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# What do we know when we know that a theory is consistent ?

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**Abstract.** Given a first-order theory and a proof that it is consistent, can we design a proof-search method for this theory that fails in finite time when it attempts to prove the formula  $\perp$ ?

## 1 Searching for proofs in a theory

### 1.1 Who knows that higher-order logic is consistent?

It is well known that higher-order logic can be presented as a first-order theory, *i.e.* that there exists a first-order theory  $H$  and a function  $\Phi$  translating closed formulas of higher-order logic to closed formulas of the language of  $H$  such that the sequent  $\vdash A$  is provable in higher-order logic if and only if the sequent  $H \vdash \Phi A$  is provable in first-order logic (see, for instance, [?]). Thus, instead of using a proof-search method specially designed for higher-order logic, such as higher-order resolution [?,?], it is possible to use a first-order method, such as resolution, to search for proofs in higher-order logic.

However, this reduction is inefficient. Indeed, if we attempt to prove the formula  $\perp$  with higher-order resolution, the clausal form of the formula  $\perp$  is the empty set of clauses, from these clauses, we can apply neither the higher-order resolution rule that requires two clauses to be applied, nor any other rule of higher-order resolution, that all require at least one clause. Thus, this attempt to prove the formula  $\perp$  fails immediately. In contrast, the axioms of  $H$  give an infinite number of opportunities to apply the resolution rule and thus when searching for a proof of  $H \vdash \perp$ , the search space is infinite.

Thus, we can say that higher-order resolution “knows” that higher-order logic is consistent, because an attempt to prove the formula  $\perp$  fails in finite time, while first-order resolution does not.

### 1.2 A proof-search method for the theory $H$

There is an link between higher-order logic and higher-order resolution and another link between higher-order logic and the first-order theory  $H$ . But can we establish a direct link between higher-order resolution and the theory  $H$ , without referring to higher-order logic?

The answer is positive because the translation  $\Phi$  can be inverted: there exists a function  $\Psi$  translating closed formulas of the language of  $H$  to closed formulas of higher-order logic such that the sequent  $H \vdash B$  is provable in first-order logic if and only if the sequent  $\vdash \Psi B$  is provable in higher-order logic. Thus, a way to search for a proof of a sequent  $H \vdash B$  is to apply higher-order resolution to the sequent  $\vdash \Psi B$ . Thus, independently of higher-order logic, higher-order resolution can be seen as special proof-search method for a the first-order theory  $H$ .

As  $\Psi \perp = \perp$  this first-order proof-search method immediately fails when attempting to prove the sequent  $H \vdash \perp$ . Thus, this method is much more efficient than applying first-order resolution to the sequent  $H \vdash \perp$  as it “knows” that the theory  $H$  is consistent.

### 1.3 A proof-search method for a theory $T$

Can we generalize this to other theories than  $H$ ? Given an arbitrary first-order theory  $T$  and a proof that  $T$  is consistent, can we always *build in* the theory  $T$ , *i.e.* exploit the consistency of  $T$  to design a proof-search method that fails in finite time when required to prove the formula  $\perp$ ?

Of course, as we are not interested in the trivial solution that first tests if the formula to be proven is  $\perp$  and then applies any method when it is not, we have to restrict to proof-search methods that do not mention the formula  $\perp$ .

It is clear that the consistency of the theory  $T$  is a necessary condition for such a method to exist: if  $T$  is inconsistent, a complete proof-search method should succeed, and not fail, when attempting to prove the formula  $\perp$ . The main problem is to know if this hypothesis is sufficient.

## 2 Resolution modulo

### 2.1 Resolution modulo

*Resolution modulo* is a proof-search method for first-order logic that generalizes higher-order resolution to other theories than the theory  $H$ .

Some axioms of the theory  $H$  are equational axioms. How to build in equational axioms is well-known: we drop equational axioms and we replace unification by equational unification modulo these axioms (see, for instance, [?,?]). Equational unification modulo the equational axioms of  $H$  is called *higher-order unification*.

From a proof-theoretical point of view, this amounts to define a congruence on formulas generated by the equational axioms and to identify congruent formulas in proofs. For instance, if we identify the terms  $2 + 2$  and  $4$ , we do not need the axiom  $2 + 2 = 4$  that is congruent to  $4 = 4$ , but when we substitute the term  $2$  for the variable  $x$  in the term  $x + 2$ , we obtain the term  $4$ . We have called *deduction modulo* the system obtained by identifying congruent formulas in proofs.

