right if } C \rightarrow^* (A \wedge B) \\
\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, C \vdash \Delta} \vee\text{-left if } C \rightarrow^* (A \vee B) \\
\frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash C, \Delta} \vee\text{-right if } C \rightarrow^* (A \vee B) \\
\frac{\Gamma, A \vdash \Delta}{\Gamma, A \vdash \Delta} \perp\text{-left if } A \rightarrow^* \perp \\
\frac{\Gamma, [t/x]A \vdash \Delta}{\Gamma, B \vdash \Delta} (x, A, t) \forall\text{-left if } B \rightarrow^* \forall x A \\
\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash B, \Delta} (x, A) \forall\text{-right if } B \rightarrow^* \forall x A \text{ and } x \notin FV(\Gamma \Delta) \\
\frac{\Gamma, A \vdash \Delta}{\Gamma, B \vdash \Delta} (x, A) \exists\text{-left if } B \rightarrow^* \exists x A \text{ and } x \notin FV(\Gamma \Delta) \\
\frac{\Gamma \vdash [t/x]A, \Delta}{\Gamma \vdash B, \Delta} (x, A, t) \exists\text{-right if } B \rightarrow^* \exists x A
\end{array}$$

**Fig. 1.** Asymmetric sequent calculus modulo

- $\Gamma$  and  $\Delta$  contain two convertible propositions or  $\Gamma$  contains a proposition convertible to  $C_1$ ,
- $\Gamma$  and  $\Delta$  contain two convertible propositions or  $\Delta$  contains a proposition convertible to  $C_2$ .

Thus, in all cases,  $\Gamma$  and  $\Delta$  contain two convertible propositions.

The next proposition shows that atomic deduction modulo has the cut elimination property, i.e. that all provable sequents have a cut free proof. It is known, in general, that deduction modulo a congruence defined by a rewrite system on terms has the cut elimination property [6], but for the case of atomic deduction modulo, this is a direct consequence of proposition 1.

**Proposition 2.** *In atomic deduction modulo, all provable sequents have a cut free proof.*

*Proof.* If a sequent  $\Gamma \vdash \Delta$  is provable then, by proposition 1,  $\Gamma$  and  $\Delta$  contain two convertible propositions and thus the sequent  $\Gamma \vdash \Delta$  can be proved with the axiom rule.

We can also define a proof reduction algorithm that reduces cuts step by step.

**Definition 1 (Proof reduction).**

*Consider a proof of the form*

$$\frac{\frac{\pi_1}{\Gamma \vdash C_1, \Delta} \quad \frac{\pi_2}{\Gamma, C_2 \vdash \Delta}}{\Gamma \vdash \Delta} \textit{Cut}$$

where  $\pi_1$  and  $\pi_2$  are cut free proofs.

*The multisets  $\Gamma$  and  $C_1, \Delta$  contain two convertible propositions. Similarly,  $\Gamma, C_2$  and  $\Delta$  contain two convertible propositions. Thus,*

- $\Gamma$  and  $\Delta$  contain two convertible propositions or  $\Gamma$  contains a proposition convertible to  $C_1$ ,
- $\Gamma$  and  $\Delta$  contain two convertible propositions or  $\Delta$  contains a proposition convertible to  $C_2$ .

*Thus, as  $C_1 \equiv C_2$ ,  $\Gamma$  and  $\Delta$  contain two convertible propositions in all cases and this proof reduces to*

$$\overline{\Gamma \vdash \Delta} \textit{Axiom}$$

*When a proof contains a cut, the proofs of the premises of the highest cut are obviously cut free and this proof reduction algorithm applies. It terminates because it removes a cut at each step. Thus, it produces a cut free proof after a finite number of steps.*

### 3 Cut elimination in atomic asymmetric deduction modulo

Let us now turn to asymmetric deduction modulo, still in the atomic case. We prove that in asymmetric atomic deduction modulo, a sequent  $P(t) \vdash P(u)$  is provable if and only if  $t \equiv u$  and that this sequent has a cut free proof if and only if  $t$  and  $u$  have a common reduct. Thus, proofs in asymmetric deduction modulo correspond to rewrite sequences and cut free proofs to valley sequences.

