

Polynomial Optimization, Moments, and Applications

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(Editors)*

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Introduction

Non-linear global optimization problems are at the heart of many technological challenges related to environment, infrastructure management, resource control, finance, and more. Mathematically speaking, they have the following simple form:

$$\inf_{x \in S} f(x) \tag{1}$$

where f , the *objective function*, is non-linear and S , the *feasible set*, can be described by equality or inequality constraints.

The last few decades have witnessed a significant advancement in the methods used to address such computational problems, dealing with non-linearity and, at the same time, providing global solutions. Traditional optimization techniques rely on convergence processes towards a local optimum of the problem, meaning that their performance is greatly influenced by the initial point. As a result, no guarantee can be given on the quality of the output.

In contrast, the new state-of-the-art approaches utilize the so-called moment method to solve Polynomial Optimization Problems (POP) globally. This method is based on exploiting semi-definite programming and computer algebra techniques. This set of techniques also enables computation of certificates for global optima, which is critical in many problems. The high level reliability of this method is achieved through a strong interplay between convex geometry, numerical analysis, algebraic techniques to handle non-linearity, and certification.

The core of the approach is to replace problem (1) with

$$\begin{aligned} \sup \lambda \\ \text{s.t. } \lambda \in \mathbb{R} \\ f(x) - \lambda \in \text{pos}(S), \end{aligned} \tag{2}$$

where $\text{pos}(S)$ is the set of functions positive on S . This is a *convex cone* of functions, since the sum of two positive functions on S is positive on S and $\text{pos}(S)$ is invariant by scaling by a positive scalar. The dual formulation reads as

$$\begin{aligned}
& \inf \Lambda(f) && (3) \\
& \text{s.t. } \Lambda(1) = 1 \\
& \Lambda \in \mathcal{M}(S) = \text{pos}(S)^\vee,
\end{aligned}$$

where $\mathcal{M}(S) = \text{pos}(S)^\vee$, the cone dual to $\text{pos}(S)$, is the *cone of measures* supported on S . In many problems, the objective function f is a polynomial and one take for $\text{pos}(S)$ the convex cone of polynomials, which are positive on S . By Riesz-Haviland's theorem, its dual cone $\mathcal{M}(S)$ is the cone of Radon measures supported on S .

As positive functions or polynomials are infinite-dimensional and difficult to characterize effectively, they are approximated by sum of squares of polynomials (SOS). These are represented by Symmetric Positive Semi-Definite (SPD) matrices. Similarly, polynomials in $\text{pos}(S)$ can be approximated by elements of truncated quadratic modules, which are sums of convex cones in finite-dimensional spaces. Since the cone of SPD matrices is self-dual, the dual problem involves cones of positive pseudo-moment sequences, which can be represented by the intersection of SPD cones. This leads to a hierarchy of so-called *SOS-Moment relaxations*, also known as *Lasserre's relaxation*. Following this approach, solving the difficult problem (1) becomes solving a sequence of convex finite-dimensional problems with SPD matrices, i.e., semidefinite optimization problems (SDP).

This approach has revolutionized the field of global optimization, going beyond the traditional paradigms of mathematical optimization by exploiting new advances in algebra and convex geometry. Specifically, over the last four years, active research at the intersection of algebra, geometry, and computer science has developed these approaches in the context of the European network POEMA (<http://poema-network.eu/>), resulting in significant scientific and technological advances. This research has stimulated the exchange of interdisciplinary and intersectoral knowledge between algebraists, geometers, computer scientists, and industrial actors facing real-world optimization problems. This book gathers eight high-quality chapters on these hot topics and presents aspects of the results of this research activity, making them accessible to a broad audience:

- Chapter 1, authored by Jean B. Lasserre, establishes an unexpected connection between two seemingly disparate fields: polynomial optimization and real algebraic geometry on the one hand, and the theory of approximation and orthogonal polynomials on the other.
- Chapter 2, written by Thorsten Theobald provides an introduction into the concepts of relative entropy programming in the context of polynomial and signomial optimization.
- In Chapter 3, Philippe Moustrou, Cordian Riener, and Hugues Verdure present techniques from representation theory and invariant theory and expose how these can be used algorithmically to leverage symmetries in polynomial optimization.
- Chapter 4, by Luis Felipe Vargas and Monique Laurent, focuses on the cone of copositive matrices and explores the development and analysis of conic inner approximations, with a specific focus on the stable set problem.

- Chapter 5, authored by Andries Steenkamp, delves into matrix factorization, the various notions of ranks associated with these factorizations, and the approximation of these ranks.
- In Chapter 6, Soodeh Habibi, Michal Kočvara, and Bernard Mourrain demonstrate applications of polynomial optimization techniques to geometric modeling problems.
- Chapter 7, written by Han Wang, Kostas Margellos, and Antonis Papachristodoulou, discusses the concept of safety for control systems in both continuous and discrete time forms.
- Finally, Chapter 8, authored by Simon Telen, discusses the theory and algorithmic practice of solving polynomial equations.

The selection of topics delivered by experts from various disciplines aims to provide a comprehensive overview of the cutting-edge research on algebraic and geometric optimization techniques and may be a valuable resource for researchers, practitioners, and graduate students interested in global optimization and its applications.