But not all axioms can be expressed as equational axioms. For instance, if the axiom of arithmetic  $S(x) = S(y) \Rightarrow x = y$  can be replaced by the equivalent

equational axiom  $Pred(S(x)) = x$ , the axiom  $\neg 0 = S(x)$ , that has no one-point model, cannot be replaced by an equational axiom.

Thus, we have extended deduction modulo by identifying some atomic formulas with not atomic ones. For instance, identifying formulas with the congruence generated by the rewrite rules  $Null(0) \longrightarrow \top$  and  $Null(S(x)) \longrightarrow \perp$  is equivalent to having the axiom  $\neg 0 = S(x)$ .

When we have such rewrite rules operating directly on formulas, equational resolution has to be extended. Besides the resolution rule, we need to add another rule called *Extended narrowing*. For instance, if we identify the formula  $P(1)$  with  $\neg P(0)$ , we can refute the set of clauses  $\{\neg P(x)\}$ , but to do so, we have to be able to substitute the term 1 for the variable  $x$  in the clause  $\neg P(x)$ , deduce the clause  $P(0)$  and conclude with the resolution rule. More generally, the *Extended narrowing* rule allows to narrow any atom in a clause with a propositional rewrite rule. The proposition obtained this way must then be put back in clausal form. Equational resolution extended with this rule is called ENAR — *Extended Narrowing and Resolution* — or *resolution modulo* for short.

When we orient the axioms of  $H$  as rewrite rules and use resolution modulo, we obtain exactly higher-order resolution.

## 2.2 Proving completeness

Proving the completeness of higher-order resolution, and more generally of resolution modulo, is not very easy. Indeed higher-order resolution knows that higher-order logic is consistent, *i.e.* it fails in finite time when attempting to prove the formula  $\perp$ . Thus, a finitary argument shows that the completeness of higher-order resolution implies the consistency of higher-order logic, and by Gödel's second incompleteness theorem, the completeness of higher-order resolution cannot be proved in higher-order logic itself. This explains that some strong proof-theoretical results are needed to prove the completeness of higher-order resolution, at least the consistency of higher-order logic. The completeness proof given by Andrews and Huet [?, ?, ?] uses a result stronger than consistency: the cut elimination theorem for higher-order logic.

In the same way, the completeness of resolution modulo rests upon the fact that deduction modulo the considered congruence has the cut elimination property. Indeed, when the congruence is defined by rules rewriting atomic formulas to non-atomic ones, deduction modulo this congruence may have the cut elimination property or not. For instance, deduction modulo the rule  $P \longrightarrow Q \wedge R$  has the cut elimination property, but not deduction modulo the rule  $P \longrightarrow Q \wedge \neg P$  [?] and resolution modulo this second rule is incomplete.

Is it possible to weaken this cut elimination hypothesis and require, for instance only consistency? The answer is negative: the rule  $P \longrightarrow Q \wedge \neg P$  is consistent, but resolution modulo this rule is incomplete. More generally, Hermant [?] has proved that the completeness of resolution modulo a congruence implies cut elimination for deduction modulo this congruence.

### 2.3 A resolution strategy

At least in the propositional case, resolution modulo can be seen as a strategy of resolution [?].

For instance, consider the rule  $P \longrightarrow Q \wedge R$ . The *Extended narrowing* rule allows to replace an atom  $P$  by  $Q \wedge R$  and to put the formula obtained this way back in clausal form. With this rule, from a clause of the form  $C \vee P$  we can derive the clauses  $C \vee Q$  and  $C \vee R$  and from a clause of the form  $C \vee \neg P$  we can derive the clause  $C \vee \neg Q \vee \neg R$ .

We can mimic this rule by adding three clauses  $\neg \underline{P} \vee Q$ ,  $\neg \underline{P} \vee R$ ,  $\underline{P} \vee \neg Q \vee \neg R$  and restricting the application of the resolution rules as follows: (1) we cannot apply the resolution rule using two clauses of the this set (2) when we apply the resolution rule using one clause of this set the eliminated atom must be the underlined atom. Notice that this set of clauses is exactly the clausal form of the formula  $\underline{P} \Leftrightarrow (Q \wedge R)$ . This strategy is in the same spirit as hyper-resolution, but the details are different.