**Proposition 3.** *In asymmetric atomic deduction modulo, the sequent  $\Gamma \vdash \Delta$  is provable if and only if  $\Gamma$  contains a proposition  $A$  and  $\Delta$  a proposition  $B$  such that  $A \equiv B$ .*

*Proof.* Obviously, if the sequent  $\Gamma \vdash \Delta$  is provable in asymmetric deduction modulo, then it is provable in symmetric deduction modulo and  $\Gamma$  and  $\Delta$  contain two convertible propositions.

The converse is slightly more difficult than in the symmetric case because the axiom rule does not apply directly. Assume there are propositions  $A$  in  $\Gamma$  and  $B$  in  $\Delta$  such that  $A \equiv B$ . Then there is a rewrite sequence  $A = C_1, \dots, C_n = B$  joining  $A$  and  $B$ . We prove, by induction on the number of peaks in this sequence that the sequent  $\Gamma \vdash \Delta$  is provable.

If the sequence contains no peak, then  $A$  and  $B$  have a common reduct and the sequent  $\Gamma \vdash \Delta$  can be proved with the axiom rule. Otherwise, there is a peak  $i$  in this sequence. The sequences  $C_1, \dots, C_i$  and  $C_i, \dots, C_n$  contain fewer peaks than  $C_1, \dots, C_n$ , thus, by induction hypothesis, the sequents  $\Gamma \vdash C_i, \Delta$  and  $\Gamma, C_i \vdash \Delta$  have proofs  $\pi_1$  and  $\pi_2$ . We build the proof

$$\frac{\frac{\pi_1}{\Gamma \vdash C_i, \Delta} \quad \frac{\pi_2}{\Gamma, C_i \vdash \Delta}}{\Gamma \vdash \Delta} (C_i) \text{ Cut}$$

**Proposition 4.** *In asymmetric deduction modulo, the sequent  $\Gamma \vdash \Delta$  has a cut free proof if and only if  $\Gamma$  contains a proposition  $A$  and  $\Delta$  a proposition  $B$  such that  $A$  and  $B$  have a common reduct.*

*Proof.* If  $\Gamma$  contains a proposition  $A$  and  $\Delta$  a proposition  $B$  such that  $A$  and  $B$  have a common reduct, then the sequent  $\Gamma \vdash \Delta$  can be proved with the axiom rule. Conversely, if the sequent  $\Gamma \vdash \Delta$  has a cut free proof, we prove, by induction over proof structure, that  $\Gamma$  contains a proposition  $A$  and  $\Delta$  a proposition  $B$  such that  $A$  and  $B$  have a common reduct. This is obvious if the last rule is an axiom rule. If the last rule is a structural rule, we apply the induction hypothesis.

We can now state our main proposition that relates confluence and cut elimination.

**Proposition 5 (Main proposition).** *The cut rule is redundant in asymmetric atomic deduction modulo a rewrite system if and only if this rewrite system is confluent.*

*Proof.* Assume that the rewrite system is confluent. If the sequent  $\Gamma \vdash \Delta$  is provable then, by proposition 3,  $\Gamma$  and  $\Delta$  contain two convertible propositions. By confluence, they have a common reduct and thus, by proposition 4, the sequent  $\Gamma \vdash \Delta$  has a cut free proof.

Conversely, assume that the cut rule is redundant. If  $A \equiv B$  (resp.  $t \equiv u$ ) then by proposition 3, the sequent  $A \vdash B$  (resp.  $P(t) \vdash P(u)$ ) is provable, thus it has a cut free proof and, by proposition 4,  $A$  and  $B$  (resp.  $t$  and  $u$ ) have a common reduct.

## 4 Proof reduction

Like in the symmetric case (definition 1) we want to design an algorithm reducing proofs and eliminating cuts step by step. In the asymmetric case however this algorithm is a little more involved as it needs to perform shorter steps to reconstruct reductions instead of conversions. This algorithm is a reformulation in sequent calculus of Newman's algorithm that reduces rewrite sequences eliminating peaks, step by step. To define it, we need to use the local confluence of the rewrite system and, to prove its termination, the termination of the system.