If we apply the same method with the formula  $\underline{P} \Leftrightarrow (Q \wedge \neg P)$ , we obtain the three clauses  $\neg \underline{P} \vee Q$ ,  $\neg \underline{P} \vee \neg P$ ,  $\underline{P} \vee \neg Q \vee P$  with the same restriction and, like resolution modulo, this strategy is incomplete: it does not refute the formula  $Q$ .

The fact that this strategy is complete for one system but not for the other is a consequence of the fact that deduction modulo the rule  $P \longrightarrow Q \wedge R$  has the cut elimination property, but not deduction modulo the rule  $P \longrightarrow Q \wedge \neg P$ .

Understanding resolution modulo as a resolution strategy seems to be more difficult when we have quantifiers. Indeed, after narrowing an atom with a rewrite rule, we have to put the formula back in clausal form and this involves skolemization.

### 3 From consistency to cut elimination

We have seen in section ?? that the theory  $\mathbb{T} = \{P \Leftrightarrow (Q \wedge \neg P)\}$  is consistent, but that resolution modulo the rule  $P \longrightarrow (Q \wedge \neg P)$  is incomplete.

Thus, it seems that the consistency hypothesis is not sufficient to design a complete proof-search method that knows that the theory is consistent. However the rule  $P \longrightarrow (Q \wedge \neg P)$  is only one among the many rewrite systems that allow to express the theory  $\mathbb{T}$  in deduction modulo. Indeed, the formula  $P \Leftrightarrow (Q \wedge \neg P)$  is equivalent to  $\neg P \wedge \neg Q$  and another solution is to take the rules  $P \longrightarrow \perp$  and  $Q \longrightarrow \perp$ . Deduction modulo this rewrite system has the cut elimination property and hence resolution modulo this rewrite system is complete. In other words, the resolution strategy above with the clauses  $\neg \underline{P}$ ,  $\neg \underline{Q}$  is complete and knows that the theory is consistent.

Thus, the goal should not be to prove that if deduction modulo a congruence is consistent then it has the cut elimination property, because this is obviously false, but to prove that a consistent set of axioms can be transformed into a congruence in such a way that deduction modulo this congruence has the cut elimination property. To stress the link with the project of Knuth and Bendix [?], we call this transformation an *orientation* of the axioms.

A first step in this direction has been made in [?] following an idea of [?]. Any consistent theory in propositional logic can be transformed into a polarized rewrite system such that deduction modulo this rewrite system has the cut elimination property.

To do so, we first put the theory  $T$  in clausal form and consider a model  $\nu$  of this theory (*i.e.* a line of a truth table).

We pick a clause. In this clause there is either a literal of the form  $P$  such that  $\nu(P) = 1$  or a literal of the form  $\neg Q$  such that  $\nu(Q) = 0$ .

In the first case, we pick all the clauses where  $P$  occurs positively  $P \vee A_1, \dots, P \vee A_n$  and replace these clauses by the formula  $(\neg A_1 \vee \dots \vee \neg A_n) \Rightarrow P$ . In the second, we pick all the clauses where  $Q$  occurs negatively  $\neg Q \vee B_1, \dots, \neg Q \vee B_n$  and replace these clauses by the formula  $Q \Rightarrow (B_1 \wedge \dots \wedge B_n)$ . We repeat this process until there are no clauses left. We obtain this way a set of axioms of the form  $A_i \Rightarrow P_i$  and  $Q_j \Rightarrow B_j$  such that the atomic formulas  $P_i$ 's and  $Q_j$ 's are disjoint.

The next step is to transform these formulas into rewrite rules and this is difficult because they are implications and not equivalences. But, this is possible if we extend deduction modulo allowing some rules to apply only to positive atoms and others to apply only to negative atoms. This extension of deduction modulo is called *polarized deduction modulo*. We get the rules  $P_i \longrightarrow_+ A_i$  and  $Q_j \longrightarrow_- B_j$ . Using the fact that the  $P_i$ 's and the  $Q_j$ 's are disjoint, it is not difficult to prove cut elimination for deduction modulo these rules [?].