**Definition 2 (Proof reduction).** Consider a proof of the form

$$\frac{\frac{\pi_1}{\Gamma \vdash C_1, \Delta} \quad \frac{\pi_2}{\Gamma, C_2 \vdash \Delta}}{\Gamma \vdash \Delta} (C) \text{ Cut}$$

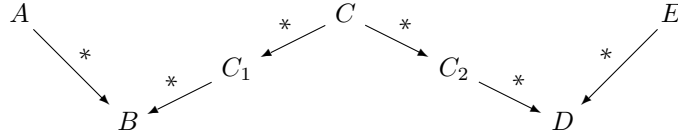
where  $\pi_1$  and  $\pi_2$  are cut free proofs.

As  $\pi_1$  is cut free,  $\Gamma, C_1$  and  $\Delta$  contain two propositions that have a common reduct. Similarly,  $\Gamma$  and  $C_2, \Delta$  contain two propositions that have a common reduct.

If  $\Gamma$  and  $\Delta$  contain two propositions that have a common reduct  $C'$ , then this proof reduces to

$$\overline{\Gamma \vdash \Delta} (C') \text{ Axiom}$$

Otherwise,  $\Gamma$  contains a proposition  $A$  that has a common reduct  $B$  with  $C_1$  and  $\Delta$  contains a proposition  $E$  that has a common reduct  $D$  with  $C_2$ . Let us write  $\Gamma = \Gamma', A$  and  $\Delta = E, \Delta'$ . We have



If  $B = C$  or  $C = D$  then either  $D$  or  $B$  is a common reduct of  $A$  and  $E$  and this proof reduces to

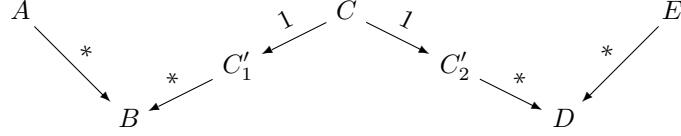
$$\overline{\Gamma \vdash \Delta} (D) \text{ Axiom}$$



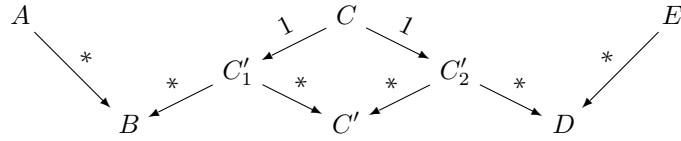
or to

$$\overline{\Gamma \vdash \Delta} (B) \text{ Axiom}$$

Otherwise we have  $B \leftarrow^+ C \rightarrow^+ D$  and there are propositions  $C'_1$  and  $C'_2$  such that



If there is a proposition  $C'$  such that  $C'_1 \rightarrow^* C' \leftarrow^* C'_2$ , then we have



and this proof reduces to

$$\frac{\frac{\overline{\Gamma', A \vdash C'_1, C'_2, E, \Delta'} (B) \text{ Axiom}}{\Gamma', A \vdash C'_2, E, \Delta'} \frac{\overline{\Gamma', A, C'_1 \vdash C'_2, E, \Delta'} (C') \text{ Axiom}}{\Gamma', A, C'_1 \vdash C', E, \Delta'} (C'_1) \text{ Cut}}{\Gamma', A \vdash E, \Delta'} \frac{\overline{\Gamma', A, C'_2 \vdash E, \Delta'} (D) \text{ Axiom}}{\Gamma', A, C'_2 \vdash E, \Delta'} (C'_2) \text{ Cut}}$$

otherwise this proof cannot be reduced.

*Example 1.* This proof reduction algorithm may fail. Consider the rewrite system

$$a \rightarrow b \quad a \rightarrow b'$$

There is no way to reduce the proof

$$\frac{\overline{P(b) \vdash P(a), P(b')} (P(b)) \text{ Axiom}}{P(b) \vdash P(b')} \frac{\overline{P(b), P(a) \vdash P(b')} (P(b')) \text{ Axiom}}{P(b) \vdash P(b')} (P(a)) \text{ Cut}$$

But this situation cannot occur if the rewrite system is locally confluent.

**Proposition 6.** *If the rewrite system is locally confluent, then the proof reduction algorithm of definition 2 does not fail.*

*Proof.* If  $C'_1 \leftarrow^1 C \rightarrow^1 C'_2$  then, by local confluence, there is a proposition  $C'$  such that  $C'_1 \rightarrow^* C' \leftarrow^* C'_2$ .

*Example 2.* The proof reduction algorithm of definition 2 may loop. Consider the rewrite system

$$a \rightarrow b \quad a \rightarrow c \quad b \rightarrow a \quad b \rightarrow d$$

Let  $A$  be the proposition  $P(a)$ ,  $B$  be the proposition  $P(b)$ ,  $C$  be the proposition  $P(c)$  and  $D$  be the proposition  $P(d)$ . We write  $D^n$  for the proposition  $D$  repeated  $n$  times. The proof

$$\frac{\frac{\overline{C \vdash A, D, D^n} (C) \text{ Axiom}}{C \vdash D, D^n} \frac{\overline{C, A \vdash D, D^n} (D) \text{ Axiom}}{(A) \text{ Cut}}}{C \vdash D, D^n}$$

reduces to

$$\frac{\frac{\overline{C \vdash C, B, D, D^n} (C) \text{ Axiom}}{C \vdash B, D, D^n} \frac{\overline{C, C \vdash B, D, D^n} (C) \text{ Axiom}}{(C) \text{ Cut}}}{C \vdash D, D^n} \frac{\overline{C, B \vdash D, D^n} (D) \text{ Axiom}}{(B) \text{ Cut}}$$

that reduces to

$$\frac{\overline{C \vdash B, D, D^n} (C) \text{ Axiom}}{C \vdash D, D^n} \frac{\overline{C, B \vdash D, D^n} (D) \text{ Axiom}}{(B) \text{ Cut}}$$

that reduces to

$$\frac{\frac{\overline{C \vdash A, D, D, D^n} (C) \text{ Axiom}}{C \vdash D, D, D^n} \frac{\overline{C, A \vdash D, D, D^n} (D) \text{ Axiom}}{(A) \text{ Cut}}}{C \vdash D, D^n} \frac{\overline{C, D \vdash D, D^n} (D) \text{ Axiom}}{(D) \text{ Cut}}$$

that contains the initial proof (for  $n + 1$ ) as a sub-proof. The proof reduction algorithm loops on this proof, replacing a cut on the proposition  $A$  by one on the proposition  $B$  and vice versa.

But this situation cannot occur if the rewrite system is terminating.

**Proposition 7.** *If the rewrite system is terminating then the proof reduction algorithm of definition 2 is terminating.*

*Proof.* As the rewrite system is terminating, its reduction ordering is well-founded and thus so is the multiset extension of this ordering.

At each step, the algorithm of definition 2 replaces a cut with the cut proposition  $C$  by two cuts with the cut propositions  $C'_1$  and  $C'_2$  where  $C \rightarrow^1 C'_1$  and  $C \rightarrow^1 C'_2$ . Thus, the multiset of cut propositions in the proof decreases for the multiset extension of the reduction ordering of the rewrite system. Therefore, the proof reduction algorithm terminates.

**Corollary 1.** *If a rewrite system is locally confluent and terminating then the proof reduction algorithm of definition 2 always succeeds.*

**Corollary 2.** *If a rewrite system is locally confluent and terminating then asymmetric deduction modulo this rewrite system has the cut elimination property.*

**Corollary 3 (Newman's theorem [9]).** *If a rewrite system is locally confluent and terminating then it is confluent.*

*Remark 1.* We have seen that confluence is equivalent to cut elimination and that local confluence and termination imply normalization (i.e. termination of the proof reduction algorithm) and hence cut elimination. But notice that confluence alone does not imply normalization. Indeed, if we add to the rewrite system of example 2 the rules  $c \rightarrow e$  and  $d \rightarrow e$ , we obtain confluence and thus cut elimination, but the counterexample to normalization still holds. We obtain this way an example of system that has the cut elimination property, but not the normalization property. Hence, in asymmetric deduction modulo, normalization is a stronger property than cut elimination.

In [9], Newman proves more than confluence (cut elimination) for terminating locally confluent rewrite systems, as he proves normalization, i.e. the termination of an algorithm that reduces peaks step by step in conversion sequences.

## 5 Cut elimination in full asymmetric deduction modulo

We consider now full asymmetric deduction modulo, and we prove that cut elimination is still equivalent to the confluence of the rewrite system.

**Proposition 8.** *The cut rule is redundant in asymmetric deduction modulo a rewrite system if and only if this rewrite system is confluent.*

*Proof.* The fact that cut elimination implies confluence is easy as cut elimination implies cut elimination for the atomic case and hence confluence.

To prove that confluence implies cut elimination, we have to extend the proof of proposition 5 to the non atomic case.