So, this result is only a partial success because resolution modulo is defined for non-polarized rewrite systems and orientation yields a polarized rewrite system. There may be two ways to bridge the gap: the first is to extend resolution modulo to polarized rewrite systems. There is no reason why this should not be possible, but this requires some work. A more ambitious goal is to produce a non-polarized rewrite system when orienting the axioms. Indeed, the axiom  $P \Rightarrow A$  can be oriented either as the polarized rewrite rule  $P \longrightarrow_- A$  or as the non-polarized rule  $P \longrightarrow (P \wedge A)$ , and similarly the axiom  $A \Rightarrow P$  can be oriented as the rule  $P \longrightarrow (P \vee A)$ . But the difficulty here is to prove that deduction modulo the rewrite system obtained this way has the cut elimination property.

Bridging this gap would solve our initial problem for the propositional case. Starting from a consistent theory, we would build a model of this theory, orient it using this model, *i.e.* define a congruence and resolution modulo this congruence would be a complete proof search method for this theory that knows that the theory is consistent.

But, this would solve only the propositional case and for full first-order logic, everything remains to be done.

We have started this note with a problem in automated deduction: given a theory  $T$  and a proof that it is consistent, can we design a complete proof-search method for  $T$  that knows that  $T$  is consistent? We have seen that this problem boils down to a problem in proof theory: given a theory  $T$  and a proof that it

is consistent, can we orient the theory into a congruence such that deduction modulo this congruence has the cut elimination property?

We seem to be quite far from a full solution to this problem, although the solution for the propositional case seems to be quite close.

Some arguments however lead to conjecture a positive answer to this problem: first the fact that the problem seems almost solved for propositional logic, then the fact that several theories such as arithmetic, higher-order logic, and some version of set theory have been oriented. Finally, we do not have examples of theories that can be proved to be non orientable (although some counter examples exist for intuitionistic logic). However, some theories still resist to being oriented, for instance higher-order logic with extensionality or set theory with the replacement scheme.

A positive answer to this problem could have some impact on automated theorem proving, as in automated theorem proving, like in logic in general, axioms are often a burden.

## References

1. P.B. Andrews, Resolution in type theory, *The Journal of Symbolic Logic*, 36, 1971, pp. 414-432.
2. G. Dowek, Axioms vs. rewrite rules: from completeness to cut elimination, H. Kirchner and Ch. Ringeissen (Eds.) *Frontiers of Combining Systems*, Lecture Notes in Artificial Intelligence 1794, Springer-Verlag, 2000, pp. 62-72.
3. G. Dowek, What is a theory?, H. Alt, A. Ferreira (Eds.), *Symposium on Theoretical Aspects of Computer Science*, Lecture Notes in Computer Science 2285, Springer-Verlag, 2002, pp. 50-64.
4. G. Dowek, Th. Hardin, and C. Kirchner, HOL-lambda-sigma: an intentional first-order expression of higher-order logic, *Mathematical Structures in Computer Science*, 11, 2001, pp. 1-25.
5. G. Dowek, Th. Hardin, and C. Kirchner, Theorem proving modulo, *Journal of Automated Reasoning*, 31, 2003, pp. 33-72.
6. G. Dowek and B. Werner, Proof normalization modulo, *The Journal of Symbolic Logic*, 68, 4, 2003, pp. 1289-1316.
7. O. Hermant, Dédution Modulo et Élimination des Coupures : Une approche syntaxique, *Mémoire de DEA*, 2002.
8. G. Huet, Constrained Resolution A Complete Method for Higher Order Logic, *Ph.D Thesis*, Case Western Reserve University, 1972.
9. G. Huet, A Mechanisation of Type Theory, *Third International Joint Conference on Artificial Inteligenge*, 1973, pp. 139-146.
10. D.E. Knuth and P.B. Bendix. Simple word problems in universal algebras. J. Leech (Ed.), *Computational Problems in Abstract Algebra*, Pergamon Press, 1970, pp. 263-297.
11. S. Negri and J. Von Plato. Cut elimination in the presence of axioms. *The Bulletin of Symbolic Logic*, 4, 4, 1998, pp. 418-435.
12. G. Peterson, and M. E. Stickel, Complete Sets of Reductions for Some Equational Theories, *Journal of the ACM* 28, 1981, pp. 233-264.
13. G. Plotkin, Building-in equational theories, *Machine Intelligence* 7, 1972, pp. 73-90.