Without loss of generality, we can restrict to proofs where the axiom rule is used on atomic propositions only.

We follow the cut elimination proof of [8]. When we have a proof containing a cut

$$\frac{\frac{\pi_1}{\Gamma, C_1 \vdash \Delta} \quad \frac{\pi_2}{\Gamma \vdash C_2, \Delta}}{\Gamma \vdash \Delta} (C) \text{ Cut } C_1 \leftarrow^* C \rightarrow^* C_2$$

then we show that from  $\pi_1$  and  $\pi_2$ , we can reconstruct a proof of  $\Gamma \vdash \Delta$  introducing cuts on smaller propositions than  $C$  (i.e. propositions with fewer connectors and quantifiers).

More generally, we prove, by induction on the structure of  $\pi_1$  and  $\pi_2$ , that from a proof  $\pi_1$  of  $\Gamma, \overline{C}_1 \vdash \Delta$  and  $\pi_2$  of  $\Gamma \vdash \overline{C}_2, \Delta$ , where  $\overline{C}_1$  and  $\overline{C}_2$  are multisets of reducts of some proposition  $C$ , we can reconstruct a proof  $\Gamma \vdash \Delta$ . Notice that, as the rewrite system rewrites terms only, rewriting does not change the logical structure of a proposition (i.e. an atomic proposition only rewrites to an atomic proposition, a conjunction to a conjunction, ...)

There are several cases to consider.

- If the last rule of  $\pi_1$  or the last rule of  $\pi_2$  is a structural rule, then we apply the induction hypothesis.

- If the last rule of  $\pi_1$  or the last rule of  $\pi_2$  is a logical rule on a proposition in  $\Gamma$  or  $\Delta$ , then we apply the induction hypothesis.
- If the last rule of  $\pi_1$  or the last rule of  $\pi_2$  is an axiom rule on propositions in  $\Gamma$  and  $\Delta$ , then  $\Gamma$  and  $\Delta$  contain two propositions that have a common reduct  $C$  and we take the proof

$$\frac{}{\Gamma \vdash \Delta} (C) \text{ Axiom}$$

- The key case in the proof of [8] is when both  $\pi_1$  and  $\pi_2$  end with a logical rule on a proposition in  $\overline{C}_1$  and  $\overline{C}_2$ . For instance, if  $C$  has the form  $A \wedge B$ ,  $A_1 \wedge B_1 \leftarrow^* A \wedge B \rightarrow^* A_2 \wedge B_2$  and the proofs  $\pi_1$  and  $\pi_2$  have the form

$$\frac{\frac{\rho_1}{\Gamma, A'_1, B'_1 \vdash \Delta}}{\Gamma, A_1 \wedge B_1 \vdash \Delta} \wedge\text{-left}$$

with  $A_1 \wedge B_1 \rightarrow^* A'_1 \wedge B'_1$  and

$$\frac{\frac{\rho_2}{\Gamma \vdash A'_2, \Delta} \quad \frac{\rho_3}{\Gamma \vdash B'_2, \Delta}}{\Gamma \vdash A_2 \wedge B_2, \Delta} \wedge\text{-right}$$

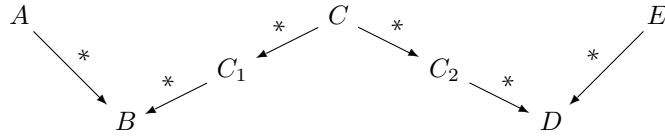
with  $A_2 \wedge B_2 \rightarrow^* A'_2 \wedge B'_2$ .

In this case, we have  $A'_1 \wedge B'_1 \leftarrow^* A_1 \wedge B_1 \leftarrow^* A \wedge B \rightarrow^* A_2 \wedge B_2 \rightarrow^* A'_2 \wedge B'_2$ , thus  $A'_1 \leftarrow^* A \rightarrow^* A'_2$  and  $B'_1 \leftarrow^* B \rightarrow^* B'_2$  and we reconstruct the proof

$$\frac{\frac{\rho_1}{\Gamma, B'_1, A'_1 \vdash \Delta} \quad \frac{\frac{\rho_2}{\Gamma \vdash A'_2, \Delta}}{\Gamma, B'_1 \vdash A'_2, \Delta} \text{ weak-left}}{\Gamma, B'_1 \vdash \Delta} (A) \text{ Cut} \quad \frac{\rho_3}{\Gamma \vdash B'_2, \Delta} (B) \text{ Cut}}{\Gamma \vdash \Delta}$$

The case of the other connectors and quantifiers is similar.

- The new case in asymmetric deduction modulo is when the last rule of both proofs is an axiom rule involving a proposition in  $\overline{C}_1$  and  $\overline{C}_2$ . Notice that the proposition  $C$  is atomic in this case, as we have restricted the Axiom rule to apply on atomic propositions only. Thus,  $\Gamma$  contains a proposition  $A$  that has a common reduct  $B$  with  $C_1$  in  $\overline{C}_1$  and  $\Delta$  contains a proposition  $E$  that has a common reduct  $D$  with  $C_2$  in  $\overline{C}_2$ . We have



This is the case where we use confluence to obtain that  $A$  and  $E$  have a common reduct  $C'$  and we take the proof

$$\frac{}{\Gamma \vdash \Delta} (C') \text{ Axiom}$$

*Remark 2.* This proof suggests a cut elimination algorithm that integrates the cut elimination algorithm of sequent calculus and Newman's algorithm: it behaves like the latter for atomic cuts and like the former for non atomic ones. Again, we need the termination of the rewrite system to prove the termination of this cut elimination algorithm.

*Remark 3.* If we consider now rules directly rewriting atomic propositions to non atomic ones: for instance a rule like  $A \rightarrow B \wedge \neg A$  then confluence [3, 6], and even confluence and termination [7], do not imply cut elimination anymore. Some propositions have proofs using the cut rule but no cut free proof.

In this case, confluence is not a sufficient analyticity condition anymore. As a consequence, with term rewrite systems, confluence is a sufficient condition for the completeness of proof search methods, such as equational resolution (i.e. resolution where some equational axioms are dropped and unification is replaced by equational unification), but with proposition rewrite systems, confluence is not a sufficient condition for the completeness of resolution modulo and this condition must be replaced by cut elimination (see [4] for a discussion on this point).

## Conclusion

When a congruence is defined by a term rewrite system, the confluence of this rewrite system and the cut elimination property for asymmetric deduction modulo this system coincide and analyticity can be defined either using one property of the other.

When the rewrite system is also terminating, then asymmetric deduction modulo not only verifies cut elimination, but also normalization.

But when a congruence is defined by rules directly rewriting atomic propositions, confluence is not a sufficient analyticity condition anymore and must be replaced by cut elimination.

## Acknowledgments

I want to thank Thérèse Hardin, Claude Kirchner and Benjamin Werner for comments on a draft of this paper and for many discussions about deduction modulo, confluence, termination and cut elimination.

## References

1. L. Bachmair and N. Dershowitz, Equational Inference, Canonical Proofs, and Proof Orderings. *Journal of the ACM* 41(2), 1994, pp. 236-276
2. A. Church and J.B. Rosser, Some properties of conversion. *Transactions of the American Mathematical Society*, 39, 1936, pp. 472-482.
3. M. Crabbé. Non-normalisation de ZF. *Manuscript*, 1974.

4. G. Dowek, Axioms vs. rewrite rules: from completeness to cut elimination. H. Kirchner and Ch. Ringeissen (Eds.) *Frontiers of Combining Systems*, Lecture Notes in Artificial Intelligence 1794, Springer-Verlag, 2000, pp. 62-72.
5. G. Dowek, Th. Hardin, and C. Kirchner. Theorem proving modulo. Rapport de Recherche 3400, Institut National de Recherche en Informatique et en Automatique, 1998. *Journal of Automated Reasoning* (to appear).
6. G. Dowek and B. Werner. Proof normalization modulo. In *Types for proofs and programs 98*, *Lecture Notes in Computer Science* 1657, Springer-Verlag, 1999, pp. 62-77.
7. G. Dowek and B. Werner. An inconsistent theory modulo defined by a confluent and terminating rewrite system. *Manuscript*, 2000.
8. J.-Y. Girard, Y. Lafont, and P. Taylor. *Proofs and Types*. Cambridge University Press, 1989.
9. M.H.A. Newman, On theories with a combinatorial definition of “equivalence”. *Annals of Mathematics*, 43, 2, 1942, pp. 223-